# Computational support for the strict golden-section bound on the second largest eigenvalue

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**Abstract**

Let $\sigma = (\sqrt{5}-1)/2$. This report records the exact computations supporting the determination of the minimal forbidden graphs for the inequality $\lambda_2 < \sigma$, where $\lambda_2$ is the second largest eigenvalue of the adjacency matrix of a finite, simple, undirected graph. For each of the nine weighted rooted-tree types representing the desired graphs, we give the parametrized graph, the inequalities tested and the corresponding finite procedure. For $T_1, T_2, \ldots, T_8$, the output is recorded by canonical parameter orders, finite bounds, exact acceptance conditions and counts; these data determine all 6717 graphs without separate tuple files. For $T_9$, we retain the complete algebraic test and all finite bounds, but do not attempt an exhaustive enumeration.

## 1 Purpose of the computation

Simić [2] initiated the determination of the minimal forbidden graphs for $\lambda_2 < \sigma$ and proved that their collection is finite. The companion paper [1] completes the theoretical part by deriving necessary and sufficient inequalities, together with finite parameter bounds, for the weighted-tree types occurring in Simić's reduction. The purpose of this report is to record the exact computations based on those results.

A graph $F$ is *minimal forbidden* if for the second largest eigenvalue $\lambda_2(F) \geq \sigma$ and $\lambda_2(F-v) < \sigma$ for every $v \in V(F)$. A *cograph* is a graph with no induced $P_4$; equivalently, it can be obtained from $K_1$ by successive applications of disjoint unions and joins. Since

$$\lambda_2(P_4) = \sigma, \quad \lambda_2(2K_2) = 1,$$

interlacing shows that every graph satisfying $\lambda_2 < \sigma$ is a cograph and is also $2K_2$-free. The graphs $P_4$ and $2K_2$ are immediate minimal forbidden graphs. Notice that $2K_2$ is itself a cograph, so its exclusion is an additional condition. Simić proved that every other minimal forbidden graph is represented by one of nine canonical weighted rooted-tree types $T_1, T_2, \ldots, T_9$.

The nine trees are displayed in both [1] and [2]. Here we work with the corresponding graphs rather than with the tree drawings. Thus, in Subsection 3.1, $T_i$ denotes the representing weighted tree from those references, and the expression following $T_i$ is the graph represented by its general weighting.

Small outputs are listed explicitly. Larger outputs are compressed by recording their canonical parameter order, exact bounds, acceptance inequalities and number of accepted tuples. Thus, the data given here determine all 6717 graphs represented by $T_1, T_2, \ldots, T_8$. For $T_9$, whose output is much larger, we give the complete finite parameter bounds and the necessary and sufficient inequalities. These yield a terminating exact procedure for determining the corresponding graphs, but no aggregate enumeration is performed.

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The computation has three stages. First, ordered parameter tuples are generated within the proved bounds. Second, the tree-specific inequalities are tested in $\mathbb{Q}(\sigma)$. Third, every accepted tuple is checked independently by computing the positive inertia of the represented graph and of every graph obtained by decreasing one non-root weight by one. The last step also applies the canonical reduction when a weight becomes zero.

# 2 Exact arithmetic and common verification

## 2.1 Arithmetic in $\mathbb{Q}(\sigma)$

Every number is written uniquely as $u + v\sigma$, where $u, v \in \mathbb{Q}$. Since $\sigma^2 = 1 - \sigma$, the arithmetic operations are

$$\begin{aligned}(u + v\sigma) + (x + y\sigma) &= (u + x) + (v + y)\sigma,\\ (u + v\sigma)(x + y\sigma) &= (ux + vy) + (uy + vx - vy)\sigma,\\ (u + v\sigma)^{-1} &= \frac{u - v}{u^2 - uv - v^2} - \frac{v}{u^2 - uv - v^2}\sigma.\end{aligned}$$

The sign is also determined exactly. Indeed, $2(u + v\sigma) = 2u - v + v\sqrt{5}$. If the two terms have opposite signs, their absolute values are compared after squaring. Thus, all comparisons reduce to integer or rational arithmetic.

