RECURRENCE FORMULAS FOR KOSTKA AND INVERSE KOSTKA NUMBERS VIA QUANTUM COHOMOLOGY OF GRASSMANNIANS

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ABSTRACT. Gröbner basis for small quantum cohomology of Grassmannian $G_{k,n}$ is constructed and used to obtain new recurrence relations for Kostka numbers and inverse Kostka numbers. Using these relations it is shown how to determine inverse Kostka numbers which are related to mod-p Wu formula.

1. INTRODUCTION

In the algebra of symmetric functions the change from the basis given by Schur functions to the basis given by elementary symmetric functions involves Kostka numbers. These numbers are known to be hard to compute (see [16]). Alternatively, these numbers may be seen in the cohomology of Grassmannians in the change from the basis given by Schubert classes to the one given by products of Chern classes. Therefore, obtaining suitable formulas for calculating in these bases produces relations between (inverse) Kostka numbers (see [17]). In this paper we use this approach toward (inverse) Kostka numbers using quantum cohomology of Grassmannians.

Schubert calculus for quantum cohomology of Grassmannians is presented in [2, 3]. It is our goal to obtain corresponding formulas for calculations in the basis given by products of Chern classes. This is done by constructing Gröbner basis for the ideal that determines this cohomology as given by Siebert and Tian (see [20]).

The presentation of results in this paper is as follows. Sections 2 and 3 are dedicated mainly to establish notation and recollect some necessary results about quantum cohomology. In Section 4, we obtain desired Gröbner bases. As a first application, in Section 5, we show how these bases can be used to determine all quantum Kostka numbers; in [3] this was done using Schubert calculus. Combining these two results, we obtain new recurrence relations between Kostka numbers, improving upon those from [17].

Final section is dedicated to inverse Kostka numbers. These numbers are even less tractable then Kostka numbers, so it is useful to obtain recurrence relations determining them (see [7, 8]). Using the similar approach as the one for Kostka numbers, we obtain such relations which completely determine inverse Kostka numbers. On the other hand, some inverse Kostka numbers appear in the formulas for expressing mod p Steenrod powers of Chern classes in terms

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of Schubert classes; this was one of main motivations for Duan to obtain recurrence formulas in [7]. From his formula, he determined these numbers for $p = 2$ (which correspond to classical Wu formulas [24]), but he had only been able to give some further recurrence formulas for $p = 3$. One of the applications of our formulas is the complete determination of these numbers for $p = 3$. Finally, it may be interesting to compare our formulas (and algorithm) for computing inverse Kostka numbers related to mod p Steenrod powers and Lenart method (from [12]) for determination of these powers.

2. Preliminaries

The Grassmann manifold $G_{k,n}$ consists of all k-dimensional linear subspaces of $\mathbb{C}^{n+k}$. Let $F : 0 = F_0 \subset F_1 \subset \cdots \subset F_i \subset \cdots \subset F_{n+k} = \mathbb{C}^{n+k}$ be a complete flag. For a partition $\lambda \subseteq k \times n$ (identified with its Young diagram), i.e., a k-tuple $\lambda = (l_1, l_2, \ldots, l_k)$ such that $n \geq l_1 \geq l_2 \geq \ldots \geq l_k \geq 0$, observe the associated *Schubert variety*

$$\Omega_\lambda(F_\bullet) = \{W \in G_{k,n} \,:\, \dim(W \cap F_{n+i-l_i}) \geq i, 1 \leq i \leq k\}.$$

Denote by Ω_λ the class of $\Omega_\lambda(F_\bullet)$ in the cohomology ring $H^*(G_{k,n};\mathbb{Z})$. The set $\{\Omega_\lambda \,:\, \lambda \subseteq k \times n\}$ forms an additive basis of this ring, in which the multiplication is given by the Littlewood-Richardson rule

$$\Omega_\lambda \cdot \Omega_\mu = \sum_{\nu \subseteq k \times n} c^\nu_{\lambda\mu} \Omega_\nu,$$

where $c^\nu_{\lambda\mu}$ is the Littlewood-Richardson coefficient (see [13]).

For $1 \leq i \leq k$, let $C_i = \Omega_{(1^i)}$, where (1^i) denotes the partition consisting of i ones (note that C_i is the i-th Chern class of the dual of the tautological vector bundle). Then $\{C_{m_1} C_{m_2} \cdots C_{m_r} \,:\, 0 \leq r \leq n,\, 1 \leq m_1, \ldots, m_r \leq k\}$ is another additive basis for $H^*(G_{k,n};\mathbb{Z})$ (cf. [13]). Multiplication in this basis was studied in [17].

The (small) quantum cohomology ring $QH^*(G_{k,n})$ of $G_{k,n}$ is a $\mathbb{Z}[q]$-algebra isomorphic to $H^*(G_{k,n};\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a $\mathbb{Z}[q]$-module. Therefore, for Schubert classes $\sigma_\lambda := \Omega_\lambda \otimes 1$ and Chern classes $c_i = C_i \otimes 1$, the following sets form additive bases of $QH^*(G_{k,n})$:

$$\Sigma_{k,n} \quad := \quad \{\sigma_\lambda \,:\, \lambda \subseteq k \times n\}; \tag{2.1}$$

$$B_{k,n} \quad := \quad \{c_{m_1} \cdots c_{m_r} \,:\, 0 \leq r \leq n,\, 1 \leq m_1, \ldots, m_r \leq k\}. \tag{2.2}$$

Multiplication in $QH^*(G_{k,n})$ is defined with

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{d \geq 0} \sum_{\nu \subseteq k \times n} q^d \langle \Omega_\lambda, \Omega_\mu, \Omega_{\nu^\vee} \rangle_d \sigma_\nu, \tag{2.3}$$

where, for $\nu = (n_1, n_2, \ldots, n_k)$, $\nu^\vee = (n - n_k, n - n_{k-1}, \ldots, n - n_1)$ is the partition for the dual Schubert class of Ω_ν, and $\langle \Omega_\lambda, \Omega_\mu, \Omega_{\nu^\vee} \rangle_d$ is a three-point, genus-zero Gromov-Witten invariant of $G_{k,n}$. It is a non-trivial fact that in this way an associative operation is defined (see [18]). Of course, specializing to $q = 0$ leads to the (classical) Littlewood-Richardson rule, i.e., $c^\nu_{\lambda\mu} = \langle \Omega_\lambda, \Omega_\mu, \Omega_{\nu^\vee} \rangle_0$. Finally, let us note that there is a purely combinatorial description of the numbers $\langle \Omega_\lambda, \Omega_\mu, \Omega_{\nu^\vee} \rangle_d$ (see [6]).

There are several algebraic descriptions of the ring $QH^*(G_{k,n})$ (see for example [2, 3, 5, 20, 23]). Let us first recall the description given in [2] and [3]. In these papers, inspired by the classical case, the authors obtained rules for calculating in the additive basis $\Sigma_{k,n}$ (which is a convenient choice of basis, having in mind the geometric implications given by (2.3)). In [2] Bertram obtained the following quantum version of Pieri's formula.

For a partition $\lambda = (l_1, \ldots, l_s)$, let $|\lambda| := l_1 + \cdots + l_s$ denote its *weight*, and $l(\lambda) := \max\{t : l_t \neq 0\}$ its *length*.

Quantum Pieri's formula. If $\lambda \subseteq k \times n$, $\lambda = (l_1, l_2, \ldots, l_k)$ and $p \leq n$, then

$$\sigma_p \cdot \sigma_\lambda = \sum \sigma_\mu + q \sum \sigma_\nu$$

where the first sum is over all partitions $\mu = (m_1, \ldots, m_k)$ such that $|\mu| = |\lambda| + p$ and $n \geq m_1 \geq l_1 \geq m_2 \geq l_2 \geq \ldots \geq m_k \geq l_k$, and the second sum is over all partitions $\nu = (n_1, \ldots, n_k)$ such that $|\nu| = |\lambda| + p - n - k$ and $l_1 - 1 \geq n_1 \geq l_2 - 1 \geq \ldots \geq l_k - 1 \geq n_k \geq 0$.

As in the classical case, this rule fully determines multiplication in the basis $\Sigma_{k,n}$. Nevertheless, in [3] Bertram, Ciocan-Fontanine and Fulton obtained a formula that may be understood as an improvement of the Pieri's formula. Here, we give it in its dual form (see also [5]). To state this result we need some definitions.

