GRÖBNER BASES FOR SOME FLAG MANIFOLDS AND APPLICATIONS

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ABSTRACT. The mod 2 cohomology of the real flag manifolds is known to be isomorphic to a polynomial algebra modulo a certain ideal. In this paper reduced Gröbner bases for these ideals are obtained in the case of manifolds $F(1,\dots,1,2,\dots,2,n)$. As an application of this result, the appropriate Stiefel-Whitney classes are calculated and some new non-embedding and non-immersion theorems for some manifolds of this type are obtained.

1. Introduction

For positive integers $n_1,\dots,n_r$, $r\geq 2$, the real flag manifold $F(n_1,\dots,n_r)$ is the set of flags of type $(n_1,\dots,n_r)$ (r–tuples $(V_1,\dots,V_r)$ of mutually orthogonal subspaces in $\mathbb{R}^m$, where $m=n_1+\ldots+n_r$ and $\dim(V_i)=n_i$, $i=\overline{1,r}$) with the manifold structure coming from the natural identification $F(n_1,\dots,n_r)=O(n_1+\cdots+n_r)/O(n_1)\times\cdots\times O(n_r)$. This identification makes $F(n_1,\dots,n_r)$ into a closed manifold of dimension $\delta(F(n_1,\dots,n_r))=\sum_{1\leq i<j\leq r}n_in_j$. By Borel's description ([2]), the mod 2 cohomology algebra of $F(n_1,n_2,\dots,n_r)$ is the polynomial algebra on the Stiefel-Whitney classes of canonical vector bundles $\gamma_1,\gamma_2,\dots,\gamma_{r-1}$ over $F(n_1,n_2,\dots,n_r)$ modulo an ideal generated by the dual classes. Although this description is simple enough, concrete calculations in cohomology of flag manifolds may be rather difficult to perform. At the same time, it is well known that a Gröbner basis can be very helpful when calculating in quotient algebra. So, in order to get a better understanding of the cohomology of flag manifolds it is natural to try to obtain a Gröbner basis for the ideal that, by Borel's description, determines this cohomology.

Gröbner bases proved useful for obtaining some topological properties of certain manifolds (see [4, 9, 10, 11, 12, 13, 14]). In [7], the authors used the software Maple V Release 4 and obtained Gröbner bases for some flag manifolds of small

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dimensions. In [10] and [11] the authors obtained Gröbner bases for flag manifolds of type $F(1,\dots,1,n)$ and $F(1,2,n)$. In [14] the authors obtained Gröbner bases for all Grassmann manifolds, that is flag manifolds of type $F(k,n-k)$. In this paper we continue research in this area.

As a main result of this paper we construct (reduced) Gröbner bases for the ideals that determine the cohomology of $F(1^{\cdots j},2^{\cdots d},n)$, thus extending results from [10, 11, 12] (we are using the notation from [5]: $F(1^{\cdots j},2^{\cdots d},n)$ stands for the flag manifold $F(\underbrace{1,\dots,1}_{j},\underbrace{2,\dots,2}_{d},n))$. As a consequence of this result we obtain an additive basis for the cohomology algebra $H^*(F(1^{\cdots j},2^{\cdots d},n);\mathbb{Z}_2)$. Finally, in Section 4 we calculate the appropriate Stiefel-Whitney classes and obtain some new non-embedding and non-immersion theorems for some manifolds of this type, thus extending results from [7] and [11].

2. Preliminaries

2.1. **Gröbner bases.** Let $\mathbb{F}$ be a field and $\mathbb{F}[x_1,\dots,x_k]$ be the polynomial algebra in k variables. A *monomial* in the variables $x_1,\dots,x_k$ is a power product $x_1^{a_1}\cdots x_k^{a_k}$, where $a_i\ge 0$, for $i=\overline{1,k}$. The set of all monomials will be denoted by M. A *term* in $\mathbb{F}[x_1,\dots,x_k]$ is a product of a coefficient $\alpha\in\mathbb{F}$ and $m\in M$. Note that in the case $\mathbb{F}=\mathbb{Z}_2$ every term is a monomial or zero.

