On self-maps of complex flag manifolds

Matej Milićević* Marko Radovanović†

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Abstract

It was conjectured in [6] that for a complex flag manifold F every endomorphism $\varphi : H^*(F;\mathbb{Z}) \to H^*(F;\mathbb{Z})$ is either a grading endomorphism or a projective endomorphism. In this paper we verify this conjecture for a new class of complex flag manifolds that captures all cases for which the conjecture was previously known to be true. This allows us to calculate the noncoincidence index (invariant that naturally generalizes the fixed-point property) for these manifolds.

1 Introduction

It is well-known that for any $n \in \mathbb{N}$ and any continuous map

$$f : \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$$

there exists $x \in \mathbb{C}P^{2n}$ such that $f(x) = x$, i.e. $\mathbb{C}P^{2n}$ has the *fixed-point property* (FPP). Moreover, $\mathbb{C}P^{2n+1}$ does not have the FPP.

A natural generalization of complex projective spaces are (complex) Grassmann manifolds $G_{k,n}$ (here $G_{k,n}$ denotes the Grassmann manifold of k-dimensional spaces in $\mathbb{C}^{n+k}$; one has $G_{1,n} = \mathbb{C}P^n$). Somewhat surprisingly, classification of Grassmann manifolds that have the FPP is still open. It was conjectured in [13] that $G_{k,n}$ has the FPP if and only if $n \neq k$ and nk is even ("only if" part is proven in Section 2 of [13]). This conjecture is proven only for $k = 2$ (in [13]), $k = 3$ (in [5]), and if $n \geq 2k^2 - k - 1$ (in [5]). (We note that there are similar conjectures and results for real and quaternionic Grassmannians – see [5].)

The *(complex) flag manifold* $F(n_1, \dots, n_r)$, where $r \geq 2$, $n_1, \dots, n_r \in \mathbb{N}$ and $n = n_1 + \cdots + n_r$, consists of complex flags in $\mathbb{C}^n$ of type $(n_1, \dots, n_r)$, that is, r-tuples $(V_1, \dots, V_r)$ of mutually orthogonal complex vector subspaces of $\mathbb{C}^n$ with $\dim_{\mathbb{C}}(V_i) = n_i$ for $1 \leq i \leq r$ (in this paper we only work with complex flag manifolds, so we omit the word "complex" when we refer to it). Note that for $r = 2$ the flag manifold $F(n_1, n_2)$ is actually the Grassmann manifold G_{n_1,n_2}, while the flag manifold with $n_1 = \cdots = n_r = 1$ is the *complete complex flag manifold*. In [7] Glover and Homer conjectured that $F(n_1, \dots, n_r)$ has the FPP if and only if the numbers n_i are distinct and at most one of them is odd (in the same paper they prove the

*University of Belgrade, Faculty of Mathematics, Studentski trg 16, Belgrade, Serbia. Partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia (Grant No: 45103-9/2021-14/200104).

†University of Belgrade, Faculty of Mathematics, Studentski trg 16, Belgrade, Serbia. Partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia (Grant No: 45103-9/2021-14/200104).

"only if" part of this conjecture). Other than for the above-mentioned cases of Grassmann manifolds, the only flag manifolds for which the "if" part of this conjecture is proven are $F(1,p,q)$ for $p \geq 2$ and $q \geq 2p^2-1$ (see [6]).

In this paper we consider a related question. Let M be a connected topological manifold. We say that self-maps f and g of M are *coincident* if there exists $x \in M$ such that $f(x)=g(x)$. Let m be the maximum number of self-maps of M such that none of them has the FPP and that no two of them are coincident. Then the *noncoincidence index* of M (defined in [9]) is

$$NI(M) = \begin{cases} m+1, & \text{if } m \text{ is finite,} \\ \infty, & \text{otherwise.} \end{cases}$$

Obviously, M has the FPP if and only if $NI(M)=1$.

Let $F := F(n_1,\ldots,n_r)$. In this paper we calculate $NI(F)$ for certain flag manifolds F. To do so we consider endomorphisms $\varphi : H^*(F;\mathbb{Z}) \to H^*(F;\mathbb{Z})$. It was conjectured in [6] that any such endomorphism is either a grading endomorphism or a projective endomorphism (these notions will be defined in Sections 5.1 and 4, respectively). We note that this conjecture is proven only in the following cases: $r=2$ and $n_1 \leq 3$ or $n_2 \geq 2n_1^2-n_1-1$ (see [5]), $r=3$, $n_1=1$ and $n_3 \geq 2n_2^2-1$ (see [6]), $n_1=\cdots=n_r=1$ (see [11]), and later for $n_1=\cdots=n_{r-1}=1$ (see [9]), while some partial results where obtained for (general) Grassmann manifolds in [8]. Assuming this conjecture, in [10] Hoffman computed $NI(F)$ (for all flag manifolds F).

The main result of this paper is the following extension of the previously mentioned results. We note that our proof works equally well for the cohomology with rational coefficients (this will be used in some of our applications).

Throughout the paper we will use the following notation: for $l \in \mathbb{N}$ and $a \in \mathbb{Z}$ we denote

$$a^{\cdots l} := \underbrace{a, a, \ldots, a}_{l}.$$

Theorem 1.1 *Let $F := F(1^{\cdots j}, k, m)$, where $k, m \in \mathbb{N}$, $j \geq 0$, $k \geq 2$ and $m \geq 2k^2-1$. Then every endomorphism $\varphi : H^*(F;\mathbb{Z}) \to H^*(F;\mathbb{Z})$ is either a grading endomorphism or a projective endomorphism.*

Remark 1.2 *In [9] this theorem is proven in the case $k=1$ (so in this paper we only consider the case $k \geq 2$).*

We note that the bound for m in the previous theorem is the same as the bound from [6] (where $j=1$).

As an immediate consequence of Theorem 1.1, and Theorems 5.2 and 5.3 from [10] we have the following result.

Theorem 1.3 *Let $F := F(1^{\cdots j}, k, m)$, where $j \geq 0$, $k \geq 2$ and $m \geq 2k^2-1$. Then*

$$NI(F) = \begin{cases} j!, & \text{if at most one of } j,k,m \text{ is odd,} \\ 2j!, & \text{otherwise.} \end{cases}$$

The paper is organized as follows. In Section 2 we recall some basic properties and identities for the cohomology of flag manifolds that are going to be used throughout the paper. In Section 3, for any flag manifold F we determine all nonzero classes in $H^2(F;\mathbb{Z})$ with the minimal height. This result will be used repeatedly in the proof of Theorem 1.1.

In Section 4 we prove that there are no j-projective endomorphisms (which we define in the same section) other than projective endomorphisms. In Section 5 we prove Theorem 1.1. In Section 6, using Theorem 1.1 we prove that for certain pairs of flag manifolds all continuous maps between them are rationally null-homotopic.

2 Cohomology of flag manifolds

Throughout the paper, for $t \in \mathbb{N}$ we denote $[t] := \{1, 2, \ldots, t\}$.

Let $F := F(n_1, \ldots, n_r, m)$ be a flag manifold. Then we denote with $\gamma_1^F, \ldots, \gamma_{r+1}^F$ (or simply with $\gamma_1, \ldots, \gamma_{r+1}$) the canonical complex vector bundles over F ($\dim_{\mathbb{C}}(\gamma_i^F) = n_i$, for $i \in [r]$, $\dim_{\mathbb{C}}(\gamma_{r+1}^F) = m$). Further, let $c_{i,j} \in H^{2j}(F; \mathbb{Z})$, for $i \in [r]$ and $j \in [n_i]$, be the j-th Chern class of the bundle γ_i^F, and $c'_j \in H^{2j}(F; \mathbb{Z})$, for $j \in [m]$, be the j-th Chern class of the bundle γ_{r+1}^F (in most of our proofs we will not use classes c'_j, and hence this asymmetry in notation; this will become more clear in the following subsection). Then

$$(1 + c_{1,1} + \cdots + c_{1,n_1}) \cdots (1 + c_{r,1} + \cdots + c_{r,n_r})(1 + c'_1 + \cdots + c'_m) = 1. \tag{2.1}$$

By Borel's description (see [1]) this relation fully determines the cohomology $H^*(F; \mathbb{Z})$, that is

$$H^*(F; \mathbb{Z}) \cong \mathbb{Z}[c_{1,1}, \ldots, c_{1,n_1}, \ldots, c_{r,1}, \ldots, c_{r,n_r}]/I_F,$$

where I_F is the ideal generated by the polynomials $\overline{c}_{m+1}, \overline{c}_{m+2}, \ldots, \overline{c}_n$, and we denote $N_F := n = n_1 + \cdots + n_r + m$.

Remark 2.1 *We abuse the notation and use $c_{i,j}$ to denote both the class of the cohomology algebra $H^*(F; \mathbb{Z})$ and the element of the polynomial ring $\mathbb{Z}[c_{1,1}, \ldots, c_{1,n_1}, \ldots, c_{r,1}, \ldots, c_{r,n_r}]$.*

The polynomials (classes) $\overline{c}_i$ for $i \geq 0$, are obtained from the equation (we denote $\overline{c}_0 = 1$)

$$(1 + \overline{c}_1 + \overline{c}_2 + \cdots) \cdot \prod_{i=1}^{r} (1 + c_{i,1} + c_{i,2} + \cdots + c_{i,n_i}) = 1, \tag{2.2}$$

which implies

$$1 + \overline{c}_1 + \overline{c}_2 + \cdots = \prod_{i=1}^{r} \sum_{d_i \geq 0} \sum_{a_{i,1} + \cdots + a_{i,n_i} = d_i} (-1)^{d_i} [a_{i,1}, \ldots, a_{i,n_i}] c_{i,1}^{a_{i,1}} \cdots c_{i,n_i}^{a_{i,n_i}}, \tag{2.3}$$

where for $\sigma = (s_1, \ldots, s_k) \in \mathbb{N}_0^k$, $[\sigma]$ denotes the corresponding multinomial coefficient, that is

$$[\sigma] = \binom{s_1 + s_2 + \cdots + s_k}{s_1} \binom{s_2 + \cdots + s_k}{s_2} \cdots \binom{s_k}{s_k}.$$

For $i \in [r]$ and an n_i-tuple $\alpha = (a_1, a_2, \ldots, a_{n_i}) \in \mathbb{N}_0^{n_i}$ we use the notation C_i^α for the monomial $c_{i,1}^{a_1} c_{i,2}^{a_2} \cdots c_{i,n_i}^{a_{n_i}}$ and $|\alpha| = a_1 + \cdots + a_{n_i}$. Also, let $S_0 := m$ and $S_i := m + n_1 + \cdots + n_i$, for $i \in [r]$.

The following theorem gives an additive basis for $H^*(F; \mathbb{Z})$ in terms of Chern classes of complex vector bundles γ_i.

Theorem 2.2 ([14]) *The set*

$$B_F := \left\{ C_1^{\alpha(1)} C_2^{\alpha(2)} \cdots C_r^{\alpha(r)} \,:\, |\alpha(i)| \le S_{i-1} \text{ for } i \in [r] \right\}$$

is an additive basis for $H^*(F;\mathbb{Z})$.