## 2.2 Common quantities and independent verification

For a cograph $H$, let $\mathbf{j}$ be the all-one column vector and $J$ the all-one matrix of order $|V(H)|$, and write

$$\eta(H) = -1 - \mathbf{j}^{\intercal}\big(A(H) - \sigma I - J\big)^{-1}\mathbf{j}$$

and let $p(H)$ be the positive inertia of $A(H) - \sigma I$. The non-singularity required in the definition of $\eta$ is proved in [1]. For a statement $\mathcal{P}$, put $[\mathcal{P}] = 1$ if $\mathcal{P}$ is true and $[\mathcal{P}] = 0$ otherwise. Put $F = H_1 \nabla H_2 \nabla \cdots \nabla H_k$. The following recursions are used by the verifier:

$$\begin{aligned}\eta(nK_1) &= -\frac{\sigma}{n + \sigma}, & p(nK_1) &= 0,\\ \eta(aK_1 \cup H) &= \frac{\sigma\eta(H)}{\sigma - a\eta(H)}, & p(aK_1 \cup H) &= p(H),\\ \eta(F) &= k - 1 + \sum_{i=1}^{k}\eta(H_i), & p(F) &= \sum_{i=1}^{k}\big(p(H_i) - [\eta(H_i) > 0]\big) + [\eta(F) > 0].\end{aligned}$$

The scalar expressions used repeatedly in Section 3 arise directly from these recursions. Since $\eta(nK_1) = n/(n + \sigma) - 1$, put

$$f(n) = \frac{n}{n + \sigma} \quad (n \geq 0).$$

Deleting a vertex from $nK_1$ changes this contribution by

$$\delta(n) = f(n) - f(n - 1) \quad (n \geq 1).$$

Moreover,

$$D(n_1, n_2, \ldots, n_r) = \sum_{i=1}^{r} f(n_i) - 1 = \eta(K_{n_1,n_2,\ldots,n_r}).$$

Finally, the threshold obtained from the term $mK_1$ is

$$t_m = \frac{\sigma f(m)}{f(m) - \sigma} = \frac{m(1 + \sigma)}{m - 1} \quad (m \geq 2),$$

and put $t_1 = +\infty$. Thus, $f$ records the contribution of an edgeless term, $\delta$ records its change under a unit deletion, $D$ records the $\eta$-parameter of a complete multipartite graph, and $t_m$ is the resulting comparison threshold.

If a vertex deletion changes a positive weight to zero, the corresponding empty term is deleted and the representing tree is reduced before the recursion. Since $\sigma$ is not an eigenvalue of a cograph [1], $p(H) \leq 1$ is equivalent to $\lambda_2(H) < \sigma$. Consequently, a candidate $G$ is accepted precisely when

$$p(G) = 2 \quad \text{and} \quad p(G - v) \leq 1 \quad \text{for every } v \in V(G).$$

The vertices counted by the same weight are interchangeable, so one unit decrease of every non-root weight represents all the vertex-deletion cases up to isomorphism. The exact sign routine and the independent verification procedure are given in Appendix A. In the direct cases of Subsection 3.2, this criterion is applied without further reformulation. For $T_4, T_6, T_8$ and $T_9$, substitution of the parametrized graphs into the recursions converts it into inequalities involving $f, \delta, D$ and $t_m$. Those inequalities generate the finite candidate sets in Section 3, after which the criterion above is applied again as an independent check.

# 3 The tree computations

## 3.1 Parametrized graphs

The symbols $T_1, T_2, \ldots, T_9$ denote the weighted rooted-tree types displayed in [1, 2]; they do not denote the graphs themselves. In each line below, $T_i$ is followed by the graph represented by a general weighting of that tree. The non-root weights are positive integers, and interchangeable weights are written in non-decreasing order. The nine input forms are

