Signed graphs of rank five

Zoran Stanić

Signed graphs of rank five

Zoran Stanić*
Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Belgrade, Serbia

Technical report, July 2026

Abstract

We characterize signed graphs of rank five by means of maximal basic extensions (MBEs) for the eigenvalue 0. The essential step is the determination, up to switching isomorphism, of all connected MBEs arising from connected non-singular signed graphs of order five. Applying the star complement algorithm, as implemented in the release of SCL 2.3, gives precisely 28 switching classes, forming 14 pairs under graph negation. These MBEs yield the full rank-five characterization: after isolated vertices are deleted, a signed graph of rank five is either a twin extension of a rank-five induced subgraph of one of the listed MBEs, or it has two non-trivial components of ranks two and three. The complete input, intermediate data and final representatives are recorded so that the finite computation can be independently checked.

1 Introduction and main achievement

The *rank* of a signed graph is the rank of its adjacency matrix, and signed graphs are considered up to switching isomorphism. This report completes the rank-five case suggested by the characterization of signed graphs with rank at most four in [1]. Its principal result is a characterization of all signed graphs of rank five, obtained through the determination of the connected maximal basic extensions for the eigenvalue 0.

The connected case is the essential part of the problem. A connected signed graph of rank five has a connected star complement for 0 of order five, by the corresponding results of [3]. Consequently, the connected rank-five graphs are controlled by the connected non-singular signed graphs of order five and their maximal basic extensions (MBEs). Once this connected case is known, the disconnected case follows from additivity of rank over components and the existing descriptions of connected signed graphs of ranks two and three [1]. Indeed, after isolated vertices are removed, the only possible decomposition of rank five into the ranks of two or more non-trivial components is $5 = 2 + 3$.

The role of an MBE is therefore structural, rather than merely computational. For $\mu = 0$, a basic extension contains neither isolated extension vertices nor prohibited twin vertices. Every basic extension is an induced subgraph of an MBE, while an arbitrary connected extension is recovered from a basic one by adding twin vertices. Thus a finite list of connected MBEs describes all connected signed graphs of the prescribed rank, including the infinite families obtained by duplicating twin vertices. Theorem 2 below makes this reduction into an explicit characterization of every signed graph of rank five.

2 Star complements, basic extensions and MBEs

We recall the part of the star complement technique needed here; a systematic treatment for signed graphs and, more generally, Hermitian matrices is given in [3]. If $\dot{G}$ has order n and the

*E-mail: zstanic@matf.bg.ac.rs. ORCID: 0000-0002-4949-4203.

eigenvalue μ has multiplicity $n-k$, an induced subgraph $\dot{H}$ of order k not having μ as an eigenvalue is a *star complement* for μ. The definition also includes the case in which the multiplicity of μ is zero. In this case, the star set is empty and $\dot{G}$ itself is regarded as its star complement for μ; this is called the trivial star complement. The vertices outside $\dot{H}$ form the corresponding *star set*. In particular, a star complement for 0 in a rank-five signed graph has order five and has a non-singular adjacency matrix. In this context, $\dot{G}$ is called an *extension* of $\dot{H}$.

Two distinct vertices u and v are called *twins* if they share the same open neighbourhood and one of the following holds: the signs of uw and vw agree for every vertex w in their common neighbourhood, or they are opposite for every such vertex w.

For $\mu = 0$, an extension $\dot{G}$ of $\dot{H}$ is called a *basic extension* if no vertex of its star set is isolated in $\dot{G}$ or is a twin of another vertex of $\dot{G}$. A basic extension is *maximal* if it is not a proper induced subgraph of any other basic extension of $\dot{H}$. Such an extension is called a *maximal basic extension*.

Starting from a connected star complement $\dot{H}$ for $\mu = 0$, the star complement technique constructs its extensions by encoding the signed neighbourhood of every new vertex by a $\{0, \pm1\}$-vector $\mathbf{b}$. For $\mu = 0$, the reconstruction conditions are expressed through the form

$$\langle \mathbf{b}, \mathbf{c} \rangle = \mathbf{b}^{\intercal}(-A_{\dot{H}})^{-1}\mathbf{c}.$$

A non-zero vector $\mathbf{b} \in \{0, \pm1\}^k$ is admissible if $\langle \mathbf{b}, \mathbf{b} \rangle = 0$. Two admissible vectors $\mathbf{b}$ and $\mathbf{c}$ are compatible if $\langle \mathbf{b}, \mathbf{c} \rangle \in \{0, \pm1\}$; their corresponding entry of the adjacency matrix is $-\langle \mathbf{b}, \mathbf{c} \rangle$. Opposite representatives and the vectors that would create prohibited twins are removed. The retained vectors are then made the vertices of a compatibility graph, with adjacency recording pairwise compatibility. A clique gives a simultaneous extension, and its maximality is precisely what produces an MBE. If there are no retained vectors, the empty clique represents the self-extended star complement.

This is the specialization to $\mu = 0$ of the construction algorithm described in [3]. It reduces the theoretical problem to two finite tasks: determining the maximal cliques of the compatibility graphs and then identifying the resulting signed graphs up to switching isomorphism. The software used below implements these steps, but the reduction itself and the significance of the output come from the preceding star complement theory.

3 Rank-five computation

The catalogue [2] contains 79 connected switching classes of signed graphs of order five. Exactly 46 have non-singular adjacency matrix, and hence qualify as star complements for 0. They form 23 pairs under graph negation, so the algorithm need only be applied to the 23 representatives in Appendix A; the negations of the resulting extensions cover the other input classes.

For each selected star complement $\dot{H}$, the release of SCL 2.3 [4] generated the admissible vectors, removed the vectors producing prohibited twins, constructed the compatibility graph while retaining its isolated vertices, and found all maximal cliques. If the compatibility graph had order zero, its empty vertex set was retained as its unique maximal clique and produced the self-extended input graph.

The retained vectors appear in Appendix B. Appendix C records the 23 compatibility graphs by their adjacency matrices, using the vector order from Appendix B, and Appendix D gives the complete maximal-clique lists. Collectively, the compatibility graphs produced 442 non-empty maximal-clique outputs and 3 empty-clique outputs. The raw extensions are described by these complete intermediate records and are not printed individually.

Finally, Appendix E records the principal computational output: the complete list of the 28 pairwise non-switching-isomorphic connected maximal basic extensions of rank five obtained after the isomorphism reduction.

3.1 Exact verification

Generation by SCL uses its documented numerical tolerance, but every accepted object was post-verified using exact integer and rational arithmetic. For each retained vector the quadratic reconstruction condition was checked. Every compatibility entry and every maximality assertion for a clique was checked exactly. For every raw extension, the reconstruction identity, symmetry, connectedness, absence of prohibited twins, rank five and characteristic polynomial were verified. All checks passed.

The global reduction used exact constrained backtracking over underlying-graph isomorphisms, with simultaneous propagation of the vertex switching signs. As an independent check, all 890 assignments of raw outputs and their negations were rechecked using a separate permutation-and-sign procedure. The 12 classes of orders at most seven were also checked by exhaustive vertex permutation followed by canonical switching normalization. In addition, the 14 graph–negation pairs were tested directly; none of the representatives is switching isomorphic to its negation.

4 Results and the rank-five characterization

The main contributions are summarized as follows.

Theorem 1. *The maximal-clique outputs obtained from the* 23 *connected non-singular star complements of order five, together with their graph negations, reduce to exactly* 28 *switching-isomorphism classes of connected rank-five MBEs. They form* 14 *negation pairs; no class is switching isomorphic to its negation. Their distribution by order is*

order	5	6	7	10	11
number of classes	6	2	4	10	6

Proof. The 79 connected classes in the catalogue were enumerated independently and tested for non-singularity, giving the 46 classes and 23 negation pairs described above. By the correspondence between maximal compatibility cliques and MBEs, the recorded clique lists exhaust the outputs belonging to the selected star complements. Exact post-verification confirms that every output has rank five and satisfies the defining basic-extension conditions. Adding graph negations covers the other member of every input pair. The exact global reduction gives 28 classes. The independent checks described above confirm both membership in the assigned classes and separation of the two cospectral pairs not already distinguished by their characteristic polynomials. □

Denote the representatives in Appendix E by $\dot{M}_1, \dot{M}_2, \ldots, \dot{M}_{28}$.

Theorem 2 (**Rank-five characterization**)**.** *A signed graph $\dot{G}$ has rank five if and only if, after deleting all isolated vertices and up to switching isomorphism, exactly one of the following holds:*

(i) *$\dot{G}$ is connected and is obtained by adding an arbitrary number of twin vertices to a rank-five induced subgraph of one of $\dot{M}_1, \dot{M}_2, \ldots, \dot{M}_{28}$;*

(ii) *$\dot{G}$ has two connected components, one obtained from K_2 by adding twin vertices and the other obtained from K_3 or $-K_3$ by adding twin vertices.*

Proof. Suppose first that $\dot{G}$ is connected and has rank five. It has a connected five-vertex star complement for 0 by [3]. Removing repeated twin vertices gives a rank-five basic extension, which is an induced subgraph of an MBE. Theorem 1 and Appendix E give all such MBEs, proving (i).

Now suppose that $\dot{G}$ is disconnected and delete its isolated vertices. The rank is additive over connected components, and every remaining component has rank at least two. Hence the only possible decomposition is $5 = 2 + 3$. The existing small-rank characterization [1] says that the connected signed graphs of rank two are obtained from K_2 by adding twins, whereas those of rank three are obtained from K_3 or $-K_3$ in the same way. This proves (ii). Conversely, adding twin or isolated vertices does not change rank, and rank is additive over components, so every signed graph described in (i) or (ii) has rank five. □

5 Software record

The computation used the last release of SCL 2.3 dated 17 July 2026, with Python 3.12.13 and NumPy 2.3.5. To identify the exact copy of the source file used in this computation, its SHA-256 checksum is recorded below. This checksum is a digital fingerprint: changing the file changes the checksum.

`4cae39fe950e809a21991485c30a36563ea15d3e84978c2a75f525cc6a99bf95`

The recording and verification scripts do not alter the SCL algorithm; they capture its intermediate objects and perform the exact post-checks described above.

References

[1] Z. Stanić, Characterization of signed graphs with rank at most four using star complements, to appear in *Ars Math. Contemp.*

[2] Z. Stanić, *Signed graphs of small order*, dataset available at https://poincare.matf.bg.ac.rs/~zstanic/siggr.htm.

[3] Z. Stanić, Star complements in Hermitian matrices, *Linear Algebra Appl.* **748** (2026), 227–248.

[4] Z. Stanić, N. Stefanović, *SCL – Star Complement Library*, software available at https://poincare.matf.bg.ac.rs/~zstanic/scl.htm.

Appendices

A The 23 connected star complements

The matrices below are the selected non-singular representatives from the 23 negation pairs.

$$\dot{H}_1 \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad \dot{H}_2 \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad \dot{H}_3 \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\dot{H}_4 \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \dot{H}_5 \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad \dot{H}_6 \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix}$$

$$\dot{H}_7 \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad \dot{H}_8 \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad \dot{H}_9 \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix}$$

$$\dot{H}_{10} \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad \dot{H}_{11} \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad \dot{H}_{12} \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix}$$

$$\dot{H}_{13} \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix}$$

$$\dot{H}_{14} \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix}$$

$$\dot{H}_{15} \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix}$$

$$\dot{H}_{16} \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\dot{H}_{17} \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\dot{H}_{18} \quad \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 1 & -1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 0 & -1 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}$$

$$\dot{H}_{19} \quad \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 1 & -1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\dot{H}_{20} \quad \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 \\ 1 & -1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 & 0 \end{bmatrix}$$

$$\dot{H}_{21} \quad \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 \\ 1 & -1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 0 & -1 \\ 1 & -1 & -1 & -1 & 0 \end{bmatrix}$$

$$\dot{H}_{22} \quad \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 \\ 1 & -1 & 0 & -1 & -1 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{bmatrix}$$

$$\dot{H}_{23} \quad \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

B Retained admissible vectors

The vectors are indexed in the displayed order; these indices are used for the compatibility matrices and maximal cliques.

