# ON REAL FLAG MANIFOLDS WITH CUP-LENGTH EQUAL TO ITS DIMENSION

Marko Radovanović, Belgrade

(Received Jun, 2018)

*Abstract.* We prove that for any positive integers $n_1, n_2, \ldots, n_k$ there exists a real flag manifold $F(1, \ldots, 1, n_1, n_2, \ldots, n_k)$ with cup-length equal to its dimension. Additionally, we give a necessary condition that an arbitrary real flag manifold needs to satisfy in order to have cup-length equal to its dimension.

*Keywords*: Cup-length, flag manifold, Lyusternik-Shnirel'man category

*MSC 2010*: 14M15, 55M30, 57N65.

## 1. Introduction

The $\mathbb{Z}_2$*-cohomology cup-length* (or *cup-length*) of a path connected space $X$, denoted by $\operatorname{cup}(X)$, is the supremum of all positive integers $m$ such that there exist classes $a_1, a_2, \ldots, a_m \in \widetilde{H}^*(X; \mathbb{Z}_2)$ with nonzero cup product, i.e., $a_1 a_2 \cdots a_m \neq 0$. It is well-known that $\operatorname{cup}(M)$ provides a lower bound for the Lyusternik-Shnirel'man category of $M$ (recall that the Lyusternik-Shnirel'man category of $M$, denoted by $\operatorname{cat}(M)$, is the minimum number of open subsets of $M$ covering $M$, each of which is contractible in $M$). In fact, one has

$$1 + \dim(M) \geq \operatorname{cat}(M) \geq 1 + \operatorname{cup}(M) \tag{1.1}$$

(in this paper dimension of a manifold $M$ will be denoted with $\dim(M)$). A trivial upper bound for the cup-length of a manifold is its dimension. Furthermore, if $\operatorname{cup}(M) = \dim(M)$, then (1.1) implies $\operatorname{cat}(M) = 1+\operatorname{cup}(M)$. In general, determining $\operatorname{cat}(M)$ poses a very difficult problem, so it is of interest to find manifolds $M$ with

---

Research supported by Ministry of Education, Science and Technological Development of Republic of Serbia Project #174008.

cup-length equal to its dimension. In this paper, if $\mathrm{cup}(M) = \dim(M)$, then we say that the cup-length of $M$ (or just $\mathrm{cup}(M)$) is *maximal*.

We consider this question for real flag manifolds (in this paper we only work with *real* flag manifolds, so we often use the term *flag manifold*). Let $n_1, n_2, \ldots, n_k \in \mathbb{N}$, $k \geq 2$. The *(real) flag manifold* $F(n_1, n_2, \ldots, n_k)$ is defined as the homogeneous space

$$O(n_1 + n_2 + \cdots + n_k)/O(n_1) \times O(n_2) \times \cdots \times O(n_k).$$

The numbers $n_i$, for $i \in [k]$, are the *steps* of this flag manifolds. Two special cases of flag manifolds are particularly important − flag manifolds with $k = 2$ are the *Grassmann manifolds*, and flag manifolds with $n_1 = \cdots = n_k = 1$ are the *complete flag manifolds*.

Although the cup-length of Grassmann manifold $F(m, n)$ is known only for $m \leq 4$ (see [3] and [10]), all Grassmann manifolds with maximal cup-length are known due to Berstein.

**Theorem 1.1** ([1]). *The cup-length of Grasmmann manifold $F(m, n)$ is maximal if and only if $m = 1$, or $m = 2$ and $n = 2^t - 1$, for some $t \in \mathbb{N}$.*

The cup-length of all complete flag manifolds is maximal. In fact, the following stronger result holds (in this paper, we use the following notation: $a^{\cdots k} := \underbrace{a, \ldots, a}_{k}$).

**Lemma 1.2** ([5]). *For all $j, n \in \mathbb{N}$, the cup-length of $F(1^{\cdots j}, n)$ is maximal.*

Having in mind the previous two results, one may think that a similar (simple) classification of all flag manifolds with maximal cup-length can be found, but it seems that this question is much more difficult. There have been attempts in the literature to solve this problem, but only some partial results were obtained. In [5] a family of flag manifolds of the form $F(1^{\cdots j}, 2^{\cdots d}, n)$ with maximal cup-length was found. This family was extended in [7], where, additionally, a necessary and sufficient condition for $\mathrm{cup}(F(1^{\cdots j}, 2^{\cdots d}, n)) = \dim(F(1^{\cdots j}, 2^{\cdots d}, n))$ in cases $d = 1$ and $d = 2$ was obtained. Up to now, no infinite family of flag manifolds with maximal cup-length and at least two steps greater than 2 was known.

