Higher topological complexities of real Grassmannians and semi-complete real flag manifolds

Marko Radovanović*

April 29, 2022

Abstract

Topological complexity and its higher analogues naturally appear in motion planning in robotics. In this paper we consider the problem of finding higher topological complexities (TC_h) of the real Grassmann manifold $G_k(\mathbb{R}^n)$ of k-dimensional subspaces in $\mathbb{R}^n$ and semi-complete real flag manifold $F(1^k, m)$ (here 1^k means that 1 appears k times). We use cohomology methods to prove some general bounds on the h-th zero-divisor cup-length (zcl_h), and then use them to obtain the exact values of $\mathrm{TC}_h(G_2(\mathbb{R}^{2^s+1}))$ for $h \geqslant 2^{s+1}-1$, and $\mathrm{TC}_h(F(1^k, 2^s-k+1))$ for $h \geqslant k \geqslant 3$. Additionally, we determine $\mathrm{zcl}_h(G_2(\mathbb{R}^n))$ for $h \geqslant 2^{s+1}-1$ (where $2^s < n \leqslant 2^{s+1}$), and resolve two questions from [9].

Mathematical Subject Classification. 55M30, 14M15.

Keywords. Higher topological complexity, real Grassmann manifold, semi-complete real flag manifold, zero-divisor cup-length.

1 Introduction

Topological complexity naturally appears in motion planning in robotics. Suppose that we are given a mechanical system $\mathcal{S}$; the *motion planning* problem on this system is to find an "algorithm" that given two states A and B describes how to transform one to the other. To find a mathematical model for this problem, one tries to associate a *configuration space* X to this system $\mathcal{S}$, i.e. a space whose points represent possible states of $\mathcal{S}$. For example, to the problem of rotating a line in $\mathbb{R}^{n+1}$ around a fixed point one can associate the projective space $X = \mathbb{R}P^n$. More generally, if instead of a line we are interested in rotations of a k-dimensional space in $\mathbb{R}^n$ ($1 \leqslant k < n$) we can take X to be the *real Grassmann manifold* $G_k(\mathbb{R}^n)$ (in this paper $G_k(\mathbb{R}^n)$ denotes the real Grassmann manifold of k-dimensional subspaces in $\mathbb{R}^n$); another well-studied generalization of real projective spaces that we consider in this paper are semi-complete real flag manifolds $F(1^k, m)$ (in this paper $F(1^k, m)$ consists of $(k+1)$-tuples $(V_1, \ldots, V_k, V_{k+1})$ of mutually orthogonal subspaces of $\mathbb{R}^{k+m}$ with $\dim(V_i) = 1$ for $1 \leqslant i \leqslant k$, and $\dim(V_{k+1}) = m$). It was proven by Farber in [6] that topological complexity of X in a certain way measures the instabilities of the system $\mathcal{S}$. (The reader can find a detailed treatment of the subject in Farber's monograph [7].)

*University of Belgrade, Faculty of Mathematics, Studentski trg 16, Belgrade, Serbia. Partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia (Grant No. 45103-9/2021-14/200104).

Topological complexity was introduced by Farber in [6] in the following way. Let X be a path-connected topological space X. We denote by $P(X)$ the space of all continuous paths $\gamma : [0,1] \to X$ and by $\pi : P(X) \to X \times X$ the evaluation map, defined with $\pi(\gamma) = (\gamma(0), \gamma(1))$. Then $\mathrm{TC}(X)$ is the Schwarz genus of the fibration π. (Note that in some papers *reduced* Schwarz genus is used to define topological complexity and its higher analogues – for example in [9]. So, the values on the topological complexity obtained there are one smaller than as defined in this paper.)

As noted above, finding topological complexity of real projective spaces is closely related to possibly the simplest form of motion planning, that is rotating a line around a fixed point. This problem was considered in [8], and it turned out to be extremely difficult. Remarkably, the authors proved that $\mathrm{TC}(\mathbb{R}P^n) = \mathrm{Imm}(\mathbb{R}P^n) + 1$ for $n \notin \{1,3,7\}$, while $\mathrm{TC}(\mathbb{R}P^n) = \mathrm{Imm}(\mathbb{R}P^n)$ for $n \in \{1,3,7\}$ (here, $\mathrm{Imm}(X)$ denotes the *immersion dimension* of a given smooth manifold X, i.e. the smallest positive integer k such that there is an immersion of X in $\mathbb{R}^k$; of course, finding the value of $\mathrm{Imm}(\mathbb{R}P^n)$ for general n is a well-studied open problem). The problem of finding topological complexity was later studied for other real Grassmannians and related manifolds (see [3, 5, 9, 12, 13, 16]). Although a number of results is obtained, up to now there is no real Grassmann manifold, other than the above mentioned real projective spaces, for which the exact value of the topological complexity is calculated.

The notion of topological complexity was extended in [17] by Rudyak. He defined the *h-th topological complexity* of X ($h \geqslant 2$), denoted by $\mathrm{TC}_h(X)$, as the Schwarz genus of the fibration $\pi_h : P(X) \to X^h$ defined with

$$\pi_h(\gamma) = \left(\gamma(0), \gamma\left(\frac{1}{h-1}\right), \gamma\left(\frac{2}{h-1}\right), \ldots, \gamma\left(\frac{h-2}{h-1}\right), \gamma(1)\right).$$

As in the case of topological complexity, higher topological complexities can be applied in motion planing in robotics. Indeed, $\mathrm{TC}_h(X)$ is closely related to the problem of moving an object through h prescribed states.

Although the h-th topological complexity is a natural generalization of the topological complexity, there are subtle differences between them. Indeed, some properties of $\mathrm{TC}(X)$ can not be extended to $\mathrm{TC}_h(X)$ for $h \geqslant 3$, but, on the other hand, it turned out that computing $\mathrm{TC}_h(X)$ for certain spaces X and $h \geqslant 3$ was easier than computing $\mathrm{TC}(X)$ (see, e.g. [9]). In this paper we will see that the same phenomenon holds for certain real Grassmann manifolds and semi-complete real flag manifolds.

To obtain our results we use the so called *cohomology method.* Let us briefly explain it. Let $\Delta_h : X \to X^h$ denote the diagonal map. Then, by analogy with the $h = 2$ case, the elements of

$$\mathrm{Ker}(\Delta_h^* : H^*(X^h; \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2))$$

are called the *h-th zero-divisors.* Further, the *h-th zero-divisor cup-length* of X, denoted by $\mathrm{zcl}_h(X)$, is defined to be the maximum number of elements from $\mathrm{Ker}\Delta_h^*$ whose product is nonzero. Then one has the following result.

Proposition 1.1 ([17]) *Let X have the homotopy type of an $(e-1)$-connected CW complex of dimension d. Then*

$$\mathrm{zcl}_h(X) + 1 \leqslant \mathrm{TC}_h(X) \leqslant \frac{hd}{e} + 1.$$

What is particularly interesting is that for certain spaces X and $h > 2$, the lower and the upper bound for $\mathrm{TC}_h(X)$ from the previous proposition become very close, and sometimes are even equal. In the latter case, we immediately get the value of $\mathrm{TC}_h(X)$. In this way, in [3, Theorem 1.1] it was proven that $\mathrm{TC}_h(\mathbb{R}P^n) = hn + 1$ when n is even and $h > n$, while in [9], $\mathrm{TC}_h(X)$ was calculated for a family of semi-complete real flag manifolds X (and certain $h \geqslant 2$). In the present paper we will use a similar method to extend some of these results and prove several similar results for real Grassmannians. In particular, we prove:

- $\mathrm{TC}_h(G_2(\mathbb{R}^{2^s+1})) = h \cdot (2^{s+1} - 2) + 1$, when $h \geqslant 2^{s+1} - 1$;
- $\mathrm{TC}_h(F(1^k, 2^s - k + 1)) = h \cdot (k \cdot 2^s - \binom{k}{2}) + 1$, when $h \geqslant k \geqslant 3$.

The paper is organized as follows. In Section 2 we fix the notation and prove several general results that are going to be used in the latter part of the paper. In particular, we prove that $(h-1) \cdot \mathrm{cup}(X) \leqslant \mathrm{zcl}_h(X) \leqslant h \cdot \mathrm{cup}(X)$ and that the sequence $\{h \cdot \mathrm{cup}(X) - \mathrm{zcl}_h(X)\}_{h \geqslant 2}$ is decreasing with h. In Section 3 we determine $\mathrm{zcl}_h(G_2(\mathbb{R}^n))$ for all $2^s < n \leqslant 2^{s+1}$ and $h \geqslant 2^{s+1} - 1$, and as a consequence determine $\mathrm{TC}_h(G_2(\mathbb{R}^{2^s+1}))$ for $h \geqslant 2^{s+1} - 1$ ($s \geqslant 1$). In Section 4 we obtain bounds and exact values of $\mathrm{TC}_h(X)$ for certain semi-complete real flag manifolds X (and certain $h \geqslant 3$), which resolve two questions from [9].

