On the topological complexity and zero-divisor cup-length of real Grassmannians

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Abstract

Topological complexity naturally appears in the motion planning in robotics. In this paper we consider the problem of finding topological complexity of real Grassmann manifolds $G_k(\mathbb{R}^n)$. We use cohomology methods to give estimates on the zero-divisor cup-length of $G_k(\mathbb{R}^n)$ for various $2 \leqslant k < n$, which in turn give us lower bounds on topological complexity. Our results correct and improve several results from [9].

1 Introduction

For a path-connected space X we denote its topological complexity by $\mathrm{TC}(X)$. In [9] the author considered the problem of finding $\mathrm{TC}(G_k(\mathbb{R}^n))$ for various $2 \leqslant k < n$ (in this paper, $G_k(\mathbb{R}^n)$ denotes the real Grassmann manifold of k-dimensional subspaces in $\mathbb{R}^n$). Unfortunately, there is a problem with the proof of the main lemma of that paper (Lemma 4.4) and the consequential results on the topological complexity (Theorems 4.5, 4.8 and 4.12); see [10]. In this paper we reconsider this problem, and as an outcome correct and improve several results from [9]. As in [9], we use the *cohomology method* to obtain our results.

This paper closely follows and builds on the ideas presented in [9] (so, for background, motivation and all undefined notions, the reader is advised to consult [9]). Throughout the paper we will use, as much as possible, the notation from [9]. In particular, we will be working with the *unreduced* topological complexity, as defined by Farber in [5] (for example, by this definition the topological complexity of a contractible space is equal to 1).

The paper is organized as follows. In Section 2 we describe the cohomology method mentioned above and give an overview of the cohomology of real Grassmannians. In Section 3 we consider the case $k = 2$. We obtain the exact value of the zero-divisor cup-length of $G_2(\mathbb{R}^n)$ (denoted by $\mathrm{zcl}(G_2(\mathbb{R}^n))$) for $s \geqslant 2$ and $n \in \{2^s+1, 2^s+2, 2^s+3\}$; additionally, for $s \geqslant 3$, $2^s + 4 \leqslant n \leqslant 2^{s+1}$ we prove a lower and upper bound for $\mathrm{zcl}(G_2(\mathbb{R}^n))$. These results show that the value of the zero-divisor cup-length given in [9, Theorem 4.5] is not correct; what is more interesting, our results improve lower bounds for topological complexity stated in the same theorem. Section 4 is devoted to the case $k = 3$. Separately, we prove lower bounds for $\mathrm{zcl}(G_3(\mathbb{R}^n))$ in the cases $n = 2^s + 1$, $n = 2^s + 2$ and $2^s + 3 \leqslant n \leqslant 2^{s+1}$ (for $s \geqslant 4$). The first two results show that the corresponding results from [9, Theorem 4.8] are not correct, and improves the stated lower bound for topological complexity of $G_3(\mathbb{R}^{2^s+1})$ (for $s \geqslant 5$). In Section 5 we give a general lower bound for $\mathrm{zcl}(G_k(\mathbb{R}^n))$ (for $k \geqslant 4$). For $k \geqslant 9$ this result improves the bounds stated in [9, Theorem 4.10].

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2 Background and notation

As mentioned in Introduction, to obtain our results we use the so called *cohomology method*, which we now (briefly) explain.

Let $\Delta : X \to X \times X$ denote the diagonal map. Then the elements of

$$\mathrm{Ker}(\Delta^* : H^*(X \times X; \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2))$$

are called *zero-divisors*. Further, the *zero-divisor cup-length* of X, denote by $\mathrm{zcl}(X)$, is defined to be the maximum number of elements from $\mathrm{Ker}\Delta^*$ whose product is non-zero. In [5], Farber proved that $\mathrm{zcl}(X)$ gives a lower bound for $\mathrm{TC}(X)$, that is $\mathrm{TC}(X) \geqslant \mathrm{zcl}(X) + 1$. Hence, a lower bound for $\mathrm{zcl}(X)$ immediately gives a lower bound for $\mathrm{TC}(X)$. Note that for every $w \in H^*(X; \mathbb{Z}_2)$ the element

$$z(w) = w \otimes 1 + 1 \otimes w \in H^*(X \times X; \mathbb{Z}_2)$$

is in $\mathrm{Ker}\Delta^*$ (since $\Delta^*(z(w)) = w \cdot 1 + 1 \cdot w = 0$). Then, by [2, Lemma 5.2], $\mathrm{Ker}\Delta^*$ is generated by these elements. So, if $\mathrm{zcl}(X) = t$, then there are classes $x_1, x_2, \ldots, x_t \in H^*(X; \mathbb{Z}_2)$ such that $z(x_1)z(x_2) \cdots z(x_t) \neq 0$.

To get the best possible results on $\mathrm{TC}(G_k(\mathbb{R}^n))$ using the cohomology method, one requires fine understanding of the cohomology algebra $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$. There are several ways to describe this algebra; in this paper we will use the one due to Borel (see [1]):

$$H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \ldots, w_k]/I_{k,n},$$

where $w_1, w_2, \ldots, w_k$ are the Stiefel-Whitey classes of the canonical k-dimensional vector bundle over $G_k(\mathbb{R}^n)$, and $I_{k,n} = (\overline{w}_{n-k+1}, \overline{w}_{n-k+2}, \ldots, \overline{w}_n)$ is the ideal generated by dual classes.

Let us observe $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ and the corresponding ideal $\mathrm{Ker}\Delta^*$. We denote by $\mathcal{Z}_{k,n}$ the ideal generated by the classes $z(w_1), z(w_2), \ldots, z(w_k)$. Obviously, $\mathcal{Z}_{k,n} \subseteq \mathrm{Ker}\Delta^*$, but we can prove more.

Lemma 2.1 $\mathcal{Z}_{k,n} = \mathrm{Ker}\Delta^*$.

PROOF — It is enough to prove that for every $p \in H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$, the class $p \otimes 1 + 1 \otimes p$ is in $\mathcal{Z}_{k,n}$. Since p is a polynomial in $w_1, w_2, \ldots, w_k$, it is enough to consider the case $p = w_1^{a_1} \cdots w_k^{a_k}$, where $a_i \geqslant 0$ for $1 \leqslant i \leqslant k$.

We prove by induction on $\deg(p) = a_1 + \cdots + a_k$, that $p \otimes 1 + 1 \otimes p \in \mathcal{Z}_{k,n}$. This is obvious when $\deg(p) = 1$. So, suppose that it is true for all q such that $\deg(q) < \ell$, and prove it for a given monomial $p = w_1^{a_1} \cdots w_k^{a_k}$ such that $\deg(p) = \ell > 1$. Then $a_i > 0$ for some $1 \leqslant i \leqslant k$; further, let $p = w_i q$. So, we have

$$p \otimes 1 + 1 \otimes p = w_i q \otimes 1 + 1 \otimes w_i q = w_i \otimes 1(q \otimes 1 + 1 \otimes q) + 1 \otimes q(w_i \otimes 1 + 1 \otimes w_i),$$

and hence the conclusion follows by induction. $\square$

So, by the previous lemma, if $\mathrm{zcl}(G_k(\mathbb{R}^n)) = t$, then it is easy to see that there are $a_1, a_2, \ldots, a_k \in \mathbb{N}_0$ such that $z(w_1)^{a_1} z(w_2)^{a_2} \cdots z(w_k)^{a_k} \neq 0$.

Although Borel's description of $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ appears simple enough, it turns out that performing concrete calculations in this algebra can be rather difficult. Hence, one usually needs to apply some additional methods and properties of $H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$. The following result gives an additive basis for this algebra (see, e.g. [7, 11]).

Proposition 2.2 *The set* $B_{k,n-k} = \{w_1^{a_1} \cdots w_k^{a_k} \,:\, 0 \leqslant a_1 + \cdots + a_k \leqslant n-k\}$ *is an additive basis for* $H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$.

The *height* of a class $c \in \widetilde{H}^*(X;\mathbb{Z}_2)$, denoted by $\mathrm{ht}(c)$, is the largest $m \in \mathbb{N}$ such that $c^m \neq 0$. For $k \geqslant 2$, the height of $w_1 \in H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$ is obtained by Stong in [12]: if $2 \leqslant k \leqslant n-k$ and s is the unique positive integer such that $2^s < n \leqslant 2^{s+1}$, then

$$\mathrm{ht}(w_1) = \begin{cases} 2^{s+1}-2, & \text{if } k=2 \text{ or } (k,n)=(3,2^s+1), \\ 2^{s+1}-1, & \text{otherwise.} \end{cases} \tag{2.1}$$

In this paper we will often use Stong's method from [12] for calculating in $H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$ (later this method was generalized by Korbaš and Lörinc to all flag manifolds, see [8]). In what follows we briefly explain this method.

Let $\mathrm{Flag}(\mathbb{R}^n)$ denote the *(real) complete flag manifold* ($n \geqslant 2$). Denote by $e_i := w_1(\gamma_i)$ the first Stiefel-Whitney class of the canonical line bundle γ_i over $\mathrm{Flag}(\mathbb{R}^n)$, for $1 \leqslant i \leqslant n$. Then we have the map $\pi : \mathrm{Flag}(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$, given by

$$\pi(S_1,\dots,S_k,S_{k+1},\dots,S_n) = (S_1 \oplus \cdots \oplus S_k, S_{k+1} \oplus \cdots \oplus S_n).$$

The following result will be very useful for our calculations in $H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$ (and $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$).

Proposition 2.3 *(1) The set* $B_n = \{e_1^{a_1}e_2^{a_2}\dots e_{n-1}^{a_{n-1}} \,:\, 0 \leqslant a_i \leqslant n-i\}$ *is an additive basis for* $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$.

(2) $\mathrm{ht}(e_i) = n-1$ *for* $1 \leqslant i \leqslant n$. *In particular* $e_i^n = 0$ *for* $1 \leqslant i \leqslant n$.

(3) A monomial $e_1^{a_1}e_2^{a_2}\cdots e_n^{a_n} \in H^{\binom{n}{2}}(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$ *is non-zero if and only if* $(a_1,a_2,\dots,a_n)$ *is a permutation of the* n*-tuple* $(n-1,n-2,\dots,1,0)$.

(4) If $u \in H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$ *and*

$$v = e_1^{k-1}e_2^{k-2}\cdots e_{k-1}\cdot e_{k+1}^{n-k-1}e_{k+2}^{n-k-2}\cdots e_{n-1} \in H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2),$$

then $\pi^*(u)\cdot v \in H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$, *and* $u \neq 0$ *if and only if* $\pi^*(u)\cdot v \neq 0$.

(5) For $1 \leqslant i \leqslant k$, $\pi^*(w_i)$ *is the* i*-th elementary symmetric polynomial in the variables* $e_1,e_2,\dots,e_k$.

Heights of the classes $z(w_1)$ and $z(w_k)$ will be very useful in our calculations. In what follows we determine these values.

It turns out that if $\mathrm{ht}(w)$ is known, then $\mathrm{ht}(z(w))$ can easily be calculated. This is proven in Lemma 4.3 from [9]. Namely, one has: if $w \in H^*(X;\mathbb{Z}_2)$ and t is the unique non-negative integer such that $2^t \leqslant \mathrm{ht}(w) < 2^{t+1}$, then

$$\mathrm{ht}(z(w)) = 2^{t+1}-1. \tag{2.2}$$

We will apply this identity for $X = G_k(\mathbb{R}^n)$, when $2 \leqslant k \leqslant n-k$. If $2^s < n \leqslant 2^{s+1}$, then (2.1) implies

$$\mathrm{ht}(z(w_1)) = 2^{s+1}-1. \tag{2.3}$$

On the other hand, Proposition 2.2 implies $w_k^{n-k} \neq 0$, so $\mathrm{ht}(w_k) = n-k$ (by observing dimension we conclude that $w_k^{n-k+1} = 0$). Hence, if t is the unique non-negative integer such that $2^t \leqslant n-k < 2^{t+1}$, then (2.2) implies

$$\mathrm{ht}(z(w_k)) = 2^{t+1}-1. \tag{2.4}$$

The following lemma will be very useful in Section 3.