Let $\preceq$ be a fixed well-ordering on M with the property that $m_1\preceq m_2$ implies $m\cdot m_1\preceq m\cdot m_2$ for all $m,m_1,m_2\in M$.

For $f=\sum_{i=1}^r\alpha_i m_i\in\mathbb{F}[x_1,\dots,x_k]$, where m_i are pairwise different monomials and $\alpha_i\in\mathbb{F}\setminus\{0\}$, let $M(f)=\{m_i\,|\,1\le i\le r\}$. We define the leading monomial of f, denoted by $\mathrm{LM}(f)$, as $\max M(f)$ (with respect to $\preceq$). The leading coefficient of f, denoted by $\mathrm{LC}(f)$, is the coefficient of $\mathrm{LM}(f)$ and the leading term of f is $\mathrm{LT}(f)=\mathrm{LC}(f)\cdot\mathrm{LM}(f)$.

For $f,g,p\in\mathbb{F}[x_1,\dots,x_k]$, we say that f reduces to g modulo p (and write $f\to_p g$) if there exists $t\in M(f)$ such that $\mathrm{LT}(p)\mid t$ and $g=f-\frac{\alpha}{\mathrm{LC}\,(p)}\cdot s\cdot p$, where $\alpha\in\mathbb{F}\setminus\{0\}$ is the coefficient of t in f and $s\in M$ is such that $t=s\cdot\mathrm{LT}(p)$. We say that f reduces to g modulo $P\subseteq\mathbb{F}[x_1,\dots,x_k]$ (and write $f\to_P g$) if there exists $p\in P$ such that $f\to_p g$. Finally, the relation $\to_P^*$ is defined as the reflexive-transitive closure of $\to_P$ in $\mathbb{F}[x_1,\dots,x_k]$.

Definition 2.1. Let $G\subseteq\mathbb{F}[x_1,\dots,x_k]\setminus\{0\}$ be a finite set of polynomials and let $I=(G)$ be the ideal in $\mathbb{F}[x_1,\dots,x_k]$ generated by the set G. We say that G is a *Gröbner basis* for I if $f\to_G^* 0$ for all $f\in I$.

To prove that a set forms a Gröbner basis of a given ideal, we will use Buchberger's criterion [3], for which we need the knowledge of S-polynomials. For nonzero polynomials $f,g\in\mathbb{F}[x_1,\dots,x_k]$, the *$S$-polynomial* of f and g is defined as

$$S(f,g) := \mathrm{LC}(g)\cdot\frac{u}{\mathrm{LT}(f)}\cdot f - \mathrm{LC}(f)\cdot\frac{u}{\mathrm{LT}(g)}\cdot g,$$

where $u = \mathrm{lcm}(\mathrm{LT}(f),\mathrm{LT}(g))$ is the least common multiple of $\mathrm{LT}(f)$ and $\mathrm{LT}(g)$.

Let G be an arbitrary subset of $\mathbb{F}[x_1,\dots,x_k]\setminus\{0\}$ and $I=(G)$, the ideal generated by G. If $m\in M$ is a fixed monomial and if for $f\in\mathbb{F}[x_1,\dots,x_k]$ we have $f=\sum_{i=1}^r t_ig_i$, where t_i are some terms and g_i some (not necessarily different) elements of G such that $\max_{1\le i\le r}\mathrm{LM}(t_ig_i)\preceq m$, we say that $\sum_{i=1}^r t_ig_i$ is an *m-representation of f with respect to G.*

In the following theorem ([1, Theorem, 5.35(x)]) we formulate Buchberger's criterion (equivalence of (i) and (ii)), as well as an important characterization of Gröbner bases.

Theorem 2.1. *Let $G\subseteq\mathbb{F}[x_1,\dots,x_k]\setminus\{0\}$, be a finite set of polynomials and let $I=(G)$ be the ideal in $\mathbb{F}[x_1,\dots,x_k]$ generated by the set G. Then the following three conditions are equivalent.*

(i) *G is a Gröbner basis for I.*

(ii) *For all $g_1,g_2\in G$, $S(g_1,g_2)\to_G^* 0$, or $S(g_1,g_2)$ has a t-representation with respect to G for some $t\prec\mathrm{lcm}(\mathrm{LT}(g_1),\mathrm{LT}(g_2))$.*

(iii) *The set of all cosets of all terms in $\mathbb{F}[x_1,\dots,x_k]$ that are not divisible by any of the leading terms $\mathrm{LT}(g)$, for $g\in G$, forms an additive basis for the quotient algebra $\mathbb{F}[x_1,\dots,x_k]/I$.*

The following lemma ([1, Lemma 5.66]) will be very useful for the proof of our main theorem. The greatest common divisor of polynomials f and g is denoted by $\gcd(f,g)$.