2 Background and some preliminary results

Throughout the paper all cohomology groups are assumed to have coefficients in $\mathbb{Z}_2$.

Let $\pi_i : X^h \to X$, for $1 \leqslant i \leqslant h$, be the i-th projection. Then for $w \in H^*(X)$ we denote $w(i) := \pi_i^*(w) \in H^*(X^h)$. Note that for every $1 \leqslant i < j \leqslant h$ the element $z_{i,j}(w) = w(i) + w(j)$ is in $\mathrm{Ker}\Delta_h^*$. We will call these elements *basic zero-divisors*, and denote by $\mathcal{Z}_B^h \subseteq \mathrm{Ker}\Delta_h^*$ the ideal generated by all these elements. In fact, we have the following result, which generalizes, to the higher realm, Lemma 5.2 of [4].

Lemma 2.1 $\mathcal{Z}_B^h = \mathrm{Ker}\Delta_h^*$.

PROOF — As noted above, $\mathcal{Z}_B^h \subseteq \mathrm{Ker}\Delta_h^*$. To prove the other inclusion, let

$$z = \sum_{i=1}^{t} a_i^{(1)} \otimes a_i^{(2)} \otimes \cdots \otimes a_i^{(h)} \in \mathrm{Ker}\Delta_h^*.$$

Then $\sum_{i=1}^{t} a_i^{(1)} a_i^{(2)} \cdots a_i^{(h)} = 0$. Now, one has:

$$\begin{aligned} z &= \sum_{i=1}^{t} \sum_{\ell=1}^{h-1} \underbrace{1 \otimes \cdots \otimes 1}_{\ell} \otimes a_i^{(\ell+1)} \otimes \cdots \otimes a_i^{(h)} \cdot z_{\ell,\ell+1}\left(a_i^{(1)} \cdots a_i^{(\ell)}\right) \\ &\qquad + \sum_{i=1}^{t} \underbrace{1 \otimes \cdots \otimes 1}_{h-1} \otimes a_i^{(1)} a_i^{(2)} \cdots a_i^{(h)} \\ &= \sum_{i=1}^{t} \sum_{\ell=1}^{h-1} \underbrace{1 \otimes \cdots \otimes 1}_{\ell} \otimes a_i^{(\ell+1)} \otimes \cdots \otimes a_i^{(h)} \cdot z_{\ell,\ell+1}\left(a_i^{(1)} \cdots a_i^{(\ell)}\right) \in \mathcal{Z}_B^h, \end{aligned}$$

which completes our proof. $\square$

Note that $z_{i,j}(w) = z_{1,i}(w) + z_{1,j}(w)$, for $2 \leqslant i < j \leqslant h$, so $\mathrm{Ker}\Delta_h^*$ is in fact generated by the elements $z_{1,i}(w)$. For simplicity we write $z_i(w) := z_{1,i}(w)$. Hence, if $\mathrm{zcl}_h(X) = t$, then there are classes $y_1, y_2, \ldots, y_t \in H^*(X)$ and $i_1, i_2, \ldots, i_t \in \{2, \ldots, h\}$ such that $z_{i_1}(y_1) z_{i_2}(y_2) \cdots z_{i_t}(y_t) \neq 0$.

Notions of height and cup-length will be very useful for obtaining our results. The *height* of a class $c \in \widetilde{H}^*(X)$, denoted by $\mathrm{ht}(c)$, is the supremum of all $m \in \mathbb{N}$ such that $c^m \neq 0$. The *cup-length* of a path connected space X, denoted by $\mathrm{cup}(X)$, is the supremum of all integers d such that there exist classes $a_1, a_2, \ldots, a_d \in \widetilde{H}^*(X)$ with nonzero cup product $a_1 a_2 \cdots a_d$.

Let n be a positive integer and $n = \sum_{i=0}^{t} \alpha_i \cdot 2^i$, where $\alpha_i \in \{0,1\}$ for $0 \leqslant i \leqslant t$ and $\alpha_t = 1$, be its representation in base 2. Then we write $n := (\alpha_t, \ldots, \alpha_1, \alpha_0)_2$. As we use $\mathbb{Z}_2$ coefficient, the following special case of Lucas' theorem will be particularly useful to us: if $n := (\alpha_t, \ldots, \alpha_1, \alpha_0)_2$ and $m := (\beta_r, \ldots, \beta_1, \beta_0)_2$, then

$$\binom{n}{m} \equiv 1 \pmod 2 \qquad \text{if and only if} \qquad t \geqslant r \quad \text{and} \quad \alpha_i \geqslant \beta_i \text{ for } 0 \leqslant i \leqslant r.$$

Let $h \geqslant j$. The previous observation immediately gives the following results that are going to be used throughout the paper. Let $w \in H^*(X)$. Then $\binom{2^m}{i}$ is even for all $1 \leqslant i \leqslant 2^m - 1$, and hence

$$z_j(w)^{2^m} = (w(1) + w(j))^{2^m} = w^{2^m} \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \cdots \otimes 1 \otimes \underbrace{w^{2^m}}_{j} \otimes 1 \otimes \cdots \otimes 1 \in H^*(X)^{\otimes h}$$

(throughout the paper, the number under the brackets indicates the coordinate). On the other hand, $\binom{2^m-1}{i}$ is odd for all $0 \leqslant i \leqslant 2^m - 1$, and hence

$$z_j(w)^{2^m-1} = (w(1) + w(j))^{2^m-1} = \sum_{i=0}^{2^m-1} w^i \otimes \cdots \otimes 1 \otimes \underbrace{w^{2^m-1-i}}_{j} \otimes 1 \otimes \cdots \otimes 1 \in H^*(X)^{\otimes h}.$$

This implies that if $\mathrm{ht}(w)$ is known, then $\mathrm{ht}(z_j(w))$ can easily be calculated (cf. [12, Lemma 4.3]). Namely, one has: if $w \in H^*(X)$ and t is the unique non-negative integer such that $2^t \leqslant \mathrm{ht}(w) < 2^{t+1}$, then

$$\mathrm{ht}(z_j(w)) = 2^{t+1} - 1. \tag{2.1}$$

Next, we prove several general results for $\mathrm{zcl}_h(X)$.

Proposition 2.2 *For $h \geqslant 2$ one has:*

$$(h-1) \cdot \mathrm{cup}(X) \leqslant \mathrm{zcl}_h(X) \leqslant h \cdot \mathrm{cup}(X).$$

PROOF — Let $\mathrm{cup}(X) = \ell$ and $u_1, \ldots, u_\ell \in \widetilde{H}^*(X)$ be such that $u_1 \ldots u_\ell \neq 0$. We prove that

$$A = z_2(u_1) \cdots z_2(u_\ell) z_3(u_1) \cdots z_3(u_\ell) \cdots z_h(u_1) \cdots z_h(u_\ell) \neq 0.$$

Indeed, after expanding, there is exactly one summand in A equal to

$$1 \otimes u_1 \cdots u_\ell \otimes u_1 \cdots u_\ell \otimes \cdots \otimes u_1 \cdots u_\ell$$

and this summand is nonzero. This proves: $\mathrm{zcl}_h(X) \geqslant (h-1) \cdot \mathrm{cup}(X)$.

To prove the other inequality, let us denote $\mathrm{zcl}_h(X) = t$ and let

$$B = z_{a_1}(v_1) z_{a_2}(v_2) \cdots z_{a_t}(v_t) \neq 0,$$

where $v_1, \dots, v_t \in \widetilde{H}^*(X)$. Then there is a nonzero summand after expanding B, and this summand is of the following form

$$\prod_{i \in S_1} v_i \otimes \prod_{i \in S_2} v_i \otimes \cdots \otimes \prod_{i \in S_h} v_i \neq 0,$$

where $(S_1, S_2, \dots, S_h)$ is some partition of the set $\{1, 2, \dots, t\}$. Now, $\prod_{i \in S_j} v_i \neq 0$ for $1 \leqslant j \leqslant h$, implies $|S_j| \leqslant \ell$ for $1 \leqslant j \leqslant h$, and hence $\mathrm{zcl}_h(X) = t \leqslant h\ell = h \cdot \mathrm{cup}(X)$. □

Let us denote

$$\gamma(X, h) := h \cdot \mathrm{cup}(X) - \mathrm{zcl}_h(X).$$

Note that, by Proposition 2.4, $\gamma(X, h) \geqslant 0$ for all $h \geqslant 2$.