$\dot{H}_1$ (18 vectors)

1. $(-1,-1,-1,1,1)^\intercal$
2. $(-1,-1,0,0,0)^\intercal$
3. $(-1,-1,1,-1,1)^\intercal$
4. $(-1,-1,1,1,0)^\intercal$
5. $(-1,0,-1,0,0)^\intercal$
6. $(-1,0,0,-1,0)^\intercal$
7. $(-1,0,0,1,-1)^\intercal$
8. $(-1,0,1,0,-1)^\intercal$
9. $(-1,1,-1,-1,1)^\intercal$
10. $(-1,1,-1,1,0)^\intercal$
11. $(-1,1,0,0,-1)^\intercal$
12. $(-1,1,1,-1,0)^\intercal$
13. $(0,-1,-1,0,-1)^\intercal$
14. $(0,-1,-1,0,0)^\intercal$
15. $(0,-1,0,-1,-1)^\intercal$
16. $(0,-1,0,-1,0)^\intercal$
17. $(0,0,-1,-1,-1)^\intercal$
18. $(0,0,-1,-1,0)^\intercal$

$\dot{H}_2$ (13 vectors)

1. $(-1,-1,-1,0,-1)^\intercal$
2. $(-1,-1,0,1,-1)^\intercal$
3. $(-1,-1,1,0,0)^\intercal$
4. $(-1,0,-1,-1,0)^\intercal$
5. $(-1,0,0,0,-1)^\intercal$
6. $(-1,0,1,-1,-1)^\intercal$
7. $(-1,1,0,-1,-1)^\intercal$
8. $(0,-1,0,1,-1)^\intercal$
9. $(0,-1,0,1,1)^\intercal$
10. $(0,-1,1,0,-1)^\intercal$
11. $(0,-1,1,0,0)^\intercal$
12. $(0,0,-1,-1,-1)^\intercal$
13. $(0,0,-1,-1,0)^\intercal$

$\dot{H}_3$ (13 vectors)

1. $(-1,-1,-1,1,-1)^\intercal$
2. $(-1,-1,0,0,0)^\intercal$
3. $(-1,-1,1,-1,-1)^\intercal$
4. $(-1,-1,1,1,0)^\intercal$
5. $(-1,0,-1,0,-1)^\intercal$
6. $(-1,0,0,-1,-1)^\intercal$
7. $(-1,1,-1,-1,-1)^\intercal$
8. $(0,-1,0,1,-1)^\intercal$
9. $(0,-1,0,1,0)^\intercal$
10. $(0,-1,1,0,-1)^\intercal$
11. $(0,-1,1,0,0)^\intercal$
12. $(0,0,-1,-1,-1)^\intercal$
13. $(0,0,-1,-1,0)^\intercal$

$\dot{H}_4$ (16 vectors)

1. $(-1,-1,-1,-1,0)^\intercal$
2. $(-1,-1,-1,1,-1)^\intercal$
3. $(-1,-1,0,0,0)^\intercal$
4. $(-1,-1,1,-1,1)^\intercal$
5. $(-1,0,-1,0,0)^\intercal$
6. $(-1,0,0,1,1)^\intercal$
7. $(-1,0,1,0,1)^\intercal$
8. $(-1,1,-1,-1,1)^\intercal$
9. $(-1,1,0,0,1)^\intercal$
10. $(-1,1,1,-1,0)^\intercal$
11. $(0,-1,-1,0,0)^\intercal$
12. $(0,-1,-1,0,1)^\intercal$
13. $(0,-1,0,1,-1)^\intercal$
14. $(0,-1,0,1,1)^\intercal$
15. $(0,0,-1,1,-1)^\intercal$
16. $(0,0,-1,1,1)^\intercal$

$\dot{H}_5$ (16 vectors)

1. $(-1,-1,0,-1,0)^\intercal$
2. $(-1,-1,0,1,1)^\intercal$
3. $(-1,-1,1,0,1)^\intercal$
4. $(-1,0,-1,-1,0)^\intercal$
5. $(-1,0,-1,1,1)^\intercal$
6. $(-1,0,0,0,0)^\intercal$
7. $(-1,0,1,-1,-1)^\intercal$
8. $(-1,1,-1,0,1)^\intercal$
9. $(-1,1,0,-1,-1)^\intercal$
10. $(-1,1,1,0,0)^\intercal$
11. $(0,-1,-1,0,-1)^\intercal$
12. $(0,-1,-1,0,1)^\intercal$
13. $(0,-1,0,1,-1)^\intercal$
14. $(0,-1,0,1,0)^\intercal$
15. $(0,0,-1,1,-1)^\intercal$
16. $(0,0,-1,1,0)^\intercal$

$\dot{H}_6$ (0 vectors)

No retained vectors.

$\dot{H}_7$ (10 vectors)

1. $(-1,-1,0,-1,1)^\intercal$
2. $(-1,-1,1,1,1)^\intercal$
3. $(-1,0,-1,-1,1)^\intercal$
4. $(-1,0,0,0,0)^\intercal$
5. $(-1,0,1,1,-1)^\intercal$
6. $(-1,1,-1,1,1)^\intercal$
7. $(-1,1,0,1,-1)^\intercal$
8. $(0,-1,-1,-1,-1)^\intercal$
9. $(0,-1,-1,0,0)^\intercal$
10. $(0,-1,-1,1,1)^\intercal$

$\dot{H}_8$ (4 vectors)

1. $(-1,-1,0,1,0)^\intercal$
2. $(-1,0,-1,1,0)^\intercal$
3. $(-1,0,1,0,1)^\intercal$
4. $(-1,1,0,0,1)^\intercal$

$\dot{H}_9$ (3 vectors)

1. $(-1,0,1,1,1)^\intercal$
2. $(-1,1,1,0,-1)^\intercal$
3. $(0,-1,0,-1,0)^\intercal$

$\dot{H}_{10}$ (4 vectors)

1. $(-1,0,-1,1,0)^\intercal$
2. $(-1,0,0,0,1)^\intercal$
3. $(-1,0,1,1,0)^\intercal$
4. $(0,-1,0,-1,0)^\intercal$

$\dot{H}_{11}$ (13 vectors)

1. $(-1,-1,0,0,1)^\intercal$
2. $(-1,-1,0,1,0)^\intercal$
3. $(-1,0,-1,-1,1)^\intercal$
4. $(-1,0,-1,0,0)^\intercal$
5. $(-1,0,0,0,0)^\intercal$
6. $(-1,0,0,1,-1)^\intercal$
7. $(-1,0,1,1,-1)^\intercal$
8. $(-1,1,-1,0,1)^\intercal$
9. $(-1,1,-1,1,0)^\intercal$
10. $(-1,1,1,1,0)^\intercal$
11. $(0,-1,-1,-1,0)^\intercal$
12. $(0,-1,-1,0,1)^\intercal$
13. $(0,-1,0,-1,0)^\intercal$

$\dot{H}_{12}$ (12 vectors)

1. $(-1,-1,-1,0,1)^\intercal$
2. $(-1,0,-1,0,1)^\intercal$
3. $(-1,0,-1,1,0)^\intercal$
4. $(-1,0,0,1,0)^\intercal$
5. $(-1,1,-1,1,0)^\intercal$
6. $(-1,1,0,0,-1)^\intercal$
7. $(-1,1,1,1,0)^\intercal$
8. $(0,-1,-1,-1,-1)^\intercal$
9. $(0,-1,-1,0,0)^\intercal$
10. $(0,-1,-1,1,1)^\intercal$
11. $(0,-1,0,-1,-1)^\intercal$
12. $(0,-1,0,0,0)^\intercal$

$\dot{H}_{13}$ (4 vectors)

1. $(-1,0,1,0,0)^\intercal$
2. $(-1,1,0,0,0)^\intercal$
3. $(0,-1,0,-1,0)^\intercal$
4. $(0,0,-1,-1,0)^\intercal$

$\dot{H}_{14}$ (15 vectors)

1. $(-1,-1,-1,0,1)^\intercal$
2. $(-1,-1,0,1,1)^\intercal$
3. $(-1,0,-1,1,0)^\intercal$
4. $(-1,0,0,0,0)^\intercal$
5. $(-1,0,0,1,-1)^\intercal$
6. $(-1,0,1,0,-1)^\intercal$
7. $(-1,1,-1,0,1)^\intercal$
8. $(-1,1,-1,1,0)^\intercal$
9. $(-1,1,0,-1,-1)^\intercal$
10. $(-1,1,1,-1,0)^\intercal$
11. $(0,-1,-1,0,-1)^\intercal$
12. $(0,-1,-1,1,0)^\intercal$
13. $(0,-1,0,-1,-1)^\intercal$
14. $(0,-1,0,0,0)^\intercal$
15. $(0,0,-1,-1,0)^\intercal$

$\dot{H}_{15}$ (16 vectors)

1. $(-1,-1,0,0,1)^\intercal$
2. $(-1,-1,0,1,0)^\intercal$
3. $(-1,-1,1,0,1)^\intercal$
4. $(-1,-1,1,1,0)^\intercal$
5. $(-1,0,-1,0,1)^\intercal$
6. $(-1,0,-1,1,0)^\intercal$
7. $(-1,0,0,-1,0)^\intercal$
8. $(-1,0,0,0,-1)^\intercal$
9. $(-1,1,-1,0,1)^\intercal$
10. $(-1,1,-1,1,0)^\intercal$
11. $(0,-1,-1,-1,1)^\intercal$
12. $(0,-1,-1,1,-1)^\intercal$
13. $(0,-1,0,-1,-1)^\intercal$
14. $(0,-1,0,0,0)^\intercal$
15. $(0,0,-1,-1,-1)^\intercal$
16. $(0,0,-1,0,0)^\intercal$

$\dot{H}_{16}$ (14 vectors)

1. $(-1,-1,0,-1,0)^\intercal$
2. $(-1,-1,0,0,-1)^\intercal$
3. $(-1,-1,1,-1,0)^\intercal$
4. $(-1,-1,1,0,-1)^\intercal$
5. $(-1,0,-1,-1,0)^\intercal$
6. $(-1,0,-1,0,-1)^\intercal$
7. $(-1,1,-1,-1,0)^\intercal$
8. $(-1,1,-1,0,-1)^\intercal$
9. $(0,-1,0,-1,-1)^\intercal$
10. $(0,-1,0,0,0)^\intercal$
11. $(0,-1,1,-1,1)^\intercal$
12. $(0,-1,1,1,-1)^\intercal$
13. $(0,0,-1,-1,-1)^\intercal$
14. $(0,0,-1,0,0)^\intercal$

$\dot{H}_{17}$ (15 vectors)