The main result of this paper is the following.

**Theorem 1.3.** *For any positive integers $n_1, n_2, \ldots, n_k$ there exists a positive integer $j$ such that $\mathrm{cup}(F(1^{\cdots j}, n_1, \ldots, n_k))$ is maximal.*

We divide the proof of this result in two parts. In Section 3, we use the method of embedding the cohomology of a flag manifold in the cohomology of a complete flag manifold, developed by Korbaš and Lörinc in [5], and prove the result for $k = 2$

and $n_1 = n_2$. In Section 4, we complete the proof using the method of "fiberings" introduced by Horanska and Korbaš in [4].

In Section 5 we give a necessary condition that a flag manifold with maximal cup-length needs to satisfy. In particular, this implies the following result.

**Theorem 1.4.** *If the cup-length $F(n_1, \dots, n_k)$ is maximal, then at least one of the numbers $n_i$, $i \in \{1, 2, \dots, k\}$, is not greater than 3.*

## 2. Preliminaries and notation

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also, for $k \in \mathbb{N}$ we denote $[k] := \{1, 2, \dots, k\}$. Furthermore, for an $m$-tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{N}_0^m$, we use the following notation

$$|\alpha| := \sum_{j=1}^{m} \alpha_j \quad \text{and} \quad \|\alpha\| := \sum_{j=1}^{m} j\alpha_j.$$

Let $F := F(n_1, n_2, \dots, n_k)$ be a flag manifold. Then $\dim(F) = \sum_{1 \le i < j \le k} n_i n_j$. There are $k$ canonical vector bundles over $F$, which we denote by $\gamma_i$, for $i \in [k]$ ($\dim \gamma_i = n_i$). By Borel's description from [2] (more precisely, its slight modification – see, for example [6]), each class in $H^*(F; \mathbb{Z}_2)$ is a polynomial in Stiefel-Whitney of the vector bundles $\gamma_i$, for $i \in [k-1]$. In this paper we denote with $w_{i,j}$ the $j$-th Stiefel-Whitney of the vector bundles $\gamma_i$, for $i \in [k-1]$ and $j \in [n_i]$.

For $i \in [k-1]$ and an $n_i$-tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n_i}) \in \mathbb{N}_0^{n_i}$ we use the notation $W_i^\alpha$ for the monomial $w_{i,1}^{\alpha_1} w_{i,2}^{\alpha_2} \cdots w_{i,n_i}^{\alpha_{n_i}}$. Also, let

$$S_0 := n_k \quad \text{and} \quad S_i := n_k + n_1 + n_2 + \dots + n_i, \ i \in [k-1].$$

Furthermore, by an abuse of notation, we denote $W_i = \{w_{i,1}, \dots, w_{i,n_i}\}$, $i \in [k-1]$. So,

$$\mathbb{Z}_2[W_1, \dots, W_{k-1}] = \mathbb{Z}_2[w_{1,1}, \dots, w_{1,n_1}, \dots, w_{k-1,1}, \dots, w_{k-1,n_{k-1}}].$$

Note that the cup-length of $F$ is maximal if and only if $a_1 \cdots a_{\dim_F} \neq 0$ for some $a_i \in \widetilde{H}^*(F; \mathbb{Z}_2)$, $i \in [\dim(F)]$. Note that the latter implies $a_i \in H^1(F; \mathbb{Z}_2)$, for all $i \in [\dim(F)]$. Hence, the cup-length of $F$ is maximal if and only if there exist $\alpha_i \in \mathbb{N}_0$, $i \in [k-1]$, such that $w_{1,1}^{\alpha_1} \cdots w_{k-1,1}^{\alpha_{k-1}} \neq 0$. Of course, a necessary condition for the last relation is that $\alpha_i \le \mathrm{ht}(w_{i,1})$, for $i \in [k-1]$, where $\mathrm{ht}(a)$ denotes the height of the class $a$ (the *height* of a class $a \in \widetilde{H}^*(X; \mathbb{Z}_2)$ is the largest positive integer $n$ such that $a^n \neq 0$).

Although the cup-length of a general flag manifold is far from being understood, the heights of the first Stiefel-Whitney classes are known by the following result of Korbaš and Lörinc (see [5]).