Remark 2.3 *For $X := F(1^k, m)$ (see Section 4) in [9] the authors defined the numbers $G(k, m, h) = h \cdot \dim(X) - \mathrm{zcl}_h(X)$ for $h \geqslant 2$. Note that in that case* $\mathrm{cup}(X) = \dim(X)$ *(by (4.3)), so in fact $G(k, m, h) = \gamma(X, h)$. Having in mind Proposition 2.2, we believe that the numbers $\gamma(X, h)$ naturally generalize the numbers $G(k, m, h)$. For example, in what follows we prove that for any fixed X the sequence $\{\gamma(X, h)\}_{h \geqslant 2}$ is monotonically decreasing; this generalizes [9, Corollary 4.8].*

Proposition 2.4 *The sequence $\{\gamma(X, h)\}_{h \geqslant 2}$ is monotonically decreasing.*

PROOF — It is enough to prove $\mathrm{zcl}_{h+1}(X) \geqslant \mathrm{zcl}_h(X) + \mathrm{cup}(X)$ (for every $h \geqslant 2$).

Let $\mathrm{zcl}_h(X) = t$ and $\mathrm{cup}(X) = \ell$. Further, let $z_1, z_2, \dots, z_t \in H^*(X)^{\otimes h}$ be the h-th zero-divisors and $u_1, u_2, \dots, u_\ell \in \widetilde{H}^*(X)$ such that $z_1 z_2 \cdots z_t \neq 0$ and $u_1 u_2 \cdots u_\ell \neq 0$. Then in the expansion of the product

$$A = (z_1 \otimes 1)(z_2 \otimes 1) \cdots (z_t \otimes 1) z_{h+1}(u_1) z_{h+1}(u_2) \cdots z_{h+1}(u_\ell)$$

there is only one summand in $H^*(X) \otimes \cdots \otimes H^*(X) \otimes H^d(X)$ with $d = \deg(u_1 \cdots u_\ell)$. Since this summand is nonzero, we have $A \neq 0$, which completes our proof. □

Having in mind the previous proposition, and the fact that $\gamma(X, h) \geqslant 0$ for $h \geqslant 2$, we define $\gamma(X, \infty) := \lim\limits_{h \to \infty} \gamma(X, h)$.

3 Real Grassmanninans

The cohomology algebra of real flag manifolds was described by Borel in [2]. In order to simplify the notation, we give this description separately in the special cases of real Grassmannians and semi-complete real flag manifolds (but we avoid the details that are not going to be used in the paper).

Let $w_1, w_2, \ldots, w_k$ be the Stiefel-Whitey classes of the canonical k-dimensional vector bundle over $G_k(\mathbb{R}^n)$. Then $H^*(G_k(\mathbb{R}^n))$ is isomorphic to the polynomial algebra $\mathbb{Z}_2[w_1, w_2, \ldots, w_k]$ modulo a certain ideal.

The height of $w_1 \in H^*(G_k(\mathbb{R}^n); \mathbb{Z}_2)$ was obtained by Stong in [18]: if s is the unique non-negative integer such that $2^s < n \leqslant 2^{s+1}$, then

$$\operatorname{ht}(w_1) = \begin{cases} n-1, & \text{if } k=1, \\ 2^{s+1}-2, & \text{if } k=2, \text{ or if } k=3 \text{ and } n=2^s+1, \\ 2^{s+1}-1, & \text{otherwise.} \end{cases} \tag{3.1}$$

Now, (3.1) implies that for $2^s < n \leqslant 2^{s+1}$ and $z_j(w_1) \in H^*(G_k(\mathbb{R}^n))^{\otimes h}$ $(j \leqslant h)$ one has:

$$\operatorname{ht}(z_j(w_1)) = 2^{s+1}-1, \tag{3.2}$$

Although the cup-length of $\mathbb{R}P^{n-1} = G_1(\mathbb{R}^n)$ is obviously equal to $n-1$, obtaining cup-length of $G_k(\mathbb{R}^n)$ for general k is a difficult task. For small k ($k \leqslant 4$), $\operatorname{cup}(G_k(\mathbb{R}^n))$ was calculated by Hiller (see [10]) and Stong (see [18]). In particular, for $k=2$ one has (see [10]): if s is the unique non-negative integer such that $2^s < n \leqslant 2^{s+1}$, then

$$\operatorname{cup}(G_2(\mathbb{R}^n)) = n + 2^s - 3. \tag{3.3}$$

We will also need the following result from [16].

Lemma 3.1 ([16]) *If $2^s < n \leqslant 2^{s+1}$ and $a, b \in \mathbb{N}_0$ are such that $a + 2b = 2(n-2)$, then $w_1^a w_2^b \neq 0$ in $H^{2n-4}(G_2(\mathbb{R}^n))$ if and only if*

$$(a,b) = (2^{l+1}-2, n-2^l-1) \quad \textit{for some } 0 \leqslant l \leqslant s.$$

Furthermore, if $w_1^a w_2^b \in H^{2n-4}(G_2(\mathbb{R}^n))$ is nonzero, then $w_1^a w_2^b = w_2^{n-2}$.

Lemma 3.2 *If $2^s < n \leqslant 2^{s+1}$ and $n = 2^s + t$, then $w_1^{2^s-1} w_2^t = 0$ (in $H^{n+t-1}(G_2(\mathbb{R}^n))$).*

PROOF — By [14, Corollary 2.3], we have

$$g_t = \sum_{a+2b=2^s+2t-1} \binom{a+b-t}{a} w_1^a w_2^b = 0 \quad \text{in } H^*(G_2(\mathbb{R}^n)),$$

where the sum is over $a, b \geqslant 0$ (note that in [14] $G_{2,n}$ denotes the Grassmannian $G_2(\mathbb{R}^{n+2})$ and hence n from [14, Definition 2.1] is replaced with $n-2$ in this proof). Let us examine the binomial coefficient $\binom{a+b-t}{a} = \binom{2^s-1+t-b}{b-t}$. Clearly, for $b < t$ we have $\binom{2^s-1+t-b}{b-t} = 0$, while for $b = t$ we have $\binom{2^s-1+t-b}{b-t} = 1$. So, let $b > t$. Since $a + 2b = 2^s + 2t - 1$, we have $b - t \leqslant 2^{s-1} - 1$. Let i, $0 \leqslant i < s-1$, be such that $b-t$ has digit 1 on position i in the binary expansion. Then $2^s - 1 - (b-t)$ has digit 0 on position i in the binary expansion (since $2^s - 1$ has digits 1 on all positions from 0 to $s-1$), and hence, by Lucas' theorem, $\binom{2^s-1+t-b}{b-t}$ is even. This implies $g_t = w_1^{2^s-1} w_2^t = 0$, as desired. □

Remark 3.3 *The previous lemma can also be proven using the method that was used to prove Lemma 3.1 in [16]. Since this would require a bit of preparation, we decided to use the shorter proof given above.*

Let us observe $H^*(G_k(\mathbb{R}^n))$ and the corresponding ideal $\mathrm{Ker}\Delta_h^*$ for some $h \geqslant 2$ (introduced in Section 2). Then, by Lemma 2.1, $\mathrm{Ker}\Delta_h^*$ is generated by the classes $z_j(w)$, where $2 \leqslant j \leqslant h$ and $w \in H^*(G_k(\mathbb{R}^n))$, but we can prove that it is in fact generated by the classes $z_j(w_i)$, where $2 \leqslant j \leqslant h$ and $1 \leqslant i \leqslant k$. This will be used throughout this section.

Lemma 3.4 *The ideal* $\mathrm{Ker}\Delta_h^*$ *is generated by the classes* $z_j(w_i)$*, where* $2 \leqslant j \leqslant h$ *and* $1 \leqslant i \leqslant k$.

PROOF — Let us denote by $\mathcal{I}_{k,n}^h$ the ideal generated by the classes $z_j(w_i)$, where $2 \leqslant j \leqslant h$ and $1 \leqslant i \leqslant k$. Then, by Lemma 2.1, it is enough to prove that for every $p \in H^*(G_k(\mathbb{R}^n))$ and $2 \leqslant j \leqslant h$, the class $z_j(p)$ is in $\mathcal{I}_{k,n}^h$. Since p is a polynomial in $w_1, w_2, \ldots, w_k$, it is enough to consider the case $p = w_1^{a_1} \cdots w_k^{a_k}$, where $a_i \geqslant 0$ for $1 \leqslant i \leqslant k$.