1. $(-1,-1,-1,1,0)^\intercal$
2. $(-1,-1,0,-1,-1)^\intercal$
3. $(-1,0,-1,1,-1)^\intercal$
4. $(-1,0,0,-1,0)^\intercal$
5. $(-1,0,0,0,-1)^\intercal$
6. $(-1,0,1,-1,-1)^\intercal$
7. $(-1,1,-1,-1,0)^\intercal$
8. $(-1,1,-1,0,-1)^\intercal$
9. $(-1,1,0,-1,1)^\intercal$
10. $(-1,1,1,0,1)^\intercal$
11. $(0,-1,-1,-1,-1)^\intercal$
12. $(0,-1,-1,0,0)^\intercal$
13. $(0,-1,-1,1,1)^\intercal$
14. $(0,-1,0,1,0)^\intercal$
15. $(0,0,-1,0,-1)^\intercal$

$\dot{H}_{18}$ (10 vectors)

1. $(-1,-1,-1,-1,0)^\intercal$
2. $(-1,-1,-1,1,1)^\intercal$
3. $(-1,-1,0,0,0)^\intercal$
4. $(-1,-1,1,-1,1)^\intercal$
5. $(-1,1,0,1,-1)^\intercal$
6. $(-1,1,0,1,0)^\intercal$
7. $(-1,1,1,0,-1)^\intercal$
8. $(-1,1,1,0,0)^\intercal$
9. $(0,0,-1,-1,0)^\intercal$
10. $(0,0,0,0,-1)^\intercal$

$\dot{H}_{19}$ (16 vectors)

1. $(-1,-1,-1,0,1)^\intercal$
2. $(-1,-1,0,-1,0)^\intercal$
3. $(-1,-1,0,1,1)^\intercal$
4. $(-1,0,0,-1,1)^\intercal$
5. $(-1,0,0,1,1)^\intercal$
6. $(-1,0,1,0,0)^\intercal$
7. $(-1,1,0,1,-1)^\intercal$
8. $(-1,1,0,1,0)^\intercal$
9. $(-1,1,1,0,-1)^\intercal$
10. $(-1,1,1,0,1)^\intercal$
11. $(0,-1,-1,-1,-1)^\intercal$
12. $(0,-1,0,0,0)^\intercal$
13. $(0,-1,1,-1,1)^\intercal$
14. $(0,-1,1,1,1)^\intercal$
15. $(0,0,-1,-1,0)^\intercal$
16. $(0,0,0,0,-1)^\intercal$

$\dot{H}_{20}$ (9 vectors)

1. $(-1,-1,-1,-1,1)^\intercal$
2. $(-1,-1,0,0,1)^\intercal$
3. $(-1,-1,1,1,1)^\intercal$
4. $(-1,1,0,1,-1)^\intercal$
5. $(-1,1,0,1,0)^\intercal$
6. $(-1,1,1,0,0)^\intercal$
7. $(-1,1,1,0,1)^\intercal$
8. $(0,0,-1,-1,0)^\intercal$
9. $(0,0,0,0,-1)^\intercal$

$\dot{H}_{21}$ (0 vectors)

No retained vectors.

$\dot{H}_{22}$ (12 vectors)

1. $(-1,-1,0,-1,1)^\intercal$
2. $(-1,-1,0,1,-1)^\intercal$
3. $(-1,0,-1,-1,1)^\intercal$
4. $(-1,0,-1,1,-1)^\intercal$
5. $(-1,0,0,0,1)^\intercal$
6. $(-1,0,0,1,0)^\intercal$
7. $(0,-1,0,-1,0)^\intercal$
8. $(0,-1,0,0,-1)^\intercal$
9. $(0,-1,1,-1,1)^\intercal$
10. $(0,-1,1,1,-1)^\intercal$
11. $(0,0,-1,-1,0)^\intercal$
12. $(0,0,-1,0,-1)^\intercal$

$\dot{H}_{23}$ (0 vectors)

No retained vectors.

C Compatibility graphs

For each input the compatibility graph is given by its adjacency matrix, with vertices ordered as in Appendix B.

$\mathcal{C}(\dot{H}_1)$ (order 18)

$$\begin{bmatrix}
0&1&0&1&1&0&1&0&0&1&0&0&0&0&1&1&1&1\\
1&0&1&1&1&1&1&1&0&0&1&0&0&1&0&1&1&1\\
0&1&0&1&0&1&0&1&0&0&0&1&1&1&0&0&1&1\\
1&1&1&0&0&0&1&1&0&0&0&0&0&0&0&0&1&1\\
1&1&0&0&0&1&1&1&1&1&1&0&0&1&1&1&0&1\\
0&1&1&0&1&0&1&1&1&0&1&1&1&1&0&1&0&1\\
1&1&0&1&1&1&0&1&0&1&1&0&0&0&0&1&0&1\\
0&1&1&1&1&1&1&0&0&0&1&1&0&1&0&0&0&1\\
0&0&0&0&1&1&0&0&0&1&1&1&1&1&1&1&0&0\\
1&0&0&0&1&0&1&0&1&0&1&0&0&0&1&1&0&0\\
0&1&0&0&1&1&1&1&1&1&0&1&0&1&0&1&0&0\\
0&0&1&0&0&1&0&1&1&0&1&0&1&1&0&0&0&0\\
0&0&1&0&0&1&0&0&1&0&0&1&0&1&1&1&1&1\\
0&1&1&0&1&1&0&1&1&0&1&1&1&0&1&1&1&1\\
1&0&0&0&1&0&0&0&1&1&0&0&1&1&0&1&1&1\\
1&1&0&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1\\
1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&0&1\\
1&1&1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_2)$ (order 13)

$$\begin{bmatrix}
0&1&1&1&1&0&0&1&0&0&1&1&1\\
1&0&1&1&1&0&1&1&1&0&0&1&1\\
1&1&0&1&1&1&1&0&1&0&1&1&1\\
1&1&1&0&1&1&1&1&0&1&1&0&1\\
1&1&1&1&0&1&1&1&1&0&1&0&1\\
0&0&1&1&1&0&1&0&1&1&1&0&1\\
0&1&1&1&1&1&0&1&1&1&1&0&0\\
1&1&0&1&1&0&1&0&1&1&1&1&1\\
0&1&1&0&1&1&1&1&0&1&1&1&1\\
0&0&0&1&0&1&1&1&1&0&1&1&1\\
1&0&1&1&1&1&1&1&1&1&0&1&1\\
1&1&1&0&0&0&0&1&1&1&1&0&1\\
1&1&1&1&1&1&0&1&1&1&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_3)$ (order 13)

$$\begin{bmatrix}
0&1&0&1&1&0&1&1&1&0&0&1&1\\
1&0&1&1&1&1&1&0&1&0&1&1&1\\
0&1&0&1&0&1&1&0&0&1&1&1&1\\
1&1&1&0&1&1&0&0&0&0&0&1&1\\
1&1&0&1&0&1&1&1&1&0&1&0&1\\
0&1&1&1&1&0&1&0&1&1&1&0&1\\
1&1&1&0&1&1&0&1&1&1&1&0&0\\
1&0&0&0&1&0&1&0&1&1&1&1&1\\
1&1&0&0&1&1&1&1&0&1&1&1&1\\
0&0&1&0&0&1&1&1&1&0&1&1&1\\
0&1&1&0&1&1&1&1&1&1&0&1&1\\
1&1&1&1&0&0&0&1&1&1&1&0&1\\
1&1&1&1&1&1&0&1&1&1&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_4)$ (order 16)

$$\begin{bmatrix}
0&1&1&1&1&0&0&1&0&1&1&1&1&0&1&0\\
1&0&1&1&1&1&1&1&1&0&0&0&1&0&1&0\\
1&1&0&1&1&1&1&0&1&1&1&0&1&1&1&0\\
1&1&1&0&0&0&1&0&0&1&1&1&0&1&1&0\\
1&1&1&0&0&1&1&1&1&1&1&0&1&0&1&1\\
0&1&1&0&1&0&1&0&1&1&0&0&1&1&1&1\\
0&1&1&1&1&1&0&0&1&1&1&0&0&1&1&1\\
1&1&0&0&1&0&0&0&1&1&1&1&1&0&0&1\\
0&1&1&0&1&1&1&1&0&1&1&0&1&1&0&1\\
1&0&1&1&1&1&1&1&1&0&1&1&0&1&0&1\\
1&0&1&1&1&0&1&1&1&1&0&1&1&1&1&1\\
1&0&0&1&0&0&0&1&0&1&1&0&1&1&1&1\\
1&1&1&0&1&1&0&1&1&0&1&1&0&1&1&1\\
0&0&1&1&0&1&1&0&1&1&1&1&1&0&1&1\\
1&1&1&1&1&1&1&0&0&0&1&1&1&1&0&1\\
0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_5)$ (order 16)

$$\begin{bmatrix}
0&1&1&1&0&1&1&0&1&0&0&1&0&1&1&1\\
1&0&1&0&1&1&1&0&1&1&1&0&0&1&1&1\\
1&1&0&0&0&1&1&0&0&1&1&1&0&0&1&1\\
1&0&0&0&1&1&1&1&1&0&0&1&1&1&0&1\\
0&1&0&1&0&1&1&1&1&1&1&0&1&1&0&1\\
1&1&1&1&1&0&1&1&1&1&1&1&0&1&0&1\\
1&1&1&1&1&1&0&0&1&1&0&1&0&0&0&1\\
0&0&0&1&1&1&0&0&1&1&1&1&1&1&0&0\\
1&1&0&1&1&1&1&1&0&1&0&1&0&1&0&0\\
0&1&1&0&1&1&1&1&1&0&1&1&0&0&0&0\\
0&1&1&0&1&1&0&1&0&1&0&1&1&1&1&1\\
1&0&1&1&0&1&1&1&1&1&1&0&1&1&1&1\\
0&0&0&1&1&0&0&1&0&0&1&1&0&1&1&1\\
1&1&0&1&1&1&0&1&1&0&1&1&1&0&1&1\\
1&1&1&0&0&0&0&0&0&0&1&1&1&1&0&1\\
1&1&1&1&1&1&1&0&0&0&1&1&1&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_6)$ (order 0)

$\mathcal{C}(\dot{H}_7)$ (order 10)

$$\begin{bmatrix}
0&1&1&1&1&0&1&1&1&0\\
1&0&0&1&1&0&0&1&1&1\\
1&0&0&1&1&1&1&1&1&0\\
1&1&1&0&1&1&1&1&1&1\\
1&1&1&1&0&0&1&0&1&1\\
0&0&1&1&0&0&1&1&1&1\\
1&0&1&1&1&1&0&0&1&1\\
1&1&1&1&0&1&0&0&1&1\\
1&1&1&1&1&1&1&1&0&1\\
0&1&0&1&1&1&1&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_8)$ (order 4)

$$\begin{bmatrix}
0&1&0&0\\
1&0&0&0\\
0&0&0&1\\
0&0&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_9)$ (order 3)

$$\begin{bmatrix}
0&1&0\\
1&0&0\\
0&0&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_{10})$ (order 4)

$$\begin{bmatrix} 0&0&0&1\\ 0&0&1&1\\ 0&1&0&0\\ 1&1&0&0 \end{bmatrix}$$

$\mathcal{C}(\dot{H}_{11})$ (order 13)

$$\begin{bmatrix}
0&1&1&0&1&1&1&0&0&1&1&0&1\\
1&0&0&1&1&1&1&0&0&0&1&1&1\\
1&0&0&1&1&1&1&1&1&1&1&0&1\\
0&1&1&0&1&1&1&1&1&0&1&1&1\\
1&1&1&1&0&1&1&1&0&1&1&1&1\\
1&1&1&1&1&0&1&0&1&0&0&1&1\\
1&1&1&1&1&1&0&0&0&1&1&1&0\\
0&0&1&1&1&0&0&0&1&1&1&1&1\\
0&0&1&1&0&1&0&1&0&0&0&1&1\\
1&0&1&0&1&0&1&1&0&0&1&1&0\\
1&1&1&1&1&0&1&1&0&1&0&1&1\\
0&1&0&1&1&1&1&1&1&1&1&0&1\\
1&1&1&1&1&1&0&1&1&0&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_{12})$ (order 12)

