On the topological complexity and zero-divisor cup-length of real Grassmannians

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Abstract

Topological complexity naturally appears in the motion planning in robotics. In this paper we consider the problem of finding topological complexity of real Grassmann manifolds $G_k(\mathbb{R}^n)$. We use cohomology methods to give estimates on the zero-divisor cup-length of $G_k(\mathbb{R}^n)$ for various $2 \leqslant k < n$, which in turn give us lower bounds on topological complexity. Our results correct and improve several results from [9].

1 Introduction

For a path-connected space X we denote its topological complexity by $\mathrm{TC}(X)$. In [9] the author considered the problem of finding $\mathrm{TC}(G_k(\mathbb{R}^n))$ for various $2 \leqslant k < n$ (in this paper, $G_k(\mathbb{R}^n)$ denotes the real Grassmann manifold of k-dimensional subspaces in $\mathbb{R}^n$). Unfortunately, there is a problem with the proof of the main lemma of that paper (Lemma 4.4) and the consequential results on the topological complexity (Theorems 4.5, 4.8 and 4.12); see [10]. In this paper we reconsider this problem, and as an outcome correct and improve several results from [9]. As in [9], we use the *cohomology method* to obtain our results.

This paper closely follows and builds on the ideas presented in [9] (so, for background, motivation and all undefined notions, the reader is advised to consult [9]). Throughout the paper we will use, as much as possible, the notation from [9]. In particular, we will be working with the *unreduced* topological complexity, as defined by Farber in [5] (for example, by this definition the topological complexity of a contractible space is equal to 1).

The paper is organized as follows. In Section 2 we describe the cohomology method mentioned above and give an overview of the cohomology of real Grassmannians. In Section 3 we consider the case $k = 2$. We obtain the exact value of the zero-divisor cup-length of $G_2(\mathbb{R}^{2^s+1})$ (denoted by $\mathrm{zcl}(G_2(\mathbb{R}^{2^s+1}))$, and defined in Section 2) for $s \geqslant 2$; additionally, for $s \geqslant 3$, $2^s + 4 \leqslant n \leqslant 2^{s+1}$ we prove a lower bound for $\mathrm{zcl}(G_2(\mathbb{R}^n))$. These results show that the value of the zero-divisor cup-length given in [9, Theorem 4.5] is not correct; what is more interesting, our results improve lower bounds for topological complexity stated in the same theorem. Section 4 is devoted to the case $k = 3$. Separately, we prove lower bounds for $\mathrm{zcl}(G_3(\mathbb{R}^n))$ in the cases $n = 2^s + 1$ (for $s \geqslant 3$), and $2^s + 3 \leqslant n \leqslant 2^{s+1}$ (for $s \geqslant 2$). The first result shows that the corresponding result from [9, Theorem 4.8] is not correct, and improves the stated lower bound for topological complexity of $G_3(\mathbb{R}^{2^s+1})$ (for $s \geqslant 5$). In Section 5 we give a general lower bound for $\mathrm{zcl}(G_k(\mathbb{R}^n))$ (for $k \geqslant 4$). For $k \geqslant 9$ this result improves the bounds stated in [9, Theorem 4.10].

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2 Background and notation

As mentioned in Introduction, to obtain our results we use the so called *cohomology method*, which we now (briefly) explain.

Let $\Delta : X \to X \times X$ denote the diagonal map. Then the elements of

$$\mathrm{Ker}(\Delta^* : H^*(X \times X; \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2))$$

are called *zero-divisors*. Further, the *zero-divisor cup-length* of X, denote by $\mathrm{zcl}(X)$, is defined to be the maximum number of elements from $\mathrm{Ker}\Delta^*$ whose product is non-zero. In [5], Farber proved that $\mathrm{zcl}(X)$ gives a lower bound for $\mathrm{TC}(X)$, that is $\mathrm{TC}(X) \geqslant \mathrm{zcl}(X) + 1$. Hence, a lower bound for $\mathrm{zcl}(X)$ immediately gives a lower bound for $\mathrm{TC}(X)$. Note that for every $w \in H^*(X; \mathbb{Z}_2)$ the element

$$z(w) = w \otimes 1 + 1 \otimes w \in H^*(X \times X; \mathbb{Z}_2)$$

is in $\mathrm{Ker}\Delta^*$ (since $\Delta^*(z(w)) = w \cdot 1 + 1 \cdot w = 0$). Then, by [2, Lemma 5.2], $\mathrm{Ker}\Delta^*$ is generated as an ideal by these elements, that is by the set $\{z(w) : w \in H^*(X; \mathbb{Z}_2)\}$. So, if $\mathrm{zcl}(X) = t$, then there are classes $x_1, x_2, \ldots, x_t \in H^*(X; \mathbb{Z}_2)$ such that $z(x_1)z(x_2)\cdots z(x_t) \neq 0$.

To get the best possible results on $\mathrm{TC}(G_k(\mathbb{R}^n))$ using the cohomology method, one requires fine understanding of the cohomology algebra $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$. There are several ways to describe this algebra; in this paper we will use the one due to Borel (see [1]):

$$H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \ldots, w_k]/I_{k,n},$$

where $w_1, w_2, \ldots, w_k$ are the Stiefel-Whitey classes of the canonical k-dimensional vector bundle over $G_k(\mathbb{R}^n)$, and $I_{k,n} = (\overline{w}_{n-k+1}, \overline{w}_{n-k+2}, \ldots, \overline{w}_n)$ is the ideal generated by dual classes.

Although Borel's description of $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ appears simple enough, it turns out that performing concrete calculations in this algebra can be rather difficult. Hence, one usually needs to apply some additional methods and properties of $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$. The following result gives an additive basis for this algebra (see, e.g. [7, 11]).

Proposition 2.1 *The set* $B_{k,n-k} = \{w_1^{a_1} \cdots w_k^{a_k} : 0 \leqslant a_1 + \cdots + a_k \leqslant n - k\}$ *is an additive basis for* $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$.

The *height* of a class $c \in \widetilde{H}^*(X; \mathbb{Z}_2)$, denoted by $\mathrm{ht}(c)$, is the largest $m \in \mathbb{N}$ such that $c^m \neq 0$. For $k \geqslant 2$, the height of $w_1 \in H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ is obtained by Stong in [12]: if $2 \leqslant k \leqslant n - k$ and s is the unique positive integer such that $2^s < n \leqslant 2^{s+1}$, then

$$\mathrm{ht}(w_1) = \begin{cases} 2^{s+1} - 2, & \text{if } k = 2 \text{ or } (k, n) = (3, 2^s + 1), \\ 2^{s+1} - 1, & \text{otherwise.} \end{cases} \tag{2.1}$$

In this paper we will often use Stong's method from [12] for calculating in $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ (later this method was generalized by Korbaš and Lörinc to all flag manifolds, see [8]). In what follows we briefly explain this method.

Let $\mathrm{Flag}(\mathbb{R}^n)$ denote the *(real) complete flag manifold* ($n \geqslant 2$). Denote by $e_i := w_1(\gamma_i)$ the first Stiefel-Whitney class of the canonical line bundle γ_i over $\mathrm{Flag}(\mathbb{R}^n)$, for $1 \leqslant i \leqslant n$. Then we have the map $\pi : \mathrm{Flag}(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$, given by

$$\pi(S_1, \ldots, S_k, S_{k+1}, \ldots, S_n) = (S_1 \oplus \cdots \oplus S_k, S_{k+1} \oplus \cdots \oplus S_n).$$

The following result will be very useful for our calculations in $H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$ (and $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$).

Proposition 2.2 *(1) The set* $B_n=\{e_1^{a_1}e_2^{a_2}\ldots e_{n-1}^{a_{n-1}}\ :\ 0\leqslant a_i\leqslant n-i\}$ *is an additive basis for* $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$.

(2) $\mathrm{ht}(e_i)=n-1$ *for* $1\leqslant i\leqslant n$. *In particular* $e_i^n=0$ *for* $1\leqslant i\leqslant n$.

(3) A monomial $e_1^{a_1}e_2^{a_2}\cdots e_n^{a_n}\in H^{\binom{n}{2}}(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$ *is non-zero if and only if* $(a_1,a_2,\ldots,a_n)$ *is a permutation of the n-tuple* $(n-1,n-2,\ldots,1,0)$.

(4) If $u\in H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$ *and*

$$v=e_1^{k-1}e_2^{k-2}\cdots e_{k-1}\cdot e_{k+1}^{n-k-1}e_{k+2}^{n-k-2}\cdots e_{n-1}\in H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2),$$

then $\pi^*(u)\cdot v\in H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$, *and* $u\neq 0$ *if and only if* $\pi^*(u)\cdot v\neq 0$.