Lemma 2.4 *Let $m, k, n \in \mathbb{N}$, $k < n$, and $d_1, \ldots, d_m \in \mathbb{N}$ be such that $d_1 + \cdots + d_m \geqslant 2k(n-k)$. If $x_i \in H^{d_i}(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ for $1 \leqslant i \leqslant m$, then*

$$z(x_1) \cdots z(x_m) = 0.$$

PROOF — Note that the product $p = z(x_1) \cdots z(x_m)$ is the sum of certain classes of the form $x \otimes y + y \otimes x$, for some $x, y \in H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$. Since p is in dimension at least $2k(n-k) = 2 \dim G_k(\mathbb{R}^n)$, so is $x \otimes y$, and hence $x, y \in H^{k(n-k)}(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ or $x \otimes y = y \otimes x = 0$. There is only one non-zero class in $H^{k(n-k)}(G_k(\mathbb{R}^n); \mathbb{Z}_2)$, namely w_k^{n-k} (by Proposition 2.2), and hence $x \otimes y = y \otimes x = 0$ or $x \otimes y = w_k^{n-k} \otimes w_k^{n-k} = y \otimes x$. In both cases $x \otimes y + y \otimes x = 0$, which implies $p = 0$. □

Also, we recall some results from [9] that will be used in our calculations.

Lemma 2.5 *a) If $2^s < n \leqslant 2^{s+1}$, then $w_1^{2^s} w_2^{n-2^s-1} \neq 0$ and $w_1^{2^s} w_2^{n-2^s} = 0$ in $H^*(G_2(\mathbb{R}^n); \mathbb{Z}_2)$.*

b) If $2^s + 3 \leqslant n \leqslant 2^{s+1}$ and $t = n - 2^s$, then $w_1^{2^s} w_2^{2^s} w_3^{t-3} \neq 0$ in $H^*(G_3(\mathbb{R}^n); \mathbb{Z}_2)$.

Throughout the paper we use the same notation as in this section.

Finally, let us say a few words on Lemma 4.4 from [9] and our strategy that bypasses the application of this lemma. In Lemma 4.4 from [9] the author assumes that $u_1, \ldots, u_n \in H^*(X; \mathbb{Z}_2)$ and $k_1, \ldots, k_n \in \mathbb{N}$ are such that $u_1^{k_1} \cdots u_n^{k_n} \neq 0$, and wants to prove that $A = z(u_1)^{2^{r_1}-1} \cdots z(u_n)^{2^{r_n}-1} \neq 0$, where r_i is the unique integer such that $2^{r_i-1} \leqslant k_i < 2^{r_i}$ for $1 \leqslant i \leqslant n$. For this he notices that after expanding A one summand is $u_1^{k_1} \cdots u_n^{k_n} \otimes u_1^{2^{r_1}-k_1-1} \cdots u_n^{2^{r_n}-k_n-1}$, which is nonzero, and from this immediately concludes that $A \neq 0$. As we will see in the proofs of our results, the problem is that the set

$$S = \{(l_1, \ldots, l_n) \,:\, 0 \leqslant l_i \leqslant 2^{r_i} - 1,\, u_1^{l_1} \cdots u_n^{l_n} = u_1^{k_1} \cdots u_n^{k_n}\}$$

can contain more than one element, and hence that the corresponding summands of A with the first coordinate equal to $u_1^{k_1} \cdots u_n^{k_n}$ may cancel out. So, in our proofs we choose the n-tuple $(k_1, \ldots, k_n)$ a bit more carefully to ensure that

$$\sum_{(l_1, \ldots, l_n) \in S} u_1^{2^{r_1}-l_1-1} \cdots u_n^{2^{r_n}-l_n-1} \neq 0$$

and that this further leads to $A \neq 0$ (note: in our applications the degree of $z(u_i)$ in A will not always be $2^{r_i} - 1$, so we will have slightly different formulas than the one given above).

3 The zero-divisor cup-length of $G_2(\mathbb{R}^n)$

Let s be the unique integer such that $2^s < n \leqslant 2^{s+1}$. In this section we consider $\mathrm{zcl}(G_2(\mathbb{R}^n))$. We note that Propositions 3.7, 3.9, 3.11 and 3.12, that we prove in this section, show that the corresponding results of [9, Theorem 4.5] are not correct. Fortunately, correct versions give better lower bounds for the topological complexity of $G_2(\mathbb{R}^n)$.

We will compare our results with the following upper bound from [9] (this result is a consequence of a general result from [3, Theorem 1]).

Proposition 3.1 *If $1 \leqslant k < n$, then $\mathrm{TC}(G_k(\mathbb{R}^n)) \leqslant 2k(n-k)$. In fact, if $k \neq 1$ and $(k, n) \neq (2, 2^d + 1)$ for all $d \in \mathbb{N}$, then $\mathrm{TC}(G_k(\mathbb{R}^n)) \leqslant 2k(n-k) - 1$.*

3.1 Preliminary lemmas

Let n be a positive integer and $n = \sum_{i=0}^{t} \alpha_i \cdot 2^i$, where $\alpha_i \in \{0,1\}$ for $0 \leqslant i \leqslant t$ and $\alpha_t = 1$, its representation in base 2. Then we write $n := (\alpha_t, \dots, \alpha_1, \alpha_0)_2$.

As we use $\mathbb{Z}_2$ coefficient the following special case of Lucas' theorem will be particularly useful to us: if $n := (\alpha_t, \dots, \alpha_1, \alpha_0)_2$ and $m := (\beta_r, \dots, \beta_1, \beta_0)_2$, then

$$\binom{n}{m} \equiv 1 \pmod 2 \qquad \text{if and only if} \qquad t \geqslant r \quad \text{and} \quad \alpha_i \geqslant \beta_i \text{ for } 0 \leqslant i \leqslant r.$$

We will use the following two consequences of Lucas' theorem throughout the paper. Let $w \in H^*(X;\mathbb{Z}_2)$. By Lucas' theorem, $\binom{2^m}{i}$ is even for $1 \leqslant i \leqslant 2^m - 1$, and so

$$z(w)^{2^m} = (w \otimes 1 + 1 \otimes w)^{2^m} = w^{2^m} \otimes 1 + 1 \otimes w^{2^m}.$$

On the other hand, by Lucas' theorem $\binom{2^m-1}{i}$ is odd for all $0 \leqslant i \leqslant 2^m - 1$, and hence

$$z(w)^{2^m-1} = (w \otimes 1 + 1 \otimes w)^{2^m-1} = \sum_{i=0}^{2^m-1} w^i \otimes w^{2^m-1-i}.$$

We will also need the following result.

Lemma 3.2 *Let n be a non-negative integer. Then:*

a) $\binom{2n}{n}$ is odd if and only if $n = 0$;

b) $\binom{2n}{n+1}$ is odd if and only if $n = 2^{t+1} - 1$ for some $t \in \mathbb{N}_0$.

PROOF — Part a) immediately follows from Lucas' theorem.

For part b) we note that $C_n = \binom{2n}{n} - \binom{2n}{n+1}$ is the n-th Catalan number. Then the result follows from part a) and the fact that C_n (for $n \geqslant 1$) is odd if and only if $n = 2^{t+1} - 1$ for some $t \in \mathbb{N}_0$ (see [4]). □

Lemma 3.3 *Let $0 \leqslant m \leqslant n-2$ and $\alpha_0, \alpha_1, \dots, \alpha_{n-1-m} \in \mathbb{Z}_2$. Then:*

a) $\displaystyle\sum_{i=0}^{n-1-m} \alpha_i e_1^{m+i} e_2^{n-1-i} = 0$ *in* $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$ *iff* $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-1-m}$*;*

b) for a polynomial $p \in H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$ in classes e_1 and e_2 one has

$$p \cdot e_3^{n-3} e_4^{n-4} \cdots e_{n-1} = 0 \quad \text{in} \quad H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$$

if and only if $p = 0$ in $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$.

PROOF —

a) By Proposition 2.1 from [6] we have $e_2^{n-1} = e_1^{n-1} + e_1^{n-2} e_2 + \cdots + e_1 e_2^{n-2}$ (we use this proposition for $m = 1$, $k = n-1$ and $i = n-2$). Since $e_1^n = 0$ (by Proposition 2.3.(2)), we have

$$\sum_{i=0}^{n-1-m} \alpha_i e_1^{m+i} e_2^{n-1-i} = \sum_{i=1}^{n-1-m} (\alpha_i + \alpha_0) e_1^{m+i} e_2^{n-1-i}.$$

Since $e_1^{m+1}e_2^{n-2}, e_1^{m+2}e_2^{n-3}, \dots, e_1^{n-1}e_2^{m}$ are in the additive basis B_n (from Proposition 2.3.(1)), the last sum is zero if and only if $\alpha_1+\alpha_0=\alpha_2+\alpha_0=\dots=\alpha_{n-1-m}+\alpha_0=0$, i.e. if and only if $\alpha_0=\alpha_1=\dots=\alpha_{n-1-m}$.

b) As in part a) we use the identities $e_2^{n-1}=e_1^{n-1}+e_1^{n-2}e_2+\dots+e_1e_2^{n-2}$ and $e_1^n=e_2^n=0$ to express p in the form $\sum\alpha_{i,j}e_1^ie_2^j$, where $\alpha_{i,j}\in\{0,1\}$, $0\leqslant i\leqslant n-1$ and $0\leqslant j\leqslant n-2$. Then $\sum\alpha_{i,j}e_1^ie_2^je_3^{n-3}e_4^{n-4}\cdots e_{n-1}$ $(=pe_3^{n-3}e_4^{n-4}\cdots e_{n-1})$ is a sum of the elements from the basis B_n from Proposition 2.3.(1); so this sum is zero if and only if $\alpha_{ij}=0$ for all i,j, i.e. if and only if $p=0$ (since p is also represented in the basis B_n). □

Remark 3.4 *We will use the following consequence of part a) of this lemma. Let* $p=\sum_{i=0}^{b-a}\alpha_ie_1^{a+i}e_2^{b-i}\in H^{a+b}(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$ *for some* $0\leqslant a\leqslant n-2$, $a\leqslant b\leqslant n-1$. *If there exist* $0\leqslant i'\neq i''\leqslant b-a$ *such that* $\alpha_{i'}=0$ *and* $\alpha_{i''}=1$, *then* $p\neq 0$.

Further, if $q\in H^c(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$, *where* $c\leqslant 2n-3$, *is written as a sum of some monomials of the form* $e_1^ie_2^j$, *then after removing all summands with* $i\geqslant n$ *or* $j\geqslant n$ *(since they are 0 by Proposition 2.3.(2)), we get that* q *is written in the same way as* p *above.*

Lemma 3.5 *If* $2^s<n\leqslant 2^{s+1}$ *and* $a,b\in\mathbb{N}_0$ *are such that* $a+2b=2(n-2)$, *then* $w_1^aw_2^b\neq 0$ *in* $H^{2n-4}(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ *if and only if*

$$(a,b)=(2^{l+1}-2,n-2^l-1)\quad \textit{for some } 0\leqslant l\leqslant s.$$

PROOF — By Proposition 2.3.(4), $w_1^aw_2^b\neq 0$ in $H^{2n-4}(G_2(\mathbb{R}^n);\mathbb{Z}_2)$ if and only if

$$\pi^*(w_1^aw_2^b)e_1e_3^{n-3}\cdots e_{n-1}=(e_1+e_2)^a(e_1e_2)^be_1e_3^{n-3}\cdots e_{n-1}\neq 0$$

in $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$. After expanding we have

$$(e_1+e_2)^a(e_1e_2)^be_1e_3^{n-3}\cdots e_{n-1}=e_3^{n-3}\cdots e_{n-1}\sum_{i=0}^{a}\binom{a}{i}e_1^{i+1+b}e_2^{a-i+b}.$$

Note that by Proposition 2.3.(3) the only non-zero monomials in this sum are the ones for i that satisfies $(i+1+b,a-i+b)\in\{(n-1,n-2),(n-2,n-1)\}$ and $\binom{a}{i}$ is odd, i.e. $i\in\{n-2-b,n-3-b\}$ and $\binom{a}{i}$ is odd.

If $i=n-2-b$, then $\binom{a}{i}=\binom{2(n-2-b)}{n-2-b}=\binom{2m}{m}$ (here $2m=2(n-2-b)=a$). By Lemma 3.2 this number is odd only if $m=0$, i.e. $(a,b)=(0,n-2)$.