**Proposition 2.1.** *Let $t$ be the unique integer such that $2^t < S_{k-1} \leq 2^{t+1}$, and let $m_i = \min\{n_i, S_{k-1} - n_i\}$. Then*

$$\mathrm{ht}(w_{i,1}) = \begin{cases} S_{k-1} - 1, & \text{if } m_i = 1, \\ 2^{t+1} - 2, & \text{if } m_i = 2, \text{ or } m_i = 3 \text{ and } S_{k-1} = 2^t + 1, \\ 2^{t+1} - 1, & \text{otherwise.} \end{cases}$$

We also denote $\mathrm{ht}(n_i) := \mathrm{ht}(w_{i,1})$, for $i \in [k-1]$.

## 3. Evaluation of cup-length and complete flags

In this section we prove our main result in the case $k = 2$ and $n_1 = n_2$. The method that we use in the proof is the one introduced by Korbaš and Lörinc in [5]. First, we explain this method.

Let $m \geq 2$ and observe the complete flag manifold $F(1^{\cdots m})$. Denote by $e_i := w_1(\gamma_i)$ the first Stiefel-Whitney class of the canonical line bundle $\gamma_i$ over $F(1^{\cdots m})$, $i \in [m]$. The following lemma is well-known (see [5, 10]).

**Lemma 3.1.** *A monomial $e_1^{a_1} \cdots e_m^{a_m} \in H^{\binom{m}{2}}(F(1^{\cdots m}); \mathbb{Z}_2) \cong \mathbb{Z}_2$ is nonzero if and only if $(a_1, a_2, \ldots, a_m)$ is a permutation of the $m$-tuple $(m-1, m-2, \ldots, 1, 0)$.*

Let $n_1, n_2, \ldots, n_k$ ($k \geq 2$) be positive integers, $\nu_i = n_1 + n_2 + \cdots + n_i$, $i \in [k]$ (it is understood that $\nu_0 = 0$), and $m = \nu_k$. For the flag manifold $F(n_1, n_2, \ldots, n_k)$ we have the map $p : F(1^{\cdots m}) \to F(n_1, n_2, \ldots, n_k)$, given by

$$p(V_1, \ldots, V_{n_1}, \ldots, V_{\nu_{k-1}+1}, \ldots, V_m) = (V_1 \oplus \cdots \oplus V_{n_1}, \ldots, V_{\nu_{k-1}+1} \oplus \cdots \oplus V_m).$$

Our proof is based on the following result from [5, p. 154].

**Lemma 3.2.** *If $F = F(n_1, n_2, \ldots, n_k)$, $u \in H^{\dim F}(F; \mathbb{Z}_2)$ and*

$$v = e_1^{n_1-1} \cdots e_{n_1-1} e_{n_1+1}^{n_2-1} \cdots e_{n_1+n_2-1} \cdots e_{\nu_{k-1}+1}^{n_k-1} \cdots e_{\nu_k-1} \in H^*(F(1^{\cdots m}); \mathbb{Z}_2),$$

*then $p^*(u) \cdot v \in H^{\binom{m}{2}}(F(1^{\cdots m}); \mathbb{Z}_2)$ and*

$$u \neq 0 \quad \text{if and only if} \quad p^*(u) \cdot v \neq 0.$$

In [5, p. 155] the authors also gave a description of the map $p^*$ from the previous lemma. If $w_{i,j}$ is the $j$-th Stiefel-Whitney class of the canonical bundle $\gamma_i$ over $F(n_1, n_2, \ldots, n_k)$, $i \in [k]$, $j \in [n_i]$, then $p^*(w_{i,j})$ is the $j$-th elementary symmetric polynomial in the variables $e_{\nu_{i-1}+1}, e_{\nu_{i-1}+2}, \ldots, e_{\nu_i}$. In our application the most important will be the case $j = 1$, when one has

$$p^*(w_{i,1}) = e_{\nu_{i-1}+1} + e_{\nu_{i-1}+2} + \cdots + e_{\nu_i}. \tag{3.1}$$

For $a_1, a_2, \dots, a_k \in \mathbb{N}_0$, we denote the multinomial coefficient with

$$\binom{a_1 + a_2 + \cdots + a_k}{a_1,\, a_2,\, \dots,\, a_k} := \frac{(a_1 + a_2 + \cdots + a_k)!}{a_1! \cdot a_2! \cdots a_k!}.$$

The following lemma is probably well-known, but we prove it for the sake of completeness.