We prove this by induction on $N(p) = a_1 + \cdots + a_k$. The claim is trivial when $N(p) = 0$. So, suppose that it is true for all q such that $N(q) < \ell$, and prove it for a given monomial $p = w_1^{a_1} \cdots w_k^{a_k}$ such that $N(p) = \ell \geqslant 1$. Then $a_i > 0$ for some $1 \leqslant i \leqslant k$; further, let $p = w_i q$. So, we have

$$z_j(p) = (q \otimes 1 \otimes \cdots \otimes 1) \cdot z_j(w_i) + (1 \otimes \cdots \otimes 1 \otimes \underbrace{w_i}_{j} \otimes 1 \otimes \cdots \otimes 1) \cdot z_j(q),$$

and hence the conclusion follows by induction. □

By the previous lemma we have: if $\mathrm{zcl}_h(G_k(\mathbb{R}^n)) = t$, then there are classes $y_1, y_2, \ldots, y_t \in \{w_1, w_2, \ldots, w_k\}$ and $i_1, i_2, \ldots, i_t \in \{2, \ldots, h\}$ such that $z_{i_1}(y_1) z_{i_2}(y_2) \cdots z_{i_t}(y_t) \neq 0$.

Remark 3.5 *Using a similar proof one can prove that an analogous result holds for all real flag manifolds.*

By the result of Berstein from [1], Grassmann manifold $X = G_k(\mathbb{R}^n)$ satisfies $\dim(X) = \mathrm{cup}(X)$ if and only if $k = 1$, or $k = 2$ and $n = 2^s + 1$ for some $s \geqslant 1$. Having in mind Proposition 2.2, we conclude that these are the only real Grassmann manifolds for which the upper bound in Proposition 1.1 can be equal to $\mathrm{TC}_h(X)$ for some $h \geqslant 2$ (in other words, these are the only cases in which the cohomology method can lead to $\mathrm{TC}_h(X) = h \cdot \dim(X) + 1$).

If $X = G_k(\mathbb{R}^n)$, then we denote $\gamma(k,n,h) := \gamma(X,h)$ and $\gamma(k,n,\infty) := \gamma(X,\infty)$.

The case $k = 1$ was resolved in [5], where the formula for $\mathrm{zcl}_h(G_1(\mathbb{R}^n))$ was obtained for every $h \geqslant 3$ and $n \geqslant 2$. In particular, their result implies that for every odd $n \geqslant 3$ one has $\gamma(1,n,\infty) = 0$, while for every even $n \geqslant 2$ one has $\gamma(1,n,\infty) > 0$ (in fact, this was first proven in [3, Theorem 5.7], where the values $\gamma(1,n,\infty)$ were obtained for all $n \geqslant 2$).

In what follows we examine the case $k = 2$. Determining the exact values of $\mathrm{zcl}_h(G_2(\mathbb{R}^n))$ proved to be very difficult; even in the case $h = 2$ these are not known for all $n \geqslant 3$ (see [16] for some partial results). In this section we consider a related problem, that is, we determine the numbers $\gamma(2,n,\infty)$ for all $n \geqslant 3$ (cf. [3, Theorem 5.7]). In particular, we prove that $\gamma(2,n,\infty) = 0$ for every odd $n \geqslant 3$, and that $\gamma(2,n,\infty) > 0$ for every even $n \geqslant 4$.

For a positive integer m, let us denote with $e(m)$ the number of consecutive ones ending the binary expansion of m. So, if m is even, then $e(m) = 0$.

Proposition 3.6 *Let* $n, s, t \in \mathbb{N}$ *be such that* $n = 2^s + t$ *and* $1 \leqslant t \leqslant 2^s$. *Then*

$$\gamma(2,n,\infty) = 2^e - 1,$$

where $e := e(t-1)$. Further, $\gamma(2,n,h) = 2^e - 1$ and $\mathrm{TC}_h(G_2(\mathbb{R}^n)) \geqslant h(2^{s+1}+t-3) - 2^e + 2$ for every $h \geqslant 2^{s+1}-1$.

PROOF — We have $t-1 = 2^e - 1 + 2^{e+1}r$, i.e. $t = 2^e(2r+1)$, for some $r \geqslant 0$. In particular, $e \leqslant s$ (since $t \leqslant 2^s$).

We begin with some general observations. Let $a_2, b_2, a_3, b_3, \ldots, a_h, b_h \geqslant 0$ be such that

$$A = \prod_{i=2}^{h} z_i(w_1)^{a_i} z_i(w_2)^{b_i} \neq 0,$$

and let $p = w_1^{c_1} w_2^{d_1} \otimes w_1^{c_2} w_2^{d_2} \otimes \cdots \otimes w_1^{c_h} w_2^{d_h}$ be a nonzero summand after expanding A. Then

$$\sum_{i=1}^{h} c_i = \sum_{i=2}^{h} a_i \qquad \text{and} \qquad \sum_{i=1}^{h} d_i = \sum_{i=2}^{h} b_i.$$

Claim 1. $\sum_{i=2}^{h} a_i \leqslant \min\{(h-1)(2^{s+1}-1), h(2^{s+1}-2)\}$.

Proof of Claim 1. Since $z_i(w_1)^{2^{s+1}} = 0$ (by (3.2)), we have $a_i \leqslant 2^{s+1}-1$ for all $2 \leqslant i \leqslant h$, and hence $\sum_{i=2}^{h} a_i \leqslant (h-1)(2^{s+1}-1)$. Also, by (3.1), $c_i \leqslant 2^{s+1}-2$, and hence $\sum_{i=2}^{h} a_i = \sum_{i=1}^{h} c_i \leqslant h(2^{s+1}-2)$, which concludes the proof of Claim 1.

Claim 2. If $d_i \leqslant t-1$ for all $1 \leqslant i \leqslant h$, then $\sum_{i=2}^{h} b_i \leqslant h(t-1) - 2^e + 1$.

Proof of Claim 2. Assume to the contrary that $\sum_{i=2}^{h} b_i \geqslant h(t-1) - 2^e + 2$. Let us denote $b_i = t - 1 + \delta_i$ for $2 \leqslant i \leqslant h$ (it is possible that $\delta_i < 0$ for some $2 \leqslant i \leqslant h$). Further, assume that exactly ℓ of the numbers δ_i, $2 \leqslant i \leqslant h$, are positive, and without loss of generality let these numbers be $\delta_2, \delta_3, \ldots, \delta_{\ell+1}$. Then

$$\sum_{i=2}^{h} b_i = \sum_{i=2}^{h} (t-1+\delta_i) \geqslant h(t-1) - 2^e + 2$$

implies

$$\sum_{i=2}^{\ell+1} \delta_i \geqslant \sum_{i=2}^{h} \delta_i \geqslant t + 1 - 2^e.$$

Now, let us prove that if $\delta_i > 2^e u$ for some $u \geqslant 0$ (and $2 \leqslant i \leqslant \ell+1$), then the term $z_i(w_2)^{b_i} = z_i(w_2)^{t-1+\delta_i}$ "contributes" to d_1 with at least $(u+1)2^e$. Note that

$$z_i(w_2)^{t-1+\delta_i} = z_i(w_2)^{2^e(2r+u+1)} z_i(w_2)^{\delta_i - 2^e u - 1} = z_i(w_2^{2^e})^{2r+u+1} z_i(w_2)^{\delta_i - 2^e u - 1},$$

and that the term $z_i(w_2^{2^e})^{2r+u+1}$ is a nonzero sum of summands of the form $w_2^{2^e m} \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{w_2^{(2r+u+1-m)2^e}}_{i} \otimes 1 \otimes \cdots \otimes 1$ $(0 \leqslant m \leqslant 2r+u+1)$. Since $d_i \leqslant t-1 = 2^e(2r+1)-1$, this implies $m \geqslant u+1$ for the summand that is used to obtain p, and hence this term "contributes" to d_1 with at least $(u+1)2^e$.

So, for $c \geqslant 1$, let n_c be the number of integers $\delta_2, \delta_3, \ldots, \delta_{\ell+1}$ that are in the interval

$$\{2^e(c-1)+1, 2^e(c-1)+2, \ldots, 2^e c\}.$$

Then

$$2^{e+1}r = t - 2^e < \sum_{i=2}^{\ell+1} \delta_i \leqslant \sum_{c\geqslant 1} n_c 2^e c, \qquad \text{i.e.,} \qquad 2r+1 \leqslant \sum_{c\geqslant 1} c n_c,$$

and from the previous observation we have

$$t-1 \geqslant d_1 \geqslant \sum_{c\geqslant 1} n_c 2^e c \geqslant 2^e(2r+1) = t,$$

a contradiction. This completes the proof of Claim 2.

Next, we prove that $\gamma(2, n, 2^{s+1}-1) \leqslant 2^e - 1$. By (3.3), $\operatorname{cup}(G_2(\mathbb{R}^n)) = 2^{s+1} + t - 3$, so we want to prove that $\operatorname{zcl}_d(G_2(\mathbb{R}^n)) \geqslant d(d+t-2) - 2^e + 1$, where $d = 2^{s+1} - 1$.