For a partition λ (not necessary contained in $k \times n$) let $\sigma_\lambda = \det(c_{l_i^* - i + j})$, where $\lambda^* = (l_1^*, l_2^*, \ldots)$ denotes the conjugate partition of λ. Note that, by Giambelli's formula, this definition is compatible with the previous (for $\lambda \subseteq k \times n$). A $(k+n)$-*rim hook* of a partition λ is a connected subset of $k+n$ boxes in its Young diagram that does not contain a 2×2 square. A $(k+n)$-rim hook of λ is *legal* if its removal gives a valid Young diagram.

We can now present the rim hook algorithm.

Rim hook algorithm. Let $\lambda = (l_1, l_2, \ldots, l_s)$ be a partition.

(a) If $l_{k+1} > 0$, or if $l_1 > n$ and λ contains no legal $(k+n)$-rim hook, then $\sigma_\lambda = 0$.
(b) If μ is the result of removing a legal $(k+n)$-rim hook from λ, then $\sigma_\lambda = (-1)^{k-r} q \sigma_\mu$, where r is the number of rows that removed $(k+n)$-rim hook occupied.

Note that similar formulas appear in many different contexts (see [10, 11, 22]).

The rim-hook algorithm allows us to easily reduce calculations in the basis $\Sigma_{k,n}$ to calculations in $H^*(G_{k,n}; \mathbb{Z})$ (see [3, 5]). In particular, it gives us a formula that represents numbers $\langle \Omega_\lambda, \Omega_\mu, \Omega_{\nu^\vee} \rangle_d$ as a sum of Littlewood-Richardson coefficients multiplied by ± 1.

Let us now turn our attention towards description of $QH^*(G_{k,n})$ by Siebert and Tian. In [20] they proved that

$$QH^*(G_{k,n}; \mathbb{Z}) \cong \mathbb{Z}[q][c_1, \ldots, c_k]/(\sigma_{n+1}, \ldots, \sigma_{n+k-1}, \sigma_{n+k} + (-1)^k q), \tag{2.4}$$

and that (polynomials) σ_i can be obtained in the following way. If we define that $\sigma_i = 0$ for $i < 0$, and $\sigma_0 = 1$, then for $i > 1$ one has

$$\sigma_i = -\sum_{j=1}^{k} (-1)^j c_j \sigma_{i-j}. \tag{2.5}$$

Although this completely determines $QH^*(G_{k,n})$, it is not clear how to perform concrete calculations in the cohomology. Since the classes of $QH^*(G_{k,n}; \mathbb{Z})$ in this description are naturally given in the basis $B_{k,n}$, we want to obtain a multiplication rule for the elements of this basis. This will be done by constructing a suitable Gröbner basis for the ideal $I_{k,n} := (\sigma_{n+1}, \ldots, \sigma_{n+k-1}, \sigma_{n+k} + (-1)^k q)$. Note that, in order to achieve this, we are not resorting to the results in [2] and [3], but only trying to "deform" Gröbner basis obtained in the classical case (see [17]) in accordance to (2.4). Interestingly enough, comparing with the classical case, multiplication rules for $B_{k,n}$ that we obtain are "deformed" similarly as is quantum Pieri's formula in comparison with the classical one.

For a k-tuple $\lambda = (l_1, l_2, \ldots, l_k)$ of nonnegative integers, let $C^\lambda = c_1^{l_1} c_2^{l_2} \cdots c_k^{l_k}$.

As for the classical case, the transition between the basis $B_{k,n}$ and $\Sigma_{k,n}$ involves the *Kostka numbers* $K_{\lambda\mu}$ (see [13]). Since C^λ can be represented in the basis $\Sigma_{k,n}$ for all k-tuples λ of nonnegative integers, in [3] the authors introduced the notion of *quantum Kostka numbers* $K^{k,n}_{\lambda\mu}$ in the following way. For a partition $\mu = (m_1, \ldots, m_r)$, such that $m_1 \leq k$, let

$$c_{m_1} c_{m_2} \cdots c_{m_r} = \sum K^{k,n}_{\lambda\mu} q^m \sigma_{\lambda^*} \tag{2.6}$$

where the sum is over all $\lambda \subseteq n \times k$ and $m \geq 0$, such that $|\lambda| + m(n+k) = |\mu|$.

Remark 1. In [3] quantum Kostka numbers are denoted in a different way. To avoid confusion, we note that in this paper $K^{k,n}_{\lambda\mu}$ stands for $K^{\lambda}_{\emptyset\mu}(n,k)$ from [3].

Remark 2. If $r \leq n$ formula (2.6) does not involve q, i.e., if $l(\mu) \leq n$ one has

$$K^{k,n}_{\lambda\mu} = \begin{cases} K_{\lambda\mu}, & |\lambda| = |\mu| \\ 0, & \text{otherwise} \end{cases}.$$

Using the rim hook algorithm, in [3] the authors obtained a formula that gives quantum Kostka numbers in terms of Kostka numbers. To state this result we need some additional notations. The *width* of a $(k+n)$-rim hook R, denoted by $width(R)$, is the number of columns it occupies. Furthermore, if $\mu \subseteq k \times n$ is obtained from a partition $\lambda = (l_1, l_2, \ldots)$ that satisfies $l_1 \leq n$, by successively removing $(k+n)$-rim hooks $R_1, R_2, \ldots, R_m$, let $\varepsilon(\lambda/\mu) := (-1)^{\sum(n - width(R_i))}$.

Now, for partitions $\lambda \subseteq n \times k$ and ν, such that $|\nu| = |\lambda| + m(k+n)$, one has

$$K^{k,n}_{\lambda\nu} = \sum \varepsilon(\rho/\lambda) K_{\rho\nu}, \tag{2.7}$$

where the sum is over all parititions $\rho = (r_1, r_2, \ldots)$, such that $r_1 \leq k$, obtained from λ by adding m times a $(k+n)$-rim hook. Additionally, this formula can be used to obtain a combinatorial description of quantum Kostka numbers (see [3]).

3. Notation

In this paper we denote by $\mathbb{N}_0$ the set of nonnegative integers and by $\mathbb{N}$ the set of positive integers.

Let $m \in \mathbb{N}$ and $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots, e_m = (0, \ldots, 0, 1)$ be the standard basis for $\mathbb{Z}^m$ and $e_0 = (0, \ldots, 0) \in \mathbb{Z}^m$. For an m-tuple λ of nonnegative integers we define the following m-tuples of integers obtained from λ (for $0 \leq i \leq m$ and $0 \leq i \leq j \leq m$):

- $\lambda^i = \lambda + e_i$ and $\lambda_i = \lambda - e_i$;
- $\lambda^{i,j} = \lambda + e_i + e_j$ and $\lambda_{i,j} = \lambda - e_i - e_j$.

Also, for $k \geq 2$, a k-tuple $\alpha = (a_1, \ldots, a_k)$ and a $(k-1)$-tuple $\mu = (m_2, \ldots, m_k)$ of integers, let:

- $|\alpha| := \sum_{j=1}^{k} a_j$, $\|\alpha\| := \sum_{j=1}^{k} j a_j$, and $|\mu| := \sum_{j=2}^{k} m_j$, $\|\mu\| := \sum_{j=2}^{k} (j-1) m_j$;
- $[\alpha,\mu]_t := \binom{\sum_{j=t-1}^{k} a_j - \sum_{j=t}^{k} m_j}{a_{t-1}}$, for $2 \le t \le k$;
- $[\alpha,\mu] := \prod_{t=2}^{k} [\alpha,\mu]_t$.

For example, $[\alpha,\mu]_2 = \binom{|\alpha| - |\mu|}{a_1}$, and if $\mathbf{0} = (\underbrace{0,\dots,0}_{k-1})$, then $[\alpha,\mathbf{0}]$ is the multinomial coefficient $[a_1,\dots,a_k] := \binom{a_1+a_2+\cdots+a_k}{a_1}\binom{a_2+\cdots+a_k}{a_2}\cdots\binom{a_{k-1}+a_k}{a_{k-1}}$.

Remark 3. The case $k = 1$ will be allowed as well. Then $\mu = \emptyset$, $|\mu| = \|\mu\| = 0$, and $[\alpha,\mu] = 1$ for any $\alpha = (a_1)$.

Remark 4. Note that the $(k-1)$-tuple $\mu = (m_2,\dots,m_k)$ is indexed by integers from 2 to k, not from 1 to $k-1$. The reason for this becomes clear in Proposition 4.1.

The following result is obtained in [17].