**Lemma 3.3.** *Let $a_1, \dots, a_k$ be nonnegative integers, and $a_i = (\alpha_{1,i}, \dots, \alpha_{s_i,i})_2$, for $i \in [k]$, their representations in base 2. Then*

$$\binom{a_1 + a_2 + \cdots + a_k}{a_1,\, a_2,\, \dots,\, a_k} \text{ is odd}$$

*if and only if for all $i, j \in [k]$, $i \neq j$, and $l \in [\max\{s_i, s_j\}]$ at least one of the numbers $\alpha_{l,i}$ and $\alpha_{l,j}$ is equal to zero.*

*Proof.* ($\Rightarrow$:) By symmetry, it is enough to prove the claim for $i = k-1$ and $j = k$. Since

$$\binom{a_1 + a_2 + \cdots + a_k}{a_1} \cdots \binom{a_{k-1} + a_k}{a_{k-1}} = \binom{a_1 + a_2 + \cdots + a_k}{a_1,\, a_2,\, \dots,\, a_k} \equiv 1 \pmod{2},$$

the number $\binom{a_{k-1}+a_k}{a_k}$ is odd. Now, the result follows from [7, Lemma 2.3].
($\Leftarrow$:) Note that the given condition implies that for every $i \in [k]$ and every $l \in \mathbb{N}_0$ at most one of the numbers $a_i$ and $a_{i+1} + \cdots + a_k$ has digit 1 in position $l$. By [7, Lemma 2.3], this implies that $\binom{a_i + a_{i+1} + \cdots + a_k}{a_i}$ is odd (for all $i \in [k-1]$), which completes our proof. $\square$

**Lemma 3.4.** *For every $n \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $\operatorname{cup}(F(1^{\cdots j}, n, n))$ is maximal.*

*Proof.* If $n = 1$, then by Lemma 1.2 we can take $j = 1$. So, we may assume that $n \geq 2$. Let $s$ be the unique integer such that $2^{s-1} < n \leq 2^s$. We will prove that $j = 2^{s+n-1}$ has the desired property. Let $a_1, a_2, \dots, a_j$ be the sequence obtained from the sequence $n, n+1, \dots, 2n+j-1$ by removing numbers $2^{s+i} + i$, for $i \in [n-1] \cup \{0\}$ (note that $2n + j - 1 > 2^{s+n-1} + n - 1 > 2^s \geq n$). Finally, let $m = 2^s + 2^{s+1} + \cdots + 2^{s+n-1} = 2^{s+n} - 2^s$.

Note that

$$m + \sum_{i=1}^{j} a_i = n^2 + 2nj + \binom{j}{2} = \dim(F(1^{\cdots j}, n, n)),$$

so it enough to prove that $\prod_{i=1}^{j} w_{i,1}^{a_i} \cdot w_{j+1,1}^{m}$ is nonzero (in $H^*(F(1^{\cdots j}, n, n); \mathbb{Z}_2))$. By Lemma 3.2 and (3.1), this is equivalent with (in $H^*(F(1^{\cdots j+2n}); \mathbb{Z}_2)$)

$$0 \neq (e_{j+1} + \cdots + e_{j+n})^m e_{j+1}^{n-1} \cdots e_{j+n-1} e_{j+n+1}^{n-1} \cdots e_{j+2n-1} \prod_{i=1}^{j} e_i^{a_i}$$

$$= \sum_{t_1+\cdots+t_n=m} \binom{m}{t_1,\ \ldots,\ t_n} e_{j+1}^{t_1+n-1} \cdots e_{j+n-1}^{t_{n-1}+1} e_{j+n}^{t_n} e_{j+n+1}^{n-1} \cdots e_{j+2n-1} \prod_{i=1}^{j} e_i^{a_i}. \tag{3.2}$$

By Lemma 3.1, a summand in the last expression is nonzero if and only if the multinomial coefficient $\binom{m}{t_1,\ldots,t_n}$ is odd and $(t_1+n-1, \ldots, t_{n-1}+1, t_n)$ is a permutation of the $n$-tuple $(2^{s+n-1}+n-1, \ldots, 2^{s+1}+1, 2^s)$.

Let $(t_1, \ldots, t_n)$ be an $n$-tuple satisfying these conditions. Since $t_i + n - i \geq 2^s > n-1$, we have that $t_i > 0$, for all $i \in [n]$, i.e. $t_i$ has at least one nonzero digit in the binary expansion. On the other hand, $m$ has exactly $n$ digits in the binary expansion, so, by Lemma 3.3, $\binom{m}{t_1,\ldots,t_n}$ is odd if and only if

$$\{t_1, \ldots, t_{n-1}, t_n\} = \{2^{s+n-1}, \ldots, 2^{s+1}, 2^s\}.$$

Additionally, $\{t_1+n-1, \ldots, t_{n-1}+1, t_n\} = \{2^{s+n-1}+n-1, \ldots, 2^{s+1}+1, 2^s\}$, so there is an index $i \in [n]$ such that $2^{s+n-1}+n-1 = t_i+n-i$. Since $2^{s+n-1} \geq t_i$ and $n-1 \geq n-i$, we have that $i=1$ and $t_1 = 2^{s+n-1}$. Continuing in the same way we conclude that $t_2 = 2^{s+n-2}, \ldots, t_n = 2^s$.