We do this by showing

$$B_1 = \left(\prod_{i=2}^{2r+1} z_i(w_1)^{2^{s+1}-1} z_i(w_2)^{t-1+2^e}\right)\left(\prod_{i=2r+2}^{2^{s+1}-1} z_i(w_1)^{2^{s+1}-1} z_i(w_2)^{t-1}\right) \neq 0$$

(indeed, this is a product of $2r(d+t-1+2^e)+(d-2r-1)(d+t-1) = d(d+t-2)-t+1+2^{e+1}r = d(d+t-2)-2^e+1$ basic d-th zero-divisors). Let

$$q_1 = w_1^{2^{s+1}-2} w_2^{t-2^e} \otimes w_1^{2^{s+1}-2} w_2^{t-1} \otimes \cdots \otimes w_1^{2^{s+1}-2} w_2^{t-1}.$$

By Lemma 3.1, $w_1^{2^{s+1}-2} w_2^{t-1} = w_2^{n-2} \neq 0$, and hence $w_1^{2^{s+1}-2} w_2^{t-2^e} \neq 0$. So, $q_1 \neq 0$. Hence, to prove that $B_1 \neq 0$, it is enough to show that after expanding B_1 the class q_1 is the only nonzero summand that has all the coordinates from the second to the last *equal* to w_2^{n-2} (indeed, by Lemma 3.1, w_2^{n-2} is the only nonzero class in $H^{2n-4}(G_2(\mathbb{R}^n))$). Note that if $e \geqslant s-1$, then $r = 0$ (since $t \leqslant 2^s$), and hence the first product is empty. By Lemma 3.1, if $w_1^a w_2^b = w_2^{n-2}$ and $b \neq t-1$, then $b \geqslant t-1+2^{s-1}$, and hence $b > t-1+2^e$ for $e < s-1$. So, when multiplying to obtain B_1 to have w_2^{n-2} on the i-th coordinate, where $2 \leqslant i \leqslant 2r+1$, we must choose $w_1 \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{w_1^{2^{s+1}-2}}_{i} \otimes 1 \otimes \cdots \otimes 1$ from $z_i(w_1)^{2^{s+1}-1}$ and $w_2^{2^e} \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{w_2^{t-1}}_{i} \otimes 1 \otimes \cdots \otimes 1$ from $z_i(w_2)^{t-1+2^e}$ (the second one has coefficient 1 since $\binom{t-1+2^e}{t-1} = \binom{2^{e+1}r+2^e+2^e-1}{2^{e+1}r+2^e-1}$ is odd by Lucas' theorem); to have w_2^{n-2} on the i-th coordinate, where $2r+2 \leqslant i \leqslant d$, we must choose $w_1 \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{w_1^{2^{s+1}-2}}_{i} \otimes 1 \otimes \cdots \otimes 1$ from $z_i(w_1)^{2^{s+1}-1}$ and $1 \otimes \cdots \otimes 1 \otimes \underbrace{w_2^{t-1}}_{i} \otimes 1 \otimes \cdots \otimes 1$ from $z_i(w_2)^{t-1}$. It follows that this summand has $w_1^{2^{s+1}-2} w_2^{2^e \cdot 2r} = w_1^{2^{s+1}-2} w_2^{t-2^e}$ on the first coordinate, and is hence equal to q_1 as desired.

To finish the proof it is enough to prove $\gamma(2, n, h) \geqslant 2^e - 1$ for all $h \geqslant 2$. We divide this proof in two cases.

Case 1: $t \neq 2^s$. Then $e \leqslant s-1$.

Assume that $N = \mathrm{zcl}_h(G_2(\mathbb{R}^n)) \geqslant h\cdot \mathrm{cup}(G_2(\mathbb{R}^n)) - 2^e + 1$ for some $h \geqslant 2$. By (3.3), $\mathrm{cup}(G_2(\mathbb{R}^n)) = 2^{s+1} + t - 3$, so $N \geqslant h\cdot(2^{s+1}+t-3) - 2^e + 1$. Let $a_2, b_2, a_3, b_3, \dots, a_h, b_h \geqslant 0$ be such that $a_2 + b_2 + a_3 + b_3 + \cdots + a_h + b_h = N$ and

$$B_2 = \prod_{i=2}^{h} z_i(w_1)^{a_i} z_i(w_2)^{b_i} \neq 0.$$

Let $q_2 = w_1^{c_1}w_2^{d_1} \otimes w_1^{c_2}w_2^{d_2} \otimes \cdots \otimes w_1^{c_h}w_2^{d_h}$ be a nonzero summand after expanding B_2.

We prove that $d_i \leqslant t-1$ for all $1 \leqslant i \leqslant h$. Suppose that this is not the case, and let $d_j \geqslant t$ for some $1 \leqslant j \leqslant h$. Then, by Lemma 3.2, $w_1^{c_j}w_2^{d_j} \neq 0$ implies $c_j \leqslant 2^s - 2$. Also, $c_j + 2d_j \leqslant 2(n-2)$, and hence $2(c_j + d_j) \leqslant 2(n-2) + 2^s - 2$, i.e. $c_j + d_j \leqslant n + 2^{s-1} - 3 = \mathrm{cup}(G_2(\mathbb{R}^n)) - 2^{s-1}$. Further, $w_1^{c_i}w_2^{d_i} \neq 0$ implies $c_i + d_i \leqslant \mathrm{cup}(G_2(\mathbb{R}^n))$ for all $1 \leqslant i \leqslant h$, and hence

$$N = c_j + d_j + \sum_{i\neq j}(c_i + d_i) \leqslant h\cdot \mathrm{cup}(G_2(\mathbb{R}^n)) - 2^{s-1} \leqslant h\cdot \mathrm{cup}(G_2(\mathbb{R}^n)) - 2^e < N,$$

a contradiction.

So, by Claims 1 and 2 we have

$$N = \sum_{i=2}^{h}(a_i + b_i) \leqslant \min\{(h-1)(2^{s+1}-1), h(2^{s+1}-2)\} + h(t-1) - 2^e + 1, \tag{3.4}$$

and hence $N = \mathrm{zcl}_h(G_2(\mathbb{R}^n)) = h\cdot(2^{s+1}+t-3) - 2^e + 1$, which completes our proof (see also Remark 3.7).

Case 2. $t = 2^s$.

Assume to the contrary that $N = \mathrm{zcl}_h(G_2(\mathbb{R}^n)) \geqslant h\cdot \mathrm{cup}(G_2(\mathbb{R}^n)) - 2^s + 2$ for some $h \geqslant 2$. By (3.3), $\mathrm{cup}(G_2(\mathbb{R}^n)) = 2^{s+1} + 2^s - 3$, so $N \geqslant h\cdot(2^{s+1} + 2^s - 3) - 2^s + 2$. Let $a_2, b_2, a_3, b_3, \dots, a_h, b_h \geqslant 0$ be such that $a_2 + b_2 + a_3 + b_3 + \cdots + a_h + b_h = N$ and

$$B_2 = \prod_{i=2}^{h} z_i(w_1)^{a_i} z_i(w_2)^{b_i} \neq 0.$$

Let $q_2 = w_1^{c_1}w_2^{d_1} \otimes w_1^{c_2}w_2^{d_2} \otimes \cdots \otimes w_1^{c_h}w_2^{d_h}$ be a nonzero summand after expanding B_2.