Lemma 3.1. *Let $\alpha = (a_1,\dots,a_k)$ be a k-tuple of nonnegative integers, and $\mu = (m_2,\dots,m_k)$ a $(k-1)$-tuple of nonnegative integers. Then*

$$[\alpha,\mu^{i,j}] = -[\alpha_i,\mu^j] + [\alpha_{j+1},\mu^{i-1}] + [\alpha,\mu^{i-1,j+1}], \text{ for } 1 \le i \le j \le k-1,$$

where it is understood that $[\alpha,\mu^{i-1,j+1}] = 0$ for $j = k-1$.

We proceed with some notation from the theory of Gröbner bases. Let R be a domain and $R[x_1,= \dots,x_k]$ the polynomial algebra over R in k variables. A *monomial* on variables $x_1,\dots,x_k$ is a product $x_1^{a_1}\cdots x_k^{a_k} \in R[x_1,x_2,\dots,x_k]$, where $a_1,\dots,a_k \in \mathbb{N}_0$. The set of all monomials in $R[x_1,\dots,x_k]$ will be denoted by M. A *term* in $R[x_1,\dots,x_k]$ is a product αm, where $\alpha \in R$ and $m \in M$.

Let $\preceq$ be a well ordering of M (a total ordering such that every nonempty subset of M has a least element) with the property that $m_1 \preceq m_2$ implies $mm_1 \preceq mm_2$, for all $m, m_1, m_2 \in M$.

For a polynomial $f = \sum_{i=1}^{r} \alpha_i m_i \in R[x_1,x_2,\dots,x_k]$, such that $\alpha_i \in R \setminus \{0\}$, let $\mathrm{M}(f) := \{m_i \,|\, 1 \le i \le r\}$. We define *leading monomial of* f, denoted by $\mathrm{LM}(f)$, to be $\max \mathrm{M}(f)$ with respect to $\preceq$. The *leading coefficient of* f, denoted by $\mathrm{LC}(f)$, is the coefficient of $\mathrm{LM}(f)$ and the *leading term of* f is $\mathrm{LT}(f){:=}\mathrm{LC}(f)\cdot\mathrm{LM}(f)$.

Strong Gröbner basis of an ideal I of $R[x_1,x_2,\dots,x_k]$ can be defined in many equivalent ways. In this paper, we will define it as it was done in [1].

Definition 3.1. Let $G \subset R[x_1,x_2,\dots,x_k]$ be a finite set of non-zero polynomials and $I_G = \langle G\rangle$ the ideal in $R[x_1,x_2,\dots,x_k]$ generated by G. We say that G is a *strong Gröbner basis* for I_G (with respect to $\preceq$) if for each $f \in I_G \setminus \{0\}$ there exists $g \in G$ such that $\mathrm{LT}(g) \mid \mathrm{LT}(f)$, i.e., $\mathrm{LT}(f) = t \cdot \mathrm{LT}(g)$, for some term t. We say that G is *minimal strong Gröbner basis* if $\mathrm{LT}(g') \nmid \mathrm{LT}(g'')$, for all distinct $g', g'' \in G$.

In the remainder of the paper, we use the grlex ordering $\preccurlyeq$ on the monomials in $R[x_1,\dots,x_k]$ with $x_1>\cdots>x_k$. It is defined by

$$x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}\prec x_1^{b_1}x_2^{b_2}\cdots x_k^{b_k}\ \text{if and only if one of the following holds:}$$

1) $a_1+a_2+\cdots+a_k<b_1+b_2+\ldots+b_k$ or
2) $a_1+a_2+\cdots+a_k=b_1+b_2+\ldots+b_k$ and $a_s<b_s$, where $s=\min\{i\,|\,a_i\neq b_i\}$.

As usual, $t_1\preceq t_2$ if and only if $t_1=t_2$ or $t_1\prec t_2$.

4. Gröbner basis for $I_{k,n}$

In this section integers $k,n\in\mathbb{N}$ are fixed.

Let us define certain polynomials in $\mathbb{Z}[q][c_1,c_2,\dots,c_k]$ which will be important in our considerations. For a $(k-1)$-tuple $\mu=(m_2,\dots,m_k)$ of nonnegative integers and $m\geq -k$, let

$$g_\mu^{(m)}:=\sum_{\|\alpha\|=m+1+\|\mu\|}(-1)^{m+1+|\alpha|}[\alpha,\mu]\;C^\alpha,$$

where the sum is taken over all k-tuples of nonnegative integers $\alpha=(a_1,a_2,\dots,a_k)$ such that $\|\alpha\|=m+1+\|\mu\|$.

Definition 4.1. For a $(k-1)$-tuple $\mu=(m_2,\dots,m_k)$ of nonnegative integers let

$$g_\mu:=g_\mu^{(n)}+(-1)^kqg_\mu^{(-k)}.$$

Moreover, let $G:=\{g_\mu\,:\,|\mu|\leq n+1\}$.

In Proposition 5 of [17] leading terms of polynomials $g_\mu^{(n)}$, for $|\mu|\leq n+1$, were determined. Slightly modifying this proof, one can obtain the following result on the leading terms of polynomials g_μ.

Proposition 4.1. *Let $\mu=(m_2,\dots,m_k)$ be a $(k-1)$-tuple of nonnegative integers such that $|\mu|\leq n+1$ (i.e., $g_\mu\in G$). Then $g_\mu\neq 0$ and $\mathrm{LT}(g_\mu)=C^{\overline{\mu}}$, where $\overline{\mu}=(n+1-|\mu|,m_2,\dots,m_k)$. Moreover, if $C^\alpha\in M(g_\mu)\setminus\{C^{\overline{\mu}}\}$, for a k–tuple α of nonnegative integers, then $|\alpha|<n+1$.*

In the following proposition we obtain a recurrence formula for the elements g_μ. The proof of this proposition is very similar to [17, Proposition 7], so we avoid some details.

Proposition 4.2. *Let $\mu=(m_2,\dots,m_k)$ be a $(k-1)$–tuple of nonnegative integers and $1\leq i\leq j\leq k-1$. Then, in polynomial algebra $\mathbb{Z}[q][c_1,c_2,\dots,c_k]$, the following identity holds*

$$g_{\mu^{i,j}}=c_ig_{\mu^j}-c_{j+1}g_{\mu^{i-1}}+g_{\mu^{i-1,j+1}},$$

where it is understood that $g_{\mu^{i-1,j+1}}$ is equal to zero for $j=k-1$.

Proof. It is enough to prove the following: if $\mu=(m_2,\dots,m_k)$ is a $(k-1)$–tuple of nonnegative integers, $1\leq i\leq j\leq k-1$ and $m\geq -k$, then the following identity holds

$$g^{(m)}_{\mu^{i,j}}=c_ig^{(m)}_{\mu^j}-c_{j+1}g^{(m)}_{\mu^{i-1}}+g^{(m)}_{\mu^{i-1,j+1}},$$

where it is understood that $g^{(m)}_{\mu^{i-1,j+1}}$ is equal to zero for $j=k-1$.

By Lemma 3.1, for $m \geq -k$ we have

$$\begin{aligned} g^{(m)}_{\mu^{i,j}} &= \sum_{\|\alpha\|=m+1+\|\mu^{i,j}\|} (-1)^{m+1+|\alpha|}[\alpha,\mu^{i,j}]C^\alpha \\ &= \sum_{\|\alpha\|=m+1+\|\mu^{i,j}\|} (-1)^{m+1+|\alpha|}\left(-[\alpha_i,\mu^j]+[\alpha_{j+1},\mu^{i-1}]+[\alpha,\mu^{i-1,j+1}]\right)C^\alpha \\ &= \sum_{\|\alpha\|=m+1+\|\mu^{i,j}\|} (-1)^{m+1+|\alpha_i|}[\alpha_i,\mu^j]C^\alpha - \sum_{\|\alpha\|=m+1+\|\mu^{i,j}\|} (-1)^{n+1+|\alpha_{j+1}|}[\alpha_{j+1},\mu^{i-1}]C^\alpha \\ &\qquad + \sum_{\|\alpha\|=m+1+\|\mu^{i,j}\|} (-1)^{m+1+|\alpha|}[\alpha,\mu^{i-1,j+1}]C^\alpha. \end{aligned}$$

First, since $\|\mu^{i,j}\| = \|\mu\| + i + j = \|\mu\| + i - 1 + j + 1 = \|\mu^{i-1,j+1}\|$ (for $j \leq k-2$), the last sum is equal to $g^{(m)}_{\mu^{i-1,j+1}}$. Also, note that $\|\alpha\| = m + 1 + \|\mu^{i,j}\|$ is equivalent to $\|\alpha_i\| = \|\alpha\| - i = m+1+\|\mu^{i,j}\| - i = m+1+\|\mu\|+j = m+1+\|\mu^j\|$, and similarly equivalent to $\|\alpha_{j+1}\| = m+1+\|\mu^{i-1}\|$. So, to finish the proof it is enough to justify the change of variable $\alpha_i \mapsto \alpha$ in the first sum, and $\alpha_{j+1} \mapsto \alpha$ in the second sum. For the first sum this can be done in the same way as in the proof of Proposition 7 of [17].