Endomorphisms $\varphi_1, \varphi_2 : H^*(F;\mathbb{Z}) \to H^*(F;\mathbb{Z})$ are equal *up to a conjugation* if there exists a permutation $\pi \in \mathrm{Sym}(r)$ that satisfies $n_i = n_{i'}$ whenever $\pi(i) = i'$, and such that

$$\varphi_1(c_{i,s}) = \varphi_2(c_{\pi(i),s}) \quad \text{for all } i \in [r] \text{ and } s \in [n_i].$$

2.1 Flag manifolds $F(1^{\cdot\cdot j}, n_{j+1}, \ldots, n_{j+t}, m)$

In this paper we consider a special case of flag manifolds, namely $F := F(1^{\cdot\cdot j}, n_{j+1}, \ldots, n_{j+t}, m)$, where $j \in \mathbb{N}$. Then $N_F := j + n_{j+1} + \cdots + n_{j+t} + m$, $n_1 = \cdots = n_j = 1$ and $n_{j+t+1} = m$. For our proofs we will need several variants of the identity (2.2), which we give in this subsection. Also, we simplify the notation, by denoting $x_i = c_{i,1}$, for $i \in [j]$, $y_{i,l} = c_{i,l}$, for $i \in [j+t] \setminus [j]$ and $l \in [n_i]$, $y_{j+t+1,l} = c_{j+t+1,l}$, for $l \in [m]$, and $z_i = \bar{c}_i$, for $i \ge 0$. In the special case $t = 1$ (which will be the most interesting to us), we denote $y_i = y_{1,i}$.

The identity (2.2) now simplifies to

$$\sum_{i\ge 0} z_i \cdot \prod_{s=1}^{j}(1+x_s) \cdot \prod_{s=j+1}^{j+t}(1 + y_{s,1} + y_{s,2} + \cdots + y_{s,n_s}) = 1$$

and in the case $t = 1$ and $n_{j+1} = k$ to

$$(1 + y_1 + \cdots + y_k) \cdot \prod_{s=1}^{j}(1+x_s) \cdot \sum_{i\ge 0} z_i = 1. \tag{2.4}$$

For $J \subset [j]$ the previous identities can also be written as

$$\sum_{i\ge 0} z_i \cdot \prod_{s\in J}(1+x_s) \cdot \prod_{s=j+1}^{j+t}(1 + y_{s,1} + y_{s,2} + \cdots + y_{s,n_s}) = \prod_{s\in[j]\setminus J} \sum_{l\ge 0}(-x_s)^l \tag{2.5}$$

and (in the case $t = 1$ and $n_{j+1} = k$)

$$(1 + y_1 + \cdots + y_k) \cdot \prod_{s\in J}(1+x_s) \cdot \sum_{i\ge 0} z_i = \prod_{s\in[j]\setminus J} \sum_{l\ge 0}(-x_s)^l. \tag{2.6}$$

If $t = 1$ and $n_{j+1} = k$, then for $\alpha = (a_1, \ldots, a_j) \in \mathbb{N}_0^j$, $\beta = (b_1, \ldots, b_k) \in \mathbb{N}_0^k$, we denote

$$x^\alpha := x_1^{a_1} x_2^{a_2} \cdots x_j^{a_j} \quad \text{and} \quad y^\beta := y_1^{b_1} y_2^{b_2} \cdots y_k^{b_k}.$$

So, in the case $t = 1$ and $n_{j+1} = k$ identity (2.3) leads to

$$\begin{aligned}\sum_{i\ge 0} z_i &= \prod_{s=1}^{j} \sum_{l\ge 0}(-x_s)^l \cdot \sum_{l\ge 0} \sum_{|\alpha|=l} (-1)^l [\alpha] y^\alpha \\ &= \sum_{\mu\in\mathbb{N}_0^j} (-1)^{|\mu|} x^\mu \sum_{\sigma\in\mathbb{N}_0^k} (-1)^{|\sigma|}[\sigma] y^\sigma. \end{aligned}\tag{2.7}$$

Recall that the *height* of a class $c \in \widetilde{H}^*(F;\mathbb{Z})$, denoted by $\mathrm{ht}(c)$, is the largest $n \in \mathbb{N}$ such that $c^n \neq 0$.

At the end of this section, we prove several technical results for $H^*(F;\mathbb{Z})$. Let $N := N_F$. First, we recall Corollary 10 from [14]. More precisely, with the notation from that paper, we apply this corollary for the flag manifold $F' := F(n_{j+1}, \ldots, n_{j+t}, 1^{\cdots j}, m)$ which is homeomorphic to F; also, let x_i be the first Chern class of $\gamma^{F'}_{j+t-i+1}$. Then, using the notation from Corollary 8 of [14], for $k = j+t-r+1$ and each $s \geq N-r+1 = S_{k-1}+1$, we have $\overline{c}^{(k)}_s = 0$ (in $H^*(F';\mathbb{Z})$), that is

$$(-1)^s \sum_{b_1+\cdots+b_r=s} x_1^{b_1}\cdots x_r^{b_r} = 0. \tag{2.8}$$

Note that the sum in the previous identity is in fact the complete symmetric polynomial on variables $x_1, \ldots, x_r$ of degree s.

Since $\mathrm{ht}(x_1) = \mathrm{ht}(x_2) = N-1$ (see Corollary 3.3), the identity (2.8) for $k = j+t-1$ and $s = 2N-3 \geq N-1 = S_{k-1}+1$ implies

$$0 = \sum_{b_1+b_2=2N-3} x_1^{b_1}x_2^{b_2} = x_1^{N-1}x_2^{N-2} + x_1^{N-2}x_2^{N-1}.$$

By symmetry, this implies that for all $1 \leq i < i' \leq j$ one has

$$x_i^{N-1}x_{i'}^{N-2} + x_i^{N-2}x_{i'}^{N-1} = 0. \tag{2.9}$$

Lemma 2.3 *For $r \in [j]$ and $a \in \mathbb{Z}$ the following identity holds in $H^*(F;\mathbb{Z})$*

$$\prod_{i=1}^{r}\sum_{l=0}^{N-1}(-ax_i)^l = \sum_{0\leq b_1+\cdots+b_r\leq N-r} (-a)^{b_1+\cdots+b_r}x_1^{b_1}\cdots x_r^{b_r}.$$

PROOF — Our proof is by induction on r. Base case $r=1$ is trivial. So, we assume that it is true for $r-1 \leq j-1$ and prove it for r. For $l \geq 0$, we denote with h_l (resp. $h_l^{(r-1)}$) the complete symmetric polynomial of degree l on the variables $x_1, \ldots, x_r$ (resp. $x_1, \ldots, x_{r-1}$). Then

$$h_l = h_l^{(r-1)} + x_r h_{l-1}^{(r-1)} + \cdots + x_r^l.$$

By the inductive hypothesis, identity (2.8) and since $x_r^l = 0$, for $l \geq N$ (in $H^*(F;\mathbb{Z})$)

$$\begin{aligned}
\prod_{i=1}^{r}\sum_{l=0}^{N-1}(-ax_i)^l &= \sum_{0\leq l\leq N-r+1}(-a)^l h_l^{(r-1)} \cdot \sum_{l=0}^{N-1}(-ax_r)^l \\
&= \sum_{0\leq l\leq N-1}(-a)^l h_l^{(r-1)} \cdot \sum_{l=0}^{N-1}(-ax_r)^l \\
&= \sum_{0\leq l\leq 2N-2}(-a)^l (h_l^{(r-1)} + x_r h_{l-1}^{(r-1)} + \cdots + x_r^l) \\
&= \sum_{0\leq l\leq 2N-2}(-a)^l h_l = \sum_{0\leq l\leq N-r}(-a)^l h_l \\
&= \sum_{0\leq b_1+\cdots+b_r\leq N-r}(-a)^{b_1+\cdots+b_r}x_1^{b_1}\ldots x_r^{b_r},
\end{aligned}$$

which completes the proof of this lemma. □

3 Heights of the classes in $H^2(F;\mathbb{Z})$

The heights of all classes in $H^2(F;\mathbb{Z})$ are known by the following result.

Theorem 3.1 ([2]) *Let $F := F(n_1,\dots,n_l)$ and $t_i = c_1(\gamma_i^F)$, for $i \in [l]$. For*

$$w = a_1t_1 + a_2t_2 + \cdots + a_lt_l \in H^*(F;\mathbb{Z}),$$

let $B_w = \{b_1 < b_2 < \cdots < b_g\}$ be the set of different values of a_i and $m_j = \sum_{a_i=b_j} n_i$, for $j \in [g]$. Then

$$\mathrm{ht}(w) = \sum_{1\le p<q\le g} m_pm_q.$$

Using this result (and keeping the same notation) we obtain all nonzero elements that have the minimal height in $H^2(F;\mathbb{Z})$.

Lemma 3.2 *Let $\mu = \min\{n_1,\dots,n_l\}$. Then the nonzero elements of the minimal height in $H^2(F;\mathbb{Z})$ are a_it_i, where $i \in [l]$ and satisfies $n_i = \mu$ and $a_i \in \mathbb{Z}\setminus\{0\}$.*

PROOF — Let $N_F := n_1 + \cdots + n_l$ and

$$w = a_1t_1 + a_2t_2 + \cdots + a_lt_l,$$

be a nonzero element of minimal height in $H^2(F;\mathbb{Z})$. Note that $t_1 + \cdots + t_l = 0$ (by (2.1)), so $|B_w| \neq 1$. Let $I \subseteq [l]$ be the set of all indices i that satisfy $a_i = b_1$. Then $m_1 = \sum_{i\in I} n_i$.

Assume that $|B_w| = s \geq 3$. Then, by Theorem 3.1

$$\mathrm{ht}(w) = m_1\sum_{i\ge 2} m_i + \sum_{1<i<i'\le s} m_im_{i'} > m_1\sum_{i\ge 2} m_i.$$

So, for $a, b \in \mathbb{Z}$, $a \neq b$, and $w' = a\sum_{i\in I} t_i + b\sum_{i\notin I} t_i$, one has $B_{w'} = \{a,b\}$ and

$$\mathrm{ht}(w') = \sum_{i\in I} n_i \sum_{i\notin I} n_i = m_1 \sum_{i\ge 2} m_i < \mathrm{ht}(w),$$

which is a contradiction.

Hence $|B_w| = 2$ and let $B_w = \{a,b\}$. Then $m_1 + m_2 = N_F$, and $\mu \le m_1, m_2 \le N_F - \mu$. Let $i \in [l]$ be such that $n_i = \mu$. Then, by Theorem 3.1, the height of $w'' = at_i + b\sum_{i'\neq i} t_{i'}$ is $\mathrm{ht}(w'') = n_i \sum_{i'\neq i} n_{i'} = \mu(N_F - \mu)$. Since $m_1(N_F - m_1) \ge \mu(N_F - \mu)$ with the equality if and only if $m_1 \in \{\mu, N_F - \mu\}$ (this inequality is equivalent to $(m_1 - \mu)(N_F - \mu - m_1) \ge 0$), by the minimality of $\mathrm{ht}(w)$ we conclude that $m_1 \in \{\mu, n_F - \mu\}$. If $m_1 = \mu$, then $m_1 = n_{i'} = \mu$, for some $i' \in [l]$, and hence $w = at_{i'} + b\sum_{i''\neq i'} t_{i''} = (a-b)t_{i'}$, as desired. Similarly, if $m_1 = N_F - \mu$, then $m_2 = n_{i'} = \mu$, for some $i' \in [l]$, and hence $w = a\sum_{i''\neq i'} t_{i''} + bt_{i'} = (b-a)t_{i'}$. □

Using the notation from Section 2.1, we have the following corollary.

Corollary 3.3 *Nonzero elements of the minimal height in the cohomology algebra $H^2(F(1^{\cdots j}, n_{j+1},\dots,n_{j+t},m);\mathbb{Z})$, where $n_i \neq 1$ for $i \in [j+t]\setminus[j]$, are a_ix_i, for all $i \in [j]$ and $a_i \in \mathbb{Z}\setminus\{0\}$. The height of these classes is $N_F - 1 = \sum_{i=j+1}^{j+t} n_i + j + m - 1$.*

4 Projective endomorphisms

Throughout this section we use the same notation as in Section 2.1.