$$\begin{aligned}
T_1 :&\quad m_1K_1 \nabla m_2K_1 \nabla (aK_1 \cup K_{x_1,x_2,x_3,x_4}),\\
T_2 :&\quad (aK_1 \cup K_{x_1,x_2}) \nabla \big(bK_1 \cup \big(zK_1 \nabla (cK_1 \cup K_{y_1,y_2})\big)\big),\\
T_3 :&\quad mK_1 \nabla (aK_1 \cup K_{x_1,x_2}) \nabla (bK_1 \cup K_{y_1,y_2,y_3}),\\
T_4 :&\quad mK_1 \nabla (aK_1 \cup K_{n_1,n_2,\ldots,n_p}),\\
T_5 :&\quad (aK_1 \cup K_{x_1,x_2,x_3}) \nabla (bK_1 \cup K_{y_1,y_2,\ldots,y_p}),\\
T_6 :&\quad m_1K_1 \nabla m_2K_1 \nabla \cdots \nabla m_pK_1 \nabla (aK_1 \cup K_{x,y,z}),\\
T_7 :&\quad m_1K_1 \nabla \cdots \nabla m_pK_1 \nabla \big(aK_1 \cup \big(xK_1 \nabla (bK_1 \cup K_{y,z})\big)\big),\\
T_8 :&\quad mK_1 \nabla \big(aK_1 \cup \big(n_1K_1 \nabla n_2K_1 \nabla \cdots \nabla n_qK_1 \nabla C_1 \nabla \cdots \nabla C_p\big)\big),\\
T_9 :&\quad H_1 \nabla H_2 \nabla \cdots \nabla H_p \nabla m_1K_1 \nabla m_2K_1 \nabla \cdots \nabla m_qK_1,
\end{aligned}$$

where $C_i \cong c_iK_1 \cup K_{x_i,y_i}$ and $H_i \cong a_iK_1 \cup K_{x_i,y_i}$.

## 3.2 The direct cases

For $T_1, T_2, T_3, T_5$ and $T_7$, all relevant parameters are small. The program increases the all-one weighting, retains the coordinatewise minimal forbidden tuples and applies the common verification. The output is

| tree | $T_1$ | $T_2$ | $T_3$ | $T_5$ | $T_7$ |
|---|---|---|---|---|---|
| number of graphs | 1 | 3 | 3 | 7 | 7 |

The complete output is recorded compactly below. The graph for $T_1$ is

$$K_1 \nabla K_1 \nabla (K_1 \cup K_4).$$

The three graphs for $T_2$ are

$$\begin{aligned}
&(2K_1 \cup K_2) \nabla \big(K_1 \cup (K_1 \nabla (K_1 \cup K_2))\big),\\
&(K_1 \cup K_{1,2}) \nabla \big(K_1 \cup (K_1 \nabla (K_1 \cup K_2))\big),\\
&(K_1 \cup K_2) \nabla \big(K_1 \cup (2K_1 \nabla (K_1 \cup K_2))\big).
\end{aligned}$$

The three graphs for $T_3$ are

$$4K_1 \,\nabla\, (K_1 \cup K_2) \,\nabla\, (K_1 \cup K_3),$$
$$K_1 \,\nabla\, (K_1 \cup K_{1,2}) \,\nabla\, (K_1 \cup K_3),$$
$$K_1 \,\nabla\, (2K_1 \cup K_2) \,\nabla\, (K_1 \cup K_3).$$

For example, in the first $T_3$ graph the common recursions reduce the condition on the weight $m$ to $f(m-1) < 3\sigma - 1 < f(m)$. The comparison is made exactly in $\mathbb{Q}(\sigma)$:

$$f(3) = \frac{6-3\sigma}{5}, \quad (3\sigma-1) - f(3) = \frac{18\sigma - 11}{5} > 0,$$
$$f(4) = \frac{12-4\sigma}{11}, \quad f(4) - (3\sigma-1) = \frac{23-37\sigma}{11} > 0.$$

Indeed, $18\sigma - 11 = 9\sqrt{5} - 20 > 0$ and $23 - 37\sigma = (83 - 37\sqrt{5})/2 > 0$. Hence, $m = 4$.

For $T_5$, the six graphs with $p = 2$ are

$$(K_1 \cup K_3) \,\nabla\, (K_1 \cup K_{1,3}), \quad (K_1 \cup K_3) \,\nabla\, (K_1 \cup K_{2,2}),$$
$$(K_1 \cup K_{1,1,2}) \,\nabla\, (K_1 \cup K_{1,2}), \quad (K_1 \cup K_{1,1,2}) \,\nabla\, (2K_1 \cup K_2),$$
$$(K_1 \cup K_{1,1,5}) \,\nabla\, (K_1 \cup K_2), \quad (K_1 \cup K_{1,2,2}) \,\nabla\, (K_1 \cup K_2),$$

and the graph with $p = 3$ is

$$(K_1 \cup K_3) \,\nabla\, (K_1 \cup K_3).$$

For $T_7$ and $p = 2$, the six tuples $(m_1, m_2; a, x; b, y, z)$ are

$$(1,1;1,1;1,1,2), \quad (1,1;1,1;2,1,1), \quad (1,1;1,4;1,1,1),$$
$$(1,2;1,2;1,1,1), \quad (2,4;1,1;1,1,1), \quad (3,3;1,1;1,1,1).$$

For $p = 3$, every weight is one.