For the second sum, it suffices to show that $a_{j+1} = 0$ implies $[\alpha_{j+1},\mu^{i-1}] = 0$, where $\alpha = (a_1, a_2, \ldots, a_k)$. If $j+1 < k$, then $a_{j+1} = 0$ implies $[\alpha_{j+1},\mu^{i-1}]_{j+2} = 0$, and therefore $[\alpha_{j+1},\mu^{i-1}] = 0$. So, we are left with the case $j = k-1$. Let us assume to the contrary that $a_k = 0$ and $[\alpha_k,\mu^{i-1}] \neq 0$. In the proof of Proposition 7 of [17], it was proven that $a_k = 0$ and $[\alpha_k,\mu^{i-1}] \neq 0$ implies $\|\alpha\| \leq \|\mu\| + i - 1$. However,

$$\|\mu\| + i - 1 = \|\mu^{i-1}\| = \|\alpha_k\| - m - 1 \leq \|\alpha_k\| + k - 1 < \|\alpha\|,$$

which is clearly a contradiction. □

One of the consequences of this proposition is that for every $(k-1)$-tuple of nonnegative integers $\mu = (m_2, \ldots, m_k)$ one has $g_\mu \in \langle G \rangle$ (see [17, Corollary 8]).

Let us now return to the description (2.4). We define

$$\widetilde{\sigma}_m = \sigma_m + (-1)^k q \sigma_{m-n-k}, \text{ for } m \geq 0, \tag{4.1}$$

with the reminder that $\sigma_j = 0$ for $j < 0$, and $\sigma_0 = 1$. Note that $\widetilde{\sigma}_{n+i} = \sigma_{n+i}$, for $1 \leq i \leq k-1$, and $\widetilde{\sigma}_{n+k} = \sigma_{n+k} + (-1)^k q$. Therefore, $\widetilde{\sigma}_{n+i} \in I_{k,n}$, for $1 \leq i \leq k$. In the following proposition we prove that $\widetilde{\sigma}_{n+i} \in I_{k,n}$, for all $i \geq 1$.

Proposition 4.3. *For $m \geq k+n+1$*

$$\widetilde{\sigma}_m = -\sum_{i=1}^{k} (-1)^i c_i \widetilde{\sigma}_{m-i}.$$

Proof. Using the formula (2.5) and the definition (4.1) we have

$$\widetilde{\sigma}_m = \sigma_m + (-1)^k q\sigma_{m-n-k} = -\sum_{i=1}^{k}(-1)^i c_i \sigma_{m-i} - (-1)^k q \sum_{i=1}^{k}(-1)^i c_i \sigma_{m-n-k-i}$$
$$= -\sum_{i=1}^{k}(-1)^i c_i \left(\sigma_{m-i} + (-1)^k q\sigma_{m-i-n-k}\right) = -\sum_{i=1}^{k}(-1)^i c_i \widetilde{\sigma}_{m-i},$$

which completes the proof. □

In the following lemma we give an identity which expresses polynomials $g_{(i,0,\dots,0)}$ in terms of polynomials $\widetilde{\sigma}_{n+1+i}$.

Lemma 4.1. *For $s \geq 0$, let $\mathbf{s}$ denote the $(k-1)$-tuple $(s, 0, \dots, 0)$. Then the following identity holds*

$$g_{\mathbf{s}} = \sum_{i=0}^{s} \binom{s}{i} (-1)^i c_1^{s-i} \widetilde{\sigma}_{n+1+i}.$$

Proof. By the definition of $\widetilde{\sigma}_{n+1+i}$, it is enough to prove that for $m \geq -k$ the following identity holds

$$g_{\mathbf{s}}^{(m)} = \sum_{i=0}^{s} \binom{s}{i} (-1)^i c_1^{s-i} \sigma_{m+1+i}.$$

This identity can be proven by induction on $s \geq 0$ similarly as in Lemma 9 of [17] – it is enough to note that $g_{\mathbf{0}}^{(m)} = \sigma_{m+1}$ and $g_{\mathbf{s}}^{(m)} = c_1 g_{\mathbf{s-1}}^{(m)} - g_{\mathbf{s-1}}^{(m+1)}$, holds for all $m \geq -k$ and $s \geq 1$. □

As in [17], using Proposition 4.2, Lemma 4.1 and Lemma 4.3, instead of the corresponding results, we obtain the main theorem of this section.

Theorem 4.4. *The set G is a minimal strong Gröbner basis, with respect to the grlex ordering $\preccurlyeq$ defined by $c_1 > \cdots > c_k$, for the ideal $I_{k,n}$ in $\mathbb{Z}[q][c_1, \dots, c_k]$.*

Let $\lambda = (l_1, \dots, l_k)$ be a k-tuple of nonnegative integers such that $|\lambda| = n + 1$. Then, for $\mu = (l_2, \dots, l_k)$ we have $g_\mu = 0$ in $QH^*(G_{k,n}; \mathbb{Z})$, and therefore

$$C^\lambda = \sum_{\substack{\|\alpha\|=n+1+\|\mu\| \\ \alpha\neq\lambda}} (-1)^{n+|\alpha|}[\alpha, \mu]C^\alpha + q \sum_{\|\alpha\|=-k+1+\|\mu\|} (-1)^{|\alpha|}[\alpha, \mu]C^\alpha. \tag{4.2}$$

By Proposition 4.1, this identity is in fact the presentation of C^λ in the additive basis $B_{k,n}$ (see Section 1). Also, note that these formulas completely determine the multiplication in $QH^*(G_{k,n})$. Therefore, (4.2) can be understood as a Pieri-type formula for the elements of the additive basis $B_{k,n}$.

Example 1. In $QH^*(G_{2,4})$ we have the following identities (which completely determine multiplication):

$$c_1^5 = 4c_1^3c_2 - 3c_1c_2^2 \qquad c_1^4c_2 = 3c_1^2c_2^2 - c_2^3 - q$$
$$c_1^3c_2^2 = 2c_1c_2^3 + qc_1 \qquad c_1^2c_2^3 = c_2^4 + qc_1^2 - qc_2$$
$$c_1c_2^4 = qc_1^3 - 2qc_1c_2 \qquad c_2^5 = qc_1^4 - 3qc_1^2c_2 + qc_2^2.$$

5. Recurrence formulas for Kostka numbers

In this section we obtain a recurrence formulas for quantum Kostka numbers using results from the previous section. Additionally, we use these formulas to obtain new recurrence formulas for (classical) Kostka numbers.

If $\alpha = (a_1, a_2, \ldots)$ and $\beta = (b_1, b_2, \ldots)$ are partitions, then we write $\alpha \geq \beta$ if and only if $a_1 + a_2 + \ldots + a_i \geq b_1 + b_2 + \ldots + b_i$, for all $i \in \mathbb{N}$. It is a well-known fact that $K_{\alpha\,\beta} \neq 0$ if and only if $\alpha \geq \beta$.

For a k-tuple $\alpha = (a_1, \ldots, a_k)$ of nonnegative integers, let $\alpha_\rightarrow$ denote the partition which has exactly a_i components equal to i, for $1 \leq i \leq k$ (for example, if $\alpha = (3, 2, 0, 3)$, then $\alpha_\rightarrow = (4, 4, 4, 2, 2, 1, 1, 1)$). Note that $|\alpha| = l(\alpha_\rightarrow)$ and $\|\alpha\| = |\alpha_\rightarrow|$.