Hence, $(t_1, \ldots, t_n) = (2^{s+n-1}, \ldots, 2^{s+1}, 2^s)$ is the only $n$-tuple for which the corresponding summand in (3.2) is nonzero. This completes our proof. $\square$

Note that $j$ constructed in the previous lemma satisfies $j = 2^{s+n-1} \leq (n-1)2^n$.

## 4. Fiberings and cup-length

To complete the proof of our main result we use the method of "fiberings" introduced by Horanska and Korbaš in [4]. This method proved very useful in cup-length calculation (see [4, 5, 9]). It is based on the following result.

**Theorem 4.1** ([4])**.** *Let $p : E \to B$ be a smooth fiber bundle with connected base $B$ and connected fiber $F$. Suppose that the fiber inclusion induces an epimorphism in $\mathbb{Z}_2$-cohomology. Then* $\mathrm{cup}(E) \geq \mathrm{cup}(F) + \mathrm{cup}(B)$.

Let us observe the following fiber bundle (see [5]):

$$\begin{tikzcd} F(n_{l+1}, \ldots, n_k) \arrow[r, hook] & F(n_1, \ldots, n_k) \arrow[d] \\ & F(n_1, \ldots, n_l, n_{l+1} + \cdots + n_k) \end{tikzcd}$$

Since the inclusion $i : F(n_{l+1}, \dots, n_k) \to F(n_1, \dots, n_k)$ induces an epimorphism in $\mathbb{Z}_2$-cohomology (see [5]), we can apply Theorem 4.1 on this fiber bundle. Additionally, we have

$$\dim(F(n_{l+1}, \dots, n_k)) + \dim(F(n_1, \dots, n_l, n_{l+1} + \dots + n_k)) = \dim(F(n_1, \dots, n_k)),$$

so from Theorem 4.1 and the fact that the upper bound for the cup-length is the dimension of the manifold, we obtain the following result:

(4.1) if $\mathrm{cup}(F(n_{l+1}, \dots, n_k))$ and $\mathrm{cup}(F(n_1, \dots, n_l, n_{l+1} + \dots + n_k))$ are maximal, then $\mathrm{cup}(F(n_1, \dots, n_k))$ is also maximal.

We are ready to prove our main result.

*Proof of Theorem 1.3.* By Lemma 1.2, for $k = 1$ it is enough to take $j = 1$. So, we may assume that $k \geq 2$. We continue our proof by induction on $k$.

Base case $k = 2$. Since $F(1^{\cdots j}, n_1, n_2)$ is homeomorphic to $F(1^{\cdots j}, n_2, n_1)$ (for any $j$), we may assume that $n_1 \leq n_2$. If $n_1 = n_2$, then the result follows from Lemma 3.4.

So, let us assume that $n_1 < n_2$. Furthermore, let $j'$ be a positive integer such that $\mathrm{cup}(F(1^{\cdots j'}, n_2, n_2))$ is maximal ($j'$ exists by Lemma 3.4) and consider the following fiber bundle

$$\begin{tikzcd} F(1^{\cdots(n_2-n_1)}, n_1) \arrow[r, hook] & F(1^{\cdots j'}, n_2, 1^{\cdots(n_2-n_1)}, n_1) \arrow[d] \\ & F(1^{\cdots j'}, n_2, n_2) \end{tikzcd}$$

By Lemma 1.2, Lemma 3.4 and (4.1), we conclude that the cup-length of the flag manifold $F(1^{\cdots j'+n_2-n_1}, n_1, n_2)$ is maximal.

Inductive step. Suppose that the claim is true for all $l \in [k] \setminus \{1\}$ and let us prove it for $k+1$.