(5) For $1\leqslant i\leqslant k$, $\pi^*(w_i)$ *is the i-th elementary symmetric polynomial in the variables* $e_1,e_2,\ldots,e_k$.

Heights of the classes $z(w_1)$ and $z(w_k)$ will be very useful in our calculations. In what follows we determine these values.

It turns out that if $\mathrm{ht}(w)$ is known, then $\mathrm{ht}(z(w))$ can easily be calculated. This is proven in Lemma 4.3 from [9]. Namely, one has: if $w\in H^*(X;\mathbb{Z}_2)$ and t is the unique non-negative integer such that $2^t\leqslant \mathrm{ht}(w)<2^{t+1}$, then

$$\mathrm{ht}(z(w))=2^{t+1}-1. \tag{2.2}$$

We will apply this identity for $X=G_k(\mathbb{R}^n)$, when $2\leqslant k\leqslant n-k$. If $2^s<n\leqslant 2^{s+1}$, then (2.1) implies

$$\mathrm{ht}(z(w_1))=2^{s+1}-1. \tag{2.3}$$

On the other hand, Proposition 2.1 implies $w_k^{n-k}\neq 0$, so $\mathrm{ht}(w_k)=n-k$ (by observing dimension we conclude that $w_k^{n-k+1}=0$). Hence, if t is the unique non-negative integer such that $2^t\leqslant n-k<2^{t+1}$, then (2.2) implies

$$\mathrm{ht}(z(w_k))=2^{t+1}-1. \tag{2.4}$$

The following lemma will be particularly useful in Section 3.

Lemma 2.3 *Let* $m,k,n\in\mathbb{N}$, $k<n$, *and* $d_1,\ldots,d_m\in\mathbb{N}$ *be such that* $d_1+\cdots+d_m\geqslant 2k(n-k)$. *If* $x_i\in H^{d_i}(G_k(\mathbb{R}^n);\mathbb{Z}_2)$ *for* $1\leqslant i\leqslant m$, *then*

$$z(x_1)\cdots z(x_m)=0.$$

PROOF — Note that the product $p=z(x_1)\cdots z(x_m)$ is the sum of certain classes of the form $x\otimes y+y\otimes x$, for some $x,y\in H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$. Since p is in dimension at least $2k(n-k)=2\dim G_k(\mathbb{R}^n)$, so is $x\otimes y$, and hence $x,y\in H^{k(n-k)}(G_k(\mathbb{R}^n);\mathbb{Z}_2)$ or $x\otimes y=y\otimes x=0$. There is only one non-zero class in $H^{k(n-k)}(G_k(\mathbb{R}^n);\mathbb{Z}_2)$, namely w_k^{n-k} (by Proposition 2.1), and hence $x\otimes y=y\otimes x=0$ or $x\otimes y=w_k^{n-k}\otimes w_k^{n-k}=y\otimes x$. In both cases $x\otimes y+y\otimes x=0$, which implies $p=0$. □

Also, we recall some results from [9] that will be used in our calculations.

Lemma 2.4 *a) If* $2^s < n \leqslant 2^{s+1}$*, then* $w_1^{2^s}w_2^{n-2^s-1} \neq 0$ *and* $w_1^{2^s}w_2^{n-2^s} = 0$ *in* $H^*(G_2(\mathbb{R}^n);\mathbb{Z}_2)$.

b) If $2^s+3 \leqslant n \leqslant 2^{s+1}$ *and* $t = n-2^s$*, then* $w_1^{2^s}w_2^{2^s}w_3^{t-3} \neq 0$ *in* $H^*(G_3(\mathbb{R}^n);\mathbb{Z}_2)$.

Throughout the paper we use the same notation as in this section.

Finally, let us say a few words on Lemma 4.4 from [9] and our strategy that bypasses the application of this lemma. In Lemma 4.4 from [9] the author assumes that $u_1,\dots,u_n \in H^*(X;\mathbb{Z}_2)$ and $k_1,\dots,k_n \in \mathbb{N}$ are such that $u_1^{k_1}\cdots u_n^{k_n} \neq 0$, and wants to prove that $A = z(u_1)^{2^{r_1}-1}\cdots z(u_n)^{2^{r_n}-1} \neq 0$, where r_i is the unique integer such that $2^{r_i-1} \leqslant k_i < 2^{r_i}$ for $1 \leqslant i \leqslant n$. For this he notices that after expanding A one summand is $u_1^{k_1}\cdots u_n^{k_n} \otimes u_1^{2^{r_1}-k_1-1}\cdots u_n^{2^{r_n}-k_n-1}$, which is nonzero, and from this immediately concludes that $A \neq 0$. As we will see in the proofs of our results, the problem is that the set

$$S = \{(l_1,\dots,l_n) \,:\, 0 \leqslant l_i \leqslant 2^{r_i}-1,\, u_1^{l_1}\cdots u_n^{l_n} = u_1^{k_1}\cdots u_n^{k_n}\}$$

can contain more than one element, and hence that the corresponding summands of A with the first coordinate equal to $u_1^{k_1}\cdots u_n^{k_n}$ may cancel out. So, in our proofs we choose the n-tuple $(k_1,\dots,k_n)$ a bit more carefully to ensure that

$$\sum_{(l_1,\dots,l_n)\in S} u_1^{2^{r_1}-l_1-1}\cdots u_n^{2^{r_n}-l_n-1} \neq 0$$

and that this further leads to $A \neq 0$ (note: in our applications the degree of $z(u_i)$ in A will not always be $2^{r_i}-1$, so we will have slightly different formulas than the one given above).

3 The basic-zero-divisor cup-length of $G_2(\mathbb{R}^n)$

Let s be the unique integer such that $2^s < n \leqslant 2^{s+1}$. In this section we consider $\mathrm{zcl}(G_2(\mathbb{R}^n))$. We note that Propositions 3.7 and 3.10, that we prove in this section, show that the corresponding results of [9, Theorem 4.5] are not correct (see also Remark 3.9). Fortunately, correct versions give better lower bounds for the topological complexity of $G_2(\mathbb{R}^n)$.

We will compare our results with the following upper bound from [9] (this result is a consequence of a general result from [3, Theorem 1]).

Proposition 3.1 *If* $1 \leqslant k < n$*, then* $\mathrm{TC}(G_k(\mathbb{R}^n)) \leqslant 2k(n-k)$*. In fact, if* $k \neq 1$ *and* $(k,n) \neq (2, 2^d+1)$ *for all* $d \in \mathbb{N}$*, then* $\mathrm{TC}(G_k(\mathbb{R}^n)) \leqslant 2k(n-k)-1$.

3.1 Preliminary lemmas

Let n be a positive integer and $n = \sum_{i=0}^{t}\alpha_i\cdot 2^i$, where $\alpha_i \in \{0,1\}$ for $0 \leqslant i \leqslant t$ and $\alpha_t = 1$, its representation in base 2. Then we write $n := (\alpha_t,\dots,\alpha_1,\alpha_0)_2$.

As we use $\mathbb{Z}_2$ coefficient the following special case of Lucas' theorem will be particularly useful to us: if $n := (\alpha_t,\dots,\alpha_1,\alpha_0)_2$ and $m := (\beta_r,\dots,\beta_1,\beta_0)_2$, then

$$\binom{n}{m} \equiv 1 \pmod 2 \qquad \text{if and only if} \qquad t \geqslant r \quad \text{and} \quad \alpha_i \geqslant \beta_i \text{ for } 0 \leqslant i \leqslant r.$$

We will use the following two consequences of Lucas' theorem throughout the paper. Let $w \in H^*(X;\mathbb{Z}_2)$. By Lucas' theorem, $\binom{2^m}{i}$ is even for $1 \leqslant i \leqslant 2^m-1$, and so

$$z(w)^{2^m} = (w\otimes 1 + 1\otimes w)^{2^m} = w^{2^m}\otimes 1 + 1\otimes w^{2^m}.$$

On the other hand, by Lucas' theorem $\binom{2^m-1}{i}$ is odd for all $0 \leqslant i \leqslant 2^m-1$, and hence

$$z(w)^{2^m-1} = (w \otimes 1 + 1 \otimes w)^{2^m-1} = \sum_{i=0}^{2^m-1} w^i \otimes w^{2^m-1-i}.$$

We will also need the following result.

Lemma 3.2 *Let n be a non-negative integer. Then:*

a) $\binom{2n}{n}$ *is odd if and only if* $n = 0$*;*

b) $\binom{2n}{n+1}$ *is odd if and only if* $n = 2^{t+1} - 1$ *for some* $t \in \mathbb{N}_0$*.*

PROOF — Part a) immediately follows from Lucas' theorem.