Using this notation, identity (2.6) can be written in the following way: if α is a k-tuple of nonnegative integers, then

$$C^\alpha = \sum_{\substack{m \geq 0 \\ |\lambda| = \|\alpha\| - m(n+k)}} q^m K^{k,n}_{\lambda\,\alpha_\rightarrow}\, \sigma_{\lambda^*}.$$

Plugging these in the expression for g_μ, where μ is a fixed $(k-1)$-tuple of nonnegative integers, and using the fact that for every k-tuple of nonnegative integers β one has $C^\beta \cdot g_\mu = 0$ (in $QH^*(G_{k,n})$), we get

$$\begin{aligned}
0 &= \sum_{\|\alpha\| = n+1+\|\mu\|} (-1)^{n+1+|\alpha|}[\alpha, \mu] C^{\alpha+\beta} + q \sum_{\|\alpha\| = -k+1+\|\mu\|} (-1)^{1+|\alpha|}[\alpha, \mu] C^{\alpha+\beta} \\
&= \sum_{\substack{m \geq 0 \\ \|\alpha\| = n+1+\|\mu\|}} \sum_{|\lambda| = \|\mu\| + \|\beta\| + n+1-m(n+k)} (-1)^{n+1+|\alpha|}[\alpha, \mu] q^m K^{k,n}_{\lambda\,(\alpha+\beta)_\rightarrow}\, \sigma_{\lambda^*} \\
&\quad + q \sum_{\substack{m \geq 0 \\ \|\alpha\| = -k+1+\|\mu\|}} \sum_{|\lambda| = \|\mu\| + \|\beta\| - k+1-m(n+k)} (-1)^{1+|\alpha|}[\alpha, \mu] q^m K^{k,n}_{\lambda\,(\alpha+\beta)_\rightarrow}\, \sigma_{\lambda^*} \\
&= (-1)^{n+1} \sum_{|\lambda| = \|\mu\| + \|\beta\| + n+1} \left(\sum_{\|\alpha\| = n+1+\|\mu\|} (-1)^{|\alpha|}[\alpha, \mu] K^{k,n}_{\lambda\,(\alpha+\beta)_\rightarrow} \right) \sigma_{\lambda^*} \\
&\quad + (-1)^{n+1} \sum_{\substack{m \geq 1 \\ |\lambda| = \|\mu\| + \|\beta\| + n+1-m(n+k)}} q^m \left(\sum_{\|\alpha\| = n+1+\|\mu\|} (-1)^{|\alpha|}[\alpha, \mu] K^{k,n}_{\lambda\,(\alpha+\beta)_\rightarrow} \right) \sigma_{\lambda^*} \\
&\quad + (-1)^{n+1} \sum_{\substack{m \geq 1 \\ |\lambda| = \|\mu\| + \|\beta\| + n+1-m(n+k)}} q^m \left(\sum_{\|\alpha\| = -k+1+\|\mu\|} (-1)^{n+|\alpha|}[\alpha, \mu] K^{k,n}_{\lambda\,(\alpha+\beta)_\rightarrow} \right) \sigma_{\lambda^*}
\end{aligned}$$

(in these sums α's denote k-tuple of nonnegative integers, and $\lambda \subseteq n \times k$ partitions).

Therefore, in view of the additive basis $\Sigma_{k,n}$, for every partition λ such that $l_1 \leq k$, $l(\lambda) \leq n$ and $|\lambda| = \|\mu\| + \|\beta\| + n + 1 - m(k+n)$, for some $m \geq 0$, we have

$$m = 0: \quad 0 = \sum_{\|\alpha\|=n+1+\|\mu\|} (-1)^{|\alpha|}[\alpha,\mu] K^{k,n}_{\lambda\,(\alpha+\beta)_{\to}}, \tag{5.1}$$

$$m \geq 1: \quad 0 = \sum_{\|\alpha\|=n+1+\|\mu\|} (-1)^{|\alpha|}[\alpha,\mu] K^{k,n}_{\lambda\,(\alpha+\beta)_{\to}} + \sum_{\|\alpha\|=-k+1+\|\mu\|} (-1)^{n+|\alpha|}[\alpha,\mu] K^{k,n}_{\lambda\,(\alpha+\beta)_{\to}}. \tag{5.2}$$

In what follows we will use these identities to obtain recurrence formulas for quantum Kostka numbers, without resorting to the rim-hook algorithm.

Note that for $m = 0$ one has $K^{k,n}_{\lambda\mu} = K_{\lambda\mu}$, and therefore the identity (5.1) is nothing else than the identity (21) from [17]. Additionally, by Remark 2, if $l(\mu) \leq n$, then $K^{k,n}_{\lambda\mu} = 0$ for any λ such that $|\lambda| \neq |\mu|$. Therefore, we may reduce the problem of computing quantum Kostka numbers $K^{k,n}_{\lambda\mu}$ to the case when $|\mu| \neq |\lambda|$ and $l(\mu) > n$.

So, we prove that the identity (5.2) can be used to obtain (all) quantum Kostka numbers $K^{k,n}_{\lambda\,\nu}$, such that $|\nu| \neq |\lambda|$ and $l(\nu) > n$, from (classical) Kostka numbers. Let $K^{k,n}_{\lambda\,\nu}$ be a quantum Kostka number that satisfies this condition. Then, there exists the unique k-tuple γ of nonnegative integers, such that $\gamma_{\to} = \nu$. Note that γ satisfies $|\gamma| > n$, and therefore there exist k-tuples τ and β of nonnegative integers, such that $\tau + \beta = \gamma$ and $|\tau| = n + 1$. Furthermore, if $\tau = (t_1, t_2, \ldots, t_k)$, let $\mu := (t_2, \ldots, t_k)$. Let us observe the identity 5.2 for these λ and μ. The summand for $\alpha = \tau$ in the first sum is $(-1)^{|\tau|}[\tau,\mu]K^{k,n}_{\lambda,\nu} = (-1)^{n+1}K^{k,n}_{\lambda,\nu}$, and therefore this identity can be rewritten as

$$K^{k,n}_{\lambda\,\nu} = \sum_{\substack{\|\alpha\|=n+1+\|\mu\| \\ \alpha\neq\tau}} (-1)^{n+|\alpha|}[\alpha,\mu] K^{k,n}_{\lambda\,(\alpha+\beta)_{\to}} + \sum_{\|\alpha\|=-k+1+\|\mu\|} (-1)^{|\alpha|}[\alpha,\mu] K^{k,n}_{\lambda\,(\alpha+\beta)_{\to}}. \tag{5.3}$$

Furthermore, as in Proposition 4.1, we conclude that for any α that appears in one of the sums on the right-hand side of the previous identity one has $|\alpha| \leq n$. Therefore, by the identity (5.3), $K^{k,n}_{\lambda\,\nu}$ is a linear combination of elements $K^{k,n}_{\lambda\,\nu'}$, such that $l(\nu') < l(\nu)$, which proves the claim stated at the beginning of this paragraph.

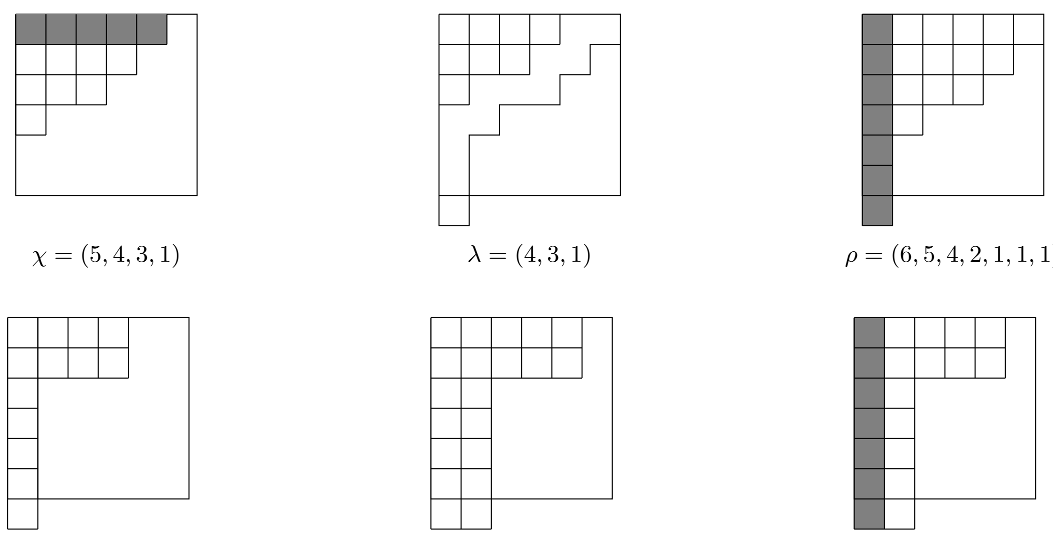
Let us compare identities (2.7) and (5.3). Note that in the identity (2.7) for $K^{k,n}_{\lambda\,\nu}$ all summands have the same ν, while in the identity (5.3) all summands have the same λ. This "duality" (used for $m = 1$ and suitable k and n) will allow us to obtain new recurrence formulas for Kostka numbers.