Let $F := F(1^{\cdots j}, n_{j+1}, \ldots, n_{j+t}, m)$ be a flag manifold, such that $j \geq 1$ (it is possible that $n_i = 1$ for some $i \in [j+t] \setminus [j]$). Then there is a natural map $F \to \mathbb{C}P^{N_F - 1}$, which induces a monomorphism of cohomology algebras. An endomorphism $\varphi : H^*(F;\mathbb{Z}) \to H^*(F;\mathbb{Z})$ is *projective* if it factors through such a monomorphism. In other words, an endomorphism $\varphi : H^*(F;\mathbb{Z}) \to H^*(F;\mathbb{Z})$ is projective if for each $z \in H^*(F;\mathbb{Z})$, $\varphi(z)$ is, up to a conjugation, a polynomial in x_1 (in $H^*(F;\mathbb{Z})$). In [6] the authors determined all projective endomorphism of flag manifolds (in this result $j = 1$ and $n_1 = 1$).

Theorem 4.1 ([6]) *Let $F := F(1, n_2, \ldots, n_{t+1}, m)$ be a flag manifold. Consider a factorization $1 - x_1^{N_F} = P_1(x_1)P_2(x_1)\cdots P_{t+2}(x_1)$ in the polynomial ring $\mathbb{Z}[x_1]$, where $\deg P_i = n_i$, $1 \leq i \leq t+1$, and $\deg P_{t+2} = m$. If $P_1(x_1) = 1 - x_1$ and $\lambda \in \mathbb{Z}$, then the formula*

$$\varphi(1 + y_{i,1} + \cdots + y_{i,n_i}) = P_i(\lambda x_1), \quad 2 \leq i \leq t+1,$$

$\varphi(1 + x_1) = P_1(\lambda x_1)$ and $\varphi(1 + y_{t+2,1} + \cdots + y_{t+2,m}) = P_{t+2}(\lambda x_1)$, gives a well-defined (projective) endomorphism of $H^*(F;\mathbb{Z})$. Conversely, every nonzero projective endomorphism (for any flag manifold with $n_1 = 1$, i.e. $j \geq 1$) has this form, up to a conjugation.

Note that in the previous theorem, if $\lambda = 0$, then φ vanishes in positive dimensions. So, if φ is a projective endomorphism, we may assume that the corresponding λ is nonzero.

In this section we extend this result for all $j \in \mathbb{N}$, that is for $F := F(1^{\cdots j}, n_{j+1}, \ldots, n_{j+t}, m)$ we classify all endomorphism $\varphi : H^*(F;\mathbb{Z}) \to H^*(F;\mathbb{Z})$ that factor through the monomorphism $H^*(F(1^{\cdots j}, N_F - j);\mathbb{Z}) \to H^*(F;\mathbb{Z})$ induced by the natural map $F \to F(1^{\cdots j}, N_F - j)$ (this map is defined with $(S_1, \ldots, S_{j+t+1}) \mapsto (S_1, \ldots, S_j, S_{j+1} \oplus \cdots \oplus S_{j+t+1})$ for a flag $(S_1, \ldots, S_{j+t+1}) \in F$). Such endomorphisms we call *j-projective*, and in fact prove that the only j-projective endomorphisms are projective endomorphisms. Of course, an endomorphism $\varphi : H^*(F;\mathbb{Z}) \to H^*(F;\mathbb{Z})$ is j-projective if and only if for each $z \in H^*(F;\mathbb{Z})$, $\varphi(z)$ is a polynomial in variables $x_1, x_2, \ldots, x_j$ (in $H^*(F;\mathbb{Z})$), up to a conjugation.

To prove this we will need the following result.

Theorem 4.2 ([15]) *For each natural number m, the plane projective curve of degree m defined by the vanishing of the polynomial*

$$G_m(x,y,z) = \sum_{\substack{a,b,c \geq 0 \\ a+b+c=m}} x^a y^b z^c$$

is non-singular in characteristic 0 and has zeros at $2m^2$ points where coordinates x, y and z are roots of unity.

Corollary 4.3 *For $m, s \in \mathbb{N}$, $s \geq 2$, the polynomial*

$$G_{m,s}(x_1, \ldots, x_s) = \sum_{0 \leq a_1 + \cdots + a_s \leq m} x_1^{a_1} \ldots x_s^{a_s}$$

is irreducible in $\mathbb{Z}[x_1, \ldots, x_s]$.

PROOF — Our proof is by induction on $s \geq 2$. First, let $s = 2$. Suppose that $G_{m,2} = PQ$, for some $P, Q \in \mathbb{Z}[x_1, x_2]$, where $\deg P = k$ and $\deg Q = l$. By Theorem 4.2, $G_{m,2}$ is non-singular. Therefore both P and Q are non-singular and do not go through the origin, and hence curves $P(x_1, x_2) = 0$ and $Q(x_1, x_2) = 0$ have at most $2k^2$ and $2l^2$, respectively, points of the form (ζ, ζ') where ζ and ζ' are roots of unity (see the bottom of page 87 in [15]). By the previous theorem this implies $2k^2 + 2l^2 \geq 2m^2 = 2(k+l)^2$, which is only possible if $k = 0$ or $l = 0$.

So, we assume that the result holds for $s - 1 \geq 2$ and prove it for s. Suppose that $G_{m,s} = PQ$, for some $P, Q \in \mathbb{Z}[x_1, \ldots, x_s]$, where P and Q are non-constant polynomials. Let a (resp. b) be maximal such that x_1^a (resp. x_1^b) is a monomial of P (resp. Q). Then $a + b = m$, $a, b > 0$, and the coefficients of these monomials are 1 or -1. But then, by letting $x_s = 0$ we obtain a non-trivial factorization of $G_{m,s-1}$, a contradiction. $\square$

Proposition 4.4 *Every j-projective endomorphism φ of the flag manifold $F := F(1^{\cdots j}, n_{j+1}, \ldots, n_{j+t}, m)$ is projective.*

PROOF — Since φ is an endomorphism, $\mathrm{ht}(c) \geq \mathrm{ht}(\varphi(c))$ for every $c \in H^2(F; \mathbb{Z})$, so, by Corollary 3.3, $\varphi(x_i) = a_i c_1(\gamma_s^F)$ for some $a_i \in \mathbb{Z}$ and $s \in [j+t]$ such that $n_s = 1$. Since φ is j-projective, this implies that for every $i \in [j]$ we have $\varphi(x_i) = a_i x_s$ for some $s \in [j]$ and $a_i \in \mathbb{Z}$. So, up to a conjugation, we may assume that $\varphi(x_i) = a_i x_i$ for $i \in [r]$, where $r \leq j$ is maximal (that is, we assume that the set $\{s \in [j] : \varphi(x_i) = a_i x_s \text{ for some } i \in [j]\}$ is equal to $[r]$; in other words, for each $i \in [j] \setminus [r]$, $\varphi(x_i)$ is not a non-zero multiple of one of the variables $x_{r+1}, x_{r+2}, \ldots, x_j$). Additionally, if at least one of $a_1, a_2, \ldots, a_r$ is non-zero, then, up to a conjugation, we may assume that all of them are non-zero (so, we assume that either $a_1 = a_2 = \cdots = a_r = 0$, or $a_i \neq 0$ for all $i \in [r]$).

Let $N := N_F$. We apply φ on (2.5) for $J = \{r+1, r+2, \ldots, j\}$ and observe this identity in $H^*(F; \mathbb{Z})$. We have $\varphi(z_{m+1}) = \varphi(z_{m+2}) = \cdots = \varphi(z_N) = 0$ and $x_1^N = \cdots = x_j^N = 0$ (since $\mathrm{ht}(x_1) = \cdots = \mathrm{ht}(x_j) = N - 1$), so (in $H^*(F; \mathbb{Z})$)

$$P_1 \cdots P_t Q_m \prod_{i=r+1}^{j} (1 + \varphi(x_i)) = \prod_{i=1}^{r} \sum_{l=0}^{N-1} (-a_i x_i)^l, \tag{4.1}$$

where $P_{i-j} = \varphi(1 + y_{i,1} + \cdots + y_{i,n_i})$, for $i \in [j+t] \setminus [j]$, and $Q_m = \varphi(1 + z_1 + \cdots + z_m)$ are polynomials in variables $x_1, \ldots, x_j$. Clearly, $\deg P_i \leq n_{j+i}$, for $i \in [t]$, and $\deg Q_m \leq m$.

Note that F is homeomorphic to $F(n_{j+1}, \ldots, n_{j+t}, 1^{\cdots j}, m)$, so, by Theorem 2.2, an additive basis for $H^*(F; \mathbb{Z})$ is the set

$$B_F = \left\{ \prod_{i=j+1}^{j+t} \prod_{l=1}^{n_i} y_{i,l}^{b_{i,l}} \prod_{i=1}^{j} x_i^{a_i} \;:\; \sum_{l=1}^{n_i} b_{s,l} \leq m + \sum_{i=j+1}^{s-1} n_i,\ a_i \leq N - i, \text{ for } s \in [j+t] \setminus [j],\ i \in [j] \right\}.$$

(Note: we have "sorted 1's" in $F(n_{j+1}, \ldots, n_{j+t}, 1^{\cdots j}, m)$ so that the class x_1 is a Chern class of the complex line bundle corresponding to the last 1, x_2 a Chern class of the complex line bundle corresponding to the second to last 1, and so on.) We prove that after expansion each monomial of the left-hand side of (4.1) is in B_F. Indeed, the degree of each variable x_l, for $1 \leq l \leq r$, is at most $\sum_{i=j+1}^{j+t} n_i + m + j - r = N - r \leq N - l$, and the degree of each variable x_l, for $r+1 \leq l \leq j$, is at most $\sum_{i=j+1}^{j+t} n_i + m = N-j \leq N-l$ (since the monomial $\varphi(x_{r+1}) \cdots \varphi(x_j)$ is not divisible by x_l).

Next, we show that it is enough to prove $a_1 = \cdots = a_r = a$. Indeed, suppose that this is the case, and apply Lemma 2.3 on the right-hand side of (4.1). From the previous paragraph we conclude that the left-hand side of (4.1) is as a polynomial equal to the right-hand side of identity from Lemma 2.3. Since the polynomial on the right-hand side of this identity is by Corollary 4.3 irreducible for $r \geq 2$, we conclude that $r = 1$. Hence, identity (4.1) implies that polynomials $P_1, \ldots, P_t, Q_m, 1+\varphi(x_2), \ldots, 1+\varphi(x_j)$ divide a polynomial in x_1, so they are also polynomial in x_1, which completes our proof.