Let us first assume that $d_i \leqslant t-1$ for all $1 \leqslant i \leqslant h$. Then, by Claims 1 and 2 we get

$$N = \sum_{i=2}^{h}(a_i + b_i) \leqslant \min\{(h-1)\cdot(2^{s+1}-1), h\cdot(2^{s+1}-2)\} + h(2^s - 1) - 2^s + 1,$$

which contradicts $N \geqslant h\cdot(2^{s+1}+2^s-3) - 2^s + 2$. So, there is at least one among $d_1, d_2, \dots, d_h$ that is at least $t = 2^s$. Suppose that there are two of them, $d_{i'}, d_{i''} \geqslant 2^s$, where $1 \leqslant i' < i'' \leqslant h$. Then, by Lemma 3.2, for $j \in \{i', i''\}$ we have $c_j \leqslant 2^s - 2$, and hence $2(c_j + d_j) = c_j + (c_j + 2d_j) \leqslant 2^s - 2 + 2(2^{s+1}-2) = 2^{s+2} + 2^s - 6$, i.e. $c_j + d_j \leqslant 2^{s+1} + 2^{s-1} - 3 = \mathrm{cup}(G_2(\mathbb{R}^n)) - 2^{s-1}$. So,

$$N = c_{i'} + d_{i'} + c_{i''} + d_{i''} + \sum_{i\notin\{i',i''\}}(c_i + d_i) \leqslant h\cdot \mathrm{cup}(G_2(\mathbb{R}^n)) - 2^s < N,$$

a contradiction. So, let $1 \leqslant j \leqslant h$ be the unique index such that $d_j \geqslant 2^s$. Again, by Lemma 3.2, $c_j \leqslant 2^s - 2$, so, by (3.1), we have

$$\sum_{i=2}^{h} a_i = \sum_{i=1}^{h} c_i = c_j + \sum_{i \neq j} c_i \leqslant 2^s - 2 + (h-1)(2^{s+1} - 2). \tag{3.5}$$

Next, we prove that at most one of $b_2, b_3, \ldots, b_h$ is greater than $2^s - 1$. Assume to the contrary that $b_{i'}, b_{i''} \geqslant 2^s$ for some $2 \leqslant i' < i'' \leqslant h$. Then, when we multiply terms from B_2 to obtain q_2 we use the summand

$$\binom{b_{i'}}{d_{i'}}\binom{b_{i''}}{d_{i''}} w_2^{b_{i'}+b_{i''}-d_{i'}-d_{i''}} \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{w_2^{d_{i'}}}_{i'} \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{w_2^{d_{i''}}}_{i''} \otimes 1 \otimes \cdots \otimes 1$$

from the product $z_{i'}(w_2)^{b_{i'}} z_{i''}(w_2)^{b_{i''}}$. Let us examine the binomial coefficients $\binom{b_{i'}}{d_{i'}}$ and $\binom{b_{i''}}{d_{i''}}$. Note that $b_{i'} \geqslant 2^s$ (resp. $b_{i''} \geqslant 2^s$) implies that $b_{i'}$ (resp. $b_{i''}$) has digit 1 on position s in the binary expansion. Also, by Lucas' theorem, $d_{i'}$ (resp. $d_{i''}$) has digits 1 in the binary expansion only on position on which so does $b_{i'}$ (resp. $b_{i''}$), so we have $d_{i'} \geqslant 2^s$ or $b_{i'} - d_{i'} \geqslant 2^s$ (resp. $d_{i''} \geqslant 2^s$ or $b_{i''} - d_{i''} \geqslant 2^s$). Now, since at most one of $d_{i'} \geqslant 2^s$ and $d_{i''} \geqslant 2^s$ holds, we have $b_{i'} - d_{i'} \geqslant 2^s$ or $b_{i''} - d_{i''} \geqslant 2^s$ and hence $d_1 \geqslant b_{i'} - d_{i'} + b_{i''} - d_{i''} \geqslant 2^s$. So, $d_{i'}, d_{i''} < 2^s$, and hence $b_{i'} - d_{i'}, b_{i''} - d_{i''} \geqslant 2^s$, i.e. $d_1 \geqslant 2^{s+1}$, which is impossible, since $\mathrm{ht}(w_2) = 2^{s+1} - 2$ (by Lemma 3.1).

Hence, at most one of $b_2, b_3, \ldots, b_h$ is greater than $2^s - 1$. Also, $\mathrm{ht}(w_2) = 2^{s+1} - 2$, and hence (2.1) implies $b_i \leqslant \mathrm{ht}(z_i(w_2)) = 2^{s+1} - 1$ for $2 \leqslant i \leqslant h$. So,

$$\sum_{i=2}^{h} b_i \leqslant 2^{s+1} - 1 + (h-2) \cdot (2^s - 1). \tag{3.6}$$

Adding (3.5) and (3.6) gives

$$N = \sum_{i=2}^{h} (a_i + b_i) \leqslant h(2^{s+1} + 2^s - 3) - 2^s + 1,$$

which is a contradiction.

Finally, by Proposition 1.1, $\gamma(2, n, h) = 2^e - 1$ for $h \geqslant 2^{s+1} - 1$ implies $\mathrm{TC}_h(G_2(\mathbb{R}^n)) \geqslant h(2^{s+1} + t - 3) - 2^e + 2$ for $h \geqslant 2^{s+1} - 1$. $\square$

Remark 3.7 *Let us observe the inequality in (3.4) (in the case $t \neq 2^s$). As noted in the following line, this inequality must be equality, and hence $(h-1)(2^{s+1} - 1) \geqslant h(2^{s+1} - 2)$, i.e. $h \geqslant 2^{s+1} - 1$. This proves that for $n = 2^s + t$ and $1 \leqslant t < 2^s$ one has*

$$2^{s+1} - 1 = \min\{h \geqslant 2 \,:\, \gamma(2, n, h) = \gamma(2, n, \infty) = 2^e - 1\}.$$

However, we were not able to determine $\min\{h \geqslant 2 \,:\, \gamma(2, 2^{s+1}, h) = \gamma(2, 2^{s+1}, \infty)\}$ *(by the previous proposition it is at most $2^{s+1} - 1$).*

Corollary 3.8 *Let $s \geqslant 1$. Then for every $h \geqslant 2^{s+1} - 1$ one has*

$$\mathrm{TC}_h(G_2(\mathbb{R}^{2^s+1})) = h \cdot (2^{s+1} - 2) + 1.$$

PROOF — Since $\mathrm{cup}(G_2(\mathbb{R}^{2^s+1})) = 2^{s+1} - 2 = \dim(G_2(\mathbb{R}^{2^s+1}))$, the result follows from Propositions 1.1 and 3.6. □

At the end of this section, let us briefly compare the lower and the upper bound for $\mathrm{TC}_h(G_k(\mathbb{R}^n))$ from Proposition 1.1 for general $n \geqslant 2k \geqslant 4$. Also, let $2^s < n \leqslant 2^{s+1}$.

To do so, we first obtain a simple upper bound for $\ell := \mathrm{cup}(G_k(\mathbb{R}^n))$. Let $a_1, \ldots, a_k \in \mathbb{N}_0$ be such that $\sum_{i=1}^{k} a_i = \ell$ and $w_1^{a_1} \cdots w_k^{a_k} \neq 0$. Then $a_1 + 2a_2 + \cdots + ka_k \leqslant \dim(G_k(\mathbb{R}^n)) = k(n-k)$ and $a_1 \leqslant \mathrm{ht}(w_1) \leqslant 2^{s+1} - 1$ (by (3.1)), and hence

$$\ell \leqslant \frac{1}{2}\left(a_1 + a_1 + 2a_2 + \cdots + ka_k\right) \leqslant \frac{k(n-k) + 2^{s+1} - 1}{2} \leqslant \frac{k(n-k) + 2n - 3}{2}.$$

Since $n \geqslant 2k$, it can be easily proven that

$$\frac{2n-3}{k(n-k)} \leqslant \frac{4k-3}{k^2},$$

which together with the previous inequality and Proposition 2.2 gives:

$$\mathrm{zcl}_h(G_k(\mathbb{R}^n)) \leqslant h\ell \leqslant \left(\frac{1}{2} + \frac{4k-3}{2k^2}\right) \cdot h \dim(G_k(\mathbb{R}^n)).$$

So, for large k there is quite a gap between the lower and the upper bound for $\mathrm{TC}_h(G_k(\mathbb{R}^n))$ from Proposition 1.1. This suggests that the cohomology method is not that efficient for finding $\mathrm{TC}_h(G_k(\mathbb{R}^n))$ for general k.

4 Semi-complete real flag manifolds

Let $k, m \in \mathbb{N}$. For $F(1^k, m)$ there are k canonical line bundles over it; let x_i, for $1 \leqslant i \leqslant k$, denote the first Stiefel-Whitney classes of these line bundles. Then, by Borel's description, $H^*(F(1^k, m))$ is isomorphic to the polynomial algebra $\mathbb{Z}_2[x_1, \ldots, x_k]$ modulo a certain ideal.

The heights of $x_i \in H^*(F(1^k, m))$ are known due to Korbaš and Lörinc (see [11]):

$$\mathrm{ht}(x_i) = m + k. \tag{4.1}$$

Let $2^s \leqslant m + k < 2^{s+1}$. The previous result together with (2.1) implies that for $z_j(x_i) \in H^*(F(1^k, m))^{\otimes h}$ ($1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant h$):

$$\mathrm{ht}(z_j(x_i)) = 2^{s+1} - 1. \tag{4.2}$$

For the cup-length of $F(1^k, m)$ we have the following result (see, e.g. [9]):

$$\mathrm{cup}(F(1^k, m)) = km + \binom{k}{2} = \dim(F(1^k, m)). \tag{4.3}$$

In the following proposition we give an additive basis for $H^*(F(1^k, m))$ in terms of the Stiefel-Whitney classes (see, e.g. [9]).