For part b) we note that $C_n = \binom{2n}{n} - \binom{2n}{n+1}$ is the n-th Catalan number. Then the result follows from part a) and the fact that C_n (for $n \geqslant 1$) is odd if and only if $n = 2^{t+1} - 1$ for some $t \in \mathbb{N}_0$ (see [4]). □

Lemma 3.3 *Let $0 \leqslant m \leqslant n-2$ and $\alpha_0, \alpha_1, \ldots, \alpha_{n-1-m} \in \mathbb{Z}_2$. Then:*

a) $\displaystyle\sum_{i=0}^{n-1-m} \alpha_i e_1^{m+i} e_2^{n-1-i} = 0$ *in* $H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$ *iff* $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-1-m}$*;*

b) for a polynomial $p \in H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$ in classes e_1 and e_2 one has

$$p \cdot e_3^{n-3} e_4^{n-4} \cdots e_{n-1} = 0 \quad \textit{in} \quad H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$$

if and only if $p = 0$ in $H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$.

PROOF —

a) By Proposition 2.1 from [6] we have $e_2^{n-1} = e_1^{n-1} + e_1^{n-2} e_2 + \cdots + e_1 e_2^{n-2}$ (we use this proposition for $m = 1$, $k = n-1$ and $i = n-2$). Since $e_1^n = 0$ (by Proposition 2.2.(2)), we have

$$\sum_{i=0}^{n-1-m} \alpha_i e_1^{m+i} e_2^{n-1-i} = \sum_{i=1}^{n-1-m} (\alpha_i + \alpha_0) e_1^{m+i} e_2^{n-1-i}.$$

Since $e_1^{m+1} e_2^{n-2}, e_1^{m+2} e_2^{n-3}, \ldots, e_1^{n-1} e_2^m$ are in the additive basis B_n (from Proposition 2.2.(1)), the last sum is zero if and only if $\alpha_1 + \alpha_0 = \alpha_2 + \alpha_0 = \cdots = \alpha_{n-1-m} + \alpha_0 = 0$, i.e. if and only if $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-1-m}$.

b) As in part a) we use the identities $e_2^{n-1} = e_1^{n-1} + e_1^{n-2} e_2 + \cdots + e_1 e_2^{n-2}$ and $e_1^n = e_2^n = 0$ to express p in the form $\sum \alpha_{i,j} e_1^i e_2^j$, where $\alpha_{i,j} \in \{0, 1\}$, $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant n-2$. Then $\sum \alpha_{i,j} e_1^i e_2^j e_3^{n-3} e_4^{n-4} \cdots e_{n-1}$ $(= p e_3^{n-3} e_4^{n-4} \cdots e_{n-1})$ is a sum of the elements from the basis B_n from Proposition 2.2.(1); so this sum is zero if and only if $\alpha_{ij} = 0$ for all i, j, i.e. if and only if $p = 0$ (since p is also represented in the basis B_n). □

Remark 3.4 *We will use the following consequence of part a) of this lemma. Let* $p = \sum_{i=0}^{b-a} \alpha_i e_1^{a+i} e_2^{b-i} \in H^{a+b}(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$ *for some* $0 \leqslant a \leqslant n-2$, $a \leqslant b \leqslant n-1$. *If there exist* $0 \leqslant i' \neq i'' \leqslant b-a$ *such that* $\alpha_{i'} = 0$ *and* $\alpha_{i''} = 1$, *then* $p \neq 0$.

Further, if $q \in H^c(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$, *where* $c \leqslant 2n-3$, *is written as a sum of some monomials of the form* $e_1^i e_2^j$, *then after removing all summands with* $i \geqslant n$ *or* $j \geqslant n$ *(since they are 0 by Proposition 2.2.(2)), we get that* q *is written in the same way as* p *above.*

Lemma 3.5 *If* $2^s < n \leqslant 2^{s+1}$ *and* $a, b \in \mathbb{N}_0$ *are such that* $a + 2b = 2(n-2)$, *then* $w_1^a w_2^b \neq 0$ *in* $H^{2n-4}(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ *if and only if*

$$(a,b) = (2^{l+1}-2, n-2^l-1) \quad \textit{for some } 0 \leqslant l \leqslant s.$$

PROOF — By Proposition 2.2.(4), $w_1^a w_2^b \neq 0$ in $H^{2n-4}(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ if and only if

$$\pi^*(w_1^a w_2^b) e_1 e_3^{n-3} \cdots e_{n-1} = (e_1+e_2)^a (e_1 e_2)^b e_1 e_3^{n-3} \cdots e_{n-1} \neq 0$$

in $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$. After expanding we have

$$(e_1+e_2)^a (e_1 e_2)^b e_1 e_3^{n-3} \cdots e_{n-1} = e_3^{n-3} \cdots e_{n-1} \sum_{i=0}^{a} \binom{a}{i} e_1^{i+1+b} e_2^{a-i+b}.$$

Note that by Proposition 2.2.(3) the only non-zero monomials in this sum are the ones for i that satisfies $(i+1+b, a-i+b) \in \{(n-1, n-2), (n-2, n-1)\}$ and $\binom{a}{i}$ is odd, i.e. $i \in \{n-2-b, n-3-b\}$ and $\binom{a}{i}$ is odd.

If $i = n-2-b$, then $\binom{a}{i} = \binom{2(n-2-b)}{n-2-b} = \binom{2m}{m}$ (here $2m = 2(n-2-b) = a$). By Lemma 3.2 this number is odd only if $m = 0$, i.e. $(a,b) = (0, n-2)$.

Let us now consider the case $i = n-3-b$. Then $\binom{a}{i} = \binom{2(n-2-b)}{n-3-b} = \binom{2m}{m-1} = \binom{2m}{m+1}$ (again $2m = 2(n-2-b) = a$). By Lemma 3.2 this number is odd if and only if $m = 2^l - 1$ for some $l \geqslant 1$. Then $a = 2^{l+1} - 2$ and $b = n - 2^l - 1 \geqslant 0$, which completes our proof. □

Remark 3.6 *If* $w_1^a w_2^b \neq 0$ *and* $a + 2b = 2(n-2)$, *then, by Proposition 2.1,* $w_1^a w_2^b = w_2^{n-2}$ *(since* w_2^{n-2} *is the only non-zero class in* $H^{2(n-2)}(G_2(\mathbb{R}^n);\mathbb{Z}_2)$*).*

3.2 Some exact values

In this section we calculate $\mathrm{zcl}(G_2(\mathbb{R}^n))$ for $n = 2^s + 1$.

In the proof of the main result we will use the following observation. Let $n \geqslant 4$. Then, by Proposition 2.1, every class in $H^1(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ is of the form αw_1, $\alpha \in \mathbb{Z}_2$, while every class in $H^2(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ is of the form $\beta w_1^2 + \gamma w_2$, $\beta, \gamma \in \mathbb{Z}_2$. Since $z(w_1^2) = z(w_1)^2$, we conclude: if $\mathrm{zcl}(G_2(\mathbb{R}^n)) = t$, then there are $a, b, c \in \mathbb{N}_0$ such that $z(w_1)^a z(w_2)^b z(x_1) \cdots z(x_c) \neq 0$, where $a + b + c = t$ and $x_1, \ldots, x_c$ are some classes of $H^*(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ each in dimension at least 3.

Proposition 3.7 *For* $s \geqslant 2$ *and* $n = 2^s + 1$ *one has*

$$\mathrm{zcl}(G_2(\mathbb{R}^n)) = 2^{s+1} + 2^s - 4 \quad \textit{and} \quad \mathrm{TC}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s - 3.$$

PROOF — First, we prove that $z(w_1)^{2^{s+1}-1}z(w_2)^{2^s-3} \neq 0$. After expanding, we consider all summands of the form $w_2^{n-2} \otimes x$, for some $x \in H^*(G_2(\mathbb{R}^n);\mathbb{Z}_2)$. By Lemma 3.5 each such summand is of the form $w_1^{2^{l+1}-2}w_2^{2^s-2^l} \otimes w_1^{2^{s+1}-2^{l+1}+1}w_2^{2^l-3}$ (for $l \geqslant 2$) with coefficient $\binom{2^{s+1}-1}{2^{l+1}-2}\binom{2^s-3}{2^s-2^l}$. By Lucas' theorem each of these binomial coefficients is odd, so $z(w_1)^{2^{s+1}-1}z(w_2)^{2^s-3}$ contains $w_2^{n-2} \otimes \sum_{l=2}^{s} w_1^{2^{s+1}-2^{l+1}+1}w_2^{2^l-3}$. Since w_2^{n-2} is the only non-zero class in $H^{2(n-2)}(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ (by Proposition 2.1), it is enough to prove $\sum_{l=2}^{s} w_1^{2^{s+1}-2^{l+1}+1}w_2^{2^l-3} \neq 0$ (in $H^*(G_2(\mathbb{R}^{2^s+1});\mathbb{Z}_2)$).