### 3.3 The tree $T_4$

The computation first gives $a = 1$ and $m \geq 2$. Put $D = D(n_1, n_2, \ldots, n_p)$. A tuple is accepted precisely when

$$t_m < D < t_{m-1}, \quad D - \delta(n_p) < t_m.$$

For fixed $p, m, n_1, n_2, \ldots, n_{p-1}$, the algorithm selects the least $n_p \geq n_{p-1}$ for which $D > t_m$. There is therefore at most one possible last coordinate. The corresponding exact enumeration is given in Appendix B.

The search boxes and outputs are given below. In the third column, $(N_1, N_2, \ldots, N_p)$ means that $n_i \leq N_i$ for every $i$.

| $p$ | range searched for $m$ | coordinate bounds | output |
|---|---|---|---|
| 4 | $2 \leq m \leq 26$ | $(3, 9, 61, 1004)$ | 125 |
| 5 | $2 \leq m \leq 5$ | $(3, 6, 12, 102, 6556)$ | 214 |
| 6 | $2 \leq m \leq 3$ | $(1, 1, 2, 2, 5, 9)$ | 7 |
| 7 | $m = 2$ | $(1, 1, 1, 1, 1, 1, 1)$ | 1 |

Let $N_{p,m}$ denote the number of accepted tuples for fixed $(p, m)$. The non-zero values are

| $m$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 17 | 26 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N_{4,m}$ | 85 | 15 | 6 | 4 | 3 | 2 | 2 | 1 | 1 | 1 | 2 | 1 | 1 | 1 |

and

| $(p, m)$ | $(5,2)$ | $(5,3)$ | $(5,4)$ | $(5,5)$ | $(6,2)$ | $(6,3)$ | $(7,2)$ |
|---|---|---|---|---|---|---|---|
| $N_{p,m}$ | 209 | 3 | 1 | 1 | 6 | 1 | 1 |

.

All unlisted pairs in the search ranges have zero output. The subtotals for $p = 4, 5, 6, 7$ are $125, 214, 7, 1$, respectively. Thus, $T_4$ gives 347 graphs.

### 3.4 The tree $T_6$

Assume first that $a = 1$, and put

$$D = D(x, y, z), \quad s = \sum_{i=1}^{p} f(m_i), \quad h = \frac{\sigma D}{D - \sigma}.$$

The basic conditions are

$$D > \sigma, \quad s > h, \quad s - \delta(m_p) < h.$$

If $D - \delta(z) > \sigma$, one also requires

$$s < \frac{\sigma(D - \delta(z))}{D - \delta(z) - \sigma}.$$

If $D - \delta(z) < \sigma$, there is no additional inequality.

For the enumeration, put

$$\tau = \frac{\sigma s}{s - \sigma}.$$

If $s - \delta(m_p) > \sigma$, put

$$\tau' = \frac{\sigma(s - \delta(m_p))}{s - \delta(m_p) - \sigma};$$

otherwise, put $\tau' = +\infty$. The conditions are equivalent to

$$D(x, y, z) - \delta(z) < \tau < D(x, y, z) < \tau'.$$

For fixed $m_1, m_2, \ldots, m_p, x, y$, the only possible value of $z$ is the least integer $z_0 \geq y$ such that $D(x, y, z_0) > \tau$. The corresponding exact enumeration is given in Appendix B.

For $p = 1$, the search gives

$$6 \leq m_1 \leq 17165, \quad 2 \leq x \leq 31, \quad x \leq y \leq 8070, \quad y \leq z \leq 244224577$$

and exactly 6150 tuples. Let $N_x$ denote their number for a fixed value of $x$. Their exact distribution is

| $x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $N_x$ | 596 | 282 | 134 | 338 | 72 | 43 | 31 | 21 | 4053 | 132 | 75 | 54 | 43 | 35 | 31 |
| $x$ | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 |
| $N_x$ | 27 | 26 | 23 | 22 | 21 | 18 | 16 | 14 | 11 | 10 | 8 | 6 | 4 | 3 | 1 |

and $\sum_{x=2}^{31} N_x = 6150$. All four bounds are attained. The four extremal tuples $(m_1; x, y, z)$ are

$$(17165; 2, 4, 51), \quad (6; 31, 31, 32), \quad (6; 10, 8070, 8071), \quad (6; 10, 4035, 244224577).$$

For $p = 2, 3, 4$, the numbers of tuples are $10, 4, 1$, respectively. There is one additional graph with $a = 2$, namely

$$K_1 \nabla (2K_1 \cup K_3).$$

Consequently, $T_6$ gives $6150 + 10 + 4 + 1 + 1 = 6166$ graphs.