Let $K_{\chi\,\eta}$ be Kostka number we want to calculate (χ and η are partitions such that $|\chi| = |\eta|$). Since $K_{\chi\,\eta} = 0$ if $l(\chi) > l(\eta)$, we may assume that $l(\chi) \leq l(\eta)$. Let $\chi = (h_1, h_2, \ldots, h_{s+1})$, $\eta = (e_1, e_2, \ldots, e_l)$, $k := h_1 + 1$ and $n := l(\eta) - 1$ (see Figure 1). Furthermore, let λ be the partition obtained from χ by removing its first row, i.e., $\lambda = (l_1, l_2, \ldots, l_s)$, where $l_i := h_{i+1}$, for $1 \leq i \leq s$, and $\nu := (e_1 + 1, e_2 + 1, \ldots, e_{n+1} + 1)$ $(l(\eta) = n + 1)$. Then $|\lambda| = |\nu| - (n + k)$, $l_1 < k$, and $l(\lambda) \leq n$, so $K^{k,n}_{\lambda\,\nu}$ is well-defined and we can apply the formula (2.7). By this formula $K^{k,n}_{\lambda\,\nu}$ is a sum of numbers $\pm K_{\tau\,\nu}$, where $\tau = (t_1, t_2, \ldots)$ is a partition obtained from

λ by adding to it a $(k+n)$–rim hook. Note that if $t_1 < k$, then $l(\tau) > n+1 = l(\nu)$, and so $K_{\tau\,\nu} = 0$. Therefore, $K^{k,n}_{\lambda\,\nu} = K_{\rho\,\nu}$, where $\rho = (r_1, r_2, \ldots, r_{n+1})$ is the unique partition that satisfies $r_1 = k$ and is obtained from λ by adding a $(k+n)$-rim hook (alternatively, this equality can be obtained using combinatorial description of quantum Kostka numbers). Additionally, since $l(\rho) = l(\nu)$, from [17, Theorem 12(iii)] we have $K_{\chi\,\eta} = K^{k,n}_{\lambda\,\nu}$.

Let us observe the identity (5.3) for $K^{k,n}_{\lambda\,\nu}$. The first sum of this identity is a linear combination of the elements $K^{k,n}_{\lambda\,\nu'}$, where $l(\nu') < l(\nu) = n+1$ and $|\nu'| = |\nu| > |\lambda|$, which are, by Remark 2, all equal to zero. Since all quantum Kosta numbers that appear in the second sum are in fact (classical) Kostka numbers, and $\beta = (0, \ldots, 0)$, we have obtained the previously announced identity

$$K_{\chi\,\eta} = \sum_{\|\alpha\| = \|\mu\| - k + 1} (-1)^{|\alpha|} [\alpha, \mu] K_{\lambda\,\alpha_{\rightarrow}}. \tag{5.4}$$



$\chi = (5,4,3,1)$ $\lambda = (4,3,1)$ $\rho = (6,5,4,2,1,1,1)$

$\eta = (4,4,1,1,1,1,1)$ $\nu = (5,5,2,2,2,2,2)$ $\nu = (5,5,2,2,2,2,2)$

Figure 1. Auxiliary partitions when calculating $K_{\chi\,\eta}$, for $\chi = (5,4,3,1)$ and $\eta = (4,4,1,1,1,1,1)$.

So, let us introduce the following notation. For a partition χ, let $\underline{\chi}$ be the partition obtained from χ by removing its first row. For a partition η let $\eta^{+}_{\leftarrow}$ denote the partition obtained from η by the following sequence of operations. Add a column of height $l(\eta)$ to η to obtain η' (this column is now the first column of η'). If $\theta = (t_1, t_2, \ldots, t_k)$ is the unique k-tuple of nonnegative integers such that $\theta_{\rightarrow} = \eta'$, then $\eta^{+}_{\leftarrow} := (t_2, \ldots, t_k)$. Since all (nonzero) entries of η' are greater than 1, we have $t_1 = 0$, and therefore $|\eta^{+}_{\leftarrow}| = |\theta| = l(\eta') = l(\eta)$. Also, note that

$$\|\eta^{+}_{\leftarrow}\| = \|\theta\| - |\theta| = |\theta_{\rightarrow}| - l(\theta_{\rightarrow}) = |\eta'| - l(\eta') = |\eta| = |\chi|.$$

Theorem 5.1. *For partitions $\chi = (h_1, h_2, \ldots)$ and η such that $|\chi| = |\eta|$ and $\chi \geq \eta$, the following identity holds*

$$K_{\chi\,\eta} = \sum_{\|\alpha\|=|\chi|-h_1} (-1)^{|\alpha|}[\alpha, \eta^+_{\leftarrow}] K_{\underline{\chi}\,\alpha_{\rightarrow}},$$

where the sum is taken over all (h_1+1)–tuples α of nonnegative integers that satisfies $\|\alpha\| = |\chi| - h_1$. □

We illustrate this result with the following two examples.

Example 2. Note that by [16, Theorem 1] computing $K_{\chi\,\eta}$ is hard even if $l(\chi) = 2$. Let us apply identity (5.4) in this case, i.e., for $\chi = (h_1, h_2)$. Since each $K_{\underline{\chi}\,\alpha_{\rightarrow}}$ that appears in the sum is equal to 1, we have

$$K_{\chi\,\eta} = \sum_{\|\alpha\|=h_2} (-1)^{|\alpha|}[\alpha, \eta^+_{\leftarrow}],$$

where the sum is taken over all (h_1+1)–tuples α of nonnegative integers.

Example 3. Let $\chi = (a+1, 1^b)$, i.e., χ is a *tableaux of hook shape.* Since $\underline{\chi} = (1^b)$, we have $K_{\underline{\chi}\,\alpha_{\rightarrow}} \neq 0$ only if $\alpha_{\rightarrow} = (1^b)$, i.e., only if $\alpha = (b)$. Therefore

$$K_{\chi\,\eta} = (-1)^b[(b), \eta^+_{\leftarrow}] = (-1)^b \binom{b - |\eta^+_{\leftarrow}|}{b} = (-1)^b \binom{b - l(\eta)}{b} = \binom{l(\eta) - 1}{b}.$$

Remark 5. The set $\Sigma'_{k,n} = \{\sigma_\lambda \,:\, l(\lambda) \leq k,\ l_1 - l_k \leq n\}$ is an additive basis (over $\mathbb{Z}$) for $QH^*(G_{k,n})$ (see [5, 10]). Additionally, the representation of elements C^μ in this basis is given by numbers $K^{cyc}_{\lambda\,\mu}$ ($K^{cyc}_{\lambda\,\mu}$ is the number of cylindric tableaux of type λ and content μ), which appear in many different contexts (see for example [9, 15]). Similarly as we did in this section, one can also obtain recurrence formulas for these numbers.

6. Recurrence formulas for inverse Kostka numbers

In this section we obtain recurrence formulas for inverse Kostka numbers. As an application of these formulas, we obtain formula for the Steenrod mod 3 operation in terms of Schubert classes. We keep the notation from the previous section.

From the definition of inverse Kostka numbers, for a partition μ, such that $l(\mu^*) \leq k$, in $QH^*(G_{k,n})$ the following holds

$$\sigma_{\mu^*} = \sum_{\|\alpha\|=|\mu|} K^{-1}_{\alpha_{\rightarrow}\,\mu} C^\alpha.$$

As in the previous section, we will use this formula for suitable μ^* and combine it with rim hook algorithm and formula (4.2).

For a partition $\chi = (h_1, h_2, \ldots)$, let $\chi_{\leftarrow}$ denote the unique h_1-tuple such that $(\chi_{\leftarrow})_{\rightarrow} = \chi$. Note that $l(\chi) = |\chi_{\leftarrow}|$.