Note that by Lemma 2.4, $w_1^{2^s}w_2 = 0$, and so $w_1^{2^{s+1}-2^{l+1}+1}w_2^{2^l-3} = 0$ for $2 \leqslant l \leqslant s-1$. Hence, it is enough to prove that $w_1w_2^{2^s-3} = w_1w_2^{n-4} \neq 0$, which follows from the fact that $w_1w_2^{n-4}$ is in the additive basis $B_{2,n-2}$ (Proposition 2.1). So, $\operatorname{zcl}(G_2(\mathbb{R}^{2^s+1})) \geqslant 2^{s+1} + 2^s - 4$.

Let us now prove that $\operatorname{zcl}(G_2(\mathbb{R}^{2^s+1})) \leqslant 2^{s+1} + 2^s - 4$. Suppose that this is not the case and let $a, b, c \in \mathbb{N}_0$ and $x_1, \ldots, x_c \in H^*(G_2(\mathbb{R}^{2^s+1});\mathbb{Z}_2)$ be some classes each in dimension at least 3, such that $a+b+c \geqslant 2^{s+1}+2^s-3$ and $z(w_1)^a z(w_2)^b z(x_1)\cdots z(x_c) \neq 0$. By Lemma 2.3, we have $a+2b+3c \leqslant 4(2^s-1)-1 = 2^{s+2}-5$, and hence $b+2c \leqslant 2^s-2$. Further, since $z(w_1)^{2^{s+1}} = 0$ (by (2.3)), we have $a \leqslant 2^{s+1}-1$ and hence $b+c = (a+b+c)-a \geqslant 2^s-2$. This implies $b = 2^s-2$ and $c = 0$. Finally, $a+b+c \geqslant 2^{s+1}+2^s-3$ and $a \leqslant 2^{s+1}-1$ imply $a = 2^{s+1}-1$.

So, it is enough to prove $A = z(w_1)^{2^{s+1}-1}z(w_2)^{2^s-2} = 0$. Suppose that this is not the case. Note that the dimension of A is $2^{s+1}-1+2(2^s-2) = 4(n-2)-1$, so every summand of A is of the form $x' \otimes x''$ where one of the classes x' and x'' has dimension $2(n-2)$ and the other $2(n-2)-1$. Note that, by Proposition 2.1, the only class in $H^*(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ of dimension $2(n-2)$ (resp. $2(n-2)-1$) is w_2^{n-2} (resp. $w_1w_2^{n-3}$). By symmetry, this and $A \neq 0$ imply $A = w_2^{n-2} \otimes w_1w_2^{n-3} + w_1w_2^{n-3} \otimes w_2^{n-2}$. Now, we proceed as in the first part of the proof to prove that the coefficient of $w_2^{n-2} \otimes w_1w_2^{n-3}$ in A is zero. By Lemma 3.5 each such summand in $A = z(w_1)^{2^{s+1}-1}z(w_2)^{2^s-2}$ is of the form $w_1^{2^{l+1}-2}w_2^{2^s-2^l} \otimes w_1^{2^{s+1}-2^{l+1}+1}w_2^{2^l-2}$ (for some $1 \leqslant l \leqslant s$) with coefficient $\binom{2^{s+1}-1}{2^{l+1}-2}\binom{2^s-2}{2^s-2^l}$. By Lucas' theorem this coefficient is 1, so it is enough to prove $\sum_{l=1}^{s} w_1^{2^{s+1}-2^{l+1}+1}w_2^{2^l-2} = 0$.

Again, by Lemma 2.4, $w_1^{2^s}w_2 = 0$, so $w_1^{2^{s+1}-2^{l+1}+1}w_2^{2^l-2} = 0$ for $2 \leqslant l \leqslant s-1$. Hence, the previous sum is equal to $w_1^{2^{s+1}-3} + w_1w_2^{2^s-2}$. By (2.1), $w_1^{2^{s+1}-3} \neq 0$, so $w_1^{2^{s+1}-3} = w_1w_2^{n-3} = w_1w_2^{2^s-2}$, and hence $A = 0$. □

Remark 3.8 *By Proposition 3.1,* $\operatorname{TC}(G_2(\mathbb{R}^{2^s+1})) \leqslant 2^{s+2}-4$, *so there is a gap of* 2^s-1 *between our lower bound and this bound. For example,* $9 \leqslant \operatorname{TC}(G_2(\mathbb{R}^5)) \leqslant 12$.

Remark 3.9 *Ideas from this paper can be used to prove the following:*

(1) If $s \geqslant 1$, *then* $\operatorname{zcl}(G_2(\mathbb{R}^{2^s+2})) = 3\cdot 2^s - 2$ *(one has* $z(w_1)^{2^{s+1}-2}z(w_2)^{2^s} \neq 0$*). So, by Proposition 3.1,* $3\cdot 2^s - 1 \leqslant \operatorname{TC}(G_2(\mathbb{R}^{2^s+2})) \leqslant 2^{s+2}-1$.

(2) If $s \geqslant 2$, *then* $\operatorname{zcl}(G_2(\mathbb{R}^{2^s+3})) = 3\cdot 2^s$ *(one has* $z(w_1)^{2^{s+1}-1}z(w_2)^{2^s+1} \neq 0$*). So, by Proposition 3.1,* $3\cdot 2^s + 1 \leqslant \operatorname{TC}(G_2(\mathbb{R}^{2^s+3})) \leqslant 2^{s+2}+3$.

Complete proofs of these results can be found in the extended version of this paper which is available on the author's website.

3.3 General bounds for $\mathrm{zcl}(G_2(\mathbb{R}^n))$

Let $2^s + 4 \leqslant n \leqslant 2^{s+1}$ and $t = n - 2^s$. Also, we assume $s \geqslant 3$ (i.e. $n \neq 8$). Further, let r be the unique integer such that $2^{r-1} < t \leqslant 2^r$. Since $t \geqslant 4$, we have $r \geqslant 2$. Let j be the smallest positive integer such that the digit on position j in the binary representation of $t-2$ is equal to 1 (j is well-defined since $t - 2 \geqslant 2$); in other words, $t - 2$ has the binary representation of the following form

$$t - 2 = 2^m + \alpha_{m-1}2^{m-1} + \cdots + \alpha_{j+1}2^{j+1} + 2^j + \alpha_0,$$

for some $\alpha_0, \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_{m-1} \in \{0, 1\}$ and $1 \leqslant j \leqslant m$. Since $2^m \leqslant t-2 \leqslant 2^r - 2 \leqslant 2^s - 2$, we additionally have $1 \leqslant j \leqslant m < r \leqslant s$.

Proposition 3.10 *If n, s, t, r and j are as above, then*

$$\mathrm{zcl}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s + 2^r - \varepsilon - 2$$

and $\mathrm{TC}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s + 2^r - \varepsilon - 1$, *where* $\varepsilon = \begin{cases} 2^j, & \text{if } t \text{ is even} \\ 2^j + 1, & \text{otherwise.} \end{cases}$

PROOF — It is enough to prove that $z(w_1)^{2^{s+1}-1} z(w_2)^{2^s+2^r-\varepsilon-1} \neq 0$. After expanding, we consider all summands of the form $w_2^{n-2} \otimes x$, for some $x \in H^*(G_2(\mathbb{R}^n); \mathbb{Z}_2)$. By Lemma 3.5 each such summand is of the form $w_1^{2^{l+1}-2} w_2^{2^s+t-2^l-1} \otimes w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^r+2^l-\varepsilon-t}$, $0 \leqslant l \leqslant s$, with coefficient $\alpha_l = \binom{2^{s+1}-1}{2^{l+1}-2}\binom{2^s+2^r-\varepsilon-1}{2^s+t-2^l-1} = \binom{2^s+2^r-\varepsilon-1}{2^s+t-2^l-1}$. (Note: if $2^r + 2^l - \varepsilon - t < 0$, then $2^s + 2^r - \varepsilon - 1 < 2^s + t - 2^l - 1$ and hence $\alpha_l = 0$, so there is no need to discard summands $\alpha_l w_1^{2^{l+1}-2} w_2^{2^s+t-2^l-1} \otimes w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^r+2^l-\varepsilon-t}$ when $2^r + 2^l - \varepsilon - t < 0$.) Since w_2^{n-2} is the only non-zero class in $H^{2(n-2)}(G_2(\mathbb{R}^n); \mathbb{Z}_2)$ (by Proposition 2.1), it is enough to prove

$$A = \sum_{l=0}^{s} \alpha_l w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^r+2^l-\varepsilon-t} \neq 0 \quad \text{in } H^*(G_2(\mathbb{R}^n); \mathbb{Z}_2).$$

Let us first consider the case when t is even. Then $\varepsilon = 2^j$. Note that $2^s + 2^r - 2^j - 1 = 2^s + 2^{r-1} + 2^{r-2} + \cdots + 2^{j+1} + 2^{j-1} + 2^{j-2} + \cdots + 1$ ($j < r$). So, by Lucas' theorem, α_0 and α_s are even (since both $2^s + t - 2$ and $t - 1$ have digit 1 on the j-th position in the binary representation), while α_j is odd (since $2^s + t - 1 - 2^j$ has digit 0 on the j-th position in the binary representation).

