Njutn-Kotesove kvadraturne formule

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$$I = \int_a^b p(x)f(x)dx \approx \frac{b-a}{2}\sum_{i=0}^{n} c_i f(x_i) = S_n(f)$$

$$c_i = \int_{-1}^{1} \overline{p}(t) \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{t-t_j}{t_i-t_j} dt$$

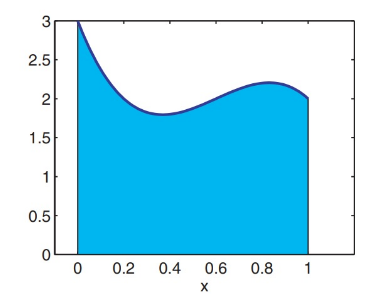
Greška: $|R_n(f)| \leq \frac{1}{(n+1)!}(\frac{b-a}{2})^{n+2} \max\limits_{x\in[a,b]} |f^{(n+1)}(x)| \int\limits_{-1}^{1} |\overline{p}(t)\overline{w}_{n+1}(t)|dt$

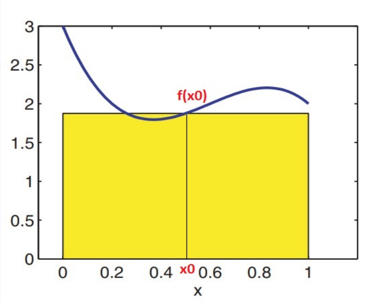
Greška povišene tačnosti (neparan broj čvorova simetričnih u odnosu na sredinu intervala, $p(x)$ parna):

$$|R_n(f)| \leq \frac{1}{(n+2)!}(\frac{b-a}{2})^{n+3} \max_{x\in[a,b]} |f^{(n+2)}(x)| \int_{-1}^{1} |\overline{p}(t)\cdot t\cdot \overline{w}_{n+1}(t)|dt$$

Osnovna formula pravougaonika ($p(x) = 1$)

$f(x) \approx L_0(x)$ $(n = 0, t_0 = 0)$





$$c_i = \int_{-1}^{1} \overline{p}(t) \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{t - t_j}{t_i - t_j} dt \overset{n=0}{\Longrightarrow} c_0 = \int_{-1}^{1} 1 dt = 2$$

$$S_n(f) = \frac{b-a}{2} \sum_{i=0}^{n} c_i f(x_i) \overset{n=0}{\Longrightarrow} S_0(f) = \frac{b-a}{2} \cdot 2 \cdot f\left(\frac{a+b}{2}\right) = (b-a) f\left(\frac{a+b}{2}\right)$$

Neparan broj čvorova simetričnih u odnosu na sredinu intervala, $p(x)$ parna $\Longrightarrow$ važi formula povišene tačnosti:

$$|R_n(f)| \leq \frac{1}{(n+2)!}\left(\frac{b-a}{2}\right)^{n+3} \max_{x\in[a,b]} |f^{(n+2)}(x)| \int_{-1}^{1} |\overline{p}(t)\cdot t \cdot \overline{w}_{n+1}(t)|dt$$

$$\overset{n=0}{\Longrightarrow} |R_0(f)| \leq \frac{1}{2!}\left(\frac{b-a}{2}\right)^{3} \max_{x\in[a,b]} |f^{(2)}(x)| \int_{-1}^{1} |t\cdot \overline{w}_1(t)|dt$$

$$= \frac{1}{2!}\left(\frac{b-a}{2}\right)^{3} \max_{x\in[a,b]} |f^{(2)}(x)| \int_{-1}^{1} |t\cdot (t-t_0)|dt$$

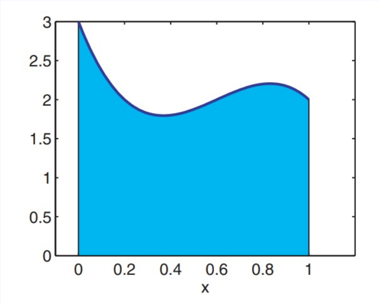
$$= \frac{1}{2!}\left(\frac{b-a}{2}\right)^{3} \max_{x\in[a,b]} |f^{(2)}(x)| \int_{-1}^{1} |t^2|dt$$