Let us now consider the case $i=n-3-b$. Then $\binom{a}{i}=\binom{2(n-2-b)}{n-3-b}=\binom{2m}{m-1}=\binom{2m}{m+1}$ (again $2m=2(n-2-b)=a$). By Lemma 3.2 this number is odd if and only if $m=2^l-1$ for some $l\geqslant 1$. Then $a=2^{l+1}-2$ and $b=n-2^l-1\geqslant 0$, which completes our proof. □

Remark 3.6 *If* $w_1^aw_2^b\neq 0$ *and* $a+2b=2(n-2)$, *then, by Proposition 2.2,* $w_1^aw_2^b=w_2^{n-2}$ *(since* w_2^{n-2} *is the only non-zero class in* $H^{2(n-2)}(G_2(\mathbb{R}^n);\mathbb{Z}_2)$*).*

3.2 Some exact values

In this section we calculate $\mathrm{zcl}(G_2(\mathbb{R}^n))$ for $s \geqslant 2$ and $n \in \{2^s+1, 2^s+2, 2^s+3\}$.

In the proofs of the main result of this section we will use the following observation. Let $n \geqslant 4$. Then, by Lemma 2.1 we conclude: if $\mathrm{zcl}(G_2(\mathbb{R}^n)) = t$, then there are $a, b, c \in \mathbb{N}_0$ such that $z(w_1)^a z(w_2)^b z(x_1) \cdots z(x_c) \neq 0$, where $a+b+c = t$ and $x_1, x_2, \ldots, x_t$ are some classes of $H^*(G_2(\mathbb{R}^n); \mathbb{Z}_2)$ each in dimension at least 3.

Proposition 3.7 *For $s \geqslant 2$ and $n = 2^s+1$ one has*

$$\mathrm{zcl}(G_2(\mathbb{R}^n)) = 2^{s+1} + 2^s - 4 \quad \textit{and} \quad \mathrm{TC}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s - 3.$$

PROOF — First, we prove that $z(w_1)^{2^{s+1}-1} z(w_2)^{2^s-3} \neq 0$. After expanding, we consider all summands of the form $w_2^{n-2} \otimes x$, for some $x \in H^*(G_2(\mathbb{R}^n); \mathbb{Z}_2)$. By Lemma 3.5 each such summand is of the form $w_1^{2^{l+1}-2} w_2^{2^s-2^l} \otimes w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^l-3}$ (for $l \geqslant 2$) with coefficient $\binom{2^{s+1}-1}{2^{l+1}-2}\binom{2^s-3}{2^s-2^l}$. By Lucas' theorem each of these binomial coefficients is odd, so $z(w_1)^{2^{s+1}-1} z(w_2)^{2^s-3}$ contains $w_2^{n-2} \otimes \sum_{l=2}^{s} w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^l-3}$. Since w_2^{n-2} is the only non-zero class in $H^{2(n-2)}(G_2(\mathbb{R}^n); \mathbb{Z}_2)$ (by Proposition 2.2), it is enough to prove $\sum_{l=2}^{s} w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^l-3} \neq 0$ (in $H^*(G_2(\mathbb{R}^{2^s+1}); \mathbb{Z}_2)$).

Note that by Lemma 2.5, $w_1^{2^s} w_2 = 0$, and so $w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^l-3} = 0$ for $2 \leqslant l \leqslant s-1$. Hence, it is enough to prove that $w_1 w_2^{2^s-3} = w_1 w_2^{n-4} \neq 0$, which follows from the fact that $w_1 w_2^{n-4}$ is in the additive basis $B_{2,n-2}$ (Proposition 2.2). So, $\mathrm{zcl}(G_2(\mathbb{R}^{2^s+1})) \geqslant 2^{s+1} + 2^s - 4$.

Let us now prove that $\mathrm{zcl}(G_2(\mathbb{R}^{2^s+1})) \leqslant 2^{s+1} + 2^s - 4$. Suppose that this is not the case and let $a, b, c \in \mathbb{N}_0$ and $x_1, \ldots, x_c \in H^*(G_2(\mathbb{R}^{2^s+1}); \mathbb{Z}_2)$ be some classes each in dimension at least 3, such that $a+b+c \geqslant 2^{s+1} + 2^s - 3$ and $z(w_1)^a z(w_2)^b z(x_1) z(x_2) \cdots z(x_c) \neq 0$. By Lemma 2.4, $a + 2b + 3c \leqslant 4(2^s-1) - 1 = 2^{s+2} - 5$, and hence $b + 2c \leqslant 2^s - 2$. Further, since $z(w_1)^{2^{s+1}} = 0$ (by (2.3)), we have $a \leqslant 2^{s+1} - 1$ and hence $b + c = (a+b+c) - a \geqslant 2^s - 2$. This implies $c = 0$ and $b = 2^s - 2$. Finally, $a+b+c \geqslant 2^{s+1} + 2^s - 3$ and $a \leqslant 2^{s+1} - 1$ imply $a = 2^{s+1} - 1$.

So, it is enough to prove $A = z(w_1)^{2^{s+1}-1} z(w_2)^{2^s-2} = 0$. Suppose that this is not the case. Note that the dimension of A is $2^{s+1} - 1 + 2(2^s-2) = 4(n-2) - 1$, so every summand of A is of the form $x' \otimes x''$ where one of the classes x' and x'' has dimension $2(n-2)$ and the other $2(n-2)-1$. Note that, by Proposition 2.2, the only class in $H^*(G_2(\mathbb{R}^n); \mathbb{Z}_2)$ of dimension $2(n-2)$ (resp. $2(n-2)-1$) is w_2^{n-2} (resp. $w_1 w_2^{n-3}$). By symmetry, this and $A \neq 0$ implies $A = w_2^{n-2} \otimes w_1 w_2^{n-3} + w_1 w_2^{n-3} \otimes w_2^{n-2}$. Now, we proceed as in the first part of the proof to prove that the coefficient of $w_2^{n-2} \otimes w_1 w_2^{n-3}$ in A is zero. By Lemma 3.5 each such summand in $A = z(w_1)^{2^{s+1}-1} z(w_2)^{2^s-2}$ is of the form $w_1^{2^{l+1}-2} w_2^{2^s-2^l} \otimes w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^l-2}$ (for some $s \geqslant l \geqslant 1$) with coefficient $\binom{2^{s+1}-1}{2^{l+1}-2}\binom{2^s-2}{2^s-2^l}$. By Lucas' theorem this coefficient is 1, so it is enough to prove $\sum_{l=1}^{s} w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^l-2} = 0$.

Again, by Lemma 2.5, $w_1^{2^s} w_2 = 0$, so $w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^l-2} = 0$ for $2 \leqslant l \leqslant s-1$. Hence, the previous sum is equal to $w_1^{2^{s+1}-3} + w_1 w_2^{2^s-2}$. By (2.1), $w_1^{2^{s+1}-3} \neq 0$, so $w_1^{2^{s+1}-3} = w_1 w_2^{n-3} = w_1 w_2^{2^s-2}$, and hence $A = 0$. $\square$

Remark 3.8 *By Proposition 3.1,* $\mathrm{TC}(G_2(\mathbb{R}^{2^s+1})) \leqslant 2^{s+2} - 4$*, so there is a gap of* $2^s - 1$ *between our lower bound and this bound. For example,* $9 \leqslant \mathrm{TC}(G_2(\mathbb{R}^5)) \leqslant 12$.

Proposition 3.9 *For* $s \geqslant 1$ *and* $n = 2^s + 2$ *one has*

$$\mathrm{zcl}(G_2(\mathbb{R}^n)) = 2^{s+1} + 2^s - 2 \quad \textit{and} \quad \mathrm{TC}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s - 1.$$

PROOF — First, we prove that $A = z(w_1)^{2^{s+1}-2} z(w_2)^{2^s} \neq 0$. Note that $w_2^{2^s} = w_2^{n-2} \neq 0$ (by Proposition 2.2) and $x \cdot w_2^{2^s} = 0$ for every $x \in \widetilde{H}^*(G_2(\mathbb{R}^n); \mathbb{Z}_2)$. Hence

$$z(w_1)^{2^{s+1}-2} z(w_2)^{2^s} = z(w_1)^{2^{s+1}-2}(w_2^{2^s} \otimes 1 + 1 \otimes w_2^{2^s}) = w_2^{2^s} \otimes w_1^{2^{s+1}-2} + w_1^{2^{s+1}-2} \otimes w_2^{2^s}.$$

Since $w_1^{2^{s+1}-2} \neq 0$ (by (2.1)) and $w_1^{2^{s+1}-2} \neq w_2^{2^s}$, we conclude that $A \neq 0$. So, $\mathrm{zcl}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s - 2$.

Let us now prove that $\mathrm{zcl}(G_2(\mathbb{R}^{2^s+2})) \leqslant 2^{s+1} + 2^s - 2$. Suppose that this is not the case and let $a, b, c \in \mathbb{N}_0$ and $x_1, \ldots, x_c \in H^*(G_2(\mathbb{R}^{2^s+1}); \mathbb{Z}_2)$ be some classes each in dimension at least 3, such that $a + b + c \geqslant 2^{s+1} + 2^s - 1$ and $B = z(w_1)^a z(w_2)^b z(x_1) z(x_2) \cdots z(x_c) \neq 0$. By Lemma 2.4, $a + 2b + 3c \leqslant 4 \cdot 2^s - 1 = 2^{s+2} - 1$, and hence $b + 2c \leqslant 2^s$. Also, since $z(w_1)^{2^{s+1}} = 0$ (by (2.3)), we have $a \leqslant 2^{s+1} - 1$, and hence $b + c = (a + b + c) - a \geqslant 2^s$, which implies $b = 2^s$ and $c = 0$. Finally, $a + b + c \geqslant 2^{s+1} + 2^s - 1$ now implies $a = 2^{s+1} - 1$.

So, $B = z(w_1)^{2^{s+1}-1} z(w_2)^{2^s}$. Since $x \cdot w_2^{2^s} = 0$ for every $x \in \widetilde{H}^*(G_2(\mathbb{R}^n); \mathbb{Z}_2)$, and $w_1^{2^{s+1}-1} = 0$ (by (2.3)), we have

$$B = z(w_1)^{2^{s+1}-1}(w_2^{2^s} \otimes 1 + 1 \otimes w_2^{2^s}) = w_2^{2^s} \otimes w_1^{2^{s+1}-1} + w_1^{2^{s+1}-1} \otimes w_2^{2^s} = 0,$$

a contradiction. □

Remark 3.10 *By Proposition 3.1,* $\mathrm{TC}(G_2(\mathbb{R}^{2^s+2})) \leqslant 2^{s+2} - 1$*, so there is a gap of* 2^s *between our lower bound and this bound. For example,* $5 \leqslant \mathrm{TC}(G_2(\mathbb{R}^4)) \leqslant 7$.

Proposition 3.11 *For* $s \geqslant 2$ *and* $n = 2^s + 3$ *one has*

$$\mathrm{zcl}(G_2(\mathbb{R}^n)) = 2^{s+1} + 2^s \quad \textit{and} \quad \mathrm{TC}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s + 1.$$

PROOF — First, we prove that $A = z(w_1)^{2^{s+1}-1} z(w_2)^{2^s+1} \neq 0$. Note that $w_2^{2^s+1} = w_2^{n-2} \neq 0$ (by Proposition 2.2), but $x \cdot w_2^{2^s+1} = 0$ for every $x \in \widetilde{H}^*(G_2(\mathbb{R}^n); \mathbb{Z}_2)$ and $w_1^k w_2^{2^s} = 0$ for all $k \geqslant 3$ (by observing dimension). Additionally, by Remark 3.6, $w_1^{2^{s+1}-2} w_2^2 = w_2^{n-2} = w_2^{2^s+1} \neq 0$ and hence $w_1^{2^{s+1}-3} w_2 \neq 0$. Now, since $w_1^{2^{s+1}-1} = 0$ (by (2.1)), and $w_1^3 w_2^{2^s} = 0$, we have

$$\begin{aligned}
A &= z(w_1)^{2^{s+1}-1}(w_2^{2^s} \otimes 1 + 1 \otimes w_2^{2^s})(w_2 \otimes 1 + 1 \otimes w_2) \\
&= z(w_1)^{2^{s+1}-1}(w_2^{2^s+1} \otimes 1 + w_2^{2^s} \otimes w_2 + w_2 \otimes w_2^{2^s} + 1 \otimes w_2^{2^s+1}) \\
&= w_1 w_2^{2^s} \otimes w_1^{2^{s+1}-2} w_2 + w_1^2 w_2^{2^s} \otimes w_1^{2^{s+1}-3} w_2 + w_1^{2^{s+1}-3} w_2 \otimes w_1^2 w_2^{2^s} + w_1^{2^{s+1}-2} w_2 \otimes w_1 w_2^{2^s}.
\end{aligned}$$

Since $w_1 w_2^{2^s} \otimes w_1^{2^{s+1}-2} w_2 \neq 0$ ($w_1 w_2^{2^s} \in B_{2,n-2}$ and $w_1^{2^{s+1}-2} w_2^2 = w_2^{n-2} \neq 0$) and $w_1 w_2^{2^s}$ is distinct from $w_1^2 w_2^{2^s}, w_1^{2^{s+1}-2} w_2, w_1^{2^{s+1}-3} w_2$ (they are in dimensions $2(n-2), 2(n-2) - 2, 2(n-2) - 3$, while $w_1 w_2^{2^s}$ is in dimension $2(n-2) - 1$), we conclude that $A \neq 0$. So, $\mathrm{zcl}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s$.