Let us denote $\tau = 2^r - 2^j - t$. Note that $t - 2 + 2^j \leqslant 2^r$, i.e. $\tau \geqslant -2$. By Proposition 2.2.(4), $A \neq 0$ if and only if

$$\sum_{l=0}^{s} \alpha_l (e_1 + e_2)^{2^{s+1}-2^{l+1}+1} (e_1e_2)^{2^l+\tau} \cdot e_1 \cdot e_3^{n-3} e_4^{n-4} \ldots e_{n-1} \neq 0,$$

and, by part b) of Lemma 3.3, if and only if

$$p_1 = \sum_{l=0}^{s} \alpha_l (e_1 + e_2)^{2^{s+1}-2^{l+1}+1} (e_1e_2)^{2^l+\tau} \cdot e_1 \neq 0.$$

To prove that $p_1 \neq 0$ we will use Remark 3.4, i.e. we write p_1 as in Remark 3.4 and find suitable indices i' and i'' (as in that remark). We denote

$$q_1 = \sum_{l=0}^{s} \alpha_l (e_1+e_2)^{2^{s+1}-2^{l+1}} (e_1e_2)^{2^l+\tau} = \sum_{l=0}^{s} \alpha_l (e_1^{2^{l+1}} + e_2^{2^{l+1}})^{2^{s-l}-1} (e_1e_2)^{2^l+\tau}$$
$$= \sum_{l=0}^{s} \alpha_l \sum_{i=0}^{2^{s-l}-1} e_1^{i\cdot 2^{l+1}+2^l+\tau} e_2^{(2^{s-l}-1-i)\cdot 2^{l+1}+2^l+\tau}.$$

Let us observe a monomial $e_1^a e_2^b$ that appears in the inner sum for l. Then $a+b = 2^{s+1}+2\tau$ and $a-b = (2i+1-2^{s-l})2^{l+1}$, i.e. $2^{l+1} \parallel a-b$ for $s \neq l$ (that is $2^{l+1} \mid a-b$ and $2^{l+2} \nmid a-b$) and $a=b$ for $s=l$; so, $e_1^a e_2^b$ appears only once in q_1 and its coefficient is α_l. Now, since α_s is even this implies that the coefficient of $(e_1e_2)^{2^s+\tau}$ in q_1 is 0, and since α_0 is even that the coefficients of $e_1^{2^s+\tau-1} e_2^{2^s+\tau+1}$ and $e_1^{2^s+\tau+2^j-1} e_2^{2^s+\tau-2^j+1}$ in q_1 are 0. On the other hand, since α_j is odd the coefficient of $e_1^{2^s+\tau+2^j} e_2^{2^s+\tau-2^j}$ in q_1 is 1.

Now, we expand $p_1 = (e_1^2 + e_1e_2)q_1$. Note that the degree of each monomial in p_1 is $2^{s+1}+2\tau+2 = 2^{s+1}+2^{r+1}-2t-2^{j+1}+2 \leqslant 2^{s+1}+4(t-1)-2t-2 = 2n-6$, and hence, after removing all monomials of the form $e_1^a e_2^b$ when $a \geqslant n$ or $b \geqslant n$, we get p_1 written as in Remark 3.4. Let us observe a monomial $e_1^a e_2^b$ in p_1. By the previous identity, its coefficient is the sum of coefficients of $e_1^{a-2} e_2^b$ and $e_1^{a-1} e_2^{b-1}$ in q_1. So, the coefficient of $(e_1e_2)^{2^s+\tau+1}$ is 0, while the coefficient of $e_1^{2^s+\tau+2^j+1} e_2^{2^s+\tau-2^j+1}$ is 1. Since $2^s+\tau+2^j+1 = 2^s+2^r-t+1 \leqslant 2^s+t-1 = n-1$, the degrees of e_1 and e_2 in these monomials are less than n, so we can apply Lemma 3.3 and Remark 3.4 to conclude $p_1 \neq 0$.

Finally, we consider the case when t is odd. Then $\varepsilon = 2^j+1$. Note that $2^s+2^r-2^j-2 = 2^s+2^{r-1}+2^{r-2}+\cdots+2^{j+1}+2^{j-1}+2^{j-2}+\cdots+2$, while $t-2 = 2^{j+1}t'+2^j+1 < 2^r \leqslant 2^s$ for some $t' \geqslant 0$. So, by Lucas' theorem, we have that $\alpha_0 = \binom{2^s+2^r-2^j-2}{2^s+2^{j+1}t'+2^j+1}$ and $\alpha_1 = \binom{2^s+2^r-2^j-2}{2^s+2^{j+1}t'+2^j}$ are even, while

$$\alpha_2 = \binom{2^s+2^r-2^j-2}{2^s+t-5} = \binom{2^s+2^{r-1}+\cdots+2^{j+1}+2^{j-1}+\cdots+2}{2^s+2^{j+1}t'+2^{j-1}+2^{j-2}+\cdots+2}$$

is odd.

Let us denote $\theta = 2^r-2^j-t-1$. Note that $2^j+t-2 \leqslant 2^r+1$, i.e. $\theta \geqslant -4$. By Proposition 2.2.(4), $A \neq 0$ if and only if

$$\sum_{l=0}^{s} \alpha_l (e_1+e_2)^{2^{s+1}-2^{l+1}+1} (e_1e_2)^{2^l+\theta} \cdot e_1 \cdot e_3^{n-3} e_4^{n-4} \ldots e_{n-1} \neq 0,$$

and, by Lemma 3.3.b), if and only if $p_2 = \sum_{l=0}^{s} \alpha_l (e_1+e_2)^{2^{s+1}-2^{l+1}+1} (e_1e_2)^{2^l+\theta} e_1$ is non-zero. Let us denote

$$q_2 = \sum_{l=0}^{s} \alpha_l (e_1+e_2)^{2^{s+1}-2^{l+1}} (e_1e_2)^{2^l+\theta} = \sum_{l=0}^{s} \alpha_l (e_1^{2^{l+1}} + e_2^{2^{l+1}})^{2^{s-l}-1} (e_1e_2)^{2^l+\theta}.$$

Now, as in the previous part of the proof we conclude: the coefficients of $e_1^{2^s+\theta-1} e_2^{2^s+\theta+1}$, $e_1^{2^s+\theta-2} e_2^{2^s+\theta+2}$ and $e_1^{2^s+\theta-3} e_2^{2^s+\theta+3}$ in q_2 are 0 (since α_0 and α_1 are even); the coefficient of

$e_1^{2^s+\theta-4}e_2^{2^s+\theta+4}$ in q_2 is 1 (since α_2 is odd). So, in the polynomial $p_2 = (e_1^2 + e_1e_2)q_2$ the coefficient of $e_1^{2^s+\theta}e_2^{2^s+\theta+2}$ is 0, while the coefficient of $e_1^{2^s+\theta-2}e_2^{2^s+\theta+4}$ is 1. Since the total degree of each monomial of p_2 is $2^{s+1}+2\theta+2 = 2^{s+1}+2^{r+1}-2^{j+1}-2t \leqslant 2^{s+1}+4t-8-2t = 2n-8$ and $2^s+\theta+4 = 2^s+2^r-2^j-t+3 \leqslant 2^s+2^r-t+1 \leqslant 2^s+t-1 = n-1$, we can apply Lemma 3.3 and Remark 3.4 to conclude $p_2 \neq 0$. □

4 The zero-divisor cup-length of $G_3(\mathbb{R}^n)$

Let s be the unique integer such that $2^s < n \leqslant 2^{s+1}$. In this section we give some bounds for $\mathrm{zcl}(G_3(\mathbb{R}^n))$.

In the following proposition we consider the case $n = 2^s+1$. This result will show that the corresponding result of [9, Theorem 4.8] is not correct (see also Remark 4.2). Fortunately, this proposition gives a better lower bound for topological complexity.