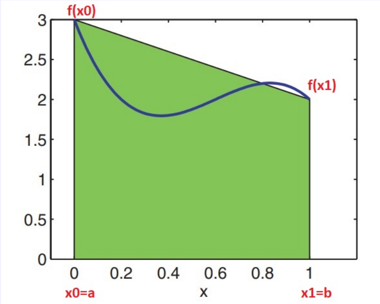
$$= \frac{1}{2!}\left(\frac{b-a}{2}\right)^{3} \max_{x\in[a,b]} |f^{(2)}(x)| \cdot \frac{2}{3}$$

$$= \frac{(b-a)^3}{24} \max_{x\in[a,b]} |f^{(2)}(x)|$$

Osnovna formula trapeza ($p(x) = 1$)

$$f(x) \approx L_1(x) \; (n = 1, t_0 = -1, t_1 = 1)$$





$$c_i = \int_{-1}^{1} \overline{p}(t) \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{t - t_j}{t_i - t_j} dt$$

$$\overset{n=1}{\Longrightarrow} c_0 = \int_{-1}^{1} \frac{t - t_1}{t_0 - t_1} dt = \int_{-1}^{1} \frac{t-1}{-2} dt = -\frac{1}{2} \frac{(t-1)^2}{2} \Bigg|_{-1}^{1} = 1$$

$$c_1 = \int_{-1}^{1} \frac{t - t_0}{t_1 - t_0} dt = \int_{-1}^{1} \frac{t+1}{2} dt = \frac{1}{2} \frac{(t+1)^2}{2} \Bigg|_{-1}^{1} = 1$$

Čvorovi su simetrično raspoređeni u odnosu na sredinu intervala, pa se može iskoristiti Lema 1 $\Longrightarrow c_0 = c_1$

$$S_n(f) = \frac{b-a}{2} \sum_{i=0}^{n} c_i f(x_i) \overset{n=1}{\Longrightarrow} S_1(f) = \frac{b-a}{2} (f(a) + f(b))$$

Paran broj čvorova (simetričnih u odnosu na sredinu intervala, $p(x)$ parna) $\Longrightarrow$ važi "obična" formula greške:

$$|R_n(f)| \leq \frac{1}{(n+1)!}\left(\frac{b-a}{2}\right)^{n+2} \max_{x\in[a,b]} |f^{(n+1)}(x)| \int_{-1}^{1} |\overline{p}(t)\overline{w}_{n+1}(t)|dt$$

$$\overset{n=1}{\Longrightarrow} |R_1(f)| \leq \frac{1}{2!}\left(\frac{b-a}{2}\right)^3 \max_{x\in[a,b]} |f^{(2)}(x)| \int_{-1}^{1} |\overline{w}_2(t)|dt$$

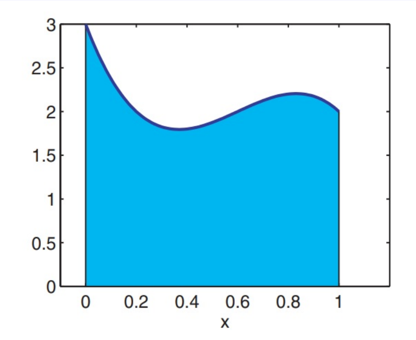
$$= \frac{1}{2!}\left(\frac{b-a}{2}\right)^3 \max_{x\in[a,b]} |f^{(2)}(x)| \int_{-1}^{1} |(t-t_0)(t-t_1)|dt$$