Let $j'$ be a positive integer such that $\mathrm{cup}(F(1^{\cdots j'}, n_k, n_{k+1}))$ is maximal and $j''$ a positive integer such that $\mathrm{cup}(F(1^{\cdots j''}, n_1, \dots, n_{k-1}, j' + n_k + n_{k+1}))$ is maximal ($j'$ and $j''$ exist by inductional hypothesis). Now, using (4.1) for the fiber bundle:

$$\begin{tikzcd} F(1^{\cdots j'}, n_k, n_{k+1}) \arrow[r, hook] & F(1^{\cdots j''}, n_1, \dots, n_{k-1}, 1^{\cdots j'}, n_k, n_{k+1}) \arrow[d] \\ & F(1^{\cdots j''}, n_1, \dots, n_{k-1}, j' + n_k + n_{k+1}) \end{tikzcd}$$

we conclude that $\mathrm{cup}\left(F(1^{\cdots j'+j''}, n_1, \dots, n_{k+1})\right)$ is also maximal. $\square$

The number $j$ constructed in the previous proof is quite large. We demonstrate this in cases $k = 2$ and $k = 3$ (we use the same notation as above).

Let $k = 2$ and w.l.o.g. $n_1 \le n_2$. Then $j = j' + n_2 - n_1$, where, by the remark after Lemma 3.4, $j' \le (n_2 - 1)2^{n_2}$. So, $j \le (n_2 - 1)2^{n_2} + n_2 - n_1 < n_2 \cdot 2^{n_2}$.

Now, let $k = 3$ and w.l.o.g. $n_1 \ge n_2 \ge n_3$. Then $j = j' + j''$, and from the case $k = 2$ one has $j' < n_2 \cdot 2^{n_2}$ and $j'' < \max\{(j' + n_2 + n_3)2^{j'+n_2+n_3}, n_1 \cdot 2^{n_1}\}$. So,

$$j < \max\{(n_2 \cdot 2^{n_2} + n_2 + n_3)2^{n_2 \cdot 2^{n_2} + n_2 + n_3}, n_1 \cdot 2^{n_1}\}.$$

At the end of this section we show that if $\operatorname{cup}(F(1^{\cdots j}, n_1, \ldots, n_k))$ is maximal, then $\operatorname{cup}(F(1^{\cdots j'}, n_1, \ldots, n_k))$ is also maximal for all $j' \ge j$. Clearly, it is enough to consider the case $j' = j + 1$. Then the proof follows from Lemma 1.2 and (4.1) applied to the following fiber bundle:

$$\begin{tikzcd} F(1^{\cdots j}, n_1, \ldots, n_k) \arrow[r, hook] & F(1^{\cdots j+1}, n_1, \ldots, n_k) \arrow[d] \\ & F(1, j + n_1 + \cdots + n_k). \end{tikzcd}$$

This construction implies that in order to obtain all flag manifolds with maximal cup-length it is enough to find (for every $k \ge 2$ and $n_1, \ldots, n_k \in \mathbb{N}$) the minimal $j = j(n_1, \ldots, n_k)$ such that $F(1^{\cdots j}, n_1, \ldots, n_k)$ has maximal cup-length.

## 5. Gröbner bases and cup-length

In this section we give a necessary condition that a flag manifold with maximal cup-length needs to satisfy. This proof is based on a result from [6] (in fact, its mod 2 variant), where Gröbner bases for all flag manifolds were constructed.

Throughout this section, let $F$ denote the flag manifold $F(n_1, n_2, \ldots, n_k)$ and $t$ the unique integer such that $2^t < S_{k-1} \le 2^{t+1}$. Also, we use notation introduced in Section 2.

**Lemma 5.1.** *For every $f \in \mathbb{Z}_2[W_1, W_2, \ldots, W_{k-1}]$ there is a polynomial $p$ such that $p = f$ in $H^*(F; \mathbb{Z}_2)$ and*

- (i) *for each monomial $W_1^{\alpha(1)} \cdots W_{k-1}^{\alpha(k-1)}$ of $p$ and $i \in [k-1]$ we have $|\alpha(i)| \le S_{i-1}$;*
- (ii) *if no monomial of $f$ contains a variable from $W_1 \cup W_2 \cup \cdots \cup W_l$, for some $l \in [k-1]$, then the same holds for $p$.*

The following lemma will be the key for obtaining the main result of this section (this lemma generalizes [8, Corollary 3.1.4.]).