Let $\nu^* = (n_1^*, n_2^*, \ldots)$ be a partition such that $l(\nu^*) \leq k-1$ and $n_1^* = n$. Additionally, let $\mu^* = (m_1^*, m_2^*, \ldots)$ be the partition obtained from ν^* by adding $(k+n)$-rim hook and such that $l(\mu^*) = k$. Note that $m_1^* = n+1$. By the rim hook algorithm we have

$$\sigma_{\mu^*} = q\sigma_{\nu^*},$$

and therefore

$$\sum_{\|\alpha\|=|\mu|} K^{-1}_{\alpha_\to\,\mu} C^\alpha = q \sum_{\|\beta\|=|\nu|} K^{-1}_{\beta_\to\,\nu} C^\beta.$$

Note that $l(\mu) = n+1$ and $l(\nu) = n$, and therefore all nonzero summands in the first sum satisfy $l(\alpha_\to) = |\alpha| \le n+1$, and $l(\beta_\to) = |\beta| \le n$ in the second sum. So, all summands in the previous identity, except the ones with $|\alpha| = n+1$, are in the basis $B_{k,n}$. Therefore, after expanding monomials C^α with $|\alpha| = n+1$ using formula (4.2), i.e., in basis $B_{k,n}$, and collecting monomials that contain q, we obtain

$$q \sum_{|\alpha|=n+1} K^{-1}_{\alpha_\to\,\mu} \sum_{\|\beta\|=-k+1+\|\underline{\alpha}\|} (-1)^{|\beta|}[\beta,\underline{\alpha}]C^\beta = q \sum_{\|\beta\|=|\nu|} K^{-1}_{\beta_\to\,\nu} C^\beta.$$

Finally, looking at the basis $B_{k,n}$, we have

$$K^{-1}_{\beta_\to\,\nu} = (-1)^{|\beta|} \sum_{\substack{|\alpha|=n+1\\ \|\underline{\alpha}\|=\|\beta\|+k-1}} [\beta,\underline{\alpha}] K^{-1}_{\alpha_\to\,\mu}. \tag{6.1}$$

If in this identity we denote $\beta_\to = (\widetilde{b}_1, \widetilde{b}_2, \ldots)$, then $\widetilde{b}_1 \le k$. So, if $\widetilde{b}_1 = k$, then identity (6.1) can be written in the following way (that does not depend on k and n).

Theorem 6.1. *Let $\chi = (h_1, h_2, \ldots)$ and $\nu = (n_1, n_2, \ldots)$ be partitions such that $|\chi| = |\nu|$ and $\chi \ge \nu$. If $\mu = (m_1, m_2, \ldots)$ is the partition obtained from ν by adding $(l(\nu)+h_1)$-rim hook and such that $m_1 = h_1$, then the following identity holds*

$$K^{-1}_{\chi\,\nu} = (-1)^{l(\chi)} \sum_\alpha [\chi_\leftarrow, \underline{\alpha}] K^{-1}_{\alpha_\to\,\mu},$$

where the sum is taken over all h_1-tuples α of nonnegative integers such that $|\alpha| = l(\nu)+1$ and $\|\underline{\alpha}\| = |\chi| + h_1 - 1$.

First, we will show that Theorem 6.1 can be used to obtain all inverse Kostka numbers. For this we will need the following result (named *Cancellation principle* in [7]), which easily follows from the definition of inverse Kostka numbers.

Proposition 6.2. *For partitions $\lambda = (l_1, l_2, \ldots, l_n)$ and $\mu = (m_1, m_2, \ldots, m_n)$ one has:*

(1) *if $l_i = m_i$, for $1 \le i \le k$, then*

$$K^{-1}_{\lambda\,\mu} = K^{-1}_{(l_{k+1}, l_{k+2}, \ldots, l_n)\,(m_{k+1}, m_{k+2}, \ldots, m_n)};$$

(2) *if $\lambda \not\ge \mu$, then $K^{-1}_{\lambda\,\mu} = 0$.*

By this proposition, it is enough to provide an algorithm for calculating $K^{-1}_{\chi\,\nu}$, for partitions $\chi = (h_1, h_2, \ldots)$ and $\nu = (n_1, n_2, \ldots)$ such that $h_1 > n_1$ and $l(\chi) \le l(\nu)$. So, let us consider the identity from Theorem 6.1 for partitions χ and ν that satisfy these conditions. We will show that this identity implies a representation of $K^{-1}_{\chi\,\nu}$ as a linear combination of numbers $K^{-1}_{\chi'\,\nu'}$, where $\chi' = (h'_1, h'_2, \ldots)$ and $\nu' = (n'_1, n'_2, \ldots)$ are partitions such that $l(\nu') < l(\nu)$, or $l(\nu') = l(\nu)$ and $h_1 - n_1 > h'_1 - n'_1 \ge 1$, which obviously gives a desired algorithm.

Let us consider $K^{-1}_{\alpha_\to\,\mu}$ from the right hand side of the identity from Theorem 6.1. If $\alpha_\to = (\widetilde{a}_1, \widetilde{a}_2, \ldots)$, then we have that $h_1 \ge \widetilde{a}_1 \ge m_1$, i.e., $\widetilde{a}_1 = h_1$, and $l(\alpha_\to) = l(\mu) = l(\nu)+1$.

So, if $i := \max\{j : \widetilde{a}_j = m_j\}$, then for $\alpha' = (\widetilde{a}_{i+1}, \widetilde{a}_{i+2}, \ldots)$ and $\mu' = (m_{i+1}, m_{i+2}, \ldots)$ by Cancellation principle we have

$$K^{-1}_{\alpha_{\rightarrow}\,\mu} = K^{-1}_{\alpha'\,\mu'}.$$

So, it is enough to consider summands that satisfy $i = 1$. Then, one has $m_2 = n_1 + 1$, i.e., $1 \leq \widetilde{a}_2 - m_2 \leq \widetilde{a}_1 - (n_1 + 1) < h_1 - n_1$, which completes our proof.

6.1. **Application to mod p Steenrod operation.** For positive integers $a_1 > a_2 > \ldots > a_m$ and $n_1, n_2, \ldots, n_m$, let $(a_1^{n_1}, a_2^{n_2}, \ldots, a_m^{n_m})$ denote the partition

$$(\underbrace{a_1, \ldots, a_1}_{n_1}, \underbrace{a_2, \ldots, a_2}_{n_2}, \ldots, \underbrace{a_m, \ldots, a_m}_{n_m}).$$

For $i \geq 0$, let $Sq^i\colon H^k(-;\mathbb{Z}_2) \longrightarrow H^{k+i}(-;\mathbb{Z}_2)$ be the Steenrod squares and, for $p \geq 3$, $\mathcal{P}^i\colon H^k(-;\mathbb{Z}_p) \longrightarrow H^{k+2i(p-1)}(-;\mathbb{Z}_p)$ be the Steenrod power operations. Using the fact that $H^*(BU(n);\mathbb{Z}_p) \cong (\mathbb{Z}_p[x_1, \ldots, x_n])^{\mathbb{S}_n}$ where Chern classes correspond to elementary symmetric functions under this isomorphism, and properties of power operations (see [21]), one arrives at the formula (see [4])

$$\mathcal{P}^i c_{k+i} = m_{(p^i, 1^k)},$$

where $m_{(p^i,1^k)}$ is the monomial symmetric function. For more information concerning research devoted to obtain explicit formulas expressing $\mathcal{P}^i c_{k+i}$ in terms of Chern classes, the reader may consult [12, 19] and the references therein.

Since these functions are related to Schur functions via Kostka numbers, we have the following equality:

$$\mathcal{P}^i c_m = \sum_{|\mu| = m + (p-1)i} K^{-1}_{(p^i, 1^{m-i}), \mu} s_\mu. \tag{6.2}$$

Corresponding formula in the cohomology of $BO(n)$ holds for application of Steenrod squares to Stiefel-Whitney classes. By formula (6.2), to express Steenrod power operations in terms of Schubert classes it is enough to determine numbers $K^{-1}_{(p^b,1^a),\,\mu}$, for $a, b \in \mathbb{N}_0$ and partition μ that satisfy $|\mu| = a + pb$.

In what follows we calculate these numbers for $p = 2$ and $p = 3$. Note that, by Cancelation principle, if $\mu = (m_1, m_2, \ldots)$ satisfies $m_1 > p$, then $K^{-1}_{(p^b,1^a),\,\mu} = 0$, so we only need to consider the case $m_1 \leq p$.

Case $p = 2$. In this case we want to find $K^{-1}_{(2^b,1^a),\,(2^d,1^c)}$, where $a, b, c, d \in \mathbb{N}_0$ are such that $a + 2b = c + 2d$.

Let $\chi = (2^b, 1^a)$ and $\nu = (2^d, 1^c)$. By Cancellation principle, if $d > b$, then $K^{-1}_{\chi\,\nu} = 0$, and if $d \leq b$ we have

$$K^{-1}_{\chi\,\nu} = K^{-1}_{(2^{b-d},1^a),\,(1^c)}.$$

Now, by Theorem 6.1, we have

$$K^{-1}_{(2^{b-d},1^a),\,(1^c)} = (-1)^{a+b-d} \sum_{\alpha} [(a, b-d), \underline{\alpha}] K^{-1}_{\alpha_{\rightarrow}\,(2^{c+1})},$$

where the sum is over all pairs α of nonnegative integers such that $|\alpha| = c+1$ and $\|\underline{\alpha}\| = c+1$. Obviously, the only α that satisfies these conditions is $\alpha = (0, c+1)$. Since $c = 2(b-d) + a$, we have

$$[(a, b-d), \underline{\alpha}] = \binom{a+b-d-c-1}{a} = \binom{d-b-1}{a} = (-1)^a \binom{a+b-d}{a},$$

and therefore the following result (cf. [7, 8]).