Proposition 4.1 *The set*

$$B_{k,m} = \{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \,:\, n_i \leqslant m+i-1 \text{ for } 1 \leqslant i \leqslant k\}$$

is an additive basis for $H^*(F(1^k, m))$.

Note that the previous proposition implies that $x_1^m x_2^{m+1} \cdots x_k^{m+k-1}$ is the only non-zero class of $H^N(F(1^k, m))$, where $N = \dim(F(1^k, m)) = mk + \binom{k}{2}$. So, by symmetry, for every permutation π of the set $\{0, 1, \ldots, k-1\}$, we have

$$x_1^{m+\pi(0)} x_2^{m+\pi(1)} \cdots x_k^{m+\pi(k-1)} = x_1^m x_2^{m+1} \cdots x_k^{m+k-1} \neq 0. \tag{4.4}$$

The detailed treatment of the algebra $H^*(F(1^k, m))$ (beyond Proposition 4.1) can be found in [9] and [15]. Here, we will need the following result (see, e.g. [9, Corollary 2.4]):

$$x_1^{m+k-1} x_2^{m+k-1} = 0. \tag{4.5}$$

Finally, by Remark 3.5, the ideal of h-th zero-divisors is generated by the classes $z_j(x_i)$ for $2 \leqslant j \leqslant h$ and $1 \leqslant i \leqslant k$. In particular, if $\mathrm{zcl}_h(F(1^k, m)) = t$, then there are classes $y_1, y_2, \ldots, y_t \in \{x_1, x_2, \ldots, x_k\}$ and $i_1, i_2, \ldots, i_t \in \{2, \ldots, h\}$ such that $z_{i_1}(y_1) z_{i_2}(y_2) \cdots z_{i_t}(y_t) \neq 0$. We use this throughout the remaining of the section.

As in [9], we denote $G(k, m, h) := \gamma(F(1^k, m), h)$. Note that, by Proposition 1.1, $G(k, m, h) = 0$ immediately implies $\mathrm{TC}_h(F(1^k, m)) = h \cdot \dim(F(1^k, m)) + 1 = h \cdot (km + \binom{k}{2}) + 1$.

Theorem 4.2 *For every* $h \geqslant k \geqslant 3$ *and* $s \in \mathbb{N}$ *such that* $2^s \geqslant k$ *one has* $G(k, 2^s - k + 1, h) = 0$, *that is*

$$\mathrm{TC}_h(F(1^k, 2^s - k + 1)) = h\left(k \cdot 2^s - \binom{k}{2}\right) + 1.$$

PROOF — By Proposition 2.4, it is enough to prove $G(k, 2^s - k + 1, k) = 0$. Let

$$A_2 = z_2(x_1)^{2^{s+1}-1} z_2(x_2)^{2^{s+1}-k+1} \prod_{i=3}^{k} z_2(x_i)^{2^s - i + 1}$$

$$A_j = z_j(x_j)^{2^{s+1}-k+j-1} \prod_{i=1}^{j-1} z_j(x_i)^{2^s - i} \prod_{i=j+1}^{k} z_j(x_i)^{2^s - i + 1} \qquad \text{for } 3 \leqslant j \leqslant k,$$

and

$$A = A_2 A_3 \cdots A_k.$$

It is easy to see that A is a product of $k(k \cdot 2^s - \binom{k}{2}) = k \cdot \dim(F(1^k, 2^s - k + 1))$ zero-divisors, so it is enough to prove $A \neq 0$.

Let $y = x_1^{2^s} x_2^{2^s - 1} \cdots x_k^{2^s - k + 1}$. By Proposition 4.1, $y \neq 0$, so for $A \neq 0$ it is enough to prove that A contains the summand

$$p = y \otimes y \otimes \cdots \otimes y \neq 0$$

with coefficient 1. To do so, we show that after expanding A there is a unique summand equal to p.

Let $m = m_1 \otimes \cdots \otimes m_k$ be some (if it exists) summand of A that is equal to p. First, for $j \geqslant 3$ let us observe the monomial m_j. The maximal degree of x_i in it is: $2^s - i$ for

$1 \leqslant i \leqslant j-1$, $2^s - i + 1$ for $j+1 \leqslant i \leqslant k$, and 2^s for $j = i$ (since $x_j^{2^s+1} = 0$ by (4.1)). Since m_j is equal to y, by comparing the dimensions of m_j and y, we conclude that the degree of each x_i must have the corresponding maximal value; further, then $m_j = y$ (by (4.4)) and $z_j(x_j)^{2^{s+1}-k+j-1}$ "contributes" with $\binom{2^s+2^s-k+j-1}{2^s} x_j^{2^s-k+j-1} = x_j^{2^s-k+j-1}$ to m_1 (by Lucas' theorem $\binom{2^s+2^s-k+j-1}{2^s}$ is odd).

Let us now observe the monomial m_2. Clearly, for $i \geqslant 3$ the maximal degree of x_i in m_2 is $2^s - i + 1$, and the maximal degree of x_1 in m_2 is 2^s (since $x_1^{2^s+1} = 0$). Let us observe the degree a of x_2. Then $\binom{2^s+2^s-k+1}{a}$ is odd, so by Lucas' theorem either $a \geqslant 2^s$ or $a \leqslant 2^s - k + 1$; further, since $x_2^{2^s+1} = 0$ (by (4.1)), we conclude that $a = 2^s$ or $a \leqslant 2^s - k + 1$. But if $a \leqslant 2^s - k + 1$, then the total degree of m_2 is at most $k \cdot 2^s - \binom{k}{2} - k + 2$, which is less that the dimension of y (which is $k \cdot 2^s - \binom{k}{2}$), and hence $a = 2^s$. Finally, let us observe the degree b of x_1 in m_2. As mentioned above $b \leqslant 2^s$; further, $x_1^{2^{s+1}-1-b}$ is a factor of m_1, and hence $2^{s+1} - 1 - b \leqslant 2^s$. So, $b \in \{2^s - 1, 2^s\}$, and since $x_1^{2^s} x_2^{2^s} = 0$ (by (4.5)), we must have $b = 2^s - 1$; additionally, the coefficient of m_2 is $\binom{2^{s+1}-1}{2^s-1}$, which is odd (by Lucas' theorem), and so $m_2 = y$ (by (4.4)). Also, we note that $z_2(x_1)^{2^{s+1}-1} z_2(x_2)^{2^{s+1}-k+1}$ "contributes" with $x_1^{2^s} x_2^{2^s-k+1}$ to m_1.

Finally, $m_1 = x_1^{2^s} x_2^{2^s-k+1} \prod_{i=3}^{k} x_i^{2^s-k+j-1}$, which is equal to y (by (4.4)). Hence, there is a unique summand equal to p after expanding A, which completes our proof. □

Remark 4.3 *This theorem extends [9, Theorem 4.3] (in a way described in Remark 4.14 of the same paper).*

In what follows we consider the semi-complete real flag manifolds $F(1,1,2^s)$ for $s \geqslant 2$. In fact, we will prove that for all $s \geqslant 2$ and $h \geqslant 3$ one has $G(2, 2^s, h) = 1$. This was conjectured in [9, p. 372].

We will need the following simple observation about $H^*(F(1,1,2^s))$:

$$x_1^a x_2^b \in H^{2^{s+1}+1}(F(1,1,2^s)) \text{ is nonzero} \quad \text{if and only if} \quad \{a,b\} = \{2^s, 2^s+1\}. \tag{4.6}$$

Indeed, since $\mathrm{ht}(x_1) = \mathrm{ht}(x_2) = 2^s + 1$ (by (4.1)), $x_1^a x_2^b \neq 0$ implies $a, b \leqslant 2^s + 1$, which, together with $a + b = 2^{s+1} + 1$, gives $\{a,b\} = \{2^s, 2^s+1\}$. The other direction follows from Proposition 4.1.

Proposition 4.4 *For every $s \geqslant 2$ and $h \geqslant 3$ one has $G(2, 2^s, h) = 1$.*

PROOF — By [9, Theorem 4.11], $G(2, 2^s, h) \leqslant 1$, so it is enough to prove that $G(2, 2^s, h) \neq 0$. Suppose that this is not the case, and let

$$A = \prod_{i=2}^{h} z_i(x_1)^{a_i} z_i(x_2)^{b_i} \neq 0$$

be such that $\sum_{i=2}^{h}(a_i + b_i) = h \cdot \dim(F(1,1,2^s)) = h(2^{s+1}+1)$. Then, by comparing dimensions and using Proposition 4.1, we have that $A = y \otimes y \otimes \cdots \otimes y \neq 0$, where $y = x_1^{2^s} x_2^{2^s+1}$. Let $p = y \otimes y \otimes \cdots \otimes y$.