Let us now prove that $\mathrm{zcl}(G_2(\mathbb{R}^{2^s+3})) \leqslant 2^{s+1} + 2^s$. Suppose that this is not the case and let $a, b, c \in \mathbb{N}_0$ and $x_1, \ldots, x_c \in H^*(G_2(\mathbb{R}^{2^s+1}); \mathbb{Z}_2)$ be some classes each in dimension at least 3, such that $a + b + c \geqslant 2^{s+1} + 2^s + 1$ and $B = z(w_1)^a z(w_2)^b z(x_1) z(x_2) \cdots z(x_c) \neq 0$. By Lemma 2.4, $a + 2b + 3c \leqslant 4(2^s + 1) - 1 = 2^{s+2} + 3$, and hence $b + 2c \leqslant 2^s + 2$. Further, since $z(w_1)^{2^{s+1}} = 0$ (by (2.3)), we have $a \leqslant 2^{s+1} - 1$ and hence $b + c = (a + b + c) - a \geqslant 2^s + 2$. This implies $c = 0$ and $b = 2^s + 2$. Finally, $a + b + c \geqslant 2^{s+1} + 2^s + 1$ and $a \leqslant 2^{s+1} - 1$ imply $a = 2^{s+1} - 1$.

So, $B = z(w_1)^{2^{s+1}-1} z(w_2)^{2^s+2}$. Since $w_2^{2^s+2} = 0$ and $w_1^3 w_2^{2^s} = 0$:

$$
\begin{aligned}
B &= z(w_1)^{2^{s+1}-1} z(w_2^{2^s}) z(w_2^2) \\
&= z(w_1)^{2^{s+1}-1} (w_2^{2^s} \otimes w_2^2 + w_2^2 \otimes w_2^{2^s}) \\
&= w_1 w_2^{2^s} \otimes w_1^{2^{s+1}-2} w_2^2 + w_1^2 w_2^{2^s} \otimes w_1^{2^{s+1}-3} w_2^2 + w_1^{2^{s+1}-3} w_2^2 \otimes w_1^2 w_2^{2^s} + w_1^{2^{s+1}-2} w_2^2 \otimes w_1 w_2^{2^s}.
\end{aligned}
$$

As mentioned above $w_1^{2^{s+1}-2} w_2^2 = w_2^{n-2} = w_2^{2^s+1} \neq 0$; also, this implies $w_1^{2^{s+1}-3} w_2^2 \neq 0$ and hence $w_1^{2^{s+1}-3} w_2^2 = w_1 w_2^{2^s}$ (by Proposition 2.2, $w_1 w_2^{2^s}$ is the only class in dimension $2(n-2)-1$). Let us prove that $w_1^2 w_2^{2^s} = w_2^{n-2} = w_2^{2^s+1} \neq 0$. By Proposition 2.3.(4), this is equivalent with

$$
p = \pi^*(w_1^2 w_2^{2^s}) \cdot e_1 \cdot e_3^{n-3} \cdots e_{n-1} = (e_1 + e_2)^2 (e_1 e_2)^{2^s} e_1 \cdot e_3^{n-3} \cdots e_{n-1} \neq 0.
$$

This follows from

$$
p = e_1^{2^s+3} e_2^{2^s} e_3^{n-3} \cdots e_{n-1} + e_1^{2^s+1} e_2^{2^s+2} e_3^{n-3} \cdots e_{n-1} = e_1^{n-2} e_2^{n-1} e_3^{n-3} \cdots e_{n-1} \neq 0
$$

(by Proposition 2.3.(2)). So,

$$
B = w_1 w_2^{2^s} \otimes w_2^{2^s+1} + w_2^{2^s+1} \otimes w_1 w_2^{2^s} + w_1 w_2^{2^s} \otimes w_2^{2^s+1} + w_2^{2^s+1} \otimes w_1 w_2^{2^s} = 0,
$$

a contradiction. □

3.3 General bounds for $\mathrm{zcl}(G_2(\mathbb{R}^n))$

Let $2^s + 4 \leqslant n \leqslant 2^{s+1}$ and $t = n - 2^s$. Also, we assume $s \geqslant 3$ (i.e. $n \neq 8$). Further, let r be the unique integer such that $2^{r-1} < t \leqslant 2^r$. Since $t \geqslant 4$, we have $r \geqslant 2$. Let j be the smallest positive integer such that the digit on position j in the binary representation of $t-2$ is equal to 1 (j is well-defined since $t - 2 \geqslant 2$); in other words, $t - 2$ has the binary representation of the following form

$$
t - 2 = 2^m + \alpha_{m-1} 2^{m-1} + \cdots + \alpha_{j+1} 2^{j+1} + 2^j + \alpha_0
$$

for some $\alpha_0, \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_{m-1} \in \{0, 1\}$ and $1 \leqslant j \leqslant m$. Since $2^m \leqslant t - 2 \leqslant 2^r - 2 \leqslant 2^s - 2$, we additionally have $1 \leqslant j \leqslant m < r \leqslant s$.

Proposition 3.12 *If n, s, t, r and j are as above, then*

$$
2^{s+1} + 2^s + 2^r - \varepsilon - 2 \leqslant \mathrm{zcl}(G_2(\mathbb{R}^n)) \leqslant 2^{s+1} + 2^s + 2^r - 2
$$

and $\mathrm{TC}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s + 2^r - \varepsilon - 1$, *where* $\varepsilon = \begin{cases} 2^j, & \text{if } t \text{ is even} \\ 2^j + 1, & \text{otherwise.} \end{cases}$

PROOF — First, we prove that $\mathrm{zcl}(G_2(\mathbb{R}^n)) \leqslant 2^{s+1} + 2^s + 2^r - 2$. Suppose that this is not the case and let $a, b \geqslant 0$ be such that $A = z(w_1)^a z(w_2)^b \neq 0$ and $a + b \geqslant 2^{s+1} + 2^s + 2^r - 1$. Since $z(w_1)^{2^{s+1}} = 0$ and $z(w_2)^{2^{s+1}} = 0$ (by (2.3) and (2.4)), we have $a, b \leqslant 2^{s+1} - 1$ and hence $a, b \geqslant 2^s + 2^r$. Then

$$\begin{aligned} A &= z(w_1)^{a-2^s} z(w_1^{2^s}) z(w_2)^{b-2^s-2^r} z(w_2^{2^s}) z(w_2^{2^r}) \\ &= z(w_1)^{a-2^s} z(w_2)^{b-2^s-2^r} (w_1^{2^s} \otimes 1 + 1 \otimes w_1^{2^s})(w_2^{2^s} \otimes w_2^{2^r} + w_2^{2^r} \otimes w_2^{2^s}) \end{aligned}$$

(the last equality holds since $2^s + 2^r \geqslant 2^s + t = n$ and $w_2^n = 0$). But $w_1^{2^s} w_2^t = 0$ (by Lemma 2.5) and $t \leqslant 2^r \leqslant 2^s$, so $(w_1^{2^s} \otimes 1 + 1 \otimes w_1^{2^s})(w_2^{2^s} \otimes w_2^{2^r} + w_2^{2^r} \otimes w_2^{2^s}) = 0$, a contradiction.

Let us now prove that $z(w_1)^{2^{s+1}-1} z(w_2)^{2^s+2^r-\varepsilon-1} \neq 0$, which would imply $\mathrm{zcl}(G_2(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s + 2^r - \varepsilon - 2$. After expanding, we consider all summands of the form $w_2^{n-2} \otimes x$, for some $x \in H^*(G_2(\mathbb{R}^n); \mathbb{Z}_2)$. By Lemma 3.5 each such summand is of the form $w_1^{2^{l+1}-2} w_2^{2^s+t-2^l-1} \otimes w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^r+2^l-\varepsilon-t}$, $0 \leqslant l \leqslant s$, with coefficient $\alpha_l = \binom{2^{s+1}-1}{2^{l+1}-2}\binom{2^s+2^r-\varepsilon-1}{2^s+t-2^l-1} = \binom{2^s+2^r-\varepsilon-1}{2^s+t-2^l-1}$. (Note: if $2^r + 2^l - \varepsilon - t < 0$, then $2^s + 2^r - \varepsilon - 1 < 2^s + t - 2^l - 1$ and hence $\alpha_l = 0$, so there is no need to discard summands $\alpha_l w_1^{2^{l+1}-2} w_2^{2^s+t-2^l-1} \otimes w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^r+2^l-\varepsilon-t}$ when $2^r + 2^l - \varepsilon - t < 0$.) Since w_2^{n-2} is the only non-zero class in $H^{2(n-2)}(G_2(\mathbb{R}^n); \mathbb{Z}_2)$ (by Proposition 2.2), it is enough to prove

$$A = \sum_{l=0}^{s} \alpha_l w_1^{2^{s+1}-2^{l+1}+1} w_2^{2^r+2^l-\varepsilon-t} \neq 0 \quad \text{in } H^*(G_2(\mathbb{R}^n); \mathbb{Z}_2).$$

Let us first consider the case when t is even. Then $\varepsilon = 2^j$. Note that $2^s + 2^r - 2^j - 1 = 2^s + 2^{r-1} + 2^{r-2} + \cdots + 2^{j+1} + 2^{j-1} + 2^{j-2} + \cdots + 1$ ($j < r$). So, by Lucas' theorem, α_0 and α_s are even (since both $2^s + t - 2$ and $t - 1$ have digit 1 on the j-th position in the binary representation), while α_j is odd (since $2^s + t - 1 - 2^j$ has digit 0 on the j-th position in the binary representation).

Let us denote $\tau = 2^r - 2^j - t$. Note that $t - 2 + 2^j \leqslant 2^r$, i.e. $\tau \geqslant -2$. By Proposition 2.3.(4), $A \neq 0$ if and only if

$$\sum_{l=0}^{s} \alpha_l (e_1 + e_2)^{2^{s+1}-2^{l+1}+1} (e_1 e_2)^{2^l+\tau} \cdot e_1 \cdot e_3^{n-3} e_4^{n-4} \ldots e_{n-1} \neq 0,$$

and, by part b) of Lemma 3.3, if and only if

$$p_1 = \sum_{l=0}^{s} \alpha_l (e_1 + e_2)^{2^{s+1}-2^{l+1}+1} (e_1 e_2)^{2^l+\tau} \cdot e_1 \neq 0.$$

To prove that $p_1 \neq 0$ we will use Remark 3.4, i.e. we write p_1 as in Remark 3.4 and find suitable indices i' and i'' (from that remark). We denote

$$\begin{aligned} q_1 &= \sum_{l=0}^{s} \alpha_l (e_1 + e_2)^{2^{s+1}-2^{l+1}} (e_1 e_2)^{2^l+\tau} = \sum_{l=0}^{s} \alpha_l (e_1^{2^{l+1}} + e_2^{2^{l+1}})^{2^{s-l}-1} (e_1 e_2)^{2^l+\tau} \\ &= \sum_{l=0}^{s} \alpha_l \sum_{i=0}^{2^{s-l}-1} e_1^{i \cdot 2^{l+1}+2^l+\tau} e_2^{(2^{s-l}-1-i) \cdot 2^{l+1}+2^l+\tau}. \end{aligned}$$

Let us observe a monomial $e_1^a e_2^b$ that appears in the inner sum for l. Then $a+b=2^{s+1}+2\tau$ and $b-a=(2i+1-2^{s-l})2^{l+1}$, i.e. $2^{l+1} \parallel a-b$ for $s\neq l$ (that is $2^{l+1}\mid a-b$ and $2^{l+2}\nmid a-b$) and $a=b$ for $s=l$; so, $e_1^a e_2^b$ appears only once in q_1 and its coefficient is α_l. Now, since α_s is even this implies that the coefficient of $(e_1e_2)^{2^s+\tau}$ in q_1 is 0, and since α_0 is even that the coefficients of $e_1^{2^s+\tau-1}e_2^{2^s+\tau+1}$ and $e_1^{2^s+\tau+2^j-1}e_2^{2^s+\tau-2^j+1}$ in q_1 are 0. On the other hand, since α_j is odd the coefficient of $e_1^{2^s+\tau+2^j}e_2^{2^s+\tau-2^j}$ in q_1 is 1.