Lemma 2.1. *Let $f,g\in\mathbb{F}[x_1,\dots,x_k]$ be nonzero polynomials and $P=\{f,g\}$. If $\gcd(\mathrm{LT}(f),\mathrm{LT}(g))=1$, then $S(f,g)\to_P^* 0$.*

2.2. **The cohomology algebra $H^*(F(1^{\cdots j},2^{\cdots d},n);\mathbb{Z}_2)$.** Throughout this paper $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

Let $j,d\in\mathbb{N}_0$, and $n\ge\min\{2,d+1\}$. By Borel's description, the cohomology algebra $H^*(F(1^{\cdots j},2^{\cdots d},n);\mathbb{Z}_2)$ is isomorphic to the quotient algebra

$$\mathbb{Z}_2[x_1,\dots,x_j,y_{1,1},y_{1,2},\dots,y_{d,1},y_{d,2}]/I_{j,d,n}.$$

Here $x_i\in H^1(F(1^{\cdots j},2^{\cdots d},n);\mathbb{Z}_2)$, $i=\overline{1,j}$, are the Stiefel-Whitney classes of the canonical line bundles over $F(1^{\cdots j},2^{\cdots d},n)$; $y_{i,l}\in H^l(F(1^{\cdots j},2^{\cdots d},n);\mathbb{Z}_2)$, $i=\overline{1,d}$, $l=\overline{1,2}$, are the Stiefel-Whitney classes of the canonical two-dimensional vector bundles over $F(1^{\cdots j},2^{\cdots d},n)$; $I_{j,d,n}\triangleleft\mathbb{Z}_2[x_1,\dots,x_j,y_{1,1},y_{1,2},\dots,y_{d,1},y_{d,2}]$ is the ideal generated by the dual classes z_{n+1}, z_{n+2},..., z_{n+j+2d}. The following identity holds for these dual classes

$$1+z_1+z_2+\cdots=\prod_{i=1}^{j}(1+x_i)^{-1}\prod_{i=1}^{d}(1+y_{i,1}+y_{i,2})^{-1},$$

from which we obtain

$$z_s = \sum_{l_1+\cdots+l_j+r_1+\cdots+r_d=s} \overline{x}_{1,l_1}\cdots\overline{x}_{j,l_j}\overline{y}_{1,r_1}\cdots\overline{y}_{d,r_d}, \tag{2.1}$$

where the sum is taken over all nonnegative integers $l_1,\dots,l_j,r_1,\dots,r_d$ such that $l_1+\cdots+l_j+r_1+\cdots+r_d = s$. Also, for $l,r\in\mathbb{N}_0$,

$$\begin{aligned}\overline{x}_{i,l} =& x_i^l,\ 1\le i\le j,\\ \overline{y}_{i,r} =& \sum_{a+2b=r}\binom{a+b}{a}y_{i,1}^a y_{i,2}^b,\ 1\le i\le d,\end{aligned}$$

where the sum is taken over all $(a,b)\in\mathbb{N}_0^2$ such that $a+2b=r$.

Remark 1. Note that for odd r every monomial of $\overline{y}_{i,r}$ has $y_{i,1}$ in a positive degree. In addition, for even r, the term for $b=r/2$ in the previous sum is $y_{i,2}^{r/2}$, and any other monomial of $\overline{y}_{i,r}$ has $y_{i,1}$ in positive degree.

Let $h_p(x_1,\dots,x_j)$ denote the complete homogeneous symmetric polynomial of degree p in the variables x_1,…,x_j ($h_{-1}(x_1,\dots,x_j)=0$). The identity (2.1) can be written as

$$z_s = \sum_{l+r_1+\cdots+r_d=s} h_l(x_1,\dots,x_j)\overline{y}_{1,r_1}\cdots\overline{y}_{d,r_d}, \tag{2.2}$$

where the sum is taken over all d-tuples $(r_1,\dots,r_d)$ of nonnegative integers and $l\in\mathbb{N}_0$, such that $l+r_1+\cdots+r_d=s$.