**Lemma 5.2.** *Let $\alpha(i)$, for $i \in [k-1]$, be an arbitrary $n_i$-tuple of non-negative integers. If $\sum_{i=l}^{k-1}\|\alpha(i)\| > \sum_{i=l}^{k-1} n_i S_{i-1}$ for some $l \in [k-1]$, then $W_1^{\alpha(1)}W_2^{\alpha(2)}\cdots W_{k-1}^{\alpha(k-1)} = 0$ in $H^*(F;\mathbb{Z}_2)$.*

*Proof.* It suffices to prove that $W_l^{\alpha(l)}\cdots W_{k-1}^{\alpha(k-1)} = 0$ in $H^*(F;\mathbb{Z}_2)$. We know that $W_l^{\alpha(l)}\cdots W_{k-1}^{\alpha(k-1)} \in H^q(F;\mathbb{Z}_2)$, where $q = \sum_{i=l}^{k-1}\|\alpha(i)\|$. Let $p$ be the polynomial from Lemma 5.1 such that $p = W_l^{\alpha(l)}\cdots W_{k-1}^{\alpha(k-1)}$ in $H^*(F;\mathbb{Z}_2)$. Suppose that $p$ is nonzero. Then an arbitrary monomial in $p$ is of the form $W_l^{\beta(l)}\cdots W_{k-1}^{\beta(k-1)}$, where $\beta(i) \in \mathbb{N}_0^{n_i}$ and $|\beta(i)| \le S_{i-1}$ for all $i \in \{l,\dots,k-1\}$. But the dimension of $W_l^{\beta(l)}\cdots W_{k-1}^{\beta(k-1)}$ (and also of $p$) is

$$\sum_{i=l}^{k-1}\|\beta(i)\| \le \sum_{i=l}^{k-1} n_i|\beta(i)| \le \sum_{i=l}^{k-1} n_i S_{i-1} < q,$$

which is a contradiction, since $p = W_l^{\alpha(l)}\cdots W_{k-1}^{\alpha(k-1)} \in H^q(F;\mathbb{Z}_2)$. $\square$

We are ready to prove the main result of this section.

**Proposition 5.3.** *Suppose that a flag manifold $F$ has maximal cup-length, and let $t$ be as above. Then, for every permutation $\pi$ of the set $\{1,2,\dots,k\}$ and every $l \in [k-1]$ we have*

$$n_{\pi(k)}\sum_{i=1}^{l} n_{\pi(i)} + \sum_{1\le i<i'\le l} n_{\pi(i)}n_{\pi(i')} \le \sum_{i=1}^{l}\mathrm{ht}(n_{\pi(i)}).$$

*Proof.* Since $F$ has the maximal cup-length, so does the flag manifold $\widetilde{F} := F(n_{\pi(1)}, n_{\pi(2)},\dots,n_{\pi(k)})$. Let $\widetilde{w}_{i,1}$, for $i \in [k-1]$, be the first Stiefel-Whitney class of the $i$-th tautological vector bundle over this manifold, and

$$\widetilde{w}_{1,1}^{a_1}\widetilde{w}_{2,1}^{a_2}\cdots\widetilde{w}_{k-1,1}^{a_{k-1}} \ne 0$$

a class in $H^*(\widetilde{F};\mathbb{Z}_2)$ such that $\sum_{i=1}^{k-1} a_i = \dim(\widetilde{F})$. Then, by Lemma 5.2, we have

$$\begin{aligned}\sum_{i=1}^{l} a_i = \dim(\widetilde{F}) - \sum_{i=l+1}^{k} a_i &\ge \dim(\widetilde{F}) - \sum_{i=l+1}^{k} n_{\pi(i)}\left(n_{\pi(k)} + \sum_{j=1}^{i-1} n_{\pi(j)}\right)\\ &= n_{\pi(k)}\sum_{i=1}^{l} n_{\pi(i)} + \sum_{1\le i<i'\le l} n_{\pi(i)}n_{\pi(i')}.\end{aligned}$$

On the other hand, $a_i \le \mathrm{ht}(n_{\pi(i)})$, for $i \in [k-1]$, which together with the previous inequality gives us the desired result. $\square$

Let us go back to the question from the previous sections. So, for the given positive integers $n_1, n_2, \dots, n_k$, $k \geq 2$, we want to find $j$ such that cup $F(1^{\cdots j}, n_1, \dots, n_k)$ is maximal. In what follows we show that if the numbers $n_i$ are large enough, then Proposition 5.3 implies that $j$ also must be large.