Now, we expand $p_1=(e_1^2+e_1e_2)q_1$. Note that the degree of each monomial in p_1 is $2^{s+1}+2\tau+2=2^{s+1}+2^{r+1}-2t-2^{j+1}+2\leqslant 2^{s+1}+4(t-1)-2t-2=2n-6$, and hence, after removing all monomials of the form $e_1^a e_2^b$ when $a\geqslant n$ or $b\geqslant n$, we get p_1 written as in Remark 3.4. Let us observe a monomial $e_1^a e_2^b$ in p_1. By the previous identity, its coefficient is the sum of coefficients of $e_1^{a-2}e_2^b$ and $e_1^{a-1}e_2^{b-1}$ in q_1. So, the coefficient of $(e_1e_2)^{2^s+\tau+1}$ is 0, while the coefficient of $e_1^{2^s+\tau+2^j+1}e_2^{2^s+\tau-2^j+1}$ is 1. Since $2^s+\tau+2^j+1=2^s+2^r-t+1\leqslant 2^s+t-1=n-1$, the degrees of e_1 and e_2 in these monomials are less than n, so we can apply Lemma 3.3 and Remark 3.4 to conclude $p_1\neq 0$.

Finally, we consider the case when t is odd. Then $\varepsilon=2^j+1$. Note that $2^s+2^r-2^j-2=2^s+2^{r-1}+2^{r-2}+\cdots+2^{j+1}+2^{j-1}+2^{j-2}+\cdots+2$, while $t-2=2^{j+1}t'+2^j+1<2^r\leqslant 2^s$ for some $t'\geqslant 0$. So, by Lucas' theorem, we have that $\alpha_0=\binom{2^s+2^r-2^j-2}{2^s+2^{j+1}t'+2^j+1}$ and $\alpha_1=\binom{2^s+2^r-2^j-2}{2^s+2^{j+1}t'+2^j}$ are even, while

$$\alpha_2=\binom{2^s+2^r-2^j-2}{2^s+t-5}=\binom{2^s+2^{r-1}+\cdots+2^{j+1}+2^{j-1}+\cdots+2}{2^s+2^{j+1}t'+2^{j-1}+2^{j-2}+\cdots+2}$$

is odd.

Let us denote $\theta=2^r-2^j-t-1$. Note that $2^j+t-2\leqslant 2^r+1$, i.e. $\theta\geqslant -4$. By Proposition 2.3.(4), $A\neq 0$ if and only if

$$\sum_{l=0}^{s}\alpha_l(e_1+e_2)^{2^{s+1}-2^{l+1}+1}(e_1e_2)^{2^l+\theta}\cdot e_1\cdot e_3^{n-3}e_4^{n-4}\dots e_{n-1}\neq 0,$$

and, by Lemma 3.3.b), if and only if $p_2=\sum_{l=0}^{s}\alpha_l(e_1+e_2)^{2^{s+1}-2^{l+1}+1}(e_1e_2)^{2^l+\theta}e_1$ is non-zero. Let us denote

$$q_2=\sum_{l=0}^{s}\alpha_l(e_1+e_2)^{2^{s+1}-2^{l+1}}(e_1e_2)^{2^l+\theta}=\sum_{l=0}^{s}\alpha_l(e_1^{2^{l+1}}+e_2^{2^{l+1}})^{2^{s-l}-1}(e_1e_2)^{2^l+\theta}.$$

Now, as in the previous part of the proof we conclude: the coefficients of $e_1^{2^s+\theta-1}e_2^{2^s+\theta+1}$, $e_1^{2^s+\theta-2}e_2^{2^s+\theta+2}$ and $e_1^{2^s+\theta-3}e_2^{2^s+\theta+3}$ in q_2 are 0 (since α_0 and α_1 are even); the coefficient of $e_1^{2^s+\theta-4}e_2^{2^s+\theta+4}$ in q_2 is 1 (since α_2 is odd). So, in the polynomial $p_2=(e_1^2+e_1e_2)q_2$ the coefficient of $e_1^{2^s+\theta}e_2^{2^s+\theta+2}$ is 0, while the coefficient of $e_1^{2^s+\theta-2}e_2^{2^s+\theta+4}$ is 1. Since the total degree of each monomial of p_2 is $2^{s+1}+2\theta+2=2^{s+1}+2^{r+1}-2^{j+1}-2t+1\leqslant 2^{s+1}+4(t-1)-2t-3=2n-7$ and $2^s+\theta+4=2^s+2^r-2^j-t+3\leqslant 2^s+2^r-t+1\leqslant 2^s+t-1=n-1$, we can apply Lemma 3.3 and Remark 3.4 to conclude $p_2\neq 0$. $\square$

4 The zero-divisor cup-length of $G_3(\mathbb{R}^n)$

Let s be the unique integer such that $2^s<n\leqslant 2^{s+1}$. In this section we give some bounds for $\mathrm{zcl}(G_3(\mathbb{R}^n))$.

In the following two propositions we consider the cases $n \in \{2^s+1, 2^s+2\}$. These results will show that the corresponding results of [9, Theorem 4.8] are not correct. Fortunately, these propositions give a better lower bounds for topological complexity.

Proposition 4.1 *Let $n = 2^s+1$, where $s \geqslant 3$. Then*

$$\mathrm{zcl}(G_3(\mathbb{R}^n)) \geqslant 2^{s+1}+2^s+2^{s-2}-7 \text{ and } \mathrm{TC}(G_3(\mathbb{R}^n)) \geqslant 2^{s+1}+2^s+2^{s-2}-6.$$

PROOF — It is enough to show $A = z(w_1)^{2^{s+1}-1} z(w_2)^{2^{s-1}+2^{s-2}-2} z(w_3)^{2^{s-1}-4} \neq 0$.

First, we prove that $w_1^{2^s} w_3 = 0$. By Proposition 2.3, this follows from

$$\begin{aligned} p_3 &= \pi^*(w_1^{2^s} w_3) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= (e_1+e_2+e_3)^{2^s}(e_1e_2e_3) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= (e_1^{2^s+3} e_2^2 e_3 + e_1^3 e_2^{2^s+2} e_3 + e_1^3 e_2^2 e_3^{2^s+1}) e_4^{n-4} \cdots e_{n-1} = 0. \end{aligned}$$

Since $w_1^{2^s} w_3 = 0$, we have

$$\begin{aligned} A &= z(w_1)^{2^s-1} z(w_2)^{2^{s-1}+2^{s-2}-2} z(w_1^{2^s}) z(w_3)^{2^{s-1}-4} \\ &= z(w_1)^{2^s-1} z(w_2)^{2^{s-1}+2^{s-2}-2} (w_1^{2^s} \otimes w_3^{2^{s-1}-4} + w_3^{2^{s-1}-4} \otimes w_1^{2^s}). \end{aligned}$$

Let us observe all classes of the form $w_3^{n-3} \otimes c$ for some $c \in H^*(G_3(\mathbb{R}^n);\mathbb{Z}_2)$ after expanding the expression for A; since w_3^{n-3} is the only non-zero class in $H^{3(n-3)}(G_3(\mathbb{R}^n);\mathbb{Z}_2)$ (by Proposition 2.2), to prove that A is non-zero it is enough to show that the sum of all such x is non-zero. To do so, we determine all monomials x' and x'' in classes w_1 and w_2, such that $w_1^{2^s} x' = w_3^{n-3} = w_3^{2^s-2}$ and $w_3^{2^{s-1}-4} x'' = w_3^{2^s-2}$.

Let $x' = w_1^a w_2^b$ be such that $w_1^{2^s+a} w_2^b = w_3^{2^s-2}$. Then $a+2b = 2(2^s-3)$. Next, we use Proposition 2.3:

$$\begin{aligned} p_1 &= \pi^*(w_1^{2^s+a} w_2^b) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= (e_1^{2^s}+e_2^{2^s}+e_3^{2^s})(e_1+e_2+e_3)^a (e_1e_2+e_2e_3+e_3e_1)^b e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= e_3^{2^s}(e_1+e_2)^a (e_1e_2)^{b+1} e_1 e_4^{n-4} \cdots e_{n-1} \\ &= e_3^{2^s} \sum_{i=0}^{a} \binom{a}{i} e_1^{i+b+2} e_2^{a-i+b+1} \cdot e_4^{n-4} \cdots e_{n-1}. \end{aligned}$$

Note that by Proposition 2.3.(3) the only non-zero monomials in this sum are the ones for i that satisfies $(i+b+2, a-i+b+1) \in \{(2^s-1, 2^s-2), (2^s-2, 2^s-1)\}$ and $\binom{a}{i}$ is odd, i.e. $i \in \{2^s-3-b, 2^s-4-b\}$ and $\binom{a}{i}$ is odd.

If $i = 2^s-3-b$, then $\binom{a}{i} = \binom{2(2^s-3-b)}{2^s-3-b} = \binom{2\delta}{\delta}$ (here $2\delta = 2(2^s-3-b) = a$). By Lemma 3.2, this number is odd only if $\delta = 0$, i.e. $(a,b) = (0, 2^s-3)$. Let us now consider the case $i = 2^s-4-b$. Then $\binom{a}{i} = \binom{2(2^s-3-b)}{2^s-4-b} = \binom{2\delta}{\delta-1} = \binom{2\delta}{\delta+1}$. Again, by Lemma 3.2, this number is odd only if $\delta = 2^l-1$, and hence $a = 2^{l+1}-2$ and $b = 2^s-2^l-2$ for some $1 \leqslant l \leqslant s-1$.

Let us now go back to our expression for A. Here we only consider pairs (a,b) that satisfy $a \leqslant 2^s-1$ and $b \leqslant 2^{s-1}+2^{s-2}-2$; hence $b = 2^s-2^l-2$ only if $l \in \{s-2, s-1\}$, so we have two pairs to consider: $(a,b) \in \{(2^{s-1}-2, 2^{s-1}+2^{s-2}-2), (2^s-2, 2^{s-1}-2)\} = P$.

Next, let $x'' = w_1^{a'} w_2^{b'}$ be such that $w_1^{a'} w_2^{b'} w_3^{2^{s-1}-4} = w_3^{2^s-2}$. We denote the set of all such pairs (a', b') with P'. Clearly, if $(a',b') \in P'$, then $a'+2b' = 3(2^{s-1}+2)$, and hence $a'+b' \geqslant 3(2^{s-2}+1)$; also, by observing A, it is clear that $a' \leqslant 2^s-1$.

Now, to prove that A is non-zero, it is enough to prove that B is non-zero, where B is equal to

$$\sum_{(a,b)\in P} w_1^{2^s-1-a} w_2^{2^{s-1}+2^{s-2}-2-b} w_3^{2^{s-1}-4} + \sum_{(a',b')\in P'} w_1^{2^s+2^s-1-a'} w_2^{2^{s-1}+2^{s-2}-2-b'}.$$

By Proposition 2.3.(4), this is equivalent to $p = \pi^*(B) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \neq 0$. In what follows we will be working with the additive basis

$$\widetilde{B}_{2^s+1} = \{e_1^{a_1} e_2^{a_2} \cdots e_{2^s}^{a_{2^s}} \mid a_1 \leqslant 2^s - 1,\ a_2 \leqslant 2^s - 2,\ a_3 \leqslant 2^s,\ a_i \leqslant 2^s + 1 - i,\ i \geqslant 4\}$$

for $H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$, given by Proposition 2.3.(1) and the canonical homeomorphism σ : $\mathrm{Flag}(\mathbb{R}^n) \to \mathrm{Flag}(\mathbb{R}^n)$ defined by

$$\sigma(L_1, L_2, L_3, L_4, L_5, \ldots, L_n) = (L_3, L_1, L_2, L_4, L_5, \ldots, L_n).$$

Let $d_{3,n-3} = e_1^2 e_2 e_4^{n-4} \cdots e_{n-1}$. Note that

$$\begin{aligned} p_2 &= \pi^* \left(\sum_{(a,b)\in P} w_1^{2^s-1-a} w_2^{2^{s-1}+2^{s-2}-2-b} w_3^{2^{s-1}-4} \right) d_{3,n-3} \\ &= \pi^*(w_1^{2^{s-1}+1} w_3^{2^{s-1}-4} + w_1 w_2^{2^{s-2}} w_3^{2^{s-1}-4}) d_{3,n-3} \\ &= ((e_1+e_2+e_3)^{2^{s-1}} + (e_1e_2 + e_2e_3 + e_3e_1)^{2^{s-2}}) \\ &\quad \cdot (e_1+e_2+e_3)(e_1e_2e_3)^{2^{s-1}-4} d_{3,n-3}. \end{aligned}$$

Note that the monomials of p_2 belong to $\widetilde{B}_{2^s+1}$; indeed, the degree of e_1 in each monomial is at most $2^{s-1}+1+2^{s-1}-4+2 = 2^s-1$, the degree of e_2 is at most $2^{s-1}+1+2^{s-1}-4+1 = 2^s-2$, and the degree of e_3 is at most $2^{s-1}+1+2^{s-1}-4 = 2^s-3$. In particular, each monomial of p_2 is not divisible by $e_3^{2^s}$. Finally, $p_2 \neq 0$ since $e_1^{2^s-1} e_2^{2^{s-1}-3} e_3^{2^{s-1}-4} e_4^{n-4} \cdots e_{n-1}$ has coefficient 1 in p_2.