$$\begin{bmatrix}
0&1&1&1&1&0&0&1&0&0&1&1\\
1&0&1&1&1&1&1&1&1&0&1&1\\
1&1&0&1&1&1&0&0&1&1&1&1\\
1&1&1&0&0&1&1&1&1&1&0&1\\
1&1&1&0&0&1&0&0&0&1&1&1\\
0&1&1&1&1&0&1&0&1&1&0&1\\
0&1&0&1&0&1&0&1&1&1&0&0\\
1&1&0&1&0&0&1&0&1&1&1&1\\
0&1&1&1&0&1&1&1&0&1&1&1\\
0&0&1&1&1&1&1&1&1&0&1&1\\
1&1&1&0&1&0&0&1&1&1&0&1\\
1&1&1&1&1&1&0&1&1&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_{13})$ (order 4)

$$\begin{bmatrix} 0&1&0&1\\ 1&0&1&0\\ 0&1&0&1\\ 1&0&1&0 \end{bmatrix}$$

$\mathcal{C}(\dot{H}_{14})$ (order 15)

$$\begin{bmatrix}
0&1&1&1&1&0&1&1&0&0&1&0&1&1&1\\
1&0&0&1&1&1&0&0&1&1&1&1&0&1&1\\
1&0&0&1&1&1&1&1&1&0&0&1&1&1&1\\
1&1&1&0&1&1&1&0&1&1&1&1&0&1&1\\
1&1&1&1&0&1&0&1&1&0&0&0&0&1&1\\
0&1&1&1&1&0&0&0&1&1&0&1&0&0&1\\
1&0&1&1&0&0&0&1&1&1&1&1&1&1&0\\
1&0&1&0&1&0&1&0&1&0&0&0&1&1&0\\
0&1&1&1&1&1&1&1&0&1&0&1&0&1&0\\
0&1&0&1&0&1&1&0&1&0&1&1&0&0&0\\
1&1&0&1&0&0&1&0&0&1&0&1&1&1&1\\
0&1&1&1&0&1&1&0&1&1&1&0&1&1&1\\
1&0&1&0&0&0&1&1&0&0&1&1&0&1&1\\
1&1&1&1&1&0&1&1&1&0&1&1&1&0&1\\
1&1&1&1&1&1&0&0&0&0&1&1&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_{15})$ (order 16)

$$\begin{bmatrix}
0&1&1&1&1&0&1&1&0&0&0&1&0&1&1&1\\
1&0&1&1&0&1&1&1&0&0&1&0&0&1&1&1\\
1&1&0&1&0&0&1&0&0&0&0&1&0&0&1&1\\
1&1&1&0&0&0&0&1&0&0&1&0&0&0&1&1\\
1&0&0&0&0&1&1&1&1&1&0&1&1&1&0&1\\
0&1&0&0&1&0&1&1&1&1&1&0&1&1&0&1\\
1&1&1&0&1&1&0&1&1&0&0&1&0&1&0&1\\
1&1&0&1&1&1&1&0&0&1&1&0&0&1&0&1\\
0&0&0&0&1&1&1&0&0&1&0&1&1&1&0&0\\
0&0&0&0&1&1&0&1&1&0&1&0&1&1&0&0\\
0&1&0&1&0&1&0&1&0&1&0&0&1&1&1&1\\
1&0&1&0&1&0&1&0&1&0&0&0&1&1&1&1\\
0&0&0&0&1&1&0&0&1&1&1&1&0&1&1&1\\
1&1&0&0&1&1&1&1&1&1&1&1&1&0&1&1\\
1&1&1&1&0&0&0&0&0&0&1&1&1&1&0&1\\
1&1&1&1&1&1&1&1&0&0&1&1&1&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_{16})$ (order 14)

$$\begin{bmatrix}
0&1&1&1&1&1&0&1&0&1&1&0&1&1\\
1&0&1&1&1&1&1&0&0&1&0&1&1&1\\
1&1&0&1&0&1&0&0&0&0&1&0&1&1\\
1&1&1&0&1&0&0&0&0&0&0&1&1&1\\
1&1&0&1&0&1&1&1&1&1&0&1&0&1\\
1&1&1&0&1&0&1&1&1&1&1&0&0&1\\
0&1&0&0&1&1&0&1&1&1&0&1&0&0\\
1&0&0&0&1&1&1&0&1&1&1&0&0&0\\
0&0&0&0&1&1&1&1&0&1&1&1&1&1\\
1&1&0&0&1&1&1&1&1&0&1&1&1&1\\
1&0&1&0&0&1&0&1&1&1&0&0&1&1\\
0&1&0&1&1&0&1&0&1&1&0&0&1&1\\
1&1&1&1&0&0&0&0&1&1&1&1&0&1\\
1&1&1&1&1&1&0&0&1&1&1&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_{17})$ (order 15)

$$\begin{bmatrix}
0&1&1&1&1&1&1&1&0&0&0&1&1&1&0\\
1&0&0&1&1&1&0&0&1&1&1&1&1&1&0\\
1&0&0&1&1&1&1&1&1&0&0&0&1&1&1\\
1&1&1&0&1&1&1&0&1&1&1&1&0&1&1\\
1&1&1&1&0&1&0&1&1&1&0&1&1&1&1\\
1&1&1&1&1&0&0&0&0&1&1&1&1&0&1\\
1&0&1&1&0&0&0&1&1&1&1&1&0&1&1\\
1&0&1&0&1&0&1&0&1&1&0&1&1&1&1\\
0&1&1&1&1&0&1&1&0&1&1&0&0&1&1\\
0&1&0&1&1&1&1&1&1&0&1&1&0&0&1\\
0&1&0&1&0&1&1&0&1&1&0&1&1&1&1\\
1&1&0&1&1&1&1&1&0&1&1&0&1&1&1\\
1&1&1&0&1&1&0&1&0&0&1&1&0&1&1\\
1&1&1&1&1&0&1&1&1&0&1&1&1&0&1\\
0&0&1&1&1&1&1&1&1&1&1&1&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_{18})$ (order 10)

$$\begin{bmatrix}
0&1&1&1&0&0&0&0&1&1\\
1&0&1&0&0&0&1&1&1&1\\
1&1&0&1&0&1&0&1&1&1\\
1&0&1&0&1&1&0&0&1&1\\
0&0&0&1&0&1&1&1&1&1\\
0&0&1&1&1&0&1&1&1&1\\
0&1&0&0&1&1&0&1&1&1\\
0&1&1&0&1&1&1&0&1&1\\
1&1&1&1&1&1&1&1&0&1\\
1&1&1&1&1&1&1&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_{19})$ (order 16)

$$\begin{bmatrix}
0&1&1&0&0&0&0&0&1&1&0&0&0&0&1&1\\
1&0&1&0&0&0&0&1&0&1&0&0&0&0&1&1\\
1&1&0&0&1&1&0&1&1&0&1&1&0&1&1&1\\
0&0&0&0&1&1&1&0&0&0&1&1&1&0&0&0\\
0&0&1&1&0&1&1&0&0&0&1&1&0&1&0&0\\
0&0&1&1&1&0&1&0&0&0&1&1&1&1&0&0\\
0&0&0&1&1&1&0&1&1&1&0&1&1&1&1&1\\
0&1&1&0&0&0&1&0&1&1&0&0&0&0&1&1\\
1&0&1&0&0&0&1&1&0&1&0&0&0&0&1&1\\
1&1&0&0&0&0&1&1&1&0&0&0&0&0&1&1\\
0&0&1&1&1&1&0&0&0&0&0&1&1&1&0&0\\
0&0&1&1&1&1&1&0&0&0&1&0&1&1&0&0\\
0&0&0&1&0&1&1&0&0&0&1&1&0&1&0&0\\
0&0&1&0&1&1&1&0&0&0&1&1&1&0&0&0\\
1&1&1&0&0&0&1&1&1&1&0&0&0&0&0&1\\
1&1&1&0&0&0&1&1&1&1&0&0&0&0&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_{20})$ (order 9)

$$\begin{bmatrix}
0&1&1&0&0&1&1&1&1\\
1&0&1&0&1&1&0&1&1\\
1&1&0&1&1&0&0&1&1\\
0&0&1&0&1&1&1&1&1\\
0&1&1&1&0&1&1&1&1\\
1&1&0&1&1&0&1&1&1\\
1&0&0&1&1&1&0&1&1\\
1&1&1&1&1&1&1&0&1\\
1&1&1&1&1&1&1&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_{21})$ (order 0)

$\mathcal{C}(\dot{H}_{22})$ (order 12)

$$\begin{bmatrix}
0&0&1&0&1&0&1&0&1&0&1&1\\
0&0&0&1&0&1&0&1&0&1&1&1\\
1&0&0&0&1&0&1&1&0&1&1&0\\
0&1&0&0&0&1&1&1&1&0&0&1\\
1&0&1&0&0&1&1&1&1&1&1&1\\
0&1&0&1&1&0&1&1&1&1&1&1\\
1&0&1&1&1&1&0&1&1&0&1&1\\
0&1&1&1&1&1&1&0&0&1&1&1\\
1&0&0&1&1&1&1&0&0&0&0&1\\
0&1&1&0&1&1&0&1&0&0&1&0\\
1&1&1&0&1&1&1&1&0&1&0&1\\
1&1&0&1&1&1&1&1&1&0&1&0
\end{bmatrix}$$

$\mathcal{C}(\dot{H}_{23})$ (order 0)

D Maximal cliques

The clique vertices refer to the vector indices in Appendix B. The empty clique is recorded for each compatibility graph of order zero.

$\mathcal{C}(\dot{H}_1)$: 39 maximal cliques

1. $\{1, 5, 7, 10, 16\}$
2. $\{1, 5, 10, 15, 16\}$
3. $\{1, 18, 2, 4, 7\}$
4. $\{1, 18, 2, 4, 17\}$
5. $\{1, 18, 2, 17, 16\}$
6. $\{1, 18, 5, 15, 16\}$
7. $\{1, 18, 17, 15, 16\}$
8. $\{3, 6, 8, 12, 14\}$
9. $\{3, 6, 12, 13, 14\}$
10. $\{5, 7, 10, 11, 16\}$
11. $\{5, 9, 10, 11, 16\}$
12. $\{5, 9, 10, 15, 16\}$
13. $\{5, 9, 14, 15, 16\}$
14. $\{6, 8, 11, 12, 14\}$
15. $\{6, 9, 11, 12, 14\}$
16. $\{6, 9, 12, 13, 14\}$
17. $\{6, 9, 13, 14, 16\}$
18. $\{9, 13, 14, 15, 16\}$
19. $\{17, 18, 2, 4, 3\}$
20. $\{17, 18, 2, 14, 16\}$
21. $\{17, 18, 3, 2, 14\}$
22. $\{17, 18, 3, 13, 14\}$
23. $\{18, 2, 4, 7, 8\}$
24. $\{18, 3, 4, 2, 8\}$
25. $\{18, 3, 6, 13, 14\}$
26. $\{18, 5, 14, 15, 16\}$
27. $\{18, 6, 13, 14, 16\}$
28. $\{1, 18, 2, 5, 7, 16\}$
29. $\{2, 5, 6, 7, 8, 11\}$
30. $\{2, 5, 6, 7, 11, 16\}$
31. $\{2, 5, 6, 8, 11, 14\}$
32. $\{2, 5, 6, 11, 14, 16\}$
33. $\{5, 6, 9, 11, 14, 16\}$
34. $\{17, 18, 13, 14, 15, 16\}$
35. $\{18, 2, 5, 6, 7, 8\}$
36. $\{18, 2, 5, 6, 7, 16\}$
37. $\{18, 2, 5, 6, 8, 14\}$
38. $\{18, 2, 5, 6, 14, 16\}$
39. $\{18, 3, 2, 6, 8, 14\}$