### 3.5 The tree $T_8$

Here, $p \geq 1$, $q \geq 0$, $p + q \geq 2$ and $2p + q \leq 6$. The weights $n_1, n_2, \ldots, n_q$ and the triples defining $C_1, C_2, \ldots, C_p$ are put in canonical non-decreasing order. The computation gives $a = 1$, $m \geq 2$ and

$$C_i \cong K_1 \cup K_{1,t} \quad (1 \leq t \leq 5), \quad \text{or} \quad C_i \cong 2K_1 \cup K_2.$$

Put

$$h = p - 1 + \sum_{j=1}^{q} f(n_j) + \sum_{i=1}^{p} \eta(C_i).$$

The tested conditions are

$$t_m < h < t_{m-1},$$

$$h - \delta(n_q) < t_m \quad (q \geq 1),$$

and

$$h - \eta(C_i) + \eta(C_i - v) < t_m \quad (1 \leq i \leq p,\ v \in V(C_i)).$$

The sharp bounds are $2 \leq m \leq 13$ and, when $q \geq 1$, $n_q \leq 6556$. The output is

| $(p,q)$ | $(1,1)$ | $(1,2)$ | $(1,3)$ | $(1,4)$ | $(2,0)$ | $(2,1)$ | $(2,2)$ | $(3,0)$ |
|---|---|---|---|---|---|---|---|---|
| number | 14 | 22 | 128 | 5 | 6 | 6 | 2 | 0 |

and hence $T_8$ gives 183 graphs. A pseudocode for this finite enumeration is given in Appendix C.

### 3.6 The tree $T_9$

Here,

$$p \geq 1, \quad q \geq 0, \quad p+q \geq 2, \quad 2p+q \leq 55,$$

and, when $q \geq 1$, $1 \leq m_1 \leq m_2 \leq \cdots \leq m_q$. The componentwise minimal triples $(a_i, x_i, y_i)$ for which $\eta(H_i) < 0$ are

$$(1,2,4), \quad (1,3,3), \quad (2,1,2), \quad (3,1,1).$$

Testing the ten unordered joins formed from these triples leaves one solution with two negative $\eta$-parameters:

$$(K_1 \cup K_{2,4}) \,\nabla\, (K_1 \cup K_{2,4}).$$

For all remaining solutions, exactly one graph $H_i$ has negative $\eta$-parameter. Denote it by

$$H_p \cong aK_1 \cup K_{x,y}$$

and put

$$h_{a,x}(y) = -\eta(H_p) = \frac{\sigma D(x,y)}{aD(x,y) - \sigma}.$$

The parameters of $H_p$ satisfy

$$1 \leq a \leq 4, \quad 1 \leq x \leq y, \quad aD(x,y) > \sigma, \quad x \leq 271,$$

with $x \leq 3$ when $a \geq 2$. The remaining graphs $H_i$ are selected from

$$K_1 \cup K_{1,t} \quad (1 \leq t \leq 51), \quad K_1 \cup K_{2,2}, \quad K_1 \cup K_{2,3}, \quad 2K_1 \cup K_2.$$