3. Gröbner basis for $I_{j,d,n}$

In this section we prove the main theorem of this paper which establishes a Gröbner basis for the ideal $I_{j,d,n}$. We will keep the notations from the previous section. All calculations are performed modulo 2.

Recall that for $\alpha,\beta\in\mathbb{Z}$ the binomial coefficient $\binom{\alpha}{\beta}$ is defined by

$$\binom{\alpha}{\beta} := \begin{cases}\frac{\alpha(\alpha-1)\cdots(\alpha-\beta+1)}{\beta!}, & \beta>0\\ 1, & \beta=0\\ 0, & \beta<0\end{cases},$$

and therefore, the following lemma is straightforward.

Lemma 3.1. *If $\binom{\alpha}{\beta}\neq 0$, then $\alpha\ge\beta$ or $\alpha\le -1$.*

Recall also the well-known formula (which holds for all $\alpha,\beta\in\mathbb{Z}$)

$$\binom{\alpha}{\beta} = \binom{\alpha-1}{\beta} + \binom{\alpha-1}{\beta-1}. \tag{3.1}$$

For $1 \le m \le d$, $-2 \le N \le n+j+2m-2$, and $r \ge 0$, let

$$g_{m,r}^{(N)} = \sum_{a+2b=N+1+r} \binom{a+b-r}{a} y_{m,1}^a y_{m,2}^b, \tag{3.2}$$

where the sum is taken over all $(a,b) \in \mathbb{N}_0^2$, such that $a+2b = N+1+r$.

For a d-tuple $R = (r_1, \dots, r_d)$ of nonnegative integers, let:

- $R(m) = \sum_{i=m}^d r_i$, for $m = \overline{1,d}$;
- $\overline{Y}^{R_m} = \overline{y}_{m,r_m} \cdots \overline{y}_{d,r_d}$.

Let $\preccurlyeq$ be the term ordering in $\mathbb{Z}_2[x_1, \dots, x_j, y_{1,1}, y_{1,2}, \dots, y_{d,1}, y_{d,2}]$ defined in the following way. For a term

$$t = x_1^{n_1} \dots x_j^{n_j} y_{1,1}^{n_{1,1}} y_{1,2}^{n_{1,2}} \dots y_{d,1}^{n_{d,1}} y_{d,2}^{n_{d,2}},$$

let $D(t) = (n_1, \dots, n_j, n_{j+1}, n_{1,1}, \dots, n_{j+d}, n_{d,1})$, where $n_{j+r} = n_{r,1} + n_{r,2}$, for $r = \overline{1,d}$.

Then $t \preccurlyeq t'$ if and only if one of the following holds:

- $t = t'$, or
- if s is the smallest integer such that the s-th coordinate m_s of $D(t)$ is not equal to the s-th coordinate m'_s of $D(t')$, then $m_s < m'_s$.

We are ready to define polynomials that form a Gröbner basis $G = G_1 \cup G_2$ for the ideal $I_{j,d,n}$ with respect to the ordering $\preccurlyeq$.

Let $G_1 = \{g_m \mid 1 \le m \le j\}$, where

$$g_m = \sum_{l+R(1)=n+m} h_l(x_m, \dots, x_j) \overline{Y}^{R_1},$$

and the sum is taken over all d-tuples $R = (r_1, \dots, r_d)$ of nonnegative integers and $l \in \mathbb{N}_0$, such that $l + R(1) = n + m$.

Let $G_2 = \{g_{m,r} \mid 1 \le m \le d,\ 0 \le r \le n+j+2m-1\}$, where

$$g_{m,r} = \sum_{R(m)=n+j+2m-1} g_{m,r}^{(r_m-1)} \overline{Y}^{R_{m+1}},$$

and the sum is taken over all d-tuples $R = (r_1, \dots, r_d)$ such that $r_m \ge -1$, $r_i \ge 0$, for $i = \overline{m+1,d}$, and $R(m) = n+j+2m-1$.