$\mathcal{C}(\dot{H}_2)$: 32 maximal cliques

1. $\{1, 2, 3, 12, 13\}$
2. $\{1, 2, 8, 12, 13\}$
3. $\{1, 3, 11, 12, 13\}$
4. $\{1, 8, 11, 12, 13\}$
5. $\{2, 3, 4, 5, 7\}$
6. $\{2, 3, 5, 7, 9\}$
7. $\{2, 3, 5, 9, 13\}$
8. $\{2, 3, 9, 12, 13\}$
9. $\{2, 4, 5, 7, 8\}$
10. $\{2, 5, 7, 8, 9\}$
11. $\{2, 5, 8, 9, 13\}$
12. $\{2, 8, 9, 12, 13\}$
13. $\{3, 9, 11, 12, 13\}$
14. $\{4, 5, 7, 8, 11\}$
15. $\{4, 6, 7, 10, 11\}$
16. $\{4, 6, 10, 11, 13\}$
17. $\{4, 7, 8, 10, 11\}$
18. $\{4, 8, 10, 11, 13\}$
19. $\{5, 7, 8, 9, 11\}$
20. $\{5, 8, 9, 11, 13\}$
21. $\{6, 7, 9, 10, 11\}$
22. $\{6, 9, 10, 11, 13\}$
23. $\{7, 8, 9, 10, 11\}$
24. $\{1, 2, 3, 4, 5, 13\}$
25. $\{1, 2, 4, 5, 8, 13\}$
26. $\{1, 3, 4, 5, 11, 13\}$
27. $\{1, 4, 5, 8, 11, 13\}$
28. $\{3, 4, 5, 6, 7, 11\}$
29. $\{3, 4, 5, 6, 11, 13\}$
30. $\{3, 5, 6, 7, 9, 11\}$
31. $\{3, 5, 6, 9, 11, 13\}$
32. $\{8, 9, 10, 11, 12, 13\}$

$\mathcal{C}(\dot{H}_3)$: 26 maximal cliques

1. $\{1, 2, 4, 5, 13\}$
2. $\{1, 2, 4, 12, 13\}$
3. $\{1, 2, 5, 7, 9\}$
4. $\{1, 2, 5, 9, 13\}$
5. $\{1, 2, 9, 12, 13\}$
6. $\{1, 5, 7, 8, 9\}$
7. $\{1, 5, 8, 9, 13\}$
8. $\{1, 8, 9, 12, 13\}$
9. $\{2, 3, 4, 6, 13\}$
10. $\{2, 3, 4, 12, 13\}$
11. $\{2, 3, 6, 7, 11\}$
12. $\{2, 3, 6, 11, 13\}$
13. $\{2, 3, 11, 12, 13\}$
14. $\{2, 4, 5, 6, 13\}$
15. $\{2, 9, 11, 12, 13\}$
16. $\{3, 6, 7, 10, 11\}$
17. $\{3, 6, 10, 11, 13\}$
18. $\{3, 10, 11, 12, 13\}$
19. $\{5, 7, 8, 9, 11\}$
20. $\{5, 8, 9, 11, 13\}$
21. $\{6, 7, 9, 10, 11\}$
22. $\{6, 9, 10, 11, 13\}$
23. $\{7, 8, 9, 10, 11\}$
24. $\{2, 5, 6, 7, 9, 11\}$
25. $\{2, 5, 6, 9, 11, 13\}$
26. $\{8, 9, 10, 11, 12, 13\}$

$\mathcal{C}(\dot{H}_4)$: 55 maximal cliques

1. $\{1,2,3,4,15\}$
2. $\{1,2,5,8,13\}$
3. $\{1,3,4,10,11\}$
4. $\{1,3,4,11,15\}$
5. $\{1,3,5,10,11\}$
6. $\{1,4,10,11,12\}$
7. $\{1,4,11,12,15\}$
8. $\{1,5,8,10,11\}$
9. $\{1,5,8,11,13\}$
10. $\{1,8,10,11,12\}$
11. $\{1,8,11,12,13\}$
12. $\{1,11,12,13,15\}$
13. $\{2,3,4,7,15\}$
14. $\{2,5,8,9,13\}$
15. $\{3,5,7,11,15\}$
16. $\{3,5,9,11,13\}$
17. $\{3,6,7,14,15\}$
18. $\{3,6,9,13,14\}$
19. $\{3,6,13,14,15\}$
20. $\{3,9,11,13,14\}$
21. $\{3,11,13,14,15\}$
22. $\{4,10,11,12,14\}$
23. $\{4,11,12,14,15\}$
24. $\{5,6,7,15,16\}$
25. $\{5,6,9,13,16\}$
26. $\{5,6,13,15,16\}$
27. $\{5,7,11,15,16\}$
28. $\{5,11,13,15,16\}$
29. $\{6,7,14,15,16\}$
30. $\{6,9,13,14,16\}$
31. $\{6,13,14,15,16\}$
32. $\{7,11,14,15,16\}$
33. $\{8,10,11,12,16\}$
34. $\{8,11,12,13,16\}$
35. $\{9,11,13,14,16\}$
36. $\{10,11,12,14,16\}$
37. $\{1,2,3,5,13,15\}$
38. $\{1,3,5,11,13,15\}$
39. $\{2,3,5,6,7,9\}$
40. $\{2,3,5,6,7,15\}$
41. $\{2,3,5,6,9,13\}$
42. $\{2,3,5,6,13,15\}$
43. $\{3,4,7,10,11,14\}$
44. $\{3,4,7,11,14,15\}$
45. $\{3,5,6,7,9,10\}$
46. $\{3,5,7,9,10,11\}$
47. $\{3,6,7,9,10,14\}$
48. $\{3,7,9,10,11,14\}$
49. $\{5,6,7,9,10,16\}$
50. $\{5,7,9,10,11,16\}$
51. $\{5,8,9,10,11,16\}$
52. $\{5,8,9,11,13,16\}$
53. $\{6,7,9,10,14,16\}$
54. $\{7,9,10,11,14,16\}$
55. $\{11,12,13,14,15,16\}$

$\mathcal{C}(\dot{H}_5)$: 48 maximal cliques

1. $\{1,2,3,15,16\}$
2. $\{1,2,6,7,9\}$
3. $\{1,2,6,9,14\}$
4. $\{1,2,6,14,16\}$
5. $\{1,2,14,15,16\}$
6. $\{1,3,12,15,16\}$
7. $\{1,12,14,15,16\}$
8. $\{2,3,6,7,10\}$
9. $\{2,3,6,10,11\}$
10. $\{2,3,6,11,16\}$
11. $\{2,3,11,15,16\}$
12. $\{2,5,6,7,16\}$
13. $\{2,5,6,9,14\}$
14. $\{2,5,6,10,11\}$
15. $\{2,11,14,15,16\}$
16. $\{3,6,7,10,12\}$
17. $\{3,6,10,11,12\}$
18. $\{3,6,11,12,16\}$
19. $\{3,11,12,15,16\}$
20. $\{4,5,6,7,9\}$
21. $\{4,5,6,7,16\}$
22. $\{4,5,6,14,16\}$
23. $\{4,5,8,13,14\}$
24. $\{4,5,13,14,16\}$
25. $\{4,8,12,13,14\}$
26. $\{4,12,13,14,16\}$
27. $\{5,6,8,9,10\}$
28. $\{5,6,8,10,11\}$
29. $\{5,6,8,11,14\}$
30. $\{5,8,11,13,14\}$
31. $\{5,11,13,14,16\}$
32. $\{6,7,9,10,12\}$
33. $\{6,8,9,10,12\}$
34. $\{6,8,10,11,12\}$
35. $\{6,8,11,12,14\}$
36. $\{6,11,12,14,16\}$
37. $\{8,11,12,13,14\}$
38. $\{1,2,3,6,7,16\}$
39. $\{1,3,6,7,12,16\}$
40. $\{1,4,6,7,9,12\}$
41. $\{1,4,6,7,12,16\}$
42. $\{1,4,6,9,12,14\}$
43. $\{1,4,6,12,14,16\}$
44. $\{2,5,6,7,9,10\}$
45. $\{2,5,6,11,14,16\}$
46. $\{4,5,6,8,9,14\}$
47. $\{4,6,8,9,12,14\}$
48. $\{11,12,13,14,15,16\}$

$\mathcal{C}(\dot{H}_6)$: 1 maximal cliques

1. $\varnothing$

$\mathcal{C}(\dot{H}_7)$: 11 maximal cliques

1. $\{1,2,4,5,9\}$
2. $\{1,2,4,8,9\}$
3. $\{1,3,4,8,9\}$
4. $\{2,4,5,9,10\}$
5. $\{2,4,8,9,10\}$
6. $\{3,4,6,7,9\}$
7. $\{3,4,6,8,9\}$
8. $\{4,5,7,9,10\}$
9. $\{4,6,7,9,10\}$
10. $\{4,6,8,9,10\}$
11. $\{1,3,4,5,7,9\}$

$\mathcal{C}(\dot{H}_8)$: 2 maximal cliques

1. $\{1,2\}$
2. $\{3,4\}$

$\mathcal{C}(\dot{H}_9)$: 2 maximal cliques

1. $\{3\}$
2. $\{1, 2\}$

$\mathcal{C}(\dot{H}_{10})$: 3 maximal cliques

1. $\{1, 4\}$
2. $\{2, 3\}$
3. $\{2, 4\}$

$\mathcal{C}(\dot{H}_{11})$: 24 maximal cliques

1. $\{1, 2, 5, 6, 7\}$
2. $\{1, 2, 5, 6, 13\}$
3. $\{1, 2, 5, 7, 11\}$
4. $\{1, 2, 5, 11, 13\}$
5. $\{1, 3, 5, 6, 7\}$
6. $\{1, 3, 5, 6, 13\}$
7. $\{1, 3, 5, 11, 13\}$
8. $\{3, 4, 5, 6, 7\}$
9. $\{3, 4, 5, 6, 13\}$
10. $\{3, 4, 5, 7, 11\}$
11. $\{3, 4, 6, 9, 13\}$
12. $\{3, 4, 8, 9, 13\}$
13. $\{3, 5, 8, 10, 11\}$
14. $\{4, 6, 9, 12, 13\}$
15. $\{4, 8, 9, 12, 13\}$
16. $\{5, 7, 10, 11, 12\}$
17. $\{5, 8, 10, 11, 12\}$
18. $\{1, 3, 5, 7, 10, 11\}$
19. $\{2, 4, 5, 6, 7, 12\}$
20. $\{2, 4, 5, 6, 12, 13\}$
21. $\{2, 4, 5, 7, 11, 12\}$
22. $\{2, 4, 5, 11, 12, 13\}$
23. $\{3, 4, 5, 8, 11, 13\}$
24. $\{4, 5, 8, 11, 12, 13\}$