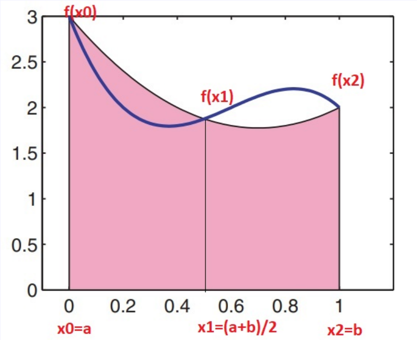
$$= \frac{1}{2!}\left(\frac{b-a}{2}\right)^3 \max_{x\in[a,b]} |f^{(2)}(x)| \int_{-1}^{1} |t^2-1|dt$$

$$= \frac{(b-a)^3}{12} \max_{x\in[a,b]} |f^{(2)}(x)|$$

Osnovna formula Simpsona ($p(x) = 1$)

$$f(x) \approx L_2(x) \ (n = 2, t_0 = -1, t_1 = 0, t_2 = 1)$$





$$c_i = \int_{-1}^{1} \overline{p}(t) \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{t-t_j}{t_i-t_j} dt$$

$$\overset{n=2}{\Longrightarrow} c_0 = \int_{-1}^{1} \frac{t-t_1}{t_0-t_1}\frac{t-t_2}{t_0-t_2} dt = \int_{-1}^{1} \frac{t}{1}\frac{t-1}{-2} dt = \frac{1}{2}\int_{-1}^{1} t(t-1) dt = \frac{1}{3}$$

$$c_1 = \int_{-1}^{1} \frac{t-t_0}{t_1-t_0}\frac{t-t_2}{t_1-t_2} dt = \int_{-1}^{1} \frac{t+1}{1}\frac{t-1}{-1} dt = -\int_{-1}^{1} (t^2-1) dt = \frac{4}{3}$$

$$c_2 = \int_{-1}^{1} \frac{t-t_0}{t_2-t_0}\frac{t-t_1}{t_2-t_1} dt = \int_{-1}^{1} \frac{t+1}{2}\frac{t}{1} dt = \frac{1}{2}\int_{-1}^{1} t(t+1) dt = \frac{1}{3}$$

Čvorovi su simetrično raspoređeni u odnosu na sredinu intervala, pa se može iskoristiti Lema 1 $\Longrightarrow c_0 = c_2$

$$S_n(f) = \frac{b-a}{2}\sum_{i=0}^{n} c_i f(x_i) \overset{n=2}{\Longrightarrow} S_2(f) = \frac{b-a}{2}\frac{1}{3}\left(f(a) + 4f(\frac{a+b}{2}) + f(b)\right)$$

(Drugi način za određivanje koeficijenata) Kvadraturna formula mora da bude tačna za polinome što je moguće većeg stepena.

$$I=\int_a^b p(x)f(x)fx \approx \frac{b-a}{2}\sum_{i=0}^{n} c_i f(x_i) = S_n(f)$$

$$I=\int_a^b p(x)P_n(x)fx = \frac{b-a}{2}\sum_{i=0}^{n} c_i f(x_i) = S_n(f)$$

Primer za Simpsonovu kvadraturnu formulu ($n=2$):

$f(x)=P_0(x)=1$:

$$\int_a^b P_0(x)dx = \int_a^b 1dx = (b-a) = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2) = c_0 + c_1 + c_2$$

$f(x)=P_1(x)=x$:

$$\int_a^b P_1(x)dx = \int_a^b xdx = \frac{1}{2}(b^2-a^2) = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2) = c_0 a + c_1 \frac{a+b}{2} + c_2 b$$

$f(x)=P_2(x)=x^2$:

$$\int_a^b P_2(x)dx = \int_a^b x^2dx = \frac{1}{3}(b^3-a^3) = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2) = c_0 a^2 + c_1 (\frac{a+b}{2})^2 + c_2 b^2$$

Simetrija $\Longrightarrow c_0 = c_2$

Rešavanjem sistema: $\int_a^b f(x)dx \approx c_0 f(a) + c_1 f(\frac{a+b}{2}) + c_2 f(b)$