So, let us prove that $a_1 = \cdots = a_r = a$. By the assumption made at the beginning of the proof, we may assume that $a_i \neq 0$ for all $i \in [r]$. Also, we may assume that $r \geq 2$, and, by symmetry, it is enough to show $a_1 = a_2$. We apply φ on the identity (2.5) for $J = \{3, 4, \ldots, j\}$ and observe this identity in $H^*(F;\mathbb{Z})$. Similarly as for (4.1), we have

$$P_1 \cdots P_t Q_m \prod_{i=3}^{j} (1+\varphi(x_i)) = \prod_{i=1}^{2} \sum_{l=0}^{N-1} (-a_i x_i)^l. \tag{4.2}$$

Suppose that $a_1 \neq a_2$. By (2.9), and since $\mathrm{ht}(x_1) = N-1$, we have

$$x_1^{N-1} x_2^{N-1} = x_1 x_1^{N-2} x_2^{N-1} = -x_1 x_1^{N-1} x_2^{N-2} = 0,$$

and also (again by (2.9))

$$\begin{aligned}(-a_1x_1)^{N-1}(-a_2x_2)^{N-2} + (-a_1x_1)^{N-2}(-a_2x_2)^{N-1} \\ = (a_1a_2)^{N-2}(-a_1+a_2)x_1^{N-1}x_2^{N-2}.\end{aligned}$$

Since $x_1^{N-1}x_2^{N-2} \in B_F$, we conclude that the nonzero monomial of the largest degree on the right-hand side of (4.2) is

$$(a_1a_2)^{N-2}(-a_1+a_2)x_1^{N-1}x_2^{N-2}.$$

On the other hand, each monomial on the left-hand side of (4.2) is in dimension at most $\sum_{i=j+1}^{j+t} n_i + m + j - 2 = N-2$, a contradiction. □

5 Endomorphisms

In this section we prove the main result of this paper, i.e. we classify all endomorphisms of $H^*(F(1^{\cdots j}, k, m);\mathbb{Z})$. Throughout this section we use the notation from Section 2.1, but, for simplicity and readability, we denote $H^*(F(1^{\cdots j}, k, m);\mathbb{Z})$ with $\mathcal{A}_{j,k,m}$ and the ideal I_F with $I_{j,k,m}$. So, by Borel's description,

$$\mathcal{A}_{j,k,m} \cong \mathbb{Z}[x_1, \ldots, x_j, y_1, \ldots, y_k]/I_{j,k,m},$$

where $I_{j,k,m} = \langle z_{m+1}, \ldots, z_{m+j+k} \rangle$.

From now on we observe $\mathbb{Z}[x_1, \ldots, x_j, y_1, \ldots, y_k]$ as a graded algebra, where $\deg x_i = 2$, for $i \in [j]$, and $\deg y_i = 2i$, for $i \in [k]$. For simplicity we denote this algebra with $\mathbb{Z}[X^{(j)}, Y^{(k)}]$. Also, for $\sigma = (s_1, \ldots, s_k) \in \mathbb{N}_0^k$, we define $\mathrm{wt}(\sigma) = \sum_{i=1}^{k} i s_i$.

Let $\theta : \mathcal{A}_{j,k,m} \to \mathcal{A}_{j,k,m}$ be an endomorphism of the (graded) algebra $\mathcal{A}_{j,k,m}$. Then θ is determined by $\theta(x_1), \ldots, \theta(x_j), \theta(y_1), \ldots, \theta(y_k)$, which are polynomials in the variables

$x_1,\dots,x_j,y_1,\dots,y_k$. Also, $\mathrm{ht}(x_i)\geq \mathrm{ht}(\theta(x_i))$, for $i\in[j]$, and hence, by Corollary 3.3, there is a function $\pi:[j]\to[j]$, such that $\theta(x_i)=a_ix_{\pi(i)}$, where $a_i\in\mathbb{Z}$. Note that there is a unique (graded) endomorphism $\varphi:\mathbb{Z}[X^{(j)},Y^{(k)}]\to\mathbb{Z}[X^{(j)},Y^{(k)}]$, such that $\varphi(x_i)=\theta(x_i)$, $i\in[j]$, and $\varphi(y_i)=\theta(y_i)$, $i\in[k]$. By the definition, φ satisfies $\varphi(I_{j,k,m})\subseteq I_{j,k,m}$. So, in order to classify all endomorphism of $\mathcal{A}_{j,k,m}$, we will find all grading endomorphism of $\varphi:\mathbb{Z}[X^{(j)},Y^{(k)}]\to\mathbb{Z}[X^{(j)},Y^{(k)}]$ that satisfy the following two conditions:

(1) $\varphi(I_{j,k,m})\subseteq I_{j,k,m}$;

(2) for $i\in[j]$, $\varphi(x_i)=a_ix_{\pi(i)}$, where $a_i\in\mathbb{Z}$ and $\pi:[j]\to[j]$ is some function.

This is what we do in the remaining of this section. In this situation we say that φ *corresponds* to θ (of course, φ uniquely determines θ).

Because of the grading, condition (1) implies that for $i\in[m+j+k]\setminus[m]$:

$$\varphi(z_i)=\alpha_iz_i+\sum_{1\leq|\mu|+\mathrm{wt}(\sigma)\leq i-m-1}\alpha^{(i)}_{\mu\sigma}x^\mu y^\sigma z_{i-|\mu|-\mathrm{wt}(\sigma)}, \tag{5.1}$$

where α_i and $\alpha^{(i)}_{\mu\sigma}$ are some integers. In particular,

$$\varphi(z_{m+1})=\alpha_{m+1}z_{m+1}. \tag{5.2}$$

Again, because of the grading, the following identities hold

$$\varphi(y_1)=\lambda y_1+\sum_{i=1}^{j}\gamma_ix_i,$$

and

$$\varphi(y_i)=\lambda_iy_i+\sum_{|\mu|+\mathrm{wt}(\sigma)=i}c^{(i)}_{\mu\sigma}x^\mu y^\sigma,\quad 2\leq i\leq k, \tag{5.3}$$

where $\lambda,\gamma_i,\lambda_i,c^{(i)}_{\mu\sigma}\in\mathbb{Z}$. We also define $c^{(1)}_{\mu\sigma}:=\gamma_i$, for $\mu=(0^{\cdots(i-1)},1,0,\dots,0)$ and $\sigma=0$.

Let $k\geq 3$ and $t=x^\mu y_1^{a_1}\cdots y_k^{a_k}$ a monomial of z_{m+i}, for some $i\in[j+k]$. Then, in $\varphi(t)$ the degree of y_k in each monomial is at most a_k, and if it is d_k, then the coefficient of this monomial is a multiple of $\lambda_k^{d_k}$ (since only $\varphi(y_k)$ can contain y_k as a monomial). Additionally, if $y_{k-1}^{b_{k-1}}y_k^{b_k}$ divides a monomial of $\varphi(t)$, then $b_k\leq a_k$ and $b_{k-1}+b_k\leq a_{k-1}+a_k$ (since only $\varphi(y_k)$ can contain y_k and only $\varphi(y_{k-1})$ and $\varphi(y_k)$ can contain y_{k-1} as a monomial); also, if $b_{k-1}\geq a_k-b_k$, then the coefficient of this monomial is a multiple of $\lambda_{k-1}^{b_{k-1}+b_k-a_k}\lambda_k^{b_k}$. We will call this *Property* $(\star)$ and use in this section.

We break our proof in two cases, $\varphi(z_{m+1})\neq 0$ and $\varphi(z_{m+1})=0$, and resolve each of them in a separate subsection.

We will need the following result from [5, Proposition 1].

Lemma 5.1 ([5]) *Let $m_0,n_0\in\mathbb{Z}$ and $p\in\mathbb{N}$. If d is an integer such that*

$$d\geq p(p-1)+n_0p+m_0(p-1),$$

then there exist integers $m\geq m_0$ and $n\geq n_0$ such that

$$m(p-1)+np=d.$$

5.1 Case $\varphi(z_{m+1}) \neq 0$

Let $\Phi : H^*(F(n_1,\dots,n_r);\mathbb{Z}) \to H^*(F(n_1,\dots,n_r);\mathbb{Z})$ be an endomorphism of $H^*(F(n_1,\dots,n_r);\mathbb{Z})$. Then Φ is a *grading endomorphism* if there is a permutation $\pi \in \mathrm{Sym}(r)$ that satisfies $n_i = n_{i'}$ whenever $\pi(i) = i'$, and $\lambda \in \mathbb{Z}$, such that

$$\Phi(c_{i,s}) = \lambda^s c_{\pi(i),s} \quad \text{for all } i \in [r] \text{ and } s \in [n_i].$$

We prove that φ corresponds to a grading endomorphism. By [6], this is true for $j = 1$, so we assume that $j \geq 2$.

Let $S_i = \{s \in [j] : \pi(s) = i\}$ for $i \in [j]$. Then for some $i \in [j]$ we have $|S_i| \leq 1$. W.l.o.g. assume that $i = 1$; additionally, if $|S_1| = 1$, then, up to a conjugation, we have $\varphi(x_1) = a_1 x_1$ (note: if $|S_1| = 0$, then $\varphi(x_1) = a_1 x_i$ for some $2 \leq i \leq j$).

Let $\mathcal{I} = \langle x_2, x_3, \dots, x_j \rangle$ and $\theta : \mathbb{Z}[X^{(1)}, Y^{(k)}] \to \mathbb{Z}[X^{(1)}, Y^{(k)}]$ be the endomorphism defined with:

$$\theta(x_1) = \varphi(x_1) \bmod \mathcal{I} \quad \text{and} \quad \theta(y_i) = \varphi(y_i) \bmod \mathcal{I} \quad \text{for } i \in [k].$$

(Note that: $\theta(x_1) = 0$ for $|S_1| = 0$, and $\theta(x_1) = a_1 x_1$ for $|S_1| = 1$.)

Let us prove that θ induces an endomorphism of $\mathcal{A}_{1,k,m}$. To prove this it is enough to show $\theta(I_{1,k,m}) \subseteq I_{1,k,m}$. By the definition we have $\theta(t) = \varphi(t) \bmod \mathcal{I}$ for every monomial $t \in \mathbb{Z}[X^{(1)}, Y^{(k)}]$. Let $\widetilde{z}_i = z_i \bmod \mathcal{I}$ for all $i \geq 1$. By (2.4), it is clear that $\widetilde{z}_{m+i}$, for $i \in [k+1]$, are the polynomials that generate $I_{1,k,m}$. Since $\varphi(x_i) \in \mathcal{I}$ for all $i \in [j] \setminus \{1\}$, we have $\varphi(x_i) \bmod \mathcal{I} = 0$ for all $i \in [j] \setminus \{1\}$, and hence

$$\theta(\widetilde{z}_i) = \varphi(\widetilde{z}_i) \bmod \mathcal{I} = \varphi(z_i) \bmod \mathcal{I} \quad \text{for } i \geq 1.$$

For $\widetilde{t} \in I_{1,k,m}$, we have $\widetilde{t} = \sum_{i=1}^{k+1} p_i \widetilde{z}_{m+i}$, and hence

$$\begin{aligned}
\theta(\widetilde{t}) &= \sum_{i=1}^{k+1} \theta(p_i)\theta(\widetilde{z}_{m+i}) = \sum_{i=1}^{k+1} \varphi(p_i)\varphi(z_{m+i}) \bmod \mathcal{I} \\
&= \varphi(t) \bmod \mathcal{I},
\end{aligned}$$

where $t = \sum_{i=1}^{k+1} p_i z_{m+i}$. Since $t \in I_{j,k,m}$ it follows $\varphi(t) \in I_{j,k,m}$, i.e. $\varphi(t) = \sum_{i=1}^{k+j} q_i z_{m+i}$. Finally,

$$\theta(\widetilde{t}) = \varphi(t) \bmod \mathcal{I} = \sum_{i=1}^{k+j} \widetilde{q}_i \widetilde{z}_{m+i} \in I_{1,k,m},$$

where $\widetilde{q}_i = q_i \bmod \mathcal{I}$ for $i \in [k+j]$ (note that, by (2.4), $\widetilde{z}_{m+i} \in I_{1,k,m}$ for all $i \geq 1$).