Proposition 4.1 *Let $n = 2^s+1$, where $s \geqslant 3$. Then*

$$\mathrm{zcl}(G_3(\mathbb{R}^n)) \geqslant 2^{s+1}+2^s+2^{s-2}-7 \text{ and } \mathrm{TC}(G_3(\mathbb{R}^n)) \geqslant 2^{s+1}+2^s+2^{s-2}-6.$$

PROOF — It is enough to show $A = z(w_1)^{2^{s+1}-1}z(w_2)^{2^{s-1}+2^{s-2}-2}z(w_3)^{2^{s-1}-4} \neq 0$.

First, we prove that $w_1^{2^s}w_3 = 0$. By Proposition 2.2, this follows from

$$\begin{aligned} p_3 &= \pi^*(w_1^{2^s}w_3)e_1^2e_2e_4^{n-4}\cdots e_{n-1} \\ &= (e_1+e_2+e_3)^{2^s}(e_1e_2e_3)e_1^2e_2e_4^{n-4}\cdots e_{n-1} \\ &= (e_1^{2^s+3}e_2^2e_3 + e_1^3e_2^{2^s+2}e_3 + e_1^3e_2^2e_3^{2^s+1})e_4^{n-4}\cdots e_{n-1} = 0. \end{aligned}$$

Since $w_1^{2^s}w_3 = 0$, we have

$$\begin{aligned} A &= z(w_1)^{2^s-1}z(w_2)^{2^{s-1}+2^{s-2}-2}z(w_1^{2^s})z(w_3)^{2^{s-1}-4} \\ &= z(w_1)^{2^s-1}z(w_2)^{2^{s-1}+2^{s-2}-2}(w_1^{2^s}\otimes w_3^{2^{s-1}-4} + w_3^{2^{s-1}-4}\otimes w_1^{2^s}). \end{aligned}$$

Let us observe all classes of the form $w_3^{n-3}\otimes x$ for some $x \in H^*(G_3(\mathbb{R}^n);\mathbb{Z}_2)$ after expanding the expression for A; since w_3^{n-3} is the only non-zero class in $H^{3(n-3)}(G_3(\mathbb{R}^n);\mathbb{Z}_2)$ (by Proposition 2.1), to prove that A is non-zero it is enough to show that the sum of all such x is non-zero. To do so, we determine all monomials x' and x'' in classes w_1 and w_2, such that $w_1^{2^s}x' = w_3^{n-3} = w_3^{2^s-2}$ and $w_3^{2^{s-1}-4}x'' = w_3^{2^s-2}$.

Let $x' = w_1^aw_2^b$ be such that $w_1^{2^s+a}w_2^b = w_3^{2^s-2}$. Then $a+2b = 2(2^s-3)$. We use Proposition 2.2:

$$\begin{aligned} p_1 &= \pi^*(w_1^{2^s+a}w_2^b)e_1^2e_2e_4^{n-4}\cdots e_{n-1} \\ &= (e_1^{2^s}+e_2^{2^s}+e_3^{2^s})(e_1+e_2+e_3)^a(e_1e_2+e_2e_3+e_3e_1)^be_1^2e_2e_4^{n-4}\cdots e_{n-1} \\ &= e_3^{2^s}(e_1+e_2)^a(e_1e_2)^{b+1}e_1e_4^{n-4}\cdots e_{n-1} \\ &= e_3^{2^s}\sum_{i=0}^{a}\binom{a}{i}e_1^{i+b+2}e_2^{a-i+b+1}\cdot e_4^{n-4}\cdots e_{n-1}. \end{aligned}$$

Note that by Proposition 2.2.(3) the only non-zero monomials in this sum are the ones for i that satisfies $(i+b+2, a-i+b+1) \in \{(2^s-1, 2^s-2), (2^s-2, 2^s-1)\}$ and $\binom{a}{i}$ is odd, i.e. $i \in \{2^s-3-b, 2^s-4-b\}$ and $\binom{a}{i}$ is odd.

If $i = 2^s-3-b$, then $\binom{a}{i} = \binom{2(2^s-3-b)}{2^s-3-b} = \binom{2\delta}{\delta}$ (here $2\delta = 2(2^s-3-b) = a$). By Lemma 3.2, this number is odd only if $\delta = 0$, i.e. $(a,b) = (0, 2^s-3)$. Let us now consider the case $i = 2^s-4-b$. Then $\binom{a}{i} = \binom{2(2^s-3-b)}{2^s-4-b} = \binom{2\delta}{\delta-1} = \binom{2\delta}{\delta+1}$. Again, by Lemma 3.2, this number is odd only if $\delta = 2^l-1$, and hence $a = 2^{l+1}-2$ and $b = 2^s-2^l-2$ for some $1 \leqslant l \leqslant s-1$.

Let us now go back to our expression for A. Here we only consider pairs (a,b) that satisfy $a \leqslant 2^s-1$ and $b \leqslant 2^{s-1}+2^{s-2}-2$; hence $b = 2^s-2^l-2$ only if $l \in \{s-2, s-1\}$, so we have two pairs to consider: $(a,b) \in \{(2^{s-1}-2, 2^{s-1}+2^{s-2}-2), (2^s-2, 2^{s-1}-2)\} = P$.

Next, let $x'' = w_1^{a'}w_2^{b'}$ be such that $w_1^{a'}w_2^{b'}w_3^{2^{s-1}-4} = w_3^{2^s-2}$. We denote the set of all such pairs (a',b') with P'. Clearly, if $(a',b') \in P'$, then $a'+2b' = 3(2^{s-1}+2)$, and hence $a'+b' \geqslant 3(2^{s-2}+1)$; also, by observing A, it is clear that $a' \leqslant 2^s-1$.

Now, to prove that A is non-zero, it is enough to prove that B is non-zero, where B is equal to

$$\sum_{(a,b)\in P} w_1^{2^s-1-a}w_2^{2^{s-1}+2^{s-2}-2-b}w_3^{2^{s-1}-4} + \sum_{(a',b')\in P'} w_1^{2^s+2^s-1-a'}w_2^{2^{s-1}+2^{s-2}-2-b'}.$$

By Proposition 2.2.(4), this is equivalent to $p = \pi^*(B)e_1^2e_2e_4^{n-4}\cdots e_{n-1} \neq 0$. In what follows we will be working with the additive basis

$$\widetilde{B}_{2^s+1} = \{e_1^{a_1}e_2^{a_2}\cdots e_{2^s}^{a_{2^s}} \mid a_1 \leqslant 2^s-1,\ a_2 \leqslant 2^s-2,\ a_3 \leqslant 2^s,\ a_i \leqslant 2^s+1-i,\ i \geqslant 4\}$$

for $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$, given by Proposition 2.2.(1) and the canonical homeomorphism $\sigma : \mathrm{Flag}(\mathbb{R}^n) \to \mathrm{Flag}(\mathbb{R}^n)$ defined by

$$\sigma(L_1, L_2, L_3, L_4, L_5, \ldots, L_n) = (L_3, L_1, L_2, L_4, L_5, \ldots, L_n).$$

Let $d_{3,n-3} = e_1^2e_2e_4^{n-4}\cdots e_{n-1}$. Then

$$\begin{aligned} p_2 &= \pi^*\left(\sum_{(a,b)\in P} w_1^{2^s-1-a}w_2^{2^{s-1}+2^{s-2}-2-b}w_3^{2^{s-1}-4}\right)d_{3,n-3} \\ &= \pi^*(w_1^{2^{s-1}+1}w_3^{2^{s-1}-4} + w_1w_2^{2^{s-2}}w_3^{2^{s-1}-4})d_{3,n-3} \\ &= ((e_1+e_2+e_3)^{2^{s-1}} + (e_1e_2+e_2e_3+e_3e_1)^{2^{s-2}}) \\ &\quad \cdot (e_1+e_2+e_3)(e_1e_2e_3)^{2^{s-1}-4}d_{3,n-3}. \end{aligned}$$

Note that the monomials of p_2 belong to $\widetilde{B}_{2^s+1}$; indeed, the degree of e_1 in each monomial is at most $2^{s-1}+1+2^{s-1}-4+2 = 2^s-1$, the degree of e_2 is at most $2^{s-1}+1+2^{s-1}-4+1 = 2^s-2$, and the degree of e_3 is at most $2^{s-1}+1+2^{s-1}-4 = 2^s-3$. In particular, each monomial of p_2 is not divisible by $e_3^{2^s}$. Finally, $p_2 \neq 0$ since $e_1^{2^s-1}e_2^{2^{s-1}-3}e_3^{2^{s-1}-4}e_4^{n-4}\cdots e_{n-1}$ has coefficient 1 in p_2.