Put

$$c = \sum_{i=1}^{p-1} (1 + \eta(H_i)), \quad b = c + \sum_{j=1}^{q} f(m_j).$$

For $p = 1$, put $c = 0$. For a positive graph $H$, let $d(H)$ denote the smallest decrease of $1 + \eta(H)$ produced by deleting one vertex. The required exact data are

| $H$ | $1+\eta(H)$ | $d(H)$ |
|---|---|---|
| $K_1 \cup K_{1,t}$ | $2 + (t-2)\sigma$ | $\begin{cases} \sigma^4, & t = 1, \\ \sigma, & 2 \leq t \leq 51, \end{cases}$ |
| $K_1 \cup K_{2,2}$ | $4 + \sigma$ | $2 + \sigma$ |
| $K_1 \cup K_{2,3}$ | $12 + 6\sigma$ | $8 + 5\sigma$ |
| $2K_1 \cup K_2$ | $2$ | $\sigma$ |

and

$$d = \min(\{d(H_i) : 1 \le i < p\} \cup \{\delta(m_q) : q \ge 1\}).$$

The weights $m_1, m_2, \dots, m_q$ are bounded successively as described in Appendix D. Once they are fixed, candidates with $ab \le \sigma$ are discarded. For every remaining candidate, put

$$\tau = \frac{\sigma b}{ab - \sigma}.$$

There is at most one possible value of $y$, determined by

$$D(x, y-1) < \tau < D(x, y).$$

The final tests are

$$b - d < h_{a,x}(y)$$

and, when $a \ge 2$ and $(a-1)D(x, y) > \sigma$,

$$b < h_{a-1,x}(y).$$

These inequalities give a terminating exact test for whether a weighting represented by $T_9$ is minimal forbidden.

To record the lower bound used in [1], consider the restricted case $p = 1$, $q = 33$, $H_1 \cong K_1 \cup K_{2,4}$ and

$$1 \le m_1 \le m_2 \le \cdots \le m_{33} \le 30.$$

Here, $-\eta(H_1) = 23 + 15\sigma$, and every vertex-deleted subgraph of $H_1$ has positive $\eta$-parameter. Consequently, a graph in this restricted family is minimal forbidden precisely when

$$23 + 15\sigma < \sum_{j=1}^{33} f(m_j) < 23 + 15\sigma + \delta(m_{33}).$$

Exact enumeration of the non-decreasing sequences gives 2 219 483 graphs. Since the exceptional graph $(K_1 \cup K_{2,4}) \nabla (K_1 \cup K_{2,4})$ does not belong to this family, $T_9$ gives at least 2 219 484 minimal forbidden graphs. No aggregate enumeration of all graphs represented by $T_9$ is made.

# 4 Exact verification and compressed results

Generation and verification are separate. The generation routines use the specialized inequalities in Section 3. The verifier uses only the recursive formulas of Section 2, reconstructs the graph expression and tests every possible unit decrease of a non-root weight. The two stages use the same exact rational-pair arithmetic but independent decision routines. All accepted tuples passed the verification.

**Theorem 4.1.** *The exact computation gives the following output.*

| *tree* | $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ | $T_8$ | *total* |
|---|---|---|---|---|---|---|---|---|---|
| *number* | 1 | 3 | 3 | 347 | 7 | 6166 | 7 | 183 | 6717 |

*For $T_9$, the procedure in Subsection 3.6 terminates for every choice of the structural parameters and gives a necessary and sufficient exact test. The only solution with two negative $\eta$-parameters is $(K_1 \cup K_{2,4}) \nabla (K_1 \cup K_{2,4})$.*

The computation was carried out in Python 3.12.13. Integers and rational numbers were represented exactly, and no floating-point value entered a decision. The finite search domains, acceptance conditions, compressed outputs and pseudocode are all contained in this report.

## References

[1] Z. Stanić, Minimal forbidden induced subgraphs for the strict golden-section bound on the second largest eigenvalue, manuscript.

[2] S.K. Simić, Some notes on graphs whose second largest eigenvalue is less than $(\sqrt{5}-1)/2$, *Linear Multilinear Algebra*, **39** (1995), 59–71.

## Appendices

# A Common verification procedure

Listing 1: Exact sign of $u + v\sigma$

```
ExactSign(u, v)
    if u = 0 and v = 0: return 0
    a ← 2u − v,   b ← v
    if b = 0: return sign(a)
    if a = 0: return sign(b)
    if a and b have the same sign: return sign(a)
    if a > 0 > b: return sign(a^2 − 5b^2)
    return sign(5b^2 − a^2)
```

Listing 2: Independent verification

```
Verify(T, ω)
    construct the graph expression G_T(ω)
    compute (η(G_T(ω)), p(G_T(ω)))
        recursively from unions and joins
    if p(G_T(ω)) ≠ 2: reject
    for every non-root coordinate i:
        decrease ω_i by one and reduce the tree
        compute p(G_T(ω − e_i))
        if this value exceeds one: reject
    accept
```