So, θ induces an endomorphism of the algebra $\mathcal{A}_{1,k,m}$, and we can use the results of [6]. Indeed, since $\theta(\widetilde{z}_{m+1}) = \alpha_{m+1}\widetilde{z}_{m+1} \neq 0$ and $m \geq 2k^2 - 1$, by Lemma 5.3 and Lemma 5.4 from [6] we have that θ corresponds to a grading endomorphism (of $\mathcal{A}_{1,k,m}$). In particular, $\theta(x_1) = \lambda x_1 \neq 0$ (and hence $|S_1| = 1$), and

$$\begin{aligned}
\theta(y_1) &= \lambda y_1 = \lambda y_1 + \gamma_1 x_1, \text{ i.e. } \gamma_1 = 0, \\
\theta(y_{k-1}) &= \lambda^{k-1} y_{k-1} = \lambda^{k-1} y_{k-1} + p_{k-1}, \\
\theta(y_k) &= \lambda^k y_k = \lambda^k y_k + c_{1,k-1} y_1 y_{k-1} + b_{1,k-1}^{(1)} x_1 y_{k-1} + p_k,
\end{aligned}$$

that is $b^{(1)}_{1,k-1} = c_{1,k-1} = 0$, where $c_{1,k-1} = c^{(k)}_{(0,\dots,0)(1,0,\dots,0,1,0)}$, $b^{(i)}_{1,k-1} = c^{(k)}_{(0\cdots(i-1),1,0\cdots(j-i))(0,\dots,0,1,0)}$ for $i \in [j]$, $\lambda_{k-1} = \lambda^{k-1}$ and $\lambda_k = \lambda^k$ (in the identities above p_{k-1} (resp. p_k) denotes some polynomial that does not have y_{k-1} (resp. y_k, $y_1 y_{k-1}$ and $x_1 y_{k-1}$) as its monomial (resp. monomials)).

Now, let us go back to φ. Since $|S_1| = 1$, we know that there exists some $\ell \in [j] \setminus \{1\}$ such that $|S_\ell| \leq 1$. Now, in a similar way as above, we conclude that $|S_\ell| = 1$. Therefore, by reiterating the proof, we have that $|S_i| = 1$ for all $i \in [j]$. We conclude that

$$\lambda_{k-1} = \lambda^{k-1}, \quad \lambda_k = \lambda^k, \quad c_{1,k-1} = 0, \tag{5.4}$$

$$a_i = \lambda, \quad \gamma_i = 0, \quad b^{(i)}_{1,k-1} = 0 \quad \text{for } i \in [j]. \tag{5.5}$$

Let l and r be positive integers such that $m + 1 = l(k-1) + rk$ (they exist by Lemma 5.1 since $m \geq 2k^2 - 1$). We now analyse the coefficient of $y^l_{k-1} y^r_k$ in (5.2). This monomial appears in z_{m+1}, so (2.7) and (5.2) imply that its coefficient is $(-1)^{l+r}[\sigma]\alpha_{m+1}$ in $\alpha_{m+1} z_{m+1}$, where $\sigma = (0, \dots, 0, l, k)$. From (2.7) and (5.3), we deduce that the coefficient of $y^l_{k-1} y^r_k$ in $\varphi(z_{m+1})$ is $(-1)^{l+r}[\sigma]\lambda^l_{k-1}\lambda^r_k$ because none of non-y^σ terms in z_{m+1} can be mapped into y^σ (for $k \geq 3$ this is clear; for $k = 2$ we also use the fact that $c_{1,k-1} = 0$ and that $\varphi(x_i)$ does not have y_1 as its monomial). Hence, by (5.4):

$$\alpha_{m+1} = \lambda^l_{k-1}\lambda^r_k = \lambda^{l(k-1)+rk} = \lambda^{m+1}. \tag{5.6}$$

By (5.4) and (5.5), $c^{(1)}_{\mu\sigma} = 0$ for $\mu = (0^{\cdots(i-1)}, 1, 0, \dots, 0)$ and $\sigma = 0$; $c^{(k)}_{\mu\sigma} = 0$ for $\mu = (0^{\cdots(i-1)}, 1, 0, \dots, 0)$ and $\sigma = (0, \dots, 0, 1, 0)$; $c^{(k)}_{\mu\sigma} = 0$ for $\mu = 0$ and $\sigma = (1, \dots, 0, 1, 0)$. Now, we arrange the $(j+k)$-tuples (μ, σ) (where μ is a j-tuple and σ a k-tuple) in lexicographical order, denoted by $\prec_{\text{lex}}$, and prove by induction on this order that $c^{(i)}_{\mu\sigma} = 0$ (note that i is uniquely determined by (μ, σ)). Let us recall that the lexicographical order is defined with: for $\alpha = (a_1, a_2, \dots, a_{j+k})$ and $\beta = (b_1, b_2, \dots, b_{j+k})$, $\alpha \neq \beta$, we have

$$\alpha \prec_{\text{lex}} \beta \quad \text{iff} \quad a_i < b_i \text{ where } i = \min\{s \mid s \in [j+k],\ a_s \neq b_s\}.$$

We note that our proof is similar to the corresponding proofs in [5] and [6], but we give it for the sake of completeness.

So, suppose that $c^{(i')}_{\mu'\sigma'} = 0$ for all $(\mu', \sigma') \prec_{lex} (\mu, \sigma)$ and prove that $c^{(i)}_{\mu\sigma} = 0$. We may assume that (μ, σ) is not one of the $(j+k)$-tuples from the previous paragraph; then $\sigma = (s_1, \dots, s_{k-2}, 0, 0)$, for some $s_1, \dots, s_{k-2} \in \mathbb{N} \cup \{0\}$. Since $m + 1 - k \geq k(k-1)$, by Lemma 5.1 there exist $l, r \geq 0$ such that $l(k-1) + rk = m + 1 - i \geq m + 1 - k$. Let us compare the coefficient of $x^\mu y^\sigma y^l_{k-1} y^r_k$ on the left-hand and right-hand side of (5.2); denote them by L and R, respectively. On the right-hand side the coefficient is (by (5.6)):

$$\alpha_{m+1}(-1)^{|\mu|+|\sigma|+l+r}[\sigma, l, r] = \lambda^{m+1}(-1)^{|\mu|+|\sigma|+l+r}[\sigma, l, r] = R$$

since $\alpha_{m+1} = \lambda^{m+1}$. Let us prove that there are two nonzero coefficients on the left-hand side; one comes from $\varphi((-1)^{|\mu|+|\sigma|+l+r}[\sigma, l, r] x^\mu y^\sigma y^l_{k-1} y^r_k)$ and is equal to R. So, let $\varphi(x^\tau y^\chi)$ for some $(\tau, \chi) \neq (\mu, \sigma, l, r)$ be such that it contains the monomial $x^\mu y^\sigma y^l_{k-1} y^r_k$ (with nonzero coefficient). Then $x^\tau \mid x^\mu$, and by (5.4), $y^l_{k-1} y^r_k \mid y^\chi$. Hence $x^{\mu-\tau} y^\sigma$ is a monomial of $\varphi(t)$

for some $t \mid y^\chi$; by the inductive hypothesis this is only possible if $t = y_i$ and $\tau = 0$. So, $y^\chi = y_i y_{k-1}^l y_k^r$ and the corresponding coefficient is

$$\begin{gathered}(-1)^{1+l+r}[0^{\cdots(j+i-1)}, 1, 0^{\cdots(k-i-2)}, l, r] c_{\mu\sigma}^{(i)} \lambda^{l(k-1)+rk}, \text{ if } i \in [k-2]; \\ (-1)^{1+l+r}[0, \ldots, 0, l+1, r] c_{\mu\sigma}^{(k-1)} (l+1) \lambda^{l(k-1)+rk}, \text{ if } i = k-1; \\ (-1)^{1+l+r}[0, \ldots, 0, l, r+1] c_{\mu\sigma}^{(k)} (r+1) \lambda^{l(k-1)+rk}, \text{ if } i = k.\end{gathered}$$

By our previous calculations this coefficient is equal to zero, and so $c_{\mu\sigma}^{(i)} = 0$.

We conclude that φ corresponds to a grading endomorphism of $\mathcal{A}_{j,k,m}$.

5.2 Case $\varphi(z_{m+1}) = 0$

We treat this case by induction on $j \geq 1$; further, we assume that φ does not vanish in positive dimensions and prove that φ corresponds to a projective endomorphism. For $j = 1$, this follows from [6]. So, let us assume that it is true for $j - 1 \geq 1$ and prove it for j.

We divide the proof in several steps (the claim in each step is valid only in the case that it is proven in).

(A) $\varphi(z_{m+i}) = 0$ for $i \in [k]$.

First, note that (5.2) and $\varphi(z_{m+1}) = 0$ imply $\alpha_{m+1} = 0$. For $i \in [k]$, let $l^{(i)} \geq 1$ and $r^{(i)} \geq k$ be integers such that

$$l^{(i)}(k-1) + r^{(i)}k = m + i$$

(they exist by Lemma 5.1 since $m + i \geq m + 1 \geq k(k-1) + k^2 + k - 1 = 2k^2 - 1$). In the remaining of the proof of (A) we assume $k \geq 3$, while the proof in the case $k = 2$ is given in Appendix.

The monomial y^σ, where $\sigma = (0^{\cdots(k-2)}, l^{(1)}, r^{(1)})$, is a monomial of z_{m+1}, so (2.7) and (5.2) imply that y^σ has the coefficient $(-1)^{l+r}[\sigma]\alpha_{m+1} = 0$ in $\varphi(z_{m+1})$. From (2.7), (5.3) and Property ($\star$) we deduce that the coefficient of y^σ in $\varphi(z_{m+1})$ is $(-1)^{l+r}[\sigma]\lambda_{k-1}^{l^{(1)}}\lambda_k^{r^{(1)}}$. Hence $\lambda_{k-1}\lambda_k = 0$.

Let us observe $\varphi(z_{m+i})$, for $i \in [k-1] \setminus \{1\}$. For $\tau = (0^{\cdots(k-2)}, l^{(i)}, r^{(i)})$, similarly as above we conclude that the only monomial of $\varphi(z_{m+i})$ that contains y^τ is also a monomial of $\varphi(y^\tau)$. The coefficient of y^τ in $\varphi(y^\tau)$ is $\lambda_{k-1}^{l^{(i)}}\lambda_k^{r^{(i)}}$, which is equal to 0. Hence, the coefficient of y^τ on the right-hand side of (5.1) is also 0. But the only polynomial there that contains y^τ is $\alpha_{m+i}z_{m+i}$ (since for each σ in the sum, one has $\mathrm{wt}(\sigma) \leq i - 1 \leq k - 2$), and hence $\alpha_{m+i} = 0$.

Finally, let us observe $\varphi(z_{m+k})$, and let $\delta = (0^{\cdots(k-2)}, l^{(k)}, r^{(k)})$. As before, we conclude that the only polynomials on the right-hand side of (5.1) that contain y^δ are $\alpha_{m+k}z_{m+k}$ and $\alpha^{(m+k)}_{(0^{\cdots j}),(0^{\cdots(k-1)},k-1)} y_{k-1} z_{m+1}$, and hence

$$0 = \alpha_{m+k} \cdot [\delta] - \alpha \cdot [0^{\cdots(k-2)}, l^{(k)} - 1, r^{(k)}],$$

where $\alpha = \alpha^{(m+k)}_{(0^{\cdots j}),(0^{\cdots(k-2)},1,0)}$. This implies $(l^{(k)} + r^{(k)})\alpha_{m+k} = l^{(k)} \cdot \alpha$. Similarly, by observing the monomial $y^{\delta'}$, for $\delta' = (0^{\cdots(k-2)}, l^{(k)} + k, r^{(k)} - k + 1)$, we obtain

$$0 = \alpha_{m+k} \cdot [\delta'] - \alpha \cdot [0^{\cdots(k-2)}, l^{(k)} + k - 1, r^{(k)} - k + 1],$$

that is $(l^{(k)}+r^{(k)}+1)\alpha_{m+k}=(l^{(k)}+k)\alpha$. Hence,

$$\begin{aligned}0&=\alpha_{m+k}((l^{(k)}+r^{(k)})(l^{(k)}+k)-l^{(k)}(l^{(k)}+r^{(k)}+1))\\&=\alpha_{m+k}(k(l^{(k)}+r^{(k)})-l^{(k)}),\end{aligned}$$

and so $\alpha_{m+k}=\alpha=0$.