On the other hand,

$$\begin{aligned} p_3 &= \pi^* \left(\sum_{(a',b')\in P'} w_1^{2^s+2^s-1-a'} w_2^{2^{s-1}+2^{s-2}-2-b'} \right) d_{3,n-3} \\ &= \sum_{(a',b')\in P'} (e_1^{2^s} + e_2^{2^s} + e_3^{2^s})(e_1+e_2+e_3)^{2^s-1-a'} \\ &\qquad \cdot (e_1e_2 + e_2e_3 + e_3e_1)^{2^{s-1}+2^{s-2}-2-b'} d_{3,n-3} \\ &= \sum_{(a',b')\in P'} e_3^{2^s} (e_1+e_2)^{2^s-1-a'} (e_1e_2)^{2^{s-1}+2^{s-2}-2-b'} d_{3,n-3}. \end{aligned}$$

Since $a' + b' \geqslant 3(2^{s-2}+1)$, the degree of e_1 (resp. e_2) in each monomial of this sum is at most $2^s + 2^{s-1} + 2^{s-2} - 1 - a' - b' \leqslant 2^s - 4$ (resp. $2^s + 2^{s-1} + 2^{s-2} - 2 - a' - b' \leqslant 2^s - 5$), and hence, after expansion, each monomial (if any) of p_3 is in $\widetilde{B}_{2^s+1}$ and divisible by $e_3^{2^s}$ (note: it is possible that $p_3 = 0$).

Hence, p_2 and p_3 do not have any common monomials from $\widetilde{B}_{2^s+1}$, and so there are no cancellations between monomials of p_2 and p_3. Now, $p_2 \neq 0$ implies $p = p_2 + p_3 \neq 0$. $\square$

Proposition 4.2 *Let $n = 2^s + 2$ where $s \geqslant 4$. Then*

$$\mathrm{zcl}(G_3(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s + 2^{s-1} \quad \textit{and} \quad \mathrm{TC}(G_3(\mathbb{R}^n)) \geqslant 2^{s+1} + 2^s + 2^{s-1} + 1.$$

PROOF — It is enough to prove $A = z(w_1)^{2^{s+1}-1} z(w_2)^{2^s+2^{s-1}} z(w_3) \neq 0$. Note that

$$A = z(w_1)^{2^s-1} z(w_1^{2^s}) z(w_2^{2^s}) z(w_2^{2^{s-1}}) z(w_3).$$

First, we prove that $w_1^{2^s} w_2^{2^s} = 0$. By Proposition 2.3.(4), to do so it is enough to prove that

$$\begin{aligned} p &= \pi^*(w_1^{2^s} w_2^{2^s}) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= (e_1 + e_2 + e_3)^{2^s} (e_1 e_2 + e_2 e_3 + e_3 e_1)^{2^s} e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= (e_1^{2^s} + e_2^{2^s} + e_3^{2^s})((e_1 e_2)^{2^s} + (e_2 e_3)^{2^s} + (e_3 e_1)^{2^s}) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \end{aligned}$$

is zero in $H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$. Since $e_1^{2^s+2} = e_1^n = 0$ (by Proposition 2.3.(2)), we have

$$p_1 = (e_2^{2^s} + e_3^{2^s})(e_2 e_3)^{2^s} e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} = 0$$

since $e_2^{2^{s+1}} = e_3^{2^{s+1}} = 0$ (by Proposition 2.3.(2)).

Now, we prove $w_2^{2^s} w_3 = 0$. Again, by Proposition 2.3.(4), to do so it is enough to prove that

$$\begin{aligned} p_2 &= \pi^*(w_2^{2^s} w_3) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= (e_1 e_2 + e_2 e_3 + e_3 e_1)^{2^s} e_1 e_2 e_3 e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= ((e_1 e_2)^{2^s} + (e_2 e_3)^{2^s} + (e_3 e_1)^{2^s}) e_1^3 e_2^2 e_3 e_4^{n-4} \cdots e_{n-1} \end{aligned}$$

is zero in $H^*(\mathrm{Flag}(\mathbb{R}^n); \mathbb{Z}_2)$. This follows from $e_1^{2^s+3} = e_1^{n+1} = 0$ and $e_2^{2^s+2} = e_2^n = 0$.

Since the dimension of $w_2^{2^s+2^{s-1}}$ is $2(2^s + 2^{s-1}) = 3 \cdot 2^s > 3(n-3)$, this class is zero, and hence

$$A = z(w_1)^{2^s-1} (w_1^{2^s} w_2^{2^{s-1}} w_3 \otimes w_2^{2^s} + w_2^{2^s} \otimes w_1^{2^s} w_2^{2^{s-1}} w_3).$$

Let us observe all classes of the form $w_3^{n-3} \otimes x$ for some $x \in H^*(G_3(\mathbb{R}^n); \mathbb{Z}_2)$ after expanding the expression for A. Since the dimension of w_3^{n-3} is $3(n-3) = 3 \cdot 2^s - 3$, the only classes of this form can be $\binom{2^s-1}{2^s-3} w_1^{2^s-3} w_2^{2^s} \otimes w_1^{2^s+2} w_2^{2^{s-1}} w_3$ and $\binom{2^s-1}{2^s-6} w_1^{2^{s+1}-6} w_2^{2^{s-1}} w_3 \otimes w_1^5 w_2^{2^s}$. By Lucas' theorem, $\binom{2^s-1}{2^s-3}$ and $\binom{2^s-1}{2^s-6}$ are odd, so to conclude our proof it is enough to prove that $w_1^{2^s-3} w_2^{2^s} = w_3^{n-3}$, $w_1^{2^s+2} w_2^{2^{s-1}} w_3 \neq 0$ and $w_1^{2^{s+1}-6} w_2^{2^{s-1}} w_3 = 0$.

Since $e_1^{2^s+2} = e_2^{2^s+2} = e_3^{2^s+2} = 0$, by Proposition 2.3.(5) we have

$$\begin{aligned} p_3 &= \pi^*(w_1^{2^s-3} w_2^{2^s}) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= (e_1 + e_2 + e_3)^{2^s-3} (e_1 e_2 + e_2 e_3 + e_3 e_1)^{2^s} e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\ &= (e_1 + e_2 + e_3)^{2^s-3} (e_1^{2^s+2} e_2^{2^s+1} + e_1^2 e_2^{2^s+1} e_3^{2^s} + e_1^{2^s+2} e_2 e_3^{2^s}) e_4^{n-4} \cdots e_{n-1} \\ &= (e_1 + e_2 + e_3)^{2^s-3} e_1^2 e_2^{2^s+1} e_3^{2^s} e_4^{n-4} \cdots e_{n-1} \\ &= e_1^{2^s-1} e_2^{2^s+1} e_3^{2^s} e_4^{n-4} \cdots e_{n-1}, \end{aligned}$$

and hence, by Proposition 2.3, $w_1^{2^s-3} w_2^{2^s} = w_3^{n-3}$.

Using similar arguments, we calculate

$$\begin{aligned}
p_4 &= \pi^*(w_1^{2^s+2}w_2^{2^{s-1}}w_3)e_1^2e_2e_4^{n-4}\cdots e_{n-1}\\
&= (e_1+e_2+e_2)^{2^s+2}(e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}}e_1e_2e_3e_1^2e_2e_4^{n-4}\cdots e_{n-1}\\
&= (e_1^{2^s}+e_2^{2^s}+e_3^{2^s})(e_1^2+e_2^2+e_3^2)((e_1e_2)^{2^{s-1}}+(e_2e_3)^{2^{s-1}}+(e_3e_1)^{2^{s-1}})e_1^3e_2^2e_3e_4^{n-4}\cdots e_{n-1}\\
&= e_3^{2^s+1}(e_1^2+e_2^2)e_1^{2^{s-1}}e_2^{2^{s-1}}e_1^3e_2^2e_4^{n-4}\cdots e_{n-1}\\
&= (e_1^{2^{s-1}+5}e_2^{2^{s-1}+2}+e_1^{2^{s-1}+3}e_2^{2^{s-1}+4})e_3^{2^s+1}e_4^{n-4}\cdots e_{n-1}.
\end{aligned}$$

Since $2^{s-1}+5<2^s$, by Proposition 2.3.(1) for the additive basis $\widetilde{B}_{2^s+2}$ (defined as $\widetilde{B}_{2^s+1}$ in Proposition 4.1), $p_4\neq 0$. Hence, $w_1^{2^s+2}w_2^{2^{s-1}}w_3\neq 0$ (by Proposition 2.3.(4)).

Finally, by Proposition 2.3.(5), we have

$$\begin{aligned}
p_5 &= \pi^*(w_1^{2^{s+1}-6}w_2^{2^{s-1}}w_3)e_1^2e_2e_4^{n-4}\cdots e_{n-1}\\
&= (e_1+e_2+e_3)^{2^{s+1}-6}(e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}}e_1e_2e_3e_1^2e_2e_4^{n-4}\cdots e_{n-1}\\
&= (e_1+e_2+e_3)^{2^s-6}(e_1^{2^s}+e_2^{2^s}+e_3^{2^s})(e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}}e_1^3e_2^2e_3e_4^{n-4}\cdots e_{n-1}\\
&= (e_1+e_2)^{2^s-6}e_3^{2^s+1}e_1^{2^{s-1}+3}e_2^{2^{s-1}+2}e_4^{n-4}\cdots e_{n-1}\\
&= \left(\binom{2^s-6}{2^{s-1}-3}e_1^{2^s}e_2^{2^s-1}+\binom{2^s-6}{2^{s-1}-4}e_1^{2^s-1}e_2^{2^s}\right)e_3^{2^s+1}e_4^{n-4}\cdots e_{n-1}.
\end{aligned}$$

Since $s\geqslant 4$, by Lucas' theorem $\binom{2^s-6}{2^{s-1}-3}$ and $\binom{2^s-6}{2^{s-1}-4}$ are even, so $p_5=0$, and hence, by Proposition 2.3.(4), $w_1^{2^{s+1}-6}w_2^{2^{s-1}}w_3=0$, which completes our proof. □

Proposition 4.3 *Let $s\geqslant 2$, $n=2^s+t\leqslant 2^{s+1}$, $t\geqslant 3$ and $2^{r-1}<t\leqslant 2^r$. Then*

$$\operatorname{zcl}(G_3(\mathbb{R}^n))\geqslant 2^{s+2}-2^r-1 \quad \textit{and} \quad \operatorname{TC}(G_3(\mathbb{R}^n))\geqslant 2^{s+2}-2^r.$$

Also, if $t-3\geqslant 2^{s-1}$, then $\operatorname{zcl}(G_3(\mathbb{R}^n))\geqslant 7\cdot 2^{s-1}-1$ and $\operatorname{TC}(G_3(\mathbb{R}^n))\geqslant 7\cdot 2^{s-1}$.

PROOF — For the first inequality it is enough to show

$$A=z(w_1)^{2^{s+1}-1}z(w_2)^{2^{s+1}-2^{r+1}}z(w_3)^{2^r}\neq 0.$$

Note that $w_1^{2^s}w_3^{2^r}=0$. Indeed, this follows from Proposition 2.3.(4), $e_i^{2^s+2^r}=0$ for $i\in\{1,2,3\}$ and the following calculations:

$$\begin{aligned}
p_1 &= \pi^*(w_1^{2^s}w_3^{2^r})e_1^2e_2e_4^{n-4}\cdots e_{n-1}\\
&= (e_1^{2^s}+e_2^{2^s}+e_3^{2^s})(e_1e_2e_3)^{2^r}e_1^2e_2e_4^{n-4}\cdots e_{n-1}\\
&= (e_1^{2^s+2^r}e_2^{2^r}e_3^{2^r}+e_1^{2^r}e_2^{2^s+2^r}e_3^{2^r}+e_1^{2^r}e_2^{2^r}e_3^{2^s+2^r})e_1^2e_2e_4^{n-4}\cdots e_{n-1}=0.
\end{aligned}$$

Similarly, one proves that $w_2^{2^s}w_3^{2^r}=0$, $w_1^{2^s}w_2^{2^s+2^r}=0$ and $w_1^{2^s+2^r}w_2^{2^s}=0$.