# B Enumeration of $T_4$ and $T_6$

Listing 3: Enumeration for $T_4$

```
for p = 4, 5, 6, 7:
    for every m in the certified search range:
        enumerate ordered prefixes n_1 ≤ ··· ≤ n_{p−1}
        find the least n_p ≥ n_{p−1} with
            D(n_1, ..., n_p) > t_m
        if no such n_p exists: continue
        if D < t_{m−1} and D − δ(n_p) < t_m:
            send the tuple to Verify
```

Listing 4: Enumeration for $T_6$ with $a = 1$

```
for p = 1, 2, 3, 4:
    enumerate ordered weights m_1 ≤ ··· ≤ m_p
    compute s, τ and τ'
    enumerate x ≤ y within the certified bounds
```

```
    if f(x) + f(y) ≤ τ: continue
    find the least z ≥ y with D(x, y, z) > τ
    if D(x, y, z) − δ(z) < τ < D(x, y, z) < τ′:
        send the tuple to Verify
test the single exceptional tuple with a = 2
```

For $p = 2$, the ten tuples $(m_1, m_2); (x, y, z)$ are

$$((1,1);(1,2,4)),\ ((1,1);(1,3,3)),\ ((1,1);(2,2,2)),\ ((1,2);(1,1,5)),\ ((1,2);(1,2,2)),$$
$$((1,3);(1,1,4)),\ ((1,4);(1,1,3)),\ ((2,2);(1,1,3)),\ ((2,4);(1,1,2)),\ ((3,3);(1,1,2)).$$

For $p = 3$, the four tuples are

$$((1,1,1);(1,1,2)),\quad ((1,2,4);(1,1,1)),\quad ((1,3,3);(1,1,1)),\quad ((2,2,2);(1,1,1)).$$

For $p = 4$, every entry is one.

# C Enumeration of $T_8$

Listing 5: Enumeration for $T_8$

```
for (p, q) = (1,1), (1,2), (1,3), (1,4), (2,0), (2,1), (2,2), (3,0):
    enumerate the ordered multiset (C_1, ..., C_p)
    for m = 2, ..., 13:
        if q = 0: compute h and test all inequalities
        if q > 0:
            recursively enumerate n_1 ≤ ··· ≤ n_{q−1}
            choose the least n_q ≥ n_{q−1} with h > t_m
            test h < t_{m−1} and h − δ(n_q) < t_m
        for every C_i and every vertex orbit of C_i:
            test h − η(C_i) + η(C_i − v) < t_m
        send every surviving tuple to Verify
```

# D Enumeration of $T_9$

For fixed $a, x$ put

$$h^{\infty}_{a,x} = \frac{\sigma f(x)}{a f(x) - \sigma}.$$

Suppose that $m_1, m_2, \ldots, m_{k-1}$ have been selected and put

$$u = q - k + 1, \quad \ell = c + \sum_{i=1}^{k-1} f(m_i) + u.$$

If $\ell \leq h^{\infty}_{a,x}$, the branch is discarded. Otherwise, let $z$ be the least integer such that $z \geq x$, $aD(x,z) > \sigma$ and $h_{a,x}(z) < \ell$, and put $\gamma = \ell - h_{a,x}(z)$. The next weight is bounded by

$$\frac{u\sigma}{m_k + \sigma} + \delta(m_k) > \gamma.$$

Listing 6: Enumeration for the one-negative case of $T_9$

```
choose p, q with p ≥ 1, q ≥ 0, p + q ≥ 2 and 2p + q ≤ 55
choose a, x in the finite ranges stated for T_9
choose the ordered positive graphs H_1, ..., H_{p−1}
BoundWeight(k, prefix):
```

```
    compute $u$, $\ell$, $h^{\infty}_{a,x}$
    if $\ell \le h^{\infty}_{a,x}$: return
    compute $z$ and $\gamma$
    enumerate $m_k \ge m_{k-1}$ while
        $u\sigma/(m_k + \sigma) + \delta(m_k) > \gamma$
    recurse until $m_1, \ldots, m_q$ are fixed
compute $b$ and $\tau$
find the unique $y \ge x$, if it exists, with
    $D(x, y-1) < \tau < D(x, y)$
test $b - d < h_{a,x}(y)$ and the conditional $a-1$ inequality
send every surviving tuple to Verify
```