Let us denote $\alpha^{(m+i)}_{(0^{\cdot\cdot j}),(0^{\cdot\cdot k})}:=\alpha_{m+i}=0$, for $i\in[k]$. We now fix i and prove that $\alpha^{(m+i)}_{\mu\sigma}=0$, where $0\le|\mu|+\mathrm{wt}(\sigma)\le i-1$, by induction on (μ,σ) w.r.t. the *graded lexicographical ordering* denoted by $\prec_{\mathrm{grlex}}$. This ordering is defined in the following way:

$$\alpha\prec_{\mathrm{grlex}}\beta\quad\text{iff}\quad|\alpha|<|\beta|,\ \text{or}\ |\alpha|=|\beta|\ \text{and}\ \alpha\prec_{\mathrm{lex}}\beta.$$

For $(\mu,\sigma)=((0^{\cdot\cdot j}),(0^{\cdot\cdot k}))$, the claim holds, so we assume that it is true for all (μ',σ'), such that $0\le|\mu'|+\mathrm{wt}(\sigma')\le i-1$ and $(\mu',\sigma')\prec_{\mathrm{grlex}}(\mu,\sigma)$, and prove it for (μ,σ). Since $\alpha^{(m+k)}_{(0^{\cdot\cdot j}),(0^{\cdot\cdot(k-2)},1,0)}=0$, we may also assume that $(\mu,\sigma)\ne((0^{\cdot\cdot j}),(0^{\cdot\cdot(k-2)},1,0))$. Let $s=i-|\mu|-\mathrm{wt}(\sigma)$. Then $1\le s\le i\le k$, so let $\tau=(0^{\cdot\cdot(k-2)},l',r')$, such that $(k-1)l'+kr'=m+s$, $l'\ge k$ and $r'>0$ (l' and r' exist by Lemma 5.1, since $m+s\ge m+1\ge k(k-1)+k(k-1)+k=2k^2-k$). Let us observe the coefficient of the monomial $x^\mu y^{\sigma+\tau}$ in $\varphi(z_{m+i})$. Since, $(\mu,\sigma)\ne((0^{\cdot\cdot j}),(0^{\cdot\cdot(k-2)},1,0))$, if $x^\mu y^{\sigma+\tau}$ is a monomial of $\varphi(t)$, where t is a monomial in z_{m+i} such that the degree of y_{k-1} (resp. y_k) in t is a (resp. b), then $b\ge r'$ and $a+b\ge l'+r'$. If $a+b\ne l'+r'$, then

$$\begin{aligned}m+i&\ge a(k-1)+bk=(a+b)(k-1)+b\\&\ge(l'+r'+1)(k-1)+r'=m+s+k-1\ge m+i,\end{aligned}$$

with the equality only if $b=r'$; hence $a\ge l'$ and the coefficient of $x^\mu y^{\sigma+\tau}$ in $\varphi(t)$ is a multiple of $\lambda_{k-1}^{l'}\lambda_k^{r'}$. If $a+b=l'+r'$, then $a\ne0$ (since otherwise $m+i\ge bk=(l'+r')k=m+s+l'\ge m+s+k>m+i$), and hence the coefficient of $x^\mu y^{\sigma+\tau}$ in $\varphi(t)$ is a multiple of $\lambda_{k-1}\lambda_k$. So, in both cases the coefficient of $x^\mu y^{\sigma+\tau}$ in $\varphi(t)$ is equal to 0. On the other hand, by (5.1), the coefficient of $x^\mu y^{\sigma+\tau}$ in $\varphi(z_{m+i})$ is the sum of the coefficients of this monomial in nonzero polynomial $\alpha^{(m+i)}_{\mu'\sigma'}x^{\mu'}y^{\sigma'}z_{m+i'}$, where $i'+|\mu'|+\mathrm{wt}(\sigma')=i$. Since $x^\mu y^{\sigma+\tau}$ is a monomial of $x^{\mu'}y^{\sigma'}z_{m+i'}$ only if a multiple of y^τ is a monomial of $z_{m+i'}$ (since $(\mu',\sigma')\notin\{(0^{\cdot\cdot j},0^{\cdot\cdot k}),(0^{\cdot\cdot j},0^{\cdot\cdot(k-2)},1,0)\}$), we have $\mathrm{wt}(\tau)\le m+i'$ and hence $|\mu'|+\mathrm{wt}(\sigma')=m+i-m-i'=|\mu|+\mathrm{wt}(\sigma)+\mathrm{wt}(\tau)-m-i'\le|\mu|+\mathrm{wt}(\sigma)$, with the equality only for $(\mu',\sigma')=(\mu,\sigma)$. Hence, the conclusion follows by induction.

So, $\varphi(z_{m+i})=0$ for $i\in[k]$, as desired.

After possibly renaming the variables, by Corollary 3.3 we may assume that, up to a conjugation, $\varphi(x_j)=a_jx_j$ (it is possible that $a_j=0$). For a polynomial $q\in\mathbb{Z}[X^{(j)},Y^{(k)}]$, let $\overline{q}$ denote its reduction modulo x_j.

(B) The map $\overline{\varphi}:\mathbb{Z}[X^{(j-1)},Y^{(k)}]\to\mathbb{Z}[X^{(j-1)},Y^{(k)}]$ defined with

$$\overline{\varphi}(p):=\overline{\varphi(p)}\ \text{for}\ p\in\mathbb{Z}[X^{(j-1)},Y^{(k)}],$$

is clearly an endomorphism. Let us prove that $\overline{\varphi}$ induces an endomorphism of $\mathcal{A}_{j-1,k,m}$. To prove this it is enough to show $\overline{\varphi}(I_{j-1,k,m})\subseteq I_{j-1,k,m}$. Since $\varphi(x_j)=a_jx_j$ and φ is an endomorphism, for $q\in\mathbb{Z}[X^{(j-1)},Y^{(k)}]$ we have $\overline{\varphi(q)}=\overline{\varphi(\overline{q})}=\overline{\varphi}(\overline{q})$. By (2.4), it is clear that the polynomials $\overline{z_{m+i}}$, for $i \in [k+j-1]$, generate $I_{j-1,k,m}$. Now, for $t' \in I_{j-1,k,m}$, we have $t' = \sum_{i=1}^{k+j-1} p_i \overline{z_{m+i}}$, and hence

$$\begin{aligned}\overline{\varphi}(t') &= \sum_{i=1}^{k+j-1} \overline{\varphi}(p_i)\overline{\varphi}(\overline{z_{m+i}}) = \sum_{i=1}^{k+j-1} \overline{\varphi(p_i)\varphi(z_{m+i})} \\ &= \overline{\varphi(t)},\end{aligned}$$

where $t = \sum_{i=1}^{k+j-1} p_i z_{m+i}$. Since $t \in I_{j,k,m}$ it follows $\varphi(t) \in I_{j,k,m}$, i.e. $\varphi(t) = \sum_{i=1}^{k+j} q_i z_{m+i}$. Finally,

$$\overline{\varphi}(t') = \overline{\varphi(t)} = \sum_{i=1}^{k+j} \overline{q}_i \overline{z_{m+i}} \in I_{j-1,k,m}$$

(note that, by (2.4), $\overline{z_{m+i}} \in I_{j-1,k,m}$ for all $i \geq 1$). By the inductive hypothesis, $\overline{\varphi}$ vanishes in positive dimensions or $\overline{\varphi}$ corresponds to a projective endomorphism of $\mathcal{A}_{j-1,k,m}$.

CASE 1. $\overline{\varphi} : \mathbb{Z}[X^{(j-1)}, Y^{(k)}] \to \mathbb{Z}[X^{(j-1)}, Y^{(k)}]$ vanishes in positive dimensions.

By Corollary 3.3, $\varphi(x_i) = a_i x_j$, for $i \in [j]$. Applying φ on the identity (2.6) (for $J = \emptyset$) gives

$$\begin{aligned}(1 + \varphi(y_1) + \cdots + \varphi(y_k)) \sum_{i \geq 0} \varphi(z_i) &= \prod_{i=1}^{j} \sum_{l \geq 0} (-a_i x_j)^l \\ &= \sum_{l \geq 0} (-1)^l h_l(a_1, \ldots, a_j) x_j^l, \end{aligned} \tag{5.7}$$

where h_l denotes the complete symmetric polynomial on j variables of degree l.

(C) $h_{m+k+i}(a_1, \ldots, a_j) = 0$ and $\varphi(z_{m+k+i}) = 0$, for $i \in [j-1]$.

The proof is by induction on i. From (5.7) and (A) we have

$$\begin{aligned}\varphi(z_{m+k+1}) &= \varphi(z_{m+k+1}) + \varphi(y_1)\varphi(z_{m+k}) + \cdots + \varphi(y_k)\varphi(z_{m+1}) \\ &= (-1)^{m+k+1} h_{m+k+1}(a_1, \ldots, a_j) x_j^{m+k+1}.\end{aligned}$$

By Corollary 3.3, $\mathrm{ht}(x_j) = m + j + k - 1 \geq m + k + 1$, and hence $x_j^{m+k+1} \notin I_{j,k,m}$; since $\varphi(z_{m+k+1}) \in I_{j,k,m}$, this implies $h_{m+k+1}(a_1, \ldots, a_j) = 0$ and $\varphi(z_{m+k+1}) = 0$. This proves the base case $i = 1$.

So, let us now assume that our claim is true for all $l \in [i-1]$, and prove it for $i \leq j-1$. From (A) and the inductive hypothesis $\varphi(z_{m+s}) = 0$ for all $1 \leq s \leq k+i-1$. So, by (5.7),

$$\begin{aligned}\varphi(z_{m+k+i}) &= \varphi(z_{m+k+i}) + \varphi(y_1)\varphi(z_{m+k+i-1}) + \cdots + \varphi(y_k)\varphi(z_{m+i}) \\ &= (-1)^{m+k+i} h_{m+k+i}(a_1, \ldots, a_j) x_j^{m+k+i}.\end{aligned}$$

Again, by Corollary 3.3, $\mathrm{ht}(x_j) = m + j + k - 1 \geq m + i + k$, and hence $x_j^{m+k+1} \notin I_{j,k,m}$, which implies $h_{m+k+i}(a_1, \ldots, a_j) = 0$ and $\varphi(z_{m+k+i}) = 0$.

To apply (C) we need the following lemma from [9].

Lemma 5.2 ([9]) *Let l, r be integers, $l \geq 0$, $r \geq 2$, and $b_1, \ldots, b_r$ some real numbers, such that*

$$h_{l+i}(b_1, \ldots, b_r) = 0 \text{ for } i \in [r-1].$$

Then unless $r = 2$ and l is even, $b_1 = \cdots = b_r = 0$. If $r = 2$ and l is even, then $b_2 = -b_1$.

We consider the following cases.

Case 1.1. $j \geq 3$.