Suppose that $n_i \geq m$, for some $m \geq 4$ and all $i \in [k]$. As usual, let $t$ be the unique integer such that $2^t < j + \sum_{i=1}^k n_i \leq 2^{t+1}$. Then, by Proposition 2.1, $\mathrm{ht}(n_i) = 2^{t+1} - 1$, so applying Proposition 5.3 for the permutation $(\pi(1), \dots, \pi(k+j)) = (n_1, \dots, n_{k-1}, 1, \dots, 1, n_k)$ and $l = k-1$, gives

$$(2^{t+1} - 1)(k-1) \geq \sum_{1 \leq i < i' \leq k} n_i n_{i'}.$$

Since $j \geq 2^t - \sum_{i=1}^k n_i + 1$, one has $2j + 2\sum_{i=1}^k n_i - 3 \geq 2^{t+1} - 1$, so by the previous inequality

$$2(k-1)j \geq \sum_{1 \leq i < i' \leq k} n_i n_{i'} - 2(k-1)\sum_{i=1}^k n_i + 3(k-1) = f(n_1, \dots, n_k).$$

Note that $f$ is a linear function in each $n_t$, $t \in [k]$. Furthermore, for $t \in [k]$

$$f(n_1, \dots, n_k) = \left(\sum_{i' \neq t} n_{i'} - 2(k-1)\right) n_t + \sum_{\substack{1 \leq i < i' \leq k \\ i, i' \neq t}} n_i n_{i'} - 2(k-1)\sum_{i' \neq t} n_{i'} + 3(k-1),$$

and since $\sum_{i' \neq t} n_{i'} - 2(k-1) \geq 2(k-1)$ this function is increasing in $n_t$, for every $t \in [k]$. This implies $f(n_1, \dots, n_k) \geq f(m, \dots, m)$, and hence

$$j \geq \frac{f(m, \dots, m)}{2(k-1)} = \frac{1}{4}\left(km^2 - 4km + 6\right) > 1. \tag{5.1}$$

This inequality immediately implies Theorem 1.4. However, we note that to obtain this result one does not need Lemma 5.2, i.e. it follows from the case $l = k-1$ of Proposition 5.3, which is in fact (obvious) inequality $\dim(F) \leq \sum_{i=1}^{k-1} \mathrm{ht}(n_i)$.

**Remark.** Of course, $j$ obtained using the proofs of Lemma 3.4 and Theorem 1.3 is much larger than the lower bound from (5.1) (see the paragraphs after the proof of Theorem 1.3), i.e. there is quite a gap between the lower and the upper bound (that we obtain in this paper) for the minimal $j$ with the desired property.

We finish this section with the following example.

**Example.** The cup-length of $F(1^{\cdots 17}, 3, 5, 7)$ is not maximal. To prove this, it is enough to apply Proposition 5.3 for the permutation $\pi = (3, 5, 1, \dots, 1, 7)$ and $l = 2$. Indeed, $\mathrm{ht}(3) = \mathrm{ht}(5) = 31$, but the left hand side of the inequality from Proposition 5.3 is equal to 71.

## References


[1] *I. Berstein:* On the Lusternik-Schnirelmann category of Grassmannians. Math. Proc. Cambridge Philos. Soc. *79* (1976), 129–134. Zbl 0315.55011, MR0400212

[2] *A. Borel:* La cohomologie mod 2 de certains espaces homogenes. Comm. Math. Helv. *27* (1953), 165–197. Zbl 0052.40301, MR0057541

[3] *H. Hiller:* On the cohomology of real Grassmanians. Trans. Amer. Math. Soc. *257* (1980), 512–533. Zbl 0462.57021, MR0552272

[4] *L. Horanska, J. Korbaš:* On cup products in some manifolds. Bull. Belg. Math. Soc. Simon Stevin *7:1* (2000), 21–28. Zbl 0956.55010, MR1741743

[5] *J. Korbaš, J. Lörinc:* The $\mathbb{Z}_2$-cohomology cup-length of real flag manifolds. Fund. Math., *178* (2003), 143–158. Zbl 1052.55006, MR2029922

[6] *Z. Z. Petrović, B. I. Prvulović, M. Radovanović:* Gröbner bases for (partial) flag manifolds. submitted for publication.

[7] *Z. Z. Petrović, B. I. Prvulović, M. Radovanović:* On maximality of the cup-length of flag manifolds. Acta Math. Hung. *149:2* (2016), 448–461. Zbl 06806118, MR3518647

[8] *M. Radovanović:* Gröbner bases for some flag manifolds and applications, Math. Slovaca *66:5* (2016), 1065–1082. Zbl 06680647, MR3602604

[9] *M. Radovanović:* On the $\mathbb{Z}_2$-cohomology cup-length of some real flag manifolds, Filomat *30:2* (2016), 1577–1590. Zbl 06749816, MR3530103

[10] *R. E. Stong:* Cup products in Grassmannians. Topology Appl., *13* (1982), 103–113. Zbl 0469.55005, MR0637432



*Authors' addresses*: *Marko Radovanović*, University of Belgrade, Faculty of Mathematics, Studentski trg 16, Belgrade, Serbia. e-mail: `markor@matf.bg.ac.rs`.