Proposition 6.3. *For nonnegative integers a, b, c, d such that $a + 2b = c + 2d$, one has*

$$K^{-1}_{(2^b,1^a),\,(2^d,1^c)} = \begin{cases} 0, & \text{if } d > b \\ (-1)^{b-d}\binom{a+b-d}{a}, & \text{otherwise} \end{cases}.$$

As mentioned in [7], this result easily implies Wu formula (see [24]).

Case $p = 3$**.** In this case we want to find $K^{-1}_{(3^b,1^a)\,(3^e,2^d,1^c)}$, where $a, b, c, d, e \in \mathbb{N}_0$ are such that $a + 3b = c + 2d + 3e$.

Let $\chi = (3^b, 1^a)$ and $\nu = (3^e, 2^d, 1^c)$. Again, by Cancellation principle, if $e > b$, then $K^{-1}_{\chi\,\nu} = 0$, and if $b \leq e$ we have

$$K^{-1}_{\chi\,\nu} = K^{-1}_{(3^t,1^a),\,(2^d,1^c)},$$

where $t = b - e$.

Let us first consider the case $c = 0$. By Theorem 6.1, we have

$$K^{-1}_{(3^t,1^a),\,(2^d)} = (-1)^{a+t} \sum_{\alpha} [(a, 0, t), \underline{\alpha}] K^{-1}_{\alpha_{\to}\,(3^{d+1})}, \tag{6.3}$$

where the sum is over triples α of nonnegative integers such that $|\alpha| = d + 1$ and $\|\underline{\alpha}\| = 3t + a + 2 = 2d + 2$. Obviously, the only α that satisfies these conditions is $\alpha = (0, 0, d + 1)$, and therefore

$$K^{-1}_{(3^t,1^a),\,(2^d)} = (-1)^{a+t}[(a, 0, t), (0, d+1)] = (-1)^{a+t}\binom{a+t-d-1}{a}.$$

Note that this formula gives a closed form expression for polynomials $h_b(t)$ considered in Corollary 6 and Example 2 of [7].

Finally, we consider the general case, i.e., $c \geq 0$. By Theorem 6.1 we have

$$K^{-1}_{(3^t,1^a),\,(2^d,1^c)} = (-1)^{a+t} \sum_{\substack{u+v+w=c+d+1 \\ v+2w=c+2d+2}} [(a, 0, t), (v, w)] K^{-1}_{(3^w,2^v,1^u),\,(3^{d+1},2^c)},$$

where $u, v, w \geq 0$. Note that by Cancellation principle, all nonzero summands in the last sum satisfy $w \geq d+1$. So, by identity (6.3) and since $a+t = c+2d-2t$, we have

$$
\begin{aligned}
&K^{-1}_{(3^t,1^a),\,(2^d,1^c)} \\
&= (-1)^{a+t} \sum_{w=d+1}^{d+1+\lfloor c/2\rfloor} [(a,0,t),(c+2d+2-2w,w)] K^{-1}_{(3^{w-d-1},2^{c+2d+2-2w},1^{w-d-1}),\,(2^c)} \\
&= (-1)^{a+t+c} \sum_{w=d+1}^{d+1+\lfloor c/2\rfloor} \binom{w-2t-2}{a}\binom{-1}{w-d-1}\binom{d-w}{c+2(d+1-w)} \\
&= (-1)^{a+t+c} \sum_{s=0}^{\lfloor c/2\rfloor} (-1)^s \binom{s+d-2t-1}{a}\binom{-s-1}{c-2s}.
\end{aligned}
$$

Proposition 6.4. *For nonnegative integers a, b, c, d, e that satisfy $a+3b = c+2d+3e$, one has:*

(1) *if $b < e$, then $K^{-1}_{(3^b,1^a),\,(3^e,2^d,1^c)} = 0$;*

(2) *if $b \geq e$, then*

$$
K^{-1}_{(3^b,1^a),\,(3^e,2^d,1^c)} = (-1)^{a+b+e} \sum_{s=0}^{\lfloor c/2\rfloor} (-1)^s \binom{s+d+2e-2b-1}{a}\binom{c-s}{c-2s}.
$$

This proposition yields formula for the Steenrod mod 3 operation in the basis given by Schubert classes, i.e., Schur functions. Since a Schubert class can be represented in terms of Chern classes by Giambelli formula (see [13]), i.e.,

$$
\sigma_{(3^e,2^d,1^c)} = \det \begin{pmatrix} c_{c+d+e} & c_{c+d+e+1} & c_{c+d+e+2} \\ c_{d+e-1} & c_{d+e} & c_{d+e+1} \\ c_{e-2} & c_{e-1} & c_e \end{pmatrix},
$$

this proposition also yield mod 3 variant of Wu formula.

Applying the algorithm explained after Theorem 6.1, in a similar way as above one can obtain formula for the Steenrod mod p operation, for any specified prime number p (cf. [12]).

References

1. W. W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*, Graduate Studies in Mathematics **3**, American Mathematical Society, Providence, 1994.
2. A. Bertram, *Quantum Schubert Calculus*, Adv. Math. **128** (1997), 289-305.
3. A. Bertram, I. Ciocan-Fontanine, W. Fulton, *Quantum Multiplication of Schur Polynomials*, J. Algebra **219** (1999), 728-746.
4. A. Borel, J.-P. Serre, *Groupes de Lie et puissances réduites de Steenrod*, Amer. J. Math. **75** (1953), 409–448.
5. A. S. Buch, *Quantum cohomology of Grassmannians*, Compositio Math. **137** (2003), 227-235.
6. I. Coskun, *A Littlewood-Richardson rule for two-step flag varieties*, Invent. Math. **176** (2009), 325-395.
7. H. Duan, *On the inverse Kostka matrix*, J. Comb. Theory A **103** (2003), 363-376.
8. O. Egecioglu, J. B. Remmel, *A combinatorial interpretation of the inverse Kostka matrix*, Linear Multilinear Algebra **26** (1990), 59-84.
9. I. Gessel, C. Krattenthaler, *Cylindric partitions*, Trans. Amer. Math. Soc. **349** (1997), 429-479.

10. F. M. Goodman, H. Wenzl, *Littlewood-Richardson coefficients for Hecke algebras at roots of unity*, Adv. Math. **82** (1990), 244-265.
11. V. G. Kac, *Infinite-dimensional Lie algebras*, Cambridge University Press 1990.
12. C. Lenart, *The combinatorics of Steenrod operations on the cohomology of Grassmannians*, Adv. Math. **136** (1998), 251-283.
13. I.G. MacDonald, *Symmetric Functions and Hall Polynomials*, Oxford Mathematical Monographs, Oxford University Press 1995.
14. J. W. Milnor, J. D. Stasheff, *Characteristic Classes*, Ann. of Math. Studies **76**, Princeton University Press, New Jersey 1974.
15. J. Morse, A. Schilling, *A combinatorial formula for fusion coefficients*, DMTCS proc **AR** (2012), 735-744.
16. H. Narayanan, *On the complexity of computing Kostka numbers and Littlewood-Richardson coefficients*, J. Algebr. Comb. **24:3** (2006) 347-354.
17. Z. Z. Petrović, B. I. Prvulović, M. Radovanović, *Multiplication in the cohomology of Grassmannians via Gröbner bases*, J. Algebra **438** (2015), 60-84.
18. Y. Ruan, G. Tian, *A mathematical theory of quantum cohomology*, Math. Res. Lett. **1** (1994), 269-278.
19. P. Shay, *mod-p Wu formulas for the Steenrod algebra and the Dyer-Lashof algebra*, Proc. Amer. Math. Soc. **63** (1977), 339-347.
20. B. Siebert, G. Tian, *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator*, Asian Journ. Math. **1** (1997), 679-695.
21. N. E. Steenrod, D. B. A. Epstein, *Cohomology Operations*, Annals of Mathematics Studies, Princeton University Press 1962.
22. M. A. Walton, *Fusion rules in Wess-Zumino-Witten models*, Nuclear Phys. B **340** (1990), 777-790.
23. E. Witten, *The Verlinde algebra and the cohomology of the Grassmannian*, in "Geometry, Topology, and Physics", Conference Proceedings and Lecture Notes in Geometric Topology, Vol. IV, 357-422, International Press, Cambridge, MA, 1995.
24. W. T. Wu, *Les i-carrés dans une variété grassmanniene*, C. R. Acad. Sci. Paris **230** (1950), 918-920.

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