So, let us observe how a summand equal to p is obtained after expanding A. In general, to obtain a summand in A, for each $2 \leqslant i \leqslant h$ one chooses a summand

$$\binom{a_i}{c_i}\binom{b_i}{d_i} x_1^{a_i-c_i} x_2^{b_i-d_i} \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{x_1^{c_i} x_2^{d_i}}_{i} \otimes 1 \otimes \cdots \otimes 1 \tag{4.7}$$

from $z_i(x_1)^{a_i} z_i(x_2)^{b_i}$ and then multiply all of them. Let us now suppose that the summand m that we obtain is equal to p. We denote by m_i the monomial on the i-th coordinate of m, for $1 \leqslant i \leqslant h$. Clearly, for $i \geqslant 2$, $m_i = x_1^{c_i} x_2^{d_i}$, and hence $\{c_i, d_i\} = \{2^s, 2^s+1\}$ (by (4.6)). In particular $a_i, b_i \geqslant 2^s$.

Suppose that for some $i \geqslant 2$, a_i is even. Since $\binom{a_i}{c_i}$ must be odd, by Lucas' theorem, c_i must also be even, i.e. $c_i = 2^s$. This further implies that $d_i = 2^s + 1$ and that b_i is odd (since $\binom{b_i}{d_i}$ is odd). Additionally, both $a_i - c_i$ and $b_i - d_i$ are even. Similarly, if b_i is even, then a_i must be odd, $d_i = 2^s$, $c_i = 2^s + 1$, and both $a_i - c_i$ and $b_i - d_i$ are even. Note that in both cases, i.e. a_i even and b_i odd, and a_i odd and b_i even, the numbers c_i and d_i are uniquely determined (also, the case when both a_i and b_i are even is not possible).

Suppose now that for some $i \geqslant 2$ both a_i and b_i are odd. Since $c_i + d_i = 2^{s+1} + 1$ is odd, then exactly one of $a_i - c_i$ and $b_i - d_i$ is odd.

Consider now m_1. We have $m_1 = x_1^a x_2^b$, where $a = \sum_{i=2}^h (a_i - c_i)$ and $b = \sum_{i=2}^h (b_i - d_i)$, and since $m_1 = y$, by (4.6), we have $\{a, b\} = \{2^s, 2^s + 1\}$. In particular, exactly one of a and b is odd, which implies that there is an odd number $2j - 1$ of indices $i \geqslant 2$ such that both a_i and b_i are odd. Without loss of generality, assume that a_i, b_i are both odd for all $2 \leqslant i \leqslant 2j \leqslant k$. Further, let $a_i = 2^s + 2\alpha_i + 1$ and $b_i = 2^s + 2\beta_i + 1$ for $2 \leqslant i \leqslant 2j$.

Finally, we prove that the number of summands in A that are equal to p is even. Suppose that we have "chosen" summands of the form (4.7) for each $i \geqslant 2j + 1$ (this can be done in the unique way). So, it is enough to prove that then the number of ways we can choose summands from $z_i(x_1)^{a_i} z_i(x_2)^{b_i}$ for $2 \leqslant i \leqslant 2j$, to obtain a product equal to p is even. Denote $\alpha = \sum_{i=2j+1}^h (a_i - c_i)$ and $\beta = \sum_{i=2j+1}^h (b_i - d_i)$ (note that α and β are even). Now, for each $2 \leqslant i \leqslant 2j$ we choose $\gamma_i \in \{0, 1\}$ and $\delta_i = 1 - \gamma_i$ such that $c_i = 2^s + \gamma_i$ and $d_i = 2^s + \delta_i$ (note that for $\gamma_i, \delta_i \in \{0, 1\}$ the numbers $\binom{2^s+2\alpha_i+1}{2^s+\gamma_i}$ and $\binom{2^s+2\beta_i+1}{2^s+\delta_i}$ are odd by Lucas' theorem). This leads to a summand m equal to p if and only if $(\alpha + \alpha', \beta + \beta') \in \{(2^s, 2^s+1), (2^s+1, 2^s)\}$, where $\alpha' = 2\sum_{i=2}^{2j} \alpha_i + \sum_{i=2}^{2j}(1 - \gamma_i) = 2\sum_{i=2}^{2j} \alpha_i + 2j - 1 - \sum_{i=2}^{2j} \gamma_i$ and $\beta' = 2\sum_{i=2}^{2j} \beta_i + \sum_{i=2}^{2j}(1 - \delta_i) = 2\sum_{i=2}^{2j} \beta_i + \sum_{i=2}^{2j} \gamma_i$. If we denote $\alpha + \alpha' = 2^s + \gamma$ and $\beta + \beta' = 2^s + 1 - \gamma$, where $\gamma \in \{0, 1\}$, then the previous identities can hold only if $\alpha + 2\sum_{i=2}^{2j} \alpha_i + 2j - 1 - 2^s = 2^s + 1 - \beta - 2\sum_{i=2}^{2j} \beta_i = c$ (note that c is odd), and then

$$\sum_{i=2}^{2j} \gamma_i + \gamma = c.$$

Hence, the numbers γ_i, for $2 \leqslant i \leqslant 2j$, can be chosen in $\binom{2j-1}{c-1} + \binom{2j-1}{c} = \binom{2j}{c}$ ways (in $\binom{2j-1}{c-1}$ ways for $\gamma = 1$, and in $\binom{2j-1}{c}$ ways for $\gamma = 0$), which is even by Lucas' theorem (since c is odd). This completes our proof. $\square$

Acknowledgement

The author would like to thank Prof. Petar Pavešić for bringing this problem to his attention and for useful discussions on the subject, and an anonymous referee for many useful comments are suggestion from which the paper benefited a lot.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] I. Berstein, *On the Lusternik-Schnirelmann category of Grassmannians*, Math. Proc. Cambridge Philos. Soc. **79** (1976), 129-134.

[2] A. Borel, *La cohomologie mod 2 de certains espaces homogenes*, Comm. Math. Helv. **27** (1953) 165–197.

[3] N. Cadavid-Aguliar, J. González, D. Gutiérrez, A. Guzmán-Sáenz, A. Lara, *Sequential motion planning algorithms in real projective spaces: an approach to their immersion dimension*, Forum Math. **30:2** (2018), 397–417.

[4] D. Cohen, A. Suciu, *Boundary manifolds of projective hypersurfaces*, Adv. Math. **206** (2006) 538–566.

[5] D. Davis, *A lower bound for higher topological complexity of real projective space*, J. Pure Appl. Algebra **222:10** (2018), 2881-2287.

[6] M. Farber, *Topological complexity of motion planning*, Discrete Comput. Geom. **29** (2003) 211–221.

[7] M. Farber, *Invitation to toplogical robotics*, EMS Publishing House, 2008.

[8] M. Farber, S. Tabachnikov, S. Yuzvinsky, *Topological Robotics: Motion planning in Projective Spaces*, Int. Math. Res. Not. **34** (2003) 1853–1870.

[9] J. González, B. Gutiérrez, D. Gutiérrez, A. Lara, *Motion planning in real flag manifolds*, Homol. Homotopy Appl. **82:2** (2016) 359-375.

[10] H. Hiller, *On the cohomology of real Grassmanians*, Trans. Amer. Math. Soc. **257** (1980), 512-533.

[11] J. Korbaš and J. Lörinc, *The $\mathbb{Z}_2$-cohomology cup-length of real flag manifolds*, Fund. Math. **178** (2003), 143-158.

[12] P. Pavešić, *Topological complexity of real Grassmannians*, Proc. Roy. Soc. Edinb. A **151:6** (2021), 2013–2029.

[13] K.J. Pearson, T. Zhang, *Toplogical Complexity and Motion Planning in Certain Real Grassmannians*, Appl. Math. Lett. **17** (2004), 499–502.

[14] Z.Z. Petrović and B.I. Prvulović, *On Groebner bases and immersions of Grassmann manifolds $G_{2,n}$*, Homol. Homotopy and Appl. **13:2** (2011), 113–128.

[15] Z.Z. Petrović and B.I. Prvulović, *On Gröbner bases for flag manifolds $F(1,1,\ldots,1,n)$*, J. Algebra Appl. **12** (2013) 113–128.

[16] M. Radovanović, *On the topological complexity and zero-divisor cup-length of real Grassmannians*, Proc. Roy. Soc. Edinb. A, to appear.

[17] Y.B. Rudyak, *On higher analogs of topological complexity*, Topol. Appl. **157:5** (2010), 916–920.

[18] R.E. Stong, *Cup products in Grassmannians*, Topol. Appl. **13** (1982), 103-113.