Neparan broj čvorova simetričnih u odnosu na sredinu intervala, $p(x)$ parna $\Longrightarrow$ važi formula povišene tačnosti:

$$|R_n(f)| \le \frac{1}{(n+2)!}\left(\frac{b-a}{2}\right)^{n+3} \max_{x\in[a,b]} |f^{(n+2)}(x)| \int_{-1}^{1} |\overline{p}(t)\cdot t \cdot \overline{w}_{n+1}(t)|dt$$

$$\overset{n=2}{\Longrightarrow} |R_2(f)| \le \frac{1}{4!}\left(\frac{b-a}{2}\right)^{5} \max_{x\in[a,b]} |f^{(4)}(x)| \int_{-1}^{1} |t\cdot \overline{w}_3(t)|dt$$

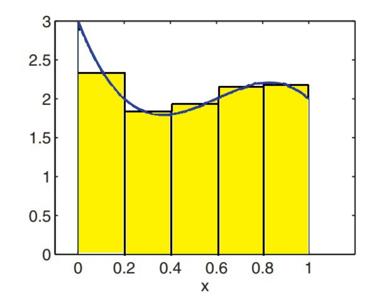
$$= \frac{1}{4!}\left(\frac{b-a}{2}\right)^{5} \max_{x\in[a,b]} |f^{(4)}(x)| \int_{-1}^{1} |t\cdot (t-t_0)(t-t_1)(t-t_2)|dt$$

$$= \frac{1}{4!}\left(\frac{b-a}{2}\right)^{5} \max_{x\in[a,b]} |f^{(4)}(x)| \int_{-1}^{1} |t^2(t^2-1)|dt$$

$$= \frac{1}{90}\left(\frac{b-a}{2}\right)^{5} \max_{x\in[a,b]} |f^{(4)}(x)|$$

Opšta formula pravougaonika ($p(x) = 1$)

$f(x) \approx L_0(x)$ na svakom od m podintervala dužine h



$$\int_a^b f(x)dx \approx h \cdot \sum_{i=1}^{m} f_{i-\frac{1}{2}} = S_0^h(f)$$

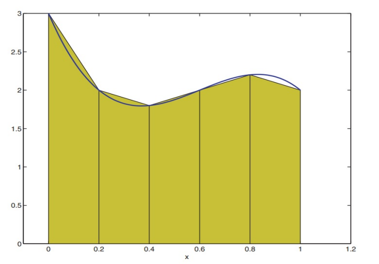
Ocena greške osnovne formule: $|R_0(f)| \leq \frac{(b-a)^3}{24} \max_{x\in[a,b]} |f^{(2)}(x)|$

Ocena greške opšte formule:

$$\begin{aligned}
|R_0^h(f)| &\leq \sum_{i=1}^{m} \frac{h^3}{24} \max_{x\in[x_{i-1},x_i]} |f^{(2)}(x)| \\
&= \frac{mh^3}{24} \left(\frac{1}{m} \sum_{i=1}^{m} \max_{x\in[x_{i-1},x_i]} |f^{(2)}(x)| \right) \\
&\leq \frac{mh^3}{24} \left(\frac{1}{m} \cdot m \cdot \max_{x\in[a,b]} |f^{(2)}(x)| \right) \\
&\overset{\text{mh=b-a}}{=} \frac{(b-a)h^2}{24} \max_{x\in[a,b]} |f^{(2)}(x)|
\end{aligned}$$

Opšta formula trapeza ($p(x) = 1$)

$f(x) \approx L_1(x)$ na svakom od m podintervala dužine h



$$\int_a^b f(x)dx \approx \sum_{i=1}^{m} \frac{h}{2}(f_{i-1} + f_i)$$

$$= \frac{h}{2}(f_0 + f_1 + f_1 + f_2 + f_2 + f_3 + \ldots + f_{m-1} + f_{m-1} + f_m)$$

$$= \frac{h}{2}(f_0 + 2\sum_{i=1}^{m-1} f_i + f_m) = S_1^h(f)$$

Ocena greške osnovne formule: $|R_1(f)| \leq \frac{(b-a)^3}{12} \max_{x\in[a,b]} |f^{(2)}(x)|$