$\mathcal{C}(\dot{H}_{12})$: 19 maximal cliques

1. $\{1, 2, 3, 4, 12\}$
2. $\{1, 2, 4, 8, 12\}$
3. $\{1, 2, 8, 11, 12\}$
4. $\{2, 3, 5, 6, 12\}$
5. $\{2, 3, 9, 11, 12\}$
6. $\{2, 4, 6, 7, 9\}$
7. $\{2, 4, 7, 8, 9\}$
8. $\{2, 4, 8, 9, 12\}$
9. $\{2, 8, 9, 11, 12\}$
10. $\{3, 5, 6, 10, 12\}$
11. $\{3, 5, 10, 11, 12\}$
12. $\{3, 9, 10, 11, 12\}$
13. $\{4, 6, 7, 9, 10\}$
14. $\{4, 7, 8, 9, 10\}$
15. $\{4, 8, 9, 10, 12\}$
16. $\{8, 9, 10, 11, 12\}$
17. $\{1, 2, 3, 5, 11, 12\}$
18. $\{2, 3, 4, 6, 9, 12\}$
19. $\{3, 4, 6, 9, 10, 12\}$

$\mathcal{C}(\dot{H}_{13})$: 4 maximal cliques

1. $\{1, 2\}$
2. $\{1, 4\}$
3. $\{2, 3\}$
4. $\{3, 4\}$

$\mathcal{C}(\dot{H}_{14})$: 34 maximal cliques

1. $\{1, 3, 4, 7, 14\}$
2. $\{1, 3, 5, 8, 14\}$
3. $\{1, 3, 13, 14, 15\}$
4. $\{1, 4, 7, 11, 14\}$
5. $\{1, 7, 11, 13, 14\}$
6. $\{1, 11, 13, 14, 15\}$
7. $\{2, 4, 5, 6, 9\}$
8. $\{2, 4, 5, 6, 15\}$
9. $\{2, 4, 5, 9, 14\}$
10. $\{2, 4, 6, 12, 15\}$
11. $\{2, 4, 9, 12, 14\}$
12. $\{2, 4, 10, 11, 12\}$
13. $\{3, 4, 5, 6, 9\}$
14. $\{3, 4, 5, 6, 15\}$
15. $\{3, 4, 5, 9, 14\}$
16. $\{3, 4, 6, 9, 12\}$
17. $\{3, 4, 6, 12, 15\}$
18. $\{3, 4, 12, 14, 15\}$
19. $\{3, 5, 8, 9, 14\}$
20. $\{3, 7, 8, 9, 14\}$
21. $\{3, 7, 12, 13, 14\}$
22. $\{3, 12, 13, 14, 15\}$
23. $\{4, 7, 9, 10, 12\}$
24. $\{4, 7, 10, 11, 12\}$
25. $\{4, 7, 11, 12, 14\}$
26. $\{7, 11, 12, 13, 14\}$
27. $\{11, 12, 13, 14, 15\}$
28. $\{1, 2, 4, 5, 14, 15\}$
29. $\{1, 2, 4, 11, 14, 15\}$
30. $\{1, 3, 4, 5, 14, 15\}$
31. $\{1, 3, 7, 8, 13, 14\}$
32. $\{2, 4, 6, 9, 10, 12\}$
33. $\{2, 4, 11, 12, 14, 15\}$
34. $\{3, 4, 7, 9, 12, 14\}$

$\mathcal{C}(\dot{H}_{15})$: 28 maximal cliques

1. $\{1,2,3,7,16\}$
2. $\{1,2,4,8,16\}$
3. $\{1,2,14,15,16\}$
4. $\{1,3,7,12,16\}$
5. $\{1,3,12,15,16\}$
6. $\{1,12,14,15,16\}$
7. $\{2,4,8,11,16\}$
8. $\{2,4,11,15,16\}$
9. $\{2,11,14,15,16\}$
10. $\{5,6,7,9,14\}$
11. $\{5,6,8,10,14\}$
12. $\{5,6,13,14,16\}$
13. $\{5,7,9,12,14\}$
14. $\{5,9,12,13,14\}$
15. $\{5,12,13,14,16\}$
16. $\{6,8,10,11,14\}$
17. $\{6,10,11,13,14\}$
18. $\{6,11,13,14,16\}$
19. $\{11,13,14,15,16\}$
20. $\{12,13,14,15,16\}$
21. $\{1,2,3,4,15,16\}$
22. $\{1,2,7,8,14,16\}$
23. $\{1,5,7,8,14,16\}$
24. $\{1,5,7,12,14,16\}$
25. $\{2,6,7,8,14,16\}$
26. $\{2,6,8,11,14,16\}$
27. $\{5,6,7,8,14,16\}$
28. $\{5,6,9,10,13,14\}$

$\mathcal{C}(\dot{H}_{16})$: 25 maximal cliques

1. $\{1,2,3,6,14\}$
2. $\{1,2,4,5,14\}$
3. $\{1,2,10,13,14\}$
4. $\{1,3,6,11,14\}$
5. $\{1,3,11,13,14\}$
6. $\{1,5,6,8,10\}$
7. $\{1,6,8,10,11\}$
8. $\{1,6,10,11,14\}$
9. $\{1,10,11,13,14\}$
10. $\{2,4,5,12,14\}$
11. $\{2,4,12,13,14\}$
12. $\{2,5,6,7,10\}$
13. $\{2,5,7,10,12\}$
14. $\{2,5,10,12,14\}$
15. $\{2,10,12,13,14\}$
16. $\{5,6,9,10,14\}$
17. $\{5,7,9,10,12\}$
18. $\{5,9,10,12,14\}$
19. $\{6,8,9,10,11\}$
20. $\{6,9,10,11,14\}$
21. $\{9,10,11,13,14\}$
22. $\{9,10,12,13,14\}$
23. $\{1,2,3,4,13,14\}$
24. $\{1,2,5,6,10,14\}$
25. $\{5,6,7,8,9,10\}$

$\mathcal{C}(\dot{H}_{17})$: 46 maximal cliques

1. $\{1,3,4,5,6\}$
2. $\{1,3,4,5,14\}$
3. $\{1,3,4,7,14\}$
4. $\{1,3,5,6,13\}$
5. $\{1,3,7,8,14\}$
6. $\{1,4,7,12,14\}$
7. $\{1,7,8,12,14\}$
8. $\{2,4,5,9,10\}$
9. $\{2,4,5,9,14\}$
10. $\{2,4,9,10,11\}$
11. $\{2,4,9,11,14\}$
12. $\{2,4,11,12,14\}$
13. $\{2,6,11,12,13\}$
14. $\{2,11,12,13,14\}$
15. $\{3,4,5,6,15\}$
16. $\{3,5,6,13,15\}$
17. $\{4,5,9,10,15\}$
18. $\{4,5,12,14,15\}$
19. $\{5,6,12,13,15\}$
20. $\{5,8,9,10,15\}$
21. $\{5,8,10,12,15\}$
22. $\{6,11,12,13,15\}$
23. $\{7,8,9,10,15\}$
24. $\{7,8,10,12,15\}$
25. $\{7,8,12,14,15\}$
26. $\{11,12,13,14,15\}$
27. $\{1,2,4,5,6,12\}$
28. $\{1,2,4,5,12,14\}$
29. $\{1,2,5,6,12,13\}$
30. $\{1,2,5,12,13,14\}$
31. $\{1,3,5,8,13,14\}$
32. $\{1,5,8,12,13,14\}$
33. $\{2,4,5,6,10,12\}$
34. $\{2,4,6,10,11,12\}$
35. $\{3,4,5,9,14,15\}$
36. $\{3,4,7,9,14,15\}$
37. $\{3,5,8,9,14,15\}$
38. $\{3,5,8,13,14,15\}$
39. $\{3,7,8,9,14,15\}$
40. $\{4,5,6,10,12,15\}$
41. $\{4,6,10,11,12,15\}$
42. $\{4,7,9,10,11,15\}$
43. $\{4,7,9,11,14,15\}$
44. $\{4,7,10,11,12,15\}$
45. $\{4,7,11,12,14,15\}$
46. $\{5,8,12,13,14,15\}$

$\mathcal{C}(\dot{H}_{18})$: 8 maximal cliques

1. $\{1,2,3,9,10\}$
2. $\{1,3,4,9,10\}$
3. $\{2,3,8,9,10\}$
4. $\{2,7,8,9,10\}$
5. $\{3,4,6,9,10\}$
6. $\{3,6,8,9,10\}$
7. $\{4,5,6,9,10\}$
8. $\{5,6,7,8,9,10\}$

$\mathcal{C}(\dot{H}_{19})$: 16 maximal cliques

1. $\{1,2,3,15,16\}$
2. $\{1,2,10,15,16\}$
3. $\{1,3,9,15,16\}$
4. $\{1,9,10,15,16\}$
5. $\{2,3,8,15,16\}$
6. $\{2,8,10,15,16\}$
7. $\{3,8,9,15,16\}$
8. $\{4,5,6,7,12\}$
9. $\{4,5,6,11,12\}$
10. $\{4,6,7,12,13\}$
11. $\{4,6,11,12,13\}$
12. $\{5,6,7,12,14\}$
13. $\{6,7,12,13,14\}$
14. $\{6,11,12,13,14\}$
15. $\{3,5,6,11,12,14\}$
16. $\{7,8,9,10,15,16\}$

$\mathcal{C}(\dot{H}_{20})$: 7 maximal cliques

1. $\{1,2,3,8,9\}$
2. $\{1,2,6,8,9\}$
3. $\{1,6,7,8,9\}$
4. $\{2,3,5,8,9\}$
5. $\{2,5,6,8,9\}$
6. $\{3,4,5,8,9\}$
7. $\{4,5,6,7,8,9\}$

$\mathcal{C}(\dot{H}_{21})$: 1 maximal cliques

1. $\varnothing$

$\mathcal{C}(\dot{H}_{22})$: 13 maximal cliques

1. $\{1,3,5,7,11\}$
2. $\{1,5,7,9,12\}$
3. $\{1,5,7,11,12\}$
4. $\{2,4,6,8,12\}$
5. $\{2,6,8,10,11\}$
6. $\{2,6,8,11,12\}$
7. $\{3,5,7,8,11\}$
8. $\{3,5,8,10,11\}$
9. $\{4,6,7,8,12\}$
10. $\{4,6,7,9,12\}$
11. $\{5,6,7,9,12\}$
12. $\{5,6,8,10,11\}$
13. $\{5,6,7,8,11,12\}$

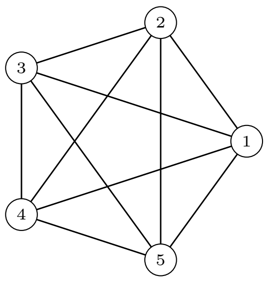
$\mathcal{C}(\dot{H}_{23})$: 1 maximal cliques

1. $\varnothing$

E The 28 switching classes

Positive edges are solid and negative edges are dashed red. A code beginning with n : represents the upper triangle of the adjacency matrix of a signed graph of order n: the entries $a_{12}, a_{13}, \ldots, a_{1n}, a_{23}, \ldots, a_{n-1,n}$ are concatenated, row by row, into a single string. The symbols `+`, `-`, and `0` stand for 1, -1, and 0, respectively. The raw multiplicity of a representative is the number of raw extensions that reduce to its switching-isomorphism class before duplicate removal.

$\dot{M}_1$ (order 5; negation $\dot{M}_4$)

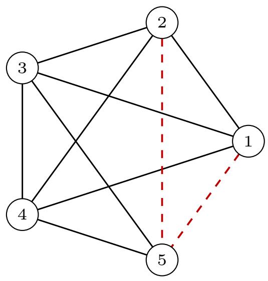


`5:++++++++++`

$\phi(x) = x^5 - 10x^3 - 20x^2 - 15x - 4$

Sources: $-\dot{H}_{21}$; raw multiplicity 1.