Note that $2^r\geqslant t\geqslant 3$ implies $r\geqslant 2$. Now, we consider the cases $2\leqslant r\leqslant s-1$ and $r=s$ separately.

Case 1: $2\leqslant r\leqslant s-1$. We have

$$A=z(w_1)^{2^s-1}z(w_1)^{2^s}z(w_2)^{2^s-2^{r+1}}z(w_2)^{2^s}z(w_3)^{2^r}$$

$$= z(w_1)^{2^s-1}z(w_2)^{2^s-2^{r+1}}(w_1^{2^s}w_2^{2^s}\otimes w_3^{2^r}+w_3^{2^r}\otimes w_1^{2^s}w_2^{2^s}).$$

Since $2^s-1=2^{s-1}+\cdots+2^{r+1}+2^r+2^r-1$ and $2^s-2^{r+1}=2^{s-1}+\cdots+2^{r+1}$, in a similar way we get

$$A=z(w_1)^{2^r-1}(w_1^{2^s}w_2^{2^s}\otimes w_1^{2^s-2^r}w_2^{2^s-2^{r+1}}w_3^{2^r}+w_1^{2^s-2^r}w_2^{2^s-2^{r+1}}w_3^{2^r}\otimes w_1^{2^s}w_2^{2^s}).$$

Since the dimension of $w_1^{2^s}w_2^{2^s}$ is greater than the dimension of the class $w_1^{2^s-2^r}w_2^{2^s-2^{r+1}}w_3^{2^r}$, after expanding the expression for A, there is only one summand with the first coordinate in dimension $3\cdot 2^s+2^r-1$, and this summand is $w_1^{2^s+2^r-1}w_2^{2^s}\otimes w_1^{2^s-2^r}w_2^{2^s-2^{r+1}}w_3^{2^r}$. Hence, it is enough to prove that $w_1^{2^s+2^r-1}w_2^{2^s}\neq 0$ and $w_1^{2^s-2^r}w_2^{2^s-2^{r+1}}w_3^{2^r}\neq 0$.

First, we prove that $w_1^{2^s+2^r-1}w_2^{2^s}\neq 0$. Since $e_i^{2^{s+1}}=0$ for $i\in\{1,2,3\}$ (by Proposition 2.3.(2)), by Proposition 2.3.(4) it is enough to prove that

$$\begin{aligned}
p_2 &= \pi^*(w_1^{2^s+2^r-1}w_2^{2^s})e_1^2e_2e_4^{n-4}\cdots e_{n-1}\\
&= (e_1+e_2+e_3)^{2^r-1}(e_1^{2^s}+e_2^{2^s}+e_3^{2^s})(e_1e_2+e_2e_3+e_3e_1)^{2^s}e_1^2e_2e_4^{n-4}\cdots e_{n-1}\\
&= (e_1+e_2+e_3)^{2^r-1}(e_1e_2e_3)^{2^s}e_1^2e_2e_4^{n-4}\cdots e_{n-1}\\
&= \pi^*(w_1^{2^r-1}w_3^{2^s})e_1^2e_2e_4^{n-4}\cdots e_{n-1}
\end{aligned}$$

is non-zero in $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$, i.e. that $w_1^{2^r-1}w_3^{2^s}$ is non-zero in $H^*(G_3(\mathbb{R}^n);\mathbb{Z}_2)$. Observe the inclusion $i:G_3(\mathbb{R}^{n-2^s})\subset G_3(\mathbb{R}^n)$. Note that the height of $i^*(w_1)$ in $H^*(G_3(\mathbb{R}^{n-2^s});\mathbb{Z}_2)$ is 2^r-1 (by (2.1)). So, let x be a class in $H^*(G_3(\mathbb{R}^{n-2^s});\mathbb{Z}_2)$ such that $i^*(w_1)^{2^r-1}x\in H^{3(n-2^s-3)}(G_3(\mathbb{R}^{n-2^s});\mathbb{Z}_2)$ is non-zero (this class exists by Poincare's duality); further, let $\widetilde{x}\in H^*(G_3(\mathbb{R}^n);\mathbb{Z}_2)$ be such that $i^*(\widetilde{x})=x$. Then, by [12, Lemma 1], the value of $w_1^{2^r-1}\widetilde{x}\cdot w_3^{2^s}$ is the same as the value of $i^*(w_1^{2^r-1}\widetilde{x})=i^*(w_1)^{2^r-1}x$, which is non-zero. Hence, $w_1^{2^r-1}w_3^{2^s}\neq 0$.

Finally, we prove that $w_1^{2^s-2^r}w_2^{2^s-2^{r+1}}w_3^{2^r}\neq 0$. This will immediately follow from $w_1^{2^s-2^r}w_2^{2^s-2^r}w_3^{2^r}=w_1^{2^s}w_2^{2^s}=w_3^{2^s}\neq 0$, which we now prove. Since $e_i^{2^s+2^r}=0$ for $i\in\{1,2,3\}$, by Proposition 2.3.(4) this follows from (here $d_{3,n-3}=e_1^2e_2e_4^{n-4}\cdots e_{n-1}$)

$$\begin{aligned}
p_3 &= \pi^*(w_1^{2^s-2^r}w_2^{2^s-2^r}w_3^{2^r})d_{3,n-3}\\
&= (e_1+e_2+e_3)^{2^s-2^r}(e_1e_2+e_2e_3+e_3e_1)^{2^s-2^r}(e_1e_2e_3)^{2^r}d_{3,n-3}\\
&= (e_1+e_2+e_3)^{2^{s-1}}(e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}}\cdot\\
&\quad\cdot(e_1+e_2+e_3)^{2^{s-1}-2^r}(e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}-2^r}(e_1e_2e_3)^{2^r}d_{3,n-3}\\
&= (e_1+e_2+e_3)^{2^{s-1}-2^r}(e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}-2^r}(e_1e_2e_3)^{2^{s-1}+2^r}d_{3,n-3}\\
&= \ \ldots\\
&= (e_1e_2e_3)^{2^{s-1}+2^{s-2}+\cdots+2^r+2^r}d_{3,n-3}\\
&= (e_1e_2e_3)^{2^s}d_{3,n-3}\\
&= (e_1+e_2+e_3)^{2^s}(e_1e_2+e_2e_3+e_3e_1)^{2^s}d_{3,n-3}\\
&= \pi^*(w_1^{2^s}w_2^{2^s})d_{3,n-3}.
\end{aligned}$$

Since $w_3^{2^s}\in B_{3,n-3}$, we have $w_3^{2^s}\neq 0$, which completes our proof.

Case 2: $r=s$. Then $A=z(w_1)^{2^s-1}(w_1^{2^s}\otimes w_3^{2^s}+w_3^{2^s}\otimes w_1^{2^s})$. Since after expanding A there is only one summand with first coordinate in dimension $2^{s+2}-1$, and this summand is

$w_1^{2^s-1}w_3^{2^s}\otimes w_1^{2^s}$, it is enough to prove $w_1^{2^s-1}w_3^{2^s}\neq 0$ and $w_1^{2^s}\neq 0$. The second follows from $w_1^{2^s}\in B_{3,n-3}$, and the first one is proven after the calculations for p_2.

Suppose now that $t-3\geqslant 2^{s-1}$. We will prove that

$$B = z(w_1)^{2^{s+1}-1}z(w_2)^{2^s}z(w_3)^{2^{s-1}}\neq 0,$$

which implies $\mathrm{zcl}(G_3(\mathbb{R}^n))\geqslant 2^{s+1}+2^s+2^{s-1}-1$.

Let us observe all summands of B with first coordinate in dimension $9\cdot 2^{s-1}$. Note that

$$B = z(w_1)^{2^s-1}z(w_1^{2^s})z(w_2^{2^s})z(w_3^{2^{s-1}}),$$

so the only monomial of this form is $w_1^{2^s}w_2^{2^s}w_3^{2^{s-1}}\otimes w_1^{2^s-1}$, and hence it is enough to prove that $w_1^{2^s}w_2^{2^s}w_3^{2^{s-1}}\neq 0$ and $w_1^{2^s-1}\neq 0$. This follows from Lemma 2.5 (indeed, since $t-3\geqslant 2^{s-1}$, both monomials divide $w_1^{2^s}w_2^{2^s}w_3^{t-3}\neq 0$). □

Proposition 4.4 *Let $n=2^s+t\leqslant 2^{s+1}$, $t\geqslant 3$ and $2^{r-1}<t\leqslant 2^r$. Then*

$$\mathrm{zcl}(G_3(\mathbb{R}^n))\leqslant 2^{s+2}+2^r-3.$$

PROOF — Suppose that this is not the case, and let $a,b,c\geqslant 0$ be such that $A = z(w_1)^a z(w_2)^b z(w_3)^c\neq 0$ and $a+b+c\geqslant 2^{s+2}+2^r-2$. Since $z(w_1)^{2^{s+1}}=0$ (by (2.3)) and $z(w_2)^{2^{s+1}}=w_2^{2^{s+1}}\otimes 1+1\otimes w_2^{2^{s+1}}=0$, we have $a,b\leqslant 2^{s+1}-1$ and hence $c\geqslant 2^r$.

Suppose that $c\geqslant 2^s+2^r$. Since $w_3^{2^s+2^r}=0$, then we have

$$A = z(w_1)^a z(w_2)^b z(w_3)^{c-2^s-2^r}(w_3^{2^s}\otimes w_3^{2^r}+w_3^{2^r}\otimes w_3^{2^s}).$$

Since $w_1^{2^s}w_3^{2^r}=0$ and $w_2^{2^s}w_3^{2^r}=0$ (see Proposition 4.3), we have $a,b\leqslant 2^s-1$, which together with $c\leqslant 2^{s+1}-1$ gives $a+b+c\leqslant 2^{s+2}-3$, a contradiction. So, $c\leqslant 2^s+2^r-1$, and hence $a,b\geqslant 2^s$. Since $w_1^{2^s}w_2^{2^s}=w_3^{2^s}$ (see Proposition 4.3):

$$\begin{aligned}
A &= z(w_1)^{a-2^s}z(w_2)^{b-2^s}z(w_3)^{c-2^r}z(w_1^{2^s})z(w_2^{2^s})z(w_3^{2^r})\\
&= z(w_1)^{a-2^s}z(w_2)^{b-2^s}z(w_3)^{c-2^r}(w_1^{2^s}w_2^{2^s}\otimes w_3^{2^r}+w_3^{2^r}\otimes w_1^{2^s}w_2^{2^s})\\
&= z(w_1)^{a-2^s}z(w_2)^{b-2^s}z(w_3)^{c-2^r}(w_3^{2^s}\otimes w_3^{2^r}+w_3^{2^r}\otimes w_3^{2^s}).
\end{aligned}$$

Next, we note that $w_1^{2^s-2^l}w_2^{2^s-2^l}w_3^{2^l}=w_3^{2^s}$ for all $l\geqslant r$. Since $e_i^{2^s+2^l}=0$ for $i\in\{1,2,3\}$, this follows as in the calculations for p_3 in Proposition 4.3.

Let $a-2^s=(\alpha_{s-1}\alpha_{s-2}\ldots\alpha_0)_2$, $b-2^s=(\beta_{s-1}\beta_{s-2}\ldots\beta_0)_2$ and $c-2^r=(\gamma_{s-1}\gamma_{s-2}\ldots\gamma_0)_2$ be the binary representations of the numbers $a-2^s$, $b-2^s$ and $c-2^r$ (we allow $\alpha_{s-1}=0$, $\beta_{s-1}=0$ and $\gamma_{s-1}=0$). Further, for $r\leqslant l\leqslant s$, let $a_l'=\sum_{i=l}^{s-1}\alpha_i 2^i$, $a_l''=a-2^s-a_l'$, $b_l'=\sum_{i=l}^{s-1}\beta_i 2^i$, $b_l''=b-2^s-b_l'$, and $c_l'=\sum_{i=l}^{s-1}\gamma_i 2^i$, $c_l''=c-2^r-c_l'$ (note that $a_s'=b_s'=c_s'=0$). We will prove that $a_l'=b_l'=2^s-2^l$ and $c_l'=0$ for every $l\geqslant r$.