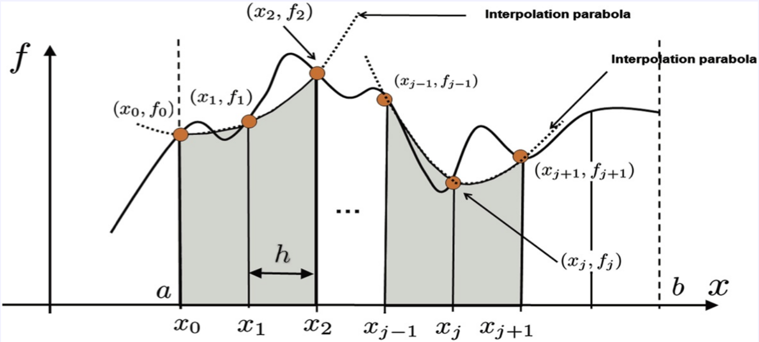
Ocena greške opšte formule:

$$
\begin{aligned}
|R_1^h(f)| &\leq \sum_{i=1}^{m} \frac{h^3}{12} \max_{x\in[x_{i-1},x_i]} |f^{(2)}(x)| \\
&= ... \\
&= \frac{(b-a)h^2}{12} \max_{x\in[a,b]} |f^{(2)}(x)|
\end{aligned}
$$

Opšta formula Simpsona ($p(x) = 1$)

$f(x) \approx L_2(x)$

Ne želimo da koristimo središnju tačku svakog intervala $[x_{i-1}, x_i]$ nego radimo na sa x_{i-1}, x_i, x_{i+1}, tj. zahtevamo paran broj intervala $2m$.



$$\int_a^b f(x)dx = \int_{x_0=a}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \dots + \int_{x_{2m-2}}^{x_{2m}=b} f(x)dx$$

$$\int_a^b f(x)dx = \sum_{i=1}^{m} \int_{x_{2(i-1)}}^{x_{2i}} f(x)dx$$

$$\approx \frac{h}{3} \sum_{i=1}^{m} (f_{2(i-1)} + 4f_{2i-1} + f_{2i})$$

$$= \frac{h}{3}(f_0 + 4\sum_{i=1}^{m} f_{2i-1} + 2\sum_{i=1}^{m-1} f_{2i} + f_{2m}) = S_2^h(f)$$

$$|R_2^h(f)| \leq \sum_{i=1}^{m} \frac{h^5}{90} \max_{x\in[x_{2i-2},x_{2i}]} |f^{(4)}(x)|$$

$$= \frac{h^4}{90} \cdot \frac{b-a}{2m} \sum_{i=1}^{m} \max_{x\in[x_{2i-2},x_{2i}]} |f^{(4)}(x)|$$

$$\leq \frac{(b-a)h^4}{180} \max_{x\in[a,b]} |f^{(4)}(x)|$$

Rungeova ocena greške

I_h - približna vrednost integrala sa korakom h
I_H - približna vrednost integrala sa korakom $H = 2h$

$$I = I_h + M_1 h^k$$
$$I = I_H + M_2 H^k$$
$$\overline{\hspace{6cm}}\ M_1 = M_2 = M \text{ (aproksimacija)}$$
$$I_h - I_H = M(H^k - h^k) \Longrightarrow M = \frac{I_h - I_H}{H^k - h^k}$$

$$R_h = Mh^k = \frac{I_h - I_H}{H^k - h^k} h^k = \frac{I_h - I_H}{h^k(\frac{H^k}{h^k} - 1)} h^k \overset{\text{H=2h}}{=} \frac{I_h - I_{2h}}{2^k - 1}$$

k - red metode

Greška računa u kvadraturnim formulama

$S_n(f) = \sum_{i=0}^{n} c_i f(x_i)$

Ako su vrednosti $f(x_i)$ date sa tačnošću ε:

$R_R \leq \sum_{i=0}^{n} |c_i|\varepsilon$

Kako je svaka kvadraturna formula tačna za $f(x) = 1$:

$\int_a^b f(x)dx = (b-a) = \sum_{i=0}^{n} c_i \cdot 1 \Longrightarrow \sum_{i=0}^{n} c_i = (b-a)$

$R_R \leq (b-a)\varepsilon$ (ako su $c_i > 0$)