In this case we prove that φ vanishes in positive dimensions. By (C) and Lemma 5.2, $a_1 = \cdots = a_j = 0$, i.e. $\varphi(x_1) = \cdots = \varphi(x_j) = 0$. Hence, it is enough to prove that $\varphi(y_i) = 0$ for $i \in [k]$. Suppose that this is not the case, and let $i \leq k$ be maximal such that $\varphi(y_i) \neq 0$. By (5.7)

$$\begin{aligned} 0 &= \varphi(z_{l+k}) + \varphi(y_1)\varphi(z_{l+k-1}) + \ldots + \varphi(y_k)\varphi(z_l) \\ &= \varphi(z_{l+k}) + \varphi(y_1)\varphi(z_{l+k-1}) + \ldots + \varphi(y_i)\varphi(z_{l+k-i}), \end{aligned}$$

for all $l \geq 0$. In particular, for $l = m + i - k$, from (A) we get $\varphi(y_i)\varphi(z_m) = 0$, i.e. $\varphi(z_m) = 0$. Now, an easy reverse induction on $s \leq m$ gives $\varphi(z_s) = 0$ for all $s \geq 1$. For $s = 1$ this gives $\varphi(-y_1) = -\varphi(-z_1 - x_1 - \ldots - x_j) = \varphi(z_1) = 0$. Also, from (2.7), we have $z_l = -y_l + p_l$, for $l \in [k] \setminus \{1\}$, where p_l is a polynomial in variables $x_1, \ldots, x_j, y_1, \ldots, y_{l-1}$. So, an easy induction on $l \geq 1$ gives $\varphi(y_l) = 0$, for all $l \in [k]$, a contradiction.

Case 1.2. $j = 2$.

By (C) and Lemma 5.2, we have two possibilities: $a_1 = a_2 = 0$, or $a_2 = -a_1 \neq 0$ and $m + k$ is even. The first one is dealt with as in Case 1.1, and leads to $\varphi \equiv 0$. So, we may assume that $a_2 = -a_1 = a \neq 0$.

Applying φ on (2.6) for $J = \emptyset$, we get

$$(1 + \varphi(y_1) + \cdots + \varphi(y_k)) \sum_{i \geq 0} \varphi(z_i) = \sum_{i \geq 0} (ax_2)^i \sum_{i \geq 0} (-ax_2)^i = \sum_{i \geq 0} (ax_2)^{2i}.$$

By (A) and (C), comparing polynomials of degree $2(m + k + 2)$ (w.r.t. the grading) in the previous identity gives

$$\varphi(z_{m+k+2}) = \varphi(z_{m+k+2}) + \cdots + \varphi(y_k)\varphi(z_{m+2}) = (ax_2)^{m+k+2}.$$

Let u_i, for $i \in [k+2]$, be the polynomial of degree $2i$ w.r.t. the grading, such that

$$1 + u_1 + \cdots + u_{k+2} = (1 + x_1)(1 + x_2)(1 + y_1 + \cdots + y_k).$$

Using this notation and applying φ on (2.4) we get

$$(1 + \varphi(u_1) + \cdots + \varphi(u_{k+2})) \cdot \sum_{i \geq 0} \varphi(z_i) = 1. \tag{5.8}$$

Let $N := m + k + 2$. We prove that

$$\varphi(z_{N+r}) = (ax_2)^N \varphi(z_r), \text{ for } N + r \geq m + 1$$

(here, $\varphi(z_r) = 0$ for $r < 0$). We prove this identity by induction on $N + r \geq m + 1$. By (A) and (C), the identity is true for r such that $m + 1 \leq N + r \leq N$. So, we assume that

$N+r \geq N+1$ and that the identity holds for $N+r-1, N+r-2, \ldots, N+r-k-2$, and prove it for $N+r$.

By (5.8) and the inductive hypothesis, we get

$$\begin{aligned}
\varphi(z_{N+r}) &= -\varphi(z_{N+r-1})\varphi(u_1) - \cdots - \varphi(z_{N+r-k-2})\varphi(u_{k+2}) \\
&= -(ax_2)^N \varphi(z_{r-1})\varphi(u_1) - \cdots - (ax_2)^N \varphi(z_{r-k-2})\varphi(u_{k+2}) \\
&= -(ax_2)^N (\varphi(z_{r-1})\varphi(u_1) + \cdots + \varphi(z_{r-k-2})\varphi(u_{k+2})) \\
&= (ax_2)^N \varphi(z_r),
\end{aligned}$$

which completes the inductive step.

Now, an easy induction on l proves that for $l \geq 0$ and $r \in [N-1] \cup \{0\}$, $\varphi(z_{lN+r}) = (ax_2)^{lN}\varphi(z_r)$.

Hence,

$$\begin{aligned}
\sum_{i \geq 0} \varphi(z_i) &= (1 + \varphi(z_1) + \ldots + \varphi(z_m)) \sum_{i \geq 0} (ax_2)^{iN} \\
&= (1 + \varphi(z_1) + \ldots + \varphi(z_m))(1 - (ax_2)^N)^{-1}.
\end{aligned}$$

By applying φ on (2.4) we get

$$(1 + \varphi(x_1))(1 + \varphi(x_2)) \left(1 + \sum_{i=1}^{k} \varphi(y_k)\right) \sum_{i=0}^{m} \varphi(z_i) = 1 - (ax_2)^N.$$

So, $1+\varphi(x_1)$, $1+\varphi(x_2)$ and $1+\varphi(y_1)+\cdots+\varphi(y_k)$ are in $\mathbb{Z}[x_2]$, and hence φ indeed corresponds to a projective endomorphism of $\mathcal{A}_{j,k,m}$.

CASE 2. $\overline{\varphi} : \mathbb{Z}[X^{(j-1)}, Y^{(k)}] \to \mathbb{Z}[X^{(j-1)}, Y^{(k)}]$ corresponds to a projective endomorphism of $\mathcal{A}_{j-1,k,m}$.

By Theorem 4.1, we have $j \in \{2, 3\}$ (indeed, for every n the polynomial $1 - x^n$ has at most two rational roots, i.e. 1 and -1, and each of them has multiplicity at most one, so $j - 1 \leq 2$).

Case 2.1. $j = 3$.

Since $\overline{\varphi}$ corresponds to a (nonzero) projective endomorphism, by Theorem 4.1, we may assume that $\overline{\varphi}(x_1) = a_1 x_1$ and $\overline{\varphi}(x_2) = -a_1 x_1$, $a_1 \neq 0$ which implies $\varphi(x_1) = a_1 x_1$ and $\varphi(x_2) = -a_1 x_1$ (by Corollary 3.3). Note that $\overline{\varphi}$ was constructed by reducing modulo x_3; in a similar way the function $\widehat{\varphi}$ that reduces polynomials modulo x_1 could be constructed. Then, if $\widehat{\varphi}$ vanishes in positive dimensions, the proof follows as in Case 1; otherwise $\widehat{\varphi}$ corresponds to a projective endomorphism, which, by Theorem 4.1, implies $\widehat{\varphi}(x_2) = -\widehat{\varphi}(x_3)$, a contradiction.

Case 2.2. $j = 2$.

First, we prove that $\varphi(1 + y_1 + \cdots + y_k), \varphi(1 + z_1 + \cdots + z_m) \in \mathbb{Z}[x_1, x_2]$. Again, let $N := m + k + 2$. Since $\overline{\varphi}$ corresponds to a projective endomorphism, by Theorem 4.1, $\overline{\varphi}(x_1) = a_1 x_1$.

If $a_1 = 0$, then $\overline{\varphi}$ vanishes in positive dimensions (by Theorem 4.1). So, we may assume that $a_1 \neq 0$. Again by Theorem 4.1, $\overline{\varphi}(y_1) \in \mathbb{Z}[x_1]$, $\overline{\varphi}(x_1) = a_1 x_1$, and hence, by Corollary

3.3, $\varphi(x_1) = a_1x_1$. Similarly, we can conclude that the reduction of $\varphi(y_1)$ modulo x_1 is in $\mathbb{Z}[x_2]$, and hence that $\varphi(y_1) \in \mathbb{Z}[x_1, x_2]$.

Applying φ to (2.6) for $J = \emptyset$, and compering polynomials of degree $2(N-1)$ (w.r.t. the grading) gives

$$\begin{aligned}\varphi(z_{N-1}) &= \varphi(z_{N-1}) + \varphi(y_1)\varphi(z_{N-2}) + \cdots + \varphi(y_k)\varphi(z_{m+1}) \\ &= \sum_{i=0}^{N-1} (-a_1x_1)^i(-a_2x_2)^{N-1-i},\end{aligned}$$

and of degree $2N$ (w.r.t. the grading)

$$\begin{aligned}\varphi(z_N) + \varphi(y_1)\varphi(z_{N-1}) &= \varphi(z_N) + \varphi(y_1)\varphi(z_{N-1}) + \cdots + \varphi(y_k)\varphi(z_{m+2}) \\ &= \sum_{i=0}^{N} (-a_1x_1)^i(-a_2x_2)^{N-i},\end{aligned}$$

which implies $\varphi(z_{N-1}), \varphi(z_N) \in \mathbb{Z}[x_1, x_2]$.

Let $t_1 := \varphi(z_{N-1})$ and

$$t_2 := \varphi(z_N) - t_1\varphi(z_1) = \varphi(z_N) + t_1\varphi(x_1 + x_2 + y_1) \in \mathbb{Z}[x_1, x_2].$$

We prove that

$$\varphi(z_{N+r}) = t_1\varphi(z_{r+1}) + t_2\varphi(z_r), \text{ for } N + r \geq m + 1 \tag{5.9}$$

(here, $\varphi(z_r) = 0$ for $r < 0$). We prove this identity by induction on $N + r \geq m + 1$. By (A) and (C), the identity is true for r such that $m + 1 \leq N + r \leq N$. So, we assume that $N + r \geq N + 1$ and that the identity holds for $N + r - 1, N + r - 2, \ldots, N + r - k - 2$, and prove it for $N + r$.

By (5.8) and inductive hypothesis, we conclude that $\varphi(z_{N+r})$ is equal to (where $r' = r - k - 2$)

$$\begin{aligned}&-\varphi(z_{N+r-1})\varphi(u_1) - \varphi(z_{N+r-2})\varphi(u_2) - \cdots - \varphi(z_{N+r'})\varphi(u_{k+2}) \\ &\quad = -(t_1\varphi(z_r) + t_2\varphi(z_{r-1}))\varphi(u_1) - \cdots - (t_1\varphi(z_{r'+1}) + t_2\varphi(z_{r'}))\varphi(u_{k+2}) \\ &\quad = -t_1(\varphi(z_r)\varphi(u_1) + \cdots + \varphi(z_{r'+1})\varphi(u_{k+2})) \\ &\qquad - t_2(\varphi(z_{r-1})\varphi(u_1) + \cdots + \varphi(z_{r'})\varphi(u_{k+2})) \\ &\quad = t_1\varphi(z_{r+1}) + t_2\varphi(z_r),\end{aligned}$$

which completes the inductive step.

Now, (5.9) implies

$$(t_1 + t_2)\sum_{i\geq 0} \varphi(z_i) = \sum_{i\geq -1} (t_1\varphi(z_{i+1}) + t_2\varphi(z_i)) = \sum_{i\geq N-1} \varphi(z_i).$$

Since, $\varphi(z_{m+1}) = \cdots = \varphi(z_{N-2}) = 0$, for $Q_m := \varphi(1 + z_1 + \cdots + z_m) \in \mathbb{Z}[x_1, x_2, y_1, \ldots, y_k]$ we have

$$(1 - t_1 - t_2)\sum_{i\geq 0} \varphi(z_i) = Q_m. \tag{5.10}$$

Let $P_k := \varphi(1 + y_1 + \cdots + y_k) \in \mathbb{Z}[x_1, x_2, y_1, \ldots, y_k]$. Applying φ on (2.4), then multiplying by $1 - t_1 - t_2$ and using (5.10) we get

$$1 - t_1 - t_2 = (1 + a_1 x_1)(1 + a_2 x_2) P_k Q_m.$$

This implies that the polynomials P_k and Q_m divide $1 - t_1 - t_2 \in \mathbb{Z}[x_1, x_2]$, and hence that $P_k, Q_m \in \mathbb{Z}[x_1, x_2]$. This implies that φ corresponds to a 2-projective endomorphism , which by Proposition 4.4 must be projective.