Our proof is by reverse induction on l, $s\geqslant l\geqslant r$. The claim is trivial for $l=s$. So, suppose that it is true for $l+1$, $s\geqslant l+1\geqslant r+1$ and let us prove it for l. First we prove that $\alpha_l+\beta_l+\gamma_l\geqslant 2$. If this were not the case, then by the inductional hypothesis

$$\begin{aligned}
a+b+c &= 2^{s+1}+2^r+a_{l+1}'+b_{l+1}'+c_{l+1}'+(\alpha_l+\beta_l+\gamma_l)\cdot 2^l+a_l''+b_l''+c_l''\\
&\leqslant 2^{s+1}+2^r+2(2^s-2^{l+1})+2^l+3(2^l-1)\\
&= 2^{s+2}+2^r-3,
\end{aligned}$$

a contradiction. So, to finish the inductional step it is enough to prove $\gamma_l = 0$. Assume this is not the case. Since $w_i^{2^l} w_3^{2^s} = 0$ for $i \in \{1,2,3\}$ (see Proposition 4.3), by the inductional hypothesis we have

$$\begin{aligned}
A &= z(w_1)^{a''_{l+1}} z(w_2)^{b''_{l+1}} z(w_3)^{c''_{l+1}} z(w_1)^{2^s-2^{l+1}} z(w_2)^{2^s-2^{l+1}} (w_3^{2^s} \otimes w_3^{2^r} + w_3^{2^r} \otimes w_3^{2^s}) \\
&= z(w_1)^{\alpha_l \cdot 2^l + a''_l} z(w_2)^{\beta_l \cdot 2^l + b''_l} z(w_3)^{\gamma_l \cdot 2^l + c''_l} \\
&\quad \cdot (w_3^{2^s} \otimes w_1^{2^s-2^{l+1}} w_2^{2^s-2^{l+1}} w_3^{2^r} + w_1^{2^s-2^{l+1}} w_2^{2^s-2^{l+1}} w_3^{2^r} \otimes w_3^{2^s}) \\
&= z(w_1)^{a''_l} z(w_2)^{b''_l} z(w_3)^{c''_l} \\
&\quad \cdot (w_3^{2^s} \otimes w_1^{2^s-2^{l+1}+\alpha_l \cdot 2^l} w_2^{2^s-2^{l+1}+\beta_l \cdot 2^l} w_3^{2^l+2^r} + w_1^{2^s-2^{l+1}+\alpha_l \cdot 2^l} w_2^{2^s-2^{l+1}+\beta_l \cdot 2^l} w_3^{2^l+2^r} \otimes w_3^{2^s}).
\end{aligned}$$

So, to obtain a contradiction, it is enough to prove $w_1^{2^s-2^l} w_2^{2^s-2^{l+1}} w_3^{2^l+2^r} = 0$ and $w_1^{2^s-2^{l+1}} w_2^{2^s-2^l} w_3^{2^l+2^r} = 0$. We prove the first identity, since the proof of the second is similar. We use Proposition 2.3:

$$\begin{aligned}
p_4 &= \pi^*(w_1^{2^s-2^l} w_2^{2^s-2^{l+1}} w_3^{2^l+2^r}) e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\
&= (e_1+e_2+e_3)^{2^s-2^l} (e_1e_2+e_2e_3+e_3e_1)^{2^s-2^{l+1}} (e_1e_2e_3)^{2^l+2^r} e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\
&= (e_1+e_2+e_3)^{2^{s-1}-2^l} (e_1e_2+e_2e_3+e_3e_1)^{2^{s-1}-2^{l+1}} (e_1e_2e_3)^{2^{s-1}+2^l+2^r} e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\
&= \ \dots \\
&= (e_1+e_2+e_3)^{2^l} (e_1e_2e_3)^{2^{s-1}+2^{s-2}+\dots+2^{l+1}+2^l+2^r} e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} \\
&= (e_1^{2^l} + e_2^{2^l} + e_3^{2^l})(e_1e_2e_3)^{2^s-2^l+2^r} e_1^2 e_2 e_4^{n-4} \cdots e_{n-1} = 0.
\end{aligned}$$

So, the proof by induction is completed. Finally, we have

$$\begin{aligned}
A &= z(w_1)^{a''_r} z(w_2)^{b''_r} z(w_3)^{c''_r} z(w_1)^{2^s-2^r} z(w_2)^{2^s-2^r} (w_3^{2^s} \otimes w_3^{2^r} + w_3^{2^r} \otimes w_3^{2^s}) \\
&= z(w_1)^{a''_r} z(w_2)^{b''_r} z(w_3)^{c''_r} (w_3^{2^s} \otimes w_1^{2^s-2^r} w_2^{2^s-2^r} w_3^{2^r} + w_1^{2^s-2^r} w_2^{2^s-2^r} w_3^{2^r} \otimes w_3^{2^s}) \\
&= z(w_1)^{a''_l} z(w_2)^{b''_l} z(w_3)^{c''_r} (w_3^{2^s} \otimes w_3^{2^s} + w_3^{2^s} \otimes w_3^{2^s}) = 0,
\end{aligned}$$

which is a contradiction. $\square$

Remark 4.5 *Note that the value of* $\mathrm{zcl}(G_3(\mathbb{R}^n))$*, where* $n \geqslant 2^s+3$*, stated in [9, Theorem 4.8] (but not proven correctly), in most cases, i.e. for* $2^{r-1}+3 \leqslant t \leqslant 2^r$*, equals the upper bound for* $\mathrm{zcl}(G_3(\mathbb{R}^n))$ *proven in Proposition 4.4.*

5 The zero-divisor cup-length of $G_k(\mathbb{R}^n)$

In this section we give a lower bound for $G_k(\mathbb{R}^n)$ for $k \geqslant 4$.

Proposition 5.1 *Let* $4 \leqslant k < n$ *and* $2^s + k \leqslant n \leqslant 2^{s+1}$*. Then*

$$\mathrm{zcl}(G_k(\mathbb{R}^n)) \geqslant (\lceil \log_2 k \rceil + 1) \cdot 2^s - 1 \quad \textit{and} \quad \mathrm{TC}(G_k(\mathbb{R}^n)) \geqslant (\lceil \log_2 k \rceil + 1) \cdot 2^s.$$

PROOF — Let $2^{r-1} < k \leqslant 2^r$. Then $\lceil \log_2 k \rceil = r$, so it is enough to prove

$$A = z(w_1)^{2^{s+1}-1} \prod_{i=1}^{r-1} z(w_{2^i})^{2^s} = z(w_1)^{2^s-1} \prod_{i=0}^{r-1} z(w_{2^i}^{2^s}) \neq 0.$$

First, let us prove that $p = \prod_{i=0}^{r-2} w_{2^i}^{2^s}$ is non-zero in $H^*(G_k(\mathbb{R}^n);\mathbb{Z}_2)$. Let $d_{k,n-k} = e_1^{k-1}\cdots e_{k-1}e_{k+1}^{n-k-1}\cdots e_{n-1}$. Since $e_i^{2^{s+1}} = 0$ for $1 \leqslant i \leqslant k$ (by Proposition 2.3.(2)) and $k' := \sum_{i=0}^{r-2} 2^i = 2^{r-1}-1 < k$ we have

$$\begin{aligned}
p_1 &= \pi^*\left(\prod_{i=0}^{r-2} w_{2^i}^{2^s}\right) d_{k,n-k}\\
&= \prod_{i=0}^{r-2}\left(\sum_{1\leqslant a_1<a_2<\cdots<a_{2^i}\leqslant k} e_{a_1}^{2^s}e_{a_2}^{2^s}\cdots e_{a_{2^i}}^{2^s}\right) d_{k,n-k}\\
&= [2^0,2^1,\ldots,2^{r-2}]\left(\sum_{1\leqslant a_1<a_2<\cdots<a_{k'}\leqslant k} e_{a_1}^{2^s}e_{a_2}^{2^s}\cdots e_{a_{k'}}^{2^s}\right) d_{k,n-k},
\end{aligned}$$

where $[2^0,2^1,\ldots,2^{r-2}] = \binom{2^0+2^1+\cdots+2^{r-2}}{2^0}\binom{2^1+\cdots+2^{r-2}}{2^1}\cdots\binom{2^{r-2}}{2^{r-2}}$ denotes the multinomial coefficient. By Lucas' theorem, this coefficient is odd. Also, for $1 \leqslant i \leqslant k$ the degree of e_i in each monomial in the last expression for p_1 is at most $2^s + k - i \leqslant n - i$, so all monomials in this expression are distinct members of the basis B_n for $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$, and hence $p_1 \neq 0$. So, by Proposition 2.3.(4), $p \neq 0$.

Now, let us observe all summands after expanding A with first coordinate in dimension $(2^{r-1}-1)\cdot 2^s$. The dimension of p is $(2^{r-1}-1)\cdot 2^s$, and it is easy to see that the only term of this form is $p \otimes w_1^{2^s-1}w_{2^{r-1}}^{2^s}$. So, to finish the proof it is enough to prove $w_1^{2^s-1}w_{2^{r-1}}^{2^s} \neq 0$. In fact, we prove that $w_1^{2^s}w_{2^{r-1}}^{2^s} \neq 0$. Since $e_i^{2^{s+1}} = 0$ for $1 \leqslant i \leqslant k$, we have

$$\begin{aligned}
p_2 &= \pi^*\left(w_1^{2^s}w_{2^{r-1}}^{2^s}\right) d_{k,n-k}\\
&= \left(e_1^{2^s}+e_2^{2^s}+\cdots+e_k^{2^s}\right)\left(\sum_{1\leqslant a_1<a_2<\cdots<a_{2^{r-1}}\leqslant k} e_{a_1}^{2^s}e_{a_2}^{2^s}\cdots e_{a_{2^{r-1}}}^{2^s}\right) d_{k,n-k}\\
&= \left(\sum_{1\leqslant a_1<a_2<\cdots<a_{2^{r-1}+1}\leqslant k} e_{a_1}^{2^s}e_{a_2}^{2^s}\cdots e_{a_{2^{r-1}+1}}^{2^s}\right) d_{k,n-k}.
\end{aligned}$$

Now, as above, $2^s + k \leqslant n$ implies that all monomials in the last expression for p_2 are distinct members of the basis B_n for $H^*(\mathrm{Flag}(\mathbb{R}^n);\mathbb{Z}_2)$, and hence $p_2 \neq 0$. By Proposition 2.3.(4), it follows that $w_1^{2^s}w_{2^{r-1}}^{2^s} \neq 0$. □

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References

[1] A. Borel, La cohomologie mod 2 de certains espaces homogenes, *Comm. Math. Helv.* **27** (1953) 165–197.

[2] D. Cohen, A. Suciu, Boundary manifolds of projective hypersurfaces, *Adv. Math.* **206** (2006) 538–566.

[3] A. Costa and M. Farber, Motion planning in spaces with small fundamental groups, *Commun. Contemp. Math.* **12** (2010) 107–119.

[4] Ö. Eğecioğlu, The parity of the Catalan numbers via lattice paths, *Fibonacci Quart.* **21** (1983) 65–66.

[5] M. Farber, Topological complexity of motion planning, *Discrete Comput. Geom.* **29** (2003) 211–221.

[6] J. González, B. Gutiérrez, D. Gutiérrez and A. Lara, Motion planning in real flag manifolds, *Homol. Homotopy Appl.* **82:2** (2016) 359–375.

[7] J. Jaworowski, An additive basis for the cohomology of real Grassmannians, *Lecture Notes in Math.* **1474**, Springer-Verlag, Berlin (1991) 231–234.

[8] J. Korbaš and J. Lörinc, The $\mathbb{Z}_2$-cohomology cup-length of real flag manifolds, *Fund. Math.* **178** (2003), 143–158.

[9] P. Pavešić, Topological complexity of real Grassmannians, *P. Roy. Soc. Edinb. A*, to appear

[10] P. Pavešić, Erratum to "Topological complexity of real Grassmannians", submitted.

[11] Z. Z. Petrović, B. I. Prvulović and M. Radovanović, Multiplication in the cohomology of Grassmannians via Gröbner bases, *J. Algebra* **438** (2015) 60–84.

[12] R.E. Stong, Cup products in Grassmannians, *Topol. Appl.* **13** (1982), 103–113.