On the other hand,

$$p_3 = \pi^*\left(\sum_{(a',b')\in P'} w_1^{2^s+2^s-1-a'}w_2^{2^{s-1}+2^{s-2}-2-b'}\right)d_{3,n-3}$$

$$= \sum_{(a',b')\in P'} (e_1^{2^s}+e_2^{2^s}+e_3^{2^s})(e_1+e_2+e_3)^{2^s-1-a'}$$

$$\cdot (e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}+2^{s-2}-2-b'}d_{3,n-3}$$

$$= \sum_{(a',b')\in P'} e_3^{2^s}(e_1+e_2)^{2^s-1-a'}(e_1e_2)^{2^{s-1}+2^{s-2}-2-b'}d_{3,n-3}.$$

Since $a'+b' \geqslant 3(2^{s-2}+1)$, the degree of e_1 (resp. e_2) in each monomial of this sum is at most $2^s+2^{s-1}+2^{s-2}-1-a'-b' \leqslant 2^s-4$ (resp. $2^s+2^{s-1}+2^{s-2}-2-a'-b' \leqslant 2^s-5$), and hence, after expansion, each monomial (if any) of p_3 is in $\widetilde{B}_{2^s+1}$ and divisible by $e_3^{2^s}$ (note: it is possible that $p_3=0$).

Hence, p_2 and p_3 do not have any common monomials from $\widetilde{B}_{2^s+1}$, and so there are no cancellations between monomials of p_2 and p_3. Now, $p_2 \neq 0$ implies $p = p_2 + p_3 \neq 0$. □

Remark 4.2 *Ideas from this paper can be used to prove the following: if* $s \geqslant 4$, *then* $\mathrm{zcl}(G_3(\mathbb{R}^{2^s+2})) \geqslant 7\cdot 2^{s-1}$ *(one has* $z(w_1)^{2^{s+1}-1}z(w_2)^{2^s+2^{s-1}}z(w_3) \neq 0$*). Hence,* $\mathrm{TC}(G_3(\mathbb{R}^{2^s+2})) \geqslant 7\cdot 2^{s-1}+1$. *Complete proof of this result can be found in the extended version of this paper which is available on the author's website.*

Proposition 4.3 *Let* $s \geqslant 2$, $n = 2^s+t \leqslant 2^{s+1}$, $t \geqslant 3$ *and* $2^{r-1} < t \leqslant 2^r$. *Then*

$$\mathrm{zcl}(G_3(\mathbb{R}^n)) \geqslant 2^{s+2}-2^r-1 \quad \textit{and} \quad \mathrm{TC}(G_3(\mathbb{R}^n)) \geqslant 2^{s+2}-2^r.$$

Also, if $t-3 \geqslant 2^{s-1}$, *then* $\mathrm{zcl}(G_3(\mathbb{R}^n)) \geqslant 7\cdot 2^{s-1}-1$ *and* $\mathrm{TC}(G_3(\mathbb{R}^n)) \geqslant 7\cdot 2^{s-1}$.

PROOF — For the first inequality it is enough to show

$$A = z(w_1)^{2^{s+1}-1}z(w_2)^{2^{s+1}-2^{r+1}}z(w_3)^{2^r} \neq 0.$$

Note that $w_1^{2^s}w_3^{2^r} = 0$. Indeed, this follows from Proposition 2.2.(4), $e_i^{2^s+2^r} = 0$ for $i \in \{1,2,3\}$ and the following calculations:

$$\begin{aligned} p_1 &= \pi^*(w_1^{2^s}w_3^{2^r})e_1^2e_2e_4^{n-4}\cdots e_{n-1} \\ &= (e_1^{2^s}+e_2^{2^s}+e_3^{2^s})(e_1e_2e_3)^{2^r}e_1^2e_2e_4^{n-4}\cdots e_{n-1} \\ &= (e_1^{2^s+2^r}e_2^{2^r}e_3^{2^r}+e_1^{2^r}e_2^{2^s+2^r}e_3^{2^r}+e_1^{2^r}e_2^{2^r}e_3^{2^s+2^r})e_1^2e_2e_4^{n-4}\cdots e_{n-1} = 0. \end{aligned}$$

Similarly, one proves that $w_2^{2^s}w_3^{2^r}=0$, $w_1^{2^s}w_2^{2^s+2^r}=0$ and $w_1^{2^s+2^r}w_2^{2^s}=0$.

Note that $2^r \geqslant t \geqslant 3$ implies $r \geqslant 2$. Now, we consider the cases $2 \leqslant r \leqslant s-1$ and $r=s$ separately.

Case 1: $2 \leqslant r \leqslant s-1$. We have

$$\begin{aligned} A &= z(w_1)^{2^s-1}z(w_1)^{2^s}z(w_2)^{2^s-2^{r+1}}z(w_2)^{2^s}z(w_3)^{2^r} \\ &= z(w_1)^{2^s-1}z(w_2)^{2^s-2^{r+1}}(w_1^{2^s}w_2^{2^s}\otimes w_3^{2^r}+w_3^{2^r}\otimes w_1^{2^s}w_2^{2^s}). \end{aligned}$$

Since $2^s-1 = 2^{s-1}+\cdots+2^{r+1}+2^r+2^r-1$ and $2^s-2^{r+1} = 2^{s-1}+\cdots+2^{r+1}$, in a similar way we get

$$A = z(w_1)^{2^r-1}(w_1^{2^s}w_2^{2^s}\otimes w_1^{2^s-2^r}w_2^{2^s-2^{r+1}}w_3^{2^r}+w_1^{2^s-2^r}w_2^{2^s-2^{r+1}}w_3^{2^r}\otimes w_1^{2^s}w_2^{2^s}).$$

Since the dimension of $w_1^{2^s}w_2^{2^s}$ is greater than the dimension of the class $w_1^{2^s-2^r}w_2^{2^s-2^{r+1}}w_3^{2^r}$, after expanding the expression for A, there is only one summand with the first coordinate in dimension $3\cdot 2^s+2^r-1$, and this summand is $w_1^{2^s+2^r-1}w_2^{2^s}\otimes w_1^{2^s-2^r}w_2^{2^s-2^{r+1}}w_3^{2^r}$. Hence, it is enough to prove that $w_1^{2^s+2^r-1}w_2^{2^s}\neq 0$ and $w_1^{2^s-2^r}w_2^{2^s-2^{r+1}}w_3^{2^r}\neq 0$.

First, we prove that $w_1^{2^s+2^r-1}w_2^{2^s}\neq 0$. Since $e_i^{2^{s+1}}=0$ for $i\in\{1,2,3\}$ (by Proposition 2.2.(2)), by Proposition 2.2.(4) it is enough to prove that

$$\begin{aligned}
p_2 &= \pi^*(w_1^{2^s+2^r-1}w_2^{2^s})e_1^2e_2e_4^{n-4}\cdots e_{n-1}\\
&= (e_1+e_2+e_3)^{2^r-1}(e_1^{2^s}+e_2^{2^s}+e_3^{2^s})(e_1e_2+e_2e_3+e_3e_1)^{2^s}e_1^2e_2e_4^{n-4}\cdots e_{n-1}\\
&= (e_1+e_2+e_3)^{2^r-1}(e_1e_2e_3)^{2^s}e_1^2e_2e_4^{n-4}\cdots e_{n-1}\\
&= \pi^*(w_1^{2^r-1}w_3^{2^s})e_1^2e_2e_4^{n-4}\cdots e_{n-1}
\end{aligned}$$

is non-zero in $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$, i.e. that $w_1^{2^r-1}w_3^{2^s}$ is non-zero in $H^*(G_3(\mathbb{R}^n);\mathbb{Z}_2)$. Observe the inclusion $i:G_3(\mathbb{R}^{n-2^s})\subset G_3(\mathbb{R}^n)$. Note that the height of $i^*(w_1)$ in $H^*(G_3(\mathbb{R}^{n-2^s});\mathbb{Z}_2)$ is 2^r-1 (by (2.1)). So, let x be a class in $H^*(G_3(\mathbb{R}^{n-2^s});\mathbb{Z}_2)$ such that $i^*(w_1)^{2^r-1}x\in H^{3(n-2^s-3)}(G_3(\mathbb{R}^{n-2^s});\mathbb{Z}_2)$ is non-zero (this class exists by Poincare's duality); further, let $\widetilde{x}\in H^*(G_3(\mathbb{R}^n);\mathbb{Z}_2)$ be such that $i^*(\widetilde{x})=x$. Then, by [12, Lemma 1], the value of $w_1^{2^r-1}\widetilde{x}\cdot w_3^{2^s}$ is the same as the value of $i^*(w_1^{2^r-1}\widetilde{x})=i^*(w_1)^{2^r-1}x$, which is non-zero. Hence, $w_1^{2^r-1}w_3^{2^s}\neq 0$.