6 Maps between certain complex flag manifolds

It was proven in Theorem 1.1 from [6] that there is a strong connection between homotopy classes of maps between flag manifolds F and F' and homomorphisms between $H^*(F'; \mathbb{Q})$ and $H^*(F; \mathbb{Q})$. Using this result, we prove that for certain flag manifolds F and F' every class in $[F, F']$ is rationally null-homotopic. We recall that similar results for Grassmann manifolds (complex and real) were obtained in [12, 3, 4]

For flag manifolds $F_a = F(a_1, \ldots, a_s, m')$ and $F_b = F(b_1, \ldots, b_s, m'')$ such that $m' \leq m''$ and $a_i \leq b_i$ for $i \in [s]$ (we allow $a_i = 0$ for some i), there is a natural imbedding $\iota : F_a \subset F_b$, which induces a homomorphism $\iota^* : H^*(F_b; \mathbb{Q}) \to H^*(F_a; \mathbb{Q})$. In this homomorphism $\iota^*(c_r(\gamma_i^{F_b})) = c_r(\gamma_i^{F_a})$, for all $i \in [s+1]$ and $r \geq 0$. In particular, if $a_i < r \leq b_i$ (where $i \in [s]$), then $\iota^*(c_r(\gamma_i^{F_b})) = 0$.

Using the same ideas as in the proof of Theorem 1.1 from [3], we prove a partial extension of this theorem.

Theorem 6.1 *Let $F_1 := F(1^{\cdots i}, k, n)$ and $F_2 := F(1^{\cdots j}, l, m)$, where j, k, l, m and n are positive integers and $i \geq 0$ such that $i \leq j$, $k \leq l$, $n \leq m$, $m \geq 2k^2 - 1$ and $\binom{i}{2} + ik + in + nk < m + l + j - 1$. Then any homomorphism $\varphi : H^*(F_1; \mathbb{Q}) \to H^*(F_2; \mathbb{Q})$ vanishes in positive dimensions.*

PROOF — It will be convenient to write F_1 as $F(0^{\cdots(j-i)}, 1^{\cdots i}, k, n)$. Let $F := F(1^{\cdots j}, k, m)$, and $\iota_1 : F \subset F_2$ and $\iota_2 : F_1 \subset F$, be the imbeddings described above. Then we have the endomorphism

$$\phi = \iota_1^* \circ \varphi \circ \iota_2^* : H^*(F; \mathbb{Q}) \to H^*(F; \mathbb{Q}).$$

To prove that φ is vanishing in positive dimensions, it is enough to prove that for $s \in [j+2]$ and $0 < r \leq \dim \gamma_s^{F_1}$, one has $\varphi(c_r(\gamma_s^{F_1})) := p_{r,s} = 0$. Let us first consider $p_{1,s} \in H^2(F_2; \mathbb{Q})$, for $s \in [j]$. If $p_{1,s} \neq 0$, then, by Corollary 3.3, $\mathrm{ht}(p_{1,s}) \geq m + l + j - 1$. On the other hand, $\mathrm{ht}(p_{1,s}) = \mathrm{ht}(\varphi(c_1(\gamma_s^{F_1}))) \leq \mathrm{ht}(c_1(\gamma_s^{F_1})) \leq \dim F_1 = \binom{i}{2} + ik + in + nk < m + l + j - 1$, a contradiction (here, $\dim F_1$ denotes the complex dimension of F_1). Hence, $p_{1,s} = 0$ for all $s \in [j]$, which implies $\phi(c_1(\gamma_s^F)) = 0$. By Theorem 1.1, ϕ is a grading endomorphism or a projective endomorphism, and hence ϕ vanishes in positive dimensions (this is clear if ϕ is a grading endomorphism; if ϕ is a projective endomorphism, then $\phi(c_1(\gamma_s^F)) = 0$ for $s \in [j]$ implies that for this projective endomorphism λ from Theorem 4.1 is equal to 0 and hence ϕ indeed vanishes in positive dimensions). So, $\iota_1^*(p_{r,s}) = 0$, for all $r > 0$ and $s \in [j+2]$. Suppose that $p_{r,s}$ has a non-zero monomial $\prod c_t(\gamma_q^{F_2})$. Then $\prod c_t(\gamma_q^F)$ is a monomial of $\iota_1^*(p_{r,s})$, and since $\dim \gamma_q^{F_1} \leq \dim \gamma_q^F$, for every q, this monomial is non-zero. Since the degree of $p_{r,s}$ is at most $r \leq \max\{k, n\} \leq m$, each of the monomials $\prod c_t(\gamma_p^F)$ is in B_F (see Theorem 2.2), and hence $\iota_1^*(p_{r,s}) \neq 0$, a contradiction. □

As in the proof of Theorem 1.2 from [3] (this proof uses Theorem 1.1 from [6]), the previous theorem implies the following.

Theorem 6.2 *Let $i, j, k, l, m, n, F_1, F_2$ be as in the previous theorem. Then the set $[F_2, F_1]$ of homotopy classes of maps is finite and moreover each homotopy class is rationally null-homotopic.*

7 Appendix

We will prove part (A) of Subsection 5.2 when $k = 2$, that is $\varphi(z_{m+2}) = 0$.

We rewrite (5.3) for $i = 2$ and $k = 2$:

$$\varphi(y_2) = \lambda_2 y_2 + c y_1^2 + \sum_{s=1}^{j} d_s x_s y_1 + \sum_{1\le s\le t\le j} d'_{st} x_s x_t, \tag{7.1}$$

where $\lambda_2, c, d_s, d'_{st}$ are some integer coefficients.

We pick two integers l' and r' such that $m+1 = l' + 2r'$:

$$(l', r') = \begin{cases} (1, m/2), & \text{if } m \text{ is even} \\ (0, (m+1)/2), & \text{if } m \text{ is odd} \end{cases}.$$

Let $\sigma' = (l', r')$. Since the degree of y_2 in $y^{\sigma'}$ is maximal, the coefficient of this monomial in $\varphi(z_{m+1})$ is the same as its coefficient in $\varphi(y^{\sigma'})$, which is equal to $(-1)^{l'+r'}[\sigma']\lambda^{l'}\lambda_2^{r'}$. Now, $\varphi(z_{m+1}) = 0$ implies

$$\alpha_{m+1} = \lambda^{l'}\lambda_2^{r'} = 0. \tag{7.2}$$

Let us rewrite (5.1) for $i = 2$:

$$\varphi(z_{m+2}) = \alpha_{m+2} z_{m+2} + \beta y_1 z_{m+1} + \sum_{i=1}^{j} \beta_i x_i z_{m+1}, \tag{7.3}$$

where $\beta := \alpha^{(m+2)}_{(0^{\cdots j})(1,0)}$ and $\beta_i := \alpha^{(m+2)}_{(0^{\cdots(i-1)},1,0^{\cdots(j-i)})(0,0)}$. Now we analyse the coefficients of monomials in $\varphi(z_{m+2})$ depending on whether m is odd or even.

First, let us consider the case when m is odd; then (7.2) immediately implies $\lambda_2 = 0$. Also, let $r := r' \ge 4$.

(1) $\mu = (0^{\cdots j})$, $\sigma = (1, r)$. The equations (7.3) and (2.7) imply that the coefficient of $y_1 y_2^r$ in $\varphi(z_{m+2})$ is $(-1)^{r+1}[1, r]\alpha_{m+2} + (-1)^r[0, r]\beta$. However, since $\lambda_2 = 0$, this coefficient equals 0 from (2.7) and (7.1). Therefore,

$$(r+1)\alpha_{m+2} = \beta. \tag{7.4}$$

(2) $\mu = (0^{\cdots j})$, $\sigma = (3, r-1)$. In the same fashion as in the previous point, we obtain $(-1)^{r+2}[3, r-1]\alpha_{m+2} + (-1)^{1+r}[2, r-1]\beta = 0$. When we combine this identity with (7.4), we deduce $\alpha_{m+2} = \beta = 0$.

(3) $\mu = (0^{\cdots(i-1)}, 1, 0^{\cdots(j-i)})$, $\sigma = (0, r)$. By comparing the coefficients of $x_i y_2^r$ in (7.3), since $\lambda_2 = 0$, it is easy to see that $\beta_i = 0$ (using (2.7) and (7.1)).

This just leaves the case when m is even. Let $r := (m+2)/2 > 4$. Then (7.2) implies $\lambda\lambda_2 = 0$.

(4) $\mu = (0^{\cdots j})$, $\sigma = (0, r)$. From equations (7.3) and (2.7) the coefficient of y_2^r in $\varphi(z_{m+2})$ is $(-1)^r[0, r]\alpha_{m+2}$. On the other hand, the equations (2.7) and (7.1) show that the coefficient is $(-1)^r[0, r]\lambda_2^r$. Therefore, we have

$$\alpha_{m+2} = \lambda_2^r. \tag{7.5}$$

(5) $\mu = (0^{\cdots j})$, $\sigma = (2, r-1)$. Now we use the equation (2.7) both for $i = m+1$ and $i = m+2$. Similarly to the previous step, using (7.3) and (2.7), we deduce that the coefficient of $y_1^2 y_2^{r-1}$ in $\varphi(z_{m+2})$ is $(-1)^{r+1}[2, r-1]\alpha_{m+2} + (-1)^r[1, r-1]\beta$. Also, from (2.7) and (7.1), it is equal to $(-1)^r[0, r]\binom{r}{1}c\lambda_2^{r-1}$ because $\lambda\lambda_2 = 0$. Using (7.5), this leads to

$$\beta = \frac{r+1}{2}\lambda_2^r + c\lambda_2^{r-1}. \tag{7.6}$$

(6) $\mu = (0^{\cdots j}), \sigma = (4, r-2)$. Assume that $\lambda_2 \neq 0$; then $\lambda = 0$. Similarly to (5) the coefficient of $y_1^4 y_2^{r-2}$ in $\varphi(z_{m+2})$ is

$$(-1)^{r+2}[4, r-2]\alpha_{m+2} + (-1)^{r+1}[3, r-2]\beta = (-1)^r[0, r]\binom{r}{2}c^2\lambda_2^{r-2}.$$

After applying (7.5) and (7.6) we get

$$\binom{r+2}{4}\lambda_2^r - \frac{r+1}{2}\binom{r+1}{3}\lambda_2^r = \binom{r+1}{3}c\lambda_2^{r-1} + \binom{r}{2}c^2\lambda_2^{r-2}.$$

Then the previous identity leads to

$$\lambda_2^2 + \frac{4}{r}c\lambda_2 + \frac{12}{r(r+1)}c^2 = 0. \tag{7.7}$$

Since

$$\lambda_2^2 + \frac{4}{r}c\lambda_2 + \frac{4}{r^2}c^2 \geq 0 \quad \text{and} \quad \frac{12}{r(r+1)} > \frac{4}{r^2},$$

we deduce that the left-hand side of (7.7) is positive, therefore we have a contradiction. This proves that $\lambda_2 = 0$, which implies $\alpha_{m+2} = \beta = 0$.

(7) $\mu = (0^{\cdots(i-1)}, 1, 0^{\cdots(j-i)})$, $\sigma = (1, r-1)$. Since $\lambda_2 = 0$, by calculating the coefficient of $x^\mu y^\sigma$ in $\varphi(z_{m+2})$ it is easy to see that $\beta_i = 0$ (using (7.3), (2.7) and (7.1)).

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