$\dot{M}_2$ (order 5; negation $\dot{M}_3$)

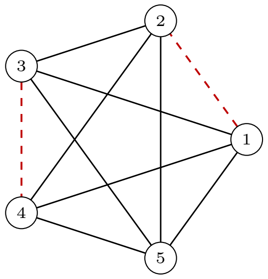


`5:+++-++-+++`

$\phi(x) = x^5 - 10x^3 - 4x^2 + 17x + 12$

Sources: $-\dot{H}_{23}$; raw multiplicity 1.

$\dot{M}_3$ (order 5; negation $\dot{M}_2$)

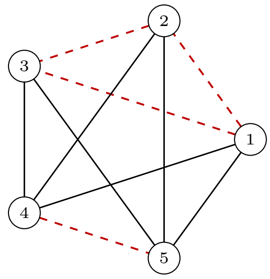


`5:-++++++-++`

$$\phi(x) = x^5 - 10x^3 + 4x^2 + 17x - 12$$

Sources: $\dot{H}_{23}$; raw multiplicity 1.

$\dot{M}_4$ (order 5; negation $\dot{M}_1$)

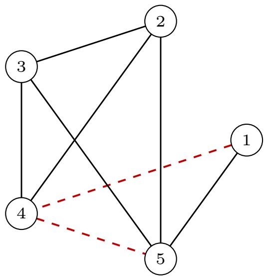


`5:-++-++++-`

$$\phi(x) = x^5 - 10x^3 + 20x^2 - 15x + 4$$

Sources: $\dot{H}_{21}$; raw multiplicity 1.

$\dot{M}_5$ (order 5; negation $\dot{M}_6$)

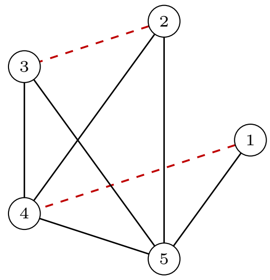


`5:00-++++++-`

$$\phi(x) = x^5 - 8x^3 - 2x^2 + 15x + 10$$

Sources: $-\dot{H}_6$; raw multiplicity 1.

$\dot{M}_6$ (order 5; negation $\dot{M}_5$)

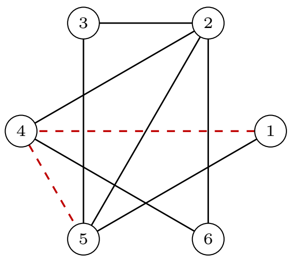


`5:00-+-+++++`

$$\phi(x) = x^5 - 8x^3 + 2x^2 + 15x - 10$$

Sources: $\dot{H}_6$; raw multiplicity 1.

$\dot{M}_7$ (order 6; negation $\dot{M}_8$)

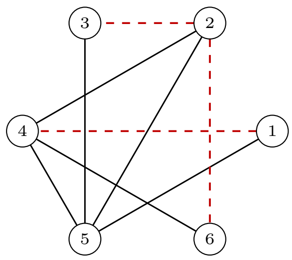


`6:00-+0++++0+0-+0`

$$\phi(x) = x\left(x^5 - 9x^3 - 4x^2 + 21x + 18\right)$$

Sources: $-\dot{H}_9$; raw multiplicity 1.

$\dot{M}_8$ (order 6; negation $\dot{M}_7$)

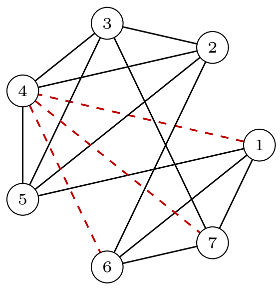


`6:00-+0-++-0+0++0`

$$\phi(x) = x\left(x^5 - 9x^3 + 4x^2 + 21x - 18\right)$$

Sources: $\dot{H}_9$; raw multiplicity 1.

$\dot{M}_9$ (order 7; negation $\dot{M}_{10}$)

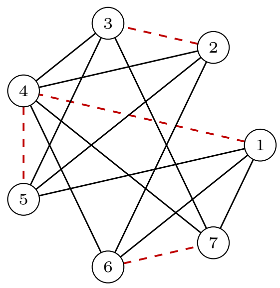


`7:00-+++++++0++0++-00+`

$$\phi(x) = x^2\left(x^5 - 15x^3 - 10x^2 + 60x + 72\right)$$

Sources: $-\dot{H}_8$, $-\dot{H}_9$; raw multiplicity 3.

$\dot{M}_{10}$ (order 7; negation $\dot{M}_9$)

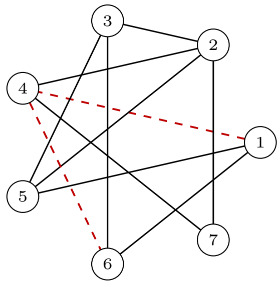


`7:00-+++-+++0++0+-++00-`

$$\phi(x) = x^2\left(x^5 - 15x^3 + 10x^2 + 60x - 72\right)$$

Sources: $\dot{H}_8$, $\dot{H}_9$; raw multiplicity 3.

$\dot{M}_{11}$ (order 7; negation $\dot{M}_{12}$)

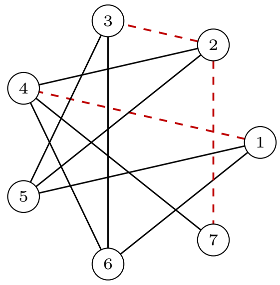


7:00-++0+++0+0++00-+000

$$\phi(x) = x^2\left(x^5 - 11x^3 - 6x^2 + 32x + 32\right)$$

Sources: $-\dot{H}_{10}$, $-\dot{H}_{13}$; raw multiplicity 7.

$\dot{M}_{12}$ (order 7; negation $\dot{M}_{11}$)

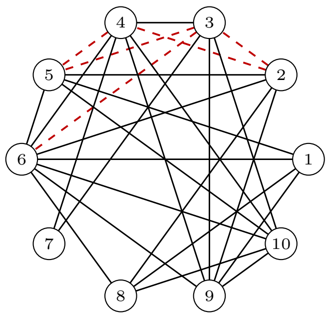


7:00-++0-++0-0++00++000

$$\phi(x) = x^2\left(x^5 - 11x^3 + 6x^2 + 32x - 32\right)$$

Sources: $\dot{H}_{10}$, $\dot{H}_{13}$; raw multiplicity 7.

$\dot{M}_{13}$ (order 10; negation $\dot{M}_{14}$)

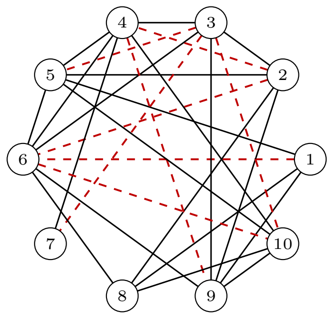


10:000++0++0-++0++0+-+0++-++0+++000+0+++0000++

$$\phi(x) = x^5\left(x^5 - 28x^3 - 26x^2 + 203x + 330\right)$$

Sources: $-\dot{H}_1$, $-\dot{H}_4$, $-\dot{H}_5$, $-\dot{H}_7$, $-\dot{H}_{11}$, $-\dot{H}_{12}$, $-\dot{H}_{14}$, $-\dot{H}_{15}$, $-\dot{H}_{18}$, $-\dot{H}_{19}$, $-\dot{H}_{20}$, $-\dot{H}_{22}$, $\dot{H}_2$, $\dot{H}_3$, $\dot{H}_{16}$; raw multiplicity 62.

$\dot{M}_{14}$ (order 10; negation $\dot{M}_{13}$)

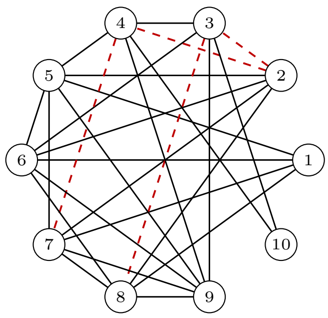


10:000+-0++0+-+-0++0+-+-0+-+++0-++000+0++-0000++

$$\phi(x) = x^5\left(x^5 - 28x^3 + 26x^2 + 203x - 330\right)$$

Sources: $-\dot{H}_2$, $-\dot{H}_3$, $-\dot{H}_{16}$, $\dot{H}_1$, $\dot{H}_4$, $\dot{H}_5$, $\dot{H}_7$, $\dot{H}_{11}$, $\dot{H}_{12}$, $\dot{H}_{14}$, $\dot{H}_{15}$, $\dot{H}_{18}$, $\dot{H}_{19}$, $\dot{H}_{20}$, $\dot{H}_{22}$; raw multiplicity 62.

$\dot{M}_{15}$ (order 10; negation $\dot{M}_{18}$)

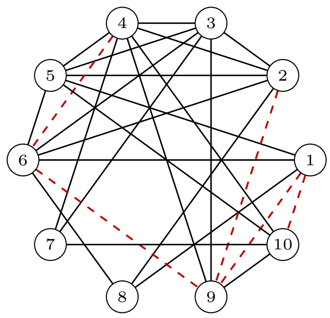


10:000++++00-++++00+0+0-+++0-0++++0+00++0++0+00

$$\phi(x) = x^5\left(x^5 - 27x^3 - 26x^2 + 180x + 288\right)$$

Sources: $-\dot{H}_4$, $-\dot{H}_5$, $-\dot{H}_7$, $-\dot{H}_{11}$, $-\dot{H}_{12}$, $-\dot{H}_{14}$, $-\dot{H}_{19}$, $-\dot{H}_{20}$, $\dot{H}_2$, $\dot{H}_3$, $\dot{H}_{16}$, $\dot{H}_{17}$; raw multiplicity 52.

$\dot{M}_{16}$ (order 10; negation $\dot{M}_{17}$)

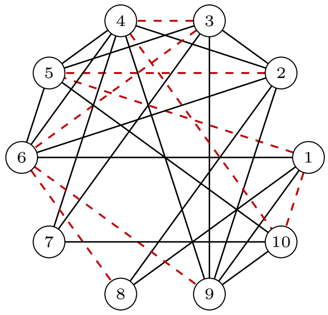


10:000++0+-++++0+-0++++0+0+-+0+++000+0+-000+00+

$$\phi(x) = x^5\left(x^5 - 27x^3 - 24x^2 + 193x + 308\right)$$

Sources: $-\dot{H}_1$, $-\dot{H}_4$, $-\dot{H}_5$, $-\dot{H}_7$, $-\dot{H}_{11}$, $-\dot{H}_{12}$, $-\dot{H}_{14}$, $-\dot{H}_{15}$, $-\dot{H}_{18}$, $-\dot{H}_{19}$, $-\dot{H}_{20}$, $-\dot{H}_{22}$, $\dot{H}_2$, $\dot{H}_3$, $\dot{H}_{16}$, $\dot{H}_{17}$; raw multiplicity 130.

$\dot{M}_{17}$ (order 10; negation $\dot{M}_{16}$)

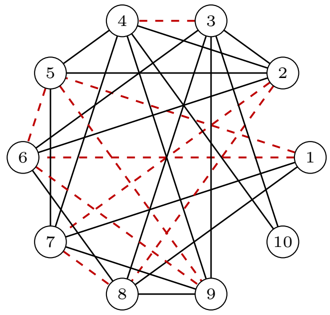


10:000-+0++-++-+0++0-+-+0+0+++0+-+000+0-000+00+

$$\phi(x) = x^5\left(x^5 - 27x^3 + 24x^2 + 193x - 308\right)$$

Sources: $-\dot{H}_2$, $-\dot{H}_3$, $-\dot{H}_{16}$, $-\dot{H}_{17}$, $\dot{H}_1$, $\dot{H}_4$, $\dot{H}_5$, $\dot{H}_7$, $\dot{H}_{11}$, $\dot{H}_{12}$, $\dot{H}_{14}$, $\dot{H}_{15}$, $\dot{H}_{18}$, $\dot{H}_{19}$, $\dot{H}_{20}$, $\dot{H}_{22}$; raw multiplicity 130.