Finally, we prove that $w_1^{2^s-2^r}w_2^{2^s-2^{r+1}}w_3^{2^r}\neq 0$. This will immediately follow from the identity $w_1^{2^s-2^r}w_2^{2^s-2^r}w_3^{2^r}=w_1^{2^s}w_2^{2^s}=w_3^{2^s}\neq 0$, which we now prove. Since $e_i^{2^s+2^r}=0$ for $i\in\{1,2,3\}$, by Proposition 2.2.(4) this follows from (here $d_{3,n-3}=e_1^2e_2e_4^{n-4}\cdots e_{n-1}$)

$$\begin{aligned}
p_3 &= \pi^*(w_1^{2^s-2^r}w_2^{2^s-2^r}w_3^{2^r})d_{3,n-3}\\
&= (e_1+e_2+e_3)^{2^s-2^r}(e_1e_2+e_2e_3+e_3e_1)^{2^s-2^r}(e_1e_2e_3)^{2^r}d_{3,n-3}\\
&= (e_1+e_2+e_3)^{2^{s-1}}(e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}}\cdot\\
&\quad\cdot(e_1+e_2+e_3)^{2^{s-1}-2^r}(e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}-2^r}(e_1e_2e_3)^{2^r}d_{3,n-3}\\
&= (e_1+e_2+e_3)^{2^{s-1}-2^r}(e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}-2^r}(e_1e_2e_3)^{2^{s-1}+2^r}d_{3,n-3}\\
&= \dots\\
&= (e_1e_2e_3)^{2^{s-1}+2^{s-2}+\cdots+2^r+2^r}d_{3,n-3}\\
&= (e_1e_2e_3)^{2^s}d_{3,n-3}\\
&= (e_1+e_2+e_3)^{2^s}(e_1e_2+e_2e_3+e_3e_1)^{2^s}d_{3,n-3}\\
&= \pi^*(w_1^{2^s}w_2^{2^s})d_{3,n-3}.
\end{aligned}$$

Since $w_3^{2^s}\in B_{3,n-3}$, we have $w_3^{2^s}\neq 0$, which completes our proof.

Case 2: $r=s$. Then $A=z(w_1)^{2^s-1}(w_1^{2^s}\otimes w_3^{2^s}+w_3^{2^s}\otimes w_1^{2^s})$. Since after expanding A there is only one summand with the first coordinate in dimension $2^{s+2}-1$, and this summand is $w_1^{2^s-1}w_3^{2^s}\otimes w_1^{2^s}$, it is enough to prove $w_1^{2^s-1}w_3^{2^s}\neq 0$ and $w_1^{2^s}\neq 0$. The second follows from $w_1^{2^s}\in B_{3,n-3}$, and the first one is proven after the calculations for p_2.

Suppose now that $t-3\geqslant 2^{s-1}$. We will prove that

$$B=z(w_1)^{2^{s+1}-1}z(w_2)^{2^s}z(w_3)^{2^{s-1}}\neq 0,$$

which implies $\mathrm{zcl}(G_3(\mathbb{R}^n))\geqslant 2^{s+1}+2^s+2^{s-1}-1$.

Let us observe all summands of B with the first coordinate in dimension $9 \cdot 2^{s-1}$. Note that

$$B = z(w_1)^{2^s-1} z(w_1^{2^s}) z(w_2^{2^s}) z(w_3^{2^{s-1}}),$$

so the only monomial of this form is $w_1^{2^s} w_2^{2^s} w_3^{2^{s-1}} \otimes w_1^{2^s-1}$, and hence it is enough to prove that $w_1^{2^s} w_2^{2^s} w_3^{2^{s-1}} \neq 0$ and $w_1^{2^s-1} \neq 0$. This follows from Lemma 2.4 (indeed, since $t-3 \geqslant 2^{s-1}$, both monomials divide $w_1^{2^s} w_2^{2^s} w_3^{t-3} \neq 0$). $\square$

5 The zero-divisor cup-length of $G_k(\mathbb{R}^n)$

In this section we give a lower bound for $G_k(\mathbb{R}^n)$ for $k \geqslant 4$.

Proposition 5.1 *Let $4 \leqslant k < n$ and $2^s + k \leqslant n \leqslant 2^{s+1}$. Then*

$$\operatorname{zcl}(G_k(\mathbb{R}^n)) \geqslant (\lceil \log_2 k \rceil + 1) \cdot 2^s - 1 \quad \textit{and} \quad \operatorname{TC}(G_k(\mathbb{R}^n)) \geqslant (\lceil \log_2 k \rceil + 1) \cdot 2^s.$$

PROOF — Let $2^{r-1} < k \leqslant 2^r$. Then $\lceil \log_2 k \rceil = r$, so it is enough to prove

$$A = z(w_1)^{2^{s+1}-1} \prod_{i=1}^{r-1} z(w_{2^i})^{2^s} = z(w_1)^{2^s-1} \prod_{i=0}^{r-1} z(w_{2^i}^{2^s}) \neq 0.$$

First, let us prove that $p = \prod_{i=0}^{r-2} w_{2^i}^{2^s}$ is non-zero in $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$. Let $d_{k,n-k} = e_1^{k-1} \cdots e_{k-1} e_{k+1}^{n-k-1} \cdots e_{n-1}$. Since $e_i^{2^{s+1}} = 0$ for $1 \leqslant i \leqslant k$ (by Proposition 2.2.(2)) and $k' := \sum_{i=0}^{r-2} 2^i = 2^{r-1} - 1 < k$ we have

$$\begin{aligned} p_1 &= \pi^* \left(\prod_{i=0}^{r-2} w_{2^i}^{2^s} \right) d_{k,n-k} \\ &= \prod_{i=0}^{r-2} \left(\sum_{1 \leqslant a_1 < a_2 < \cdots < a_{2^i} \leqslant k} e_{a_1}^{2^s} e_{a_2}^{2^s} \cdots e_{a_{2^i}}^{2^s} \right) d_{k,n-k} \\ &= [2^0, 2^1, \ldots, 2^{r-2}] \left(\sum_{1 \leqslant a_1 < a_2 < \cdots < a_{k'} \leqslant k} e_{a_1}^{2^s} e_{a_2}^{2^s} \cdots e_{a_{k'}}^{2^s} \right) d_{k,n-k}, \end{aligned}$$

where $[2^0, 2^1, \ldots, 2^{r-2}] = \binom{2^0+2^1+\cdots+2^{r-2}}{2^0} \binom{2^1+\cdots+2^{r-2}}{2^1} \cdots \binom{2^{r-2}}{2^{r-2}}$ denotes the multinomial coefficient. By Lucas' theorem, this coefficient is odd. Also, for $1 \leqslant i \leqslant k$ the degree of e_i in each monomial in the last expression for p_1 is at most $2^s + k - i \leqslant n - i$, so all monomials in this expression are distinct members of the basis B_n for $H^*(\operatorname{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$, and hence $p_1 \neq 0$. So, by Proposition 2.2.(4), $p \neq 0$.

Now, let us observe all summands after expanding A with first coordinate in dimension $(2^{r-1} - 1) \cdot 2^s$. The dimension of p is $(2^{r-1} - 1) \cdot 2^s$, and it is easy to see that the only term of this form is $p \otimes w_1^{2^s-1} w_{2^r-1}^{2^s}$. So, to finish the proof it is enough to prove $w_1^{2^s-1} w_{2^r-1}^{2^s} \neq 0$. In fact, we prove that $w_1^{2^s} w_{2^r-1}^{2^s} \neq 0$. Since $e_i^{2^{s+1}} = 0$ for $1 \leqslant i \leqslant k$, we have

$$\begin{aligned} p_2 &= \pi^* \left(w_1^{2^s} w_{2^r-1}^{2^s} \right) d_{k,n-k} \\ &= \left(e_1^{2^s} + e_2^{2^s} + \cdots + e_k^{2^s} \right) \left(\sum_{1 \leqslant a_1 < a_2 < \cdots < a_{2^r-1} \leqslant k} e_{a_1}^{2^s} e_{a_2}^{2^s} \cdots e_{a_{2^r-1}}^{2^s} \right) d_{k,n-k} \\ &= \left(\sum_{1 \leqslant a_1 < a_2 < \cdots < a_{2^r-1+1} \leqslant k} e_{a_1}^{2^s} e_{a_2}^{2^s} \cdots e_{a_{2^r-1+1}}^{2^s} \right) d_{k,n-k}. \end{aligned}$$

Now, as above, $2^s + k \leqslant n$ implies that all monomials in the last expression for p_2 are distinct members of the basis B_n for $H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$, and hence $p_2 \neq 0$. By Proposition 2.2.(4), it follows that $w_1^{2^s} w_{2^r-1}^{2^s} \neq 0$. □

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