$\dot{M}_{18}$ (order 10; negation $\dot{M}_{15}$)

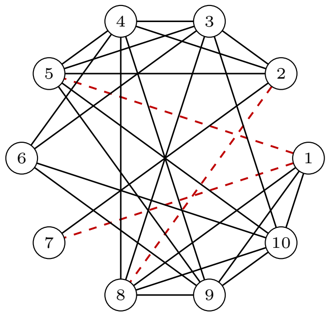


10:000-++00++++-00-0+0++++0+0++-+0-00+-0-+0+00

$$\phi(x) = x^5\left(x^5 - 27x^3 + 26x^2 + 180x - 288\right)$$

Sources: $-\dot{H}_2$, $-\dot{H}_3$, $-\dot{H}_{16}$, $-\dot{H}_{17}$, $\dot{H}_4$, $\dot{H}_5$, $\dot{H}_7$, $\dot{H}_{11}$, $\dot{H}_{12}$, $\dot{H}_{14}$, $\dot{H}_{19}$, $\dot{H}_{20}$; raw multiplicity 52.

$\dot{M}_{19}$ (order 10; negation $\dot{M}_{20}$)

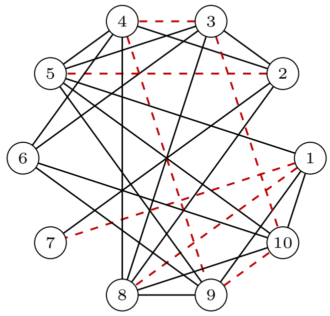


10:000-0-++++++0+-00+++0+0+++0++0000++00++000+++

$$\phi(x) = x^5\left(x^5 - 26x^3 - 28x^2 + 144x + 224\right)$$

Sources: $-\dot{H}_1$, $-\dot{H}_4$, $-\dot{H}_5$, $-\dot{H}_7$, $-\dot{H}_{11}$, $-\dot{H}_{12}$, $-\dot{H}_{14}$, $-\dot{H}_{15}$, $-\dot{H}_{18}$, $-\dot{H}_{19}$, $\dot{H}_2$, $\dot{H}_3$, $\dot{H}_{16}$, $\dot{H}_{17}$; raw multiplicity 52.

$\dot{M}_{20}$ (order 10; negation $\dot{M}_{19}$)

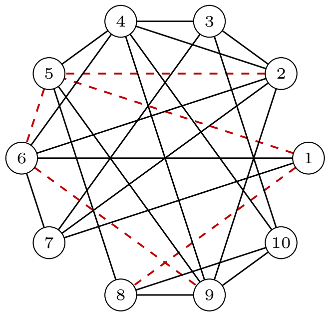


10:000+0-++++-0++00-++0+0-++0+-0000++00++000++-

$$\phi(x) = x^5\left(x^5 - 26x^3 + 28x^2 + 144x - 224\right)$$

Sources: $-\dot{H}_2$, $-\dot{H}_3$, $-\dot{H}_{16}$, $-\dot{H}_{17}$, $\dot{H}_1$, $\dot{H}_4$, $\dot{H}_5$, $\dot{H}_7$, $\dot{H}_{11}$, $\dot{H}_{12}$, $\dot{H}_{14}$, $\dot{H}_{15}$, $\dot{H}_{18}$, $\dot{H}_{19}$; raw multiplicity 52.

$\dot{M}_{21}$ (order 10; negation $\dot{M}_{22}$)

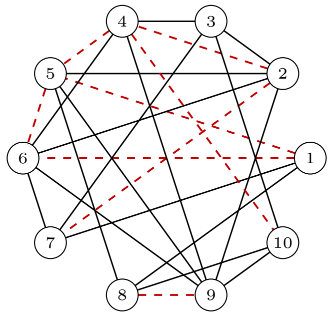


10:000-++-00++-++0+0+00+00+++00++-0++0+0-0000+++

$$\phi(x) = x^5\left(x^5 - 25x^3 - 20x^2 + 165x + 242\right)$$

Sources: $-\dot{H}_4$, $-\dot{H}_5$, $-\dot{H}_{11}$, $-\dot{H}_{12}$, $-\dot{H}_{14}$, $\dot{H}_2$, $\dot{H}_3$, $\dot{H}_{16}$, $\dot{H}_{17}$; raw multiplicity 27.

$\dot{M}_{22}$ (order 10; negation $\dot{M}_{21}$)

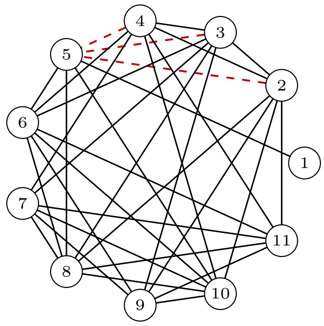


10:000-++00+-++-0+0+00+00+-+00+-0++0+0+0000-++

$$\phi(x) = x^5\left(x^5 - 25x^3 + 20x^2 + 165x - 242\right)$$

Sources: $-\dot{H}_2$, $-\dot{H}_3$, $-\dot{H}_{16}$, $-\dot{H}_{17}$, $\dot{H}_4$, $\dot{H}_5$, $\dot{H}_{11}$, $\dot{H}_{12}$, $\dot{H}_{14}$; raw multiplicity 27.

$\dot{M}_{23}$ (order 11; negation $\dot{M}_{24}$)

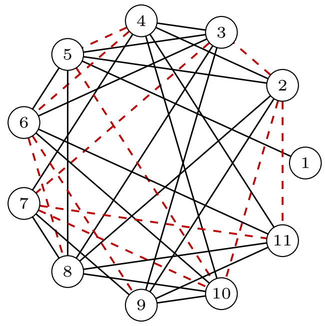


11:000+000000++-00+++++-++++00-++00+++0+0+00++++++++0++++0

$$\phi(x) = x^6\left(x^5 - 34x^3 - 54x^2 + 189x + 378\right)$$

Sources: $-\dot{H}_1$, $-\dot{H}_4$, $-\dot{H}_5$, $-\dot{H}_7$, $-\dot{H}_{11}$, $-\dot{H}_{12}$, $-\dot{H}_{14}$, $-\dot{H}_{15}$, $-\dot{H}_{18}$, $-\dot{H}_{19}$, $-\dot{H}_{20}$, $\dot{H}_2$, $\dot{H}_3$, $\dot{H}_{16}$; raw multiplicity 19.

$\dot{M}_{24}$ (order 11; negation $\dot{M}_{23}$)

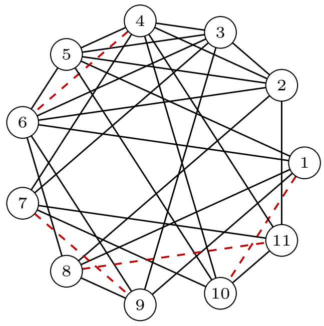


11:000+000000-++00++-+++-++00-+00+++0+0-00-++++-0++++0

$$\phi(x) = x^6\left(x^5 - 34x^3 + 54x^2 + 189x - 378\right)$$

Sources: $-\dot{H}_2$, $-\dot{H}_3$, $-\dot{H}_{16}$, $\dot{H}_1$, $\dot{H}_4$, $\dot{H}_5$, $\dot{H}_7$, $\dot{H}_{11}$, $\dot{H}_{12}$, $\dot{H}_{14}$, $\dot{H}_{15}$, $\dot{H}_{18}$, $\dot{H}_{19}$, $\dot{H}_{20}$; raw multiplicity 19.

$\dot{M}_{25}$ (order 11; negation $\dot{M}_{27}$)

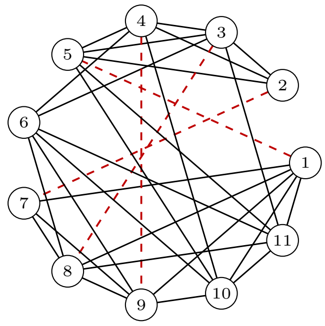


11:000++0++-0++++0+00+++++0+00+-+00+++000+00++000-+++0-00+

$$\phi(x) = x^6\left(x^5 - 31x^3 - 30x^2 + 252x + 432\right)$$

Sources: $-\dot{H}_1$, $-\dot{H}_4$, $-\dot{H}_5$, $-\dot{H}_{11}$, $-\dot{H}_{12}$, $-\dot{H}_{14}$, $-\dot{H}_{15}$, $-\dot{H}_{22}$, $\dot{H}_2$, $\dot{H}_3$, $\dot{H}_{17}$; raw multiplicity 46.

$\dot{M}_{26}$ (order 11; negation $\dot{M}_{28}$)

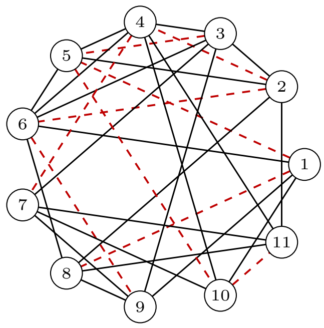


11:000-0++++++++0-0000+++0-00+++00-+00000++0++++++00+0++0+

$$\phi(x) = x^6\left(x^5 - 31x^3 - 30x^2 + 252x + 432\right)$$

Sources: $-\dot{H}_1$, $-\dot{H}_4$, $-\dot{H}_5$, $-\dot{H}_{11}$, $-\dot{H}_{12}$, $-\dot{H}_{14}$, $-\dot{H}_{15}$, $\dot{H}_2$, $\dot{H}_3$, $\dot{H}_{16}$, $\dot{H}_{17}$; raw multiplicity 43.

$\dot{M}_{27}$ (order 11; negation $\dot{M}_{25}$)

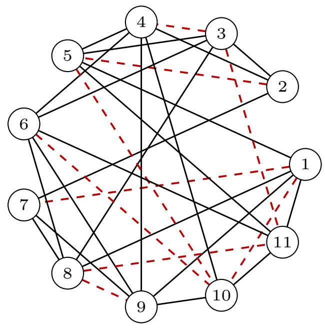


11:000-+0-++0+-+-0+00++-++0+00++-00+++000-00+-000++++0+00-

$$\phi(x) = x^6\left(x^5 - 31x^3 + 30x^2 + 252x - 432\right)$$

Sources: $-\dot{H}_2$, $-\dot{H}_3$, $-\dot{H}_{17}$, $\dot{H}_1$, $\dot{H}_4$, $\dot{H}_5$, $\dot{H}_{11}$, $\dot{H}_{12}$, $\dot{H}_{14}$, $\dot{H}_{15}$, $\dot{H}_{22}$; raw multiplicity 46.

$\dot{M}_{28}$ (order 11; negation $\dot{M}_{26}$)



11:000+0-++-+++-0+0000-++0+00-++00++00000-+0++-+++00-0-+0+

$$\phi(x) = x^6\left(x^5 - 31x^3 + 30x^2 + 252x - 432\right)$$

Sources: $-\dot{H}_2$, $-\dot{H}_3$, $-\dot{H}_{16}$, $-\dot{H}_{17}$, $\dot{H}_1$, $\dot{H}_4$, $\dot{H}_5$, $\dot{H}_{11}$, $\dot{H}_{12}$, $\dot{H}_{14}$, $\dot{H}_{15}$; raw multiplicity 43.