Note that for $d = 0$, G is the Gröbner basis obtained in [10]; for $j = 0$ and $d = 1$, G is the Gröbner basis obtained in [12]; for $j = 1$, $d = 1$, G is the Gröbner basis obtained in [11]. Having in mind these results, we construct a new generating set for $I_{j,d,n}$, by successively removing members of the generating set and replacing them with appropriate polynomials with smaller number of variables.

First, we prove that $I_{j,d,n}$ is generated by $G_1 \cup \{z'_{n+j+1}, z'_{n+j+2}, \dots, z'_{n+j+2d}\}$, where

$$z'_{n+j+m} = \sum_{R(1)=n+j+m} \overline{Y}^{R_1},\ m = \overline{1, 2d},$$

and the sum is taken over all d-tuples of nonnegative integers $R = (r_1, \dots, r_d)$ such that $R(1) = n + j + m$.

In order to do so, let us define

$$z_{n+m,i} = \sum_{l+R(1)=n+m} h_l(x_i, \dots, x_j)\overline{Y}^{R_1},\ m = \overline{1, j+2d},\ i = \overline{1, j},$$

where the sum is taken over all d-tuples $R = (r_1, \dots, r_d)$ of nonnegative integers and $l \in \mathbb{N}_0$, such that $l + R(1) = n + m$.

From (2.2) we have $z_{n+m} = z_{n+m,1}$, $m = \overline{1, j+2d}$. Also, $g_m = z_{n+m,m}$, $m = \overline{1, j+2d}$.

Note that for $l \geq -1$, $1 \leq i \leq j-1$,

$$h_{l+1}(x_{i+1}, \dots, x_j) = h_{l+1}(x_i, x_{i+1}, \dots, x_j) - x_i h_l(x_i, x_{i+1}, \dots, x_j),$$

and therefore for $m = \overline{1, j+2d-1}$, $i = \overline{1, j-1}$, since $h_{-1}(x_i, \dots, x_j) = 0$,

$$\begin{aligned} z_{n+m+1,i+1} &= \sum_{l+R(1)=n+m+1} h_l(x_{i+1}, \dots, x_j)\overline{Y}^{R_1} \\ &= \sum_{l+R(1)=n+m+1} \left(h_l(x_i, \dots, x_j) - x_i h_{l-1}(x_i, \dots, x_j)\right)\overline{Y}^{R_1} \\ &= z_{n+m+1,i} + x_i \sum_{l-1+R(1)=n+m} h_{l-1}(x_i, \dots, x_j)\overline{Y}^{R_1} \\ &= z_{n+m+1,i} + x_i z_{n+m,i}. \end{aligned} \tag{3.3}$$

Also, for $m = \overline{j, j+2d-1}$, we have

$$\begin{aligned} z_{n+m+1,j} - x_j z_{n+m,j} &= \sum_{l+R(1)=n+m+1} x_j^l \overline{Y}^{R_1} - x_j \sum_{l+R(1)=n+m} x_j^l \overline{Y}^{R_1} \\ &= \sum_{l+R(1)=n+m+1} x_j^l \overline{Y}^{R_1} - \sum_{\substack{l+R(1)=n+m+1 \\ l \geq 1}} x_j^l \overline{Y}^{R_1} \\ &= z'_{n+m+1}, \end{aligned} \tag{3.4}$$

Using identities (3.3) and (3.4) for the terms in boxes, we have the following

$$\begin{aligned}
I_{j,d,n} &= \Big(z_{n+1,1}, z_{n+2,1}, \dots, z_{n+j+2d-2,1}, \boxed{z_{n+j+2d-1,1}}, \boxed{z_{n+j+2d,1}}\Big)\\
&= \Big(z_{n+1,1}, z_{n+2,1}, \dots, \boxed{z_{n+j+2d-2,1}}, \boxed{z_{n+j+2d-1,1}}, z_{n+j+2d,2}\Big)\\
&\quad \dots\\
&= \Big(g_1, z_{n+2,2}, \dots, z_{n+j+2d-2,2}, \boxed{z_{n+j+2d-1,2}}, \boxed{z_{n+j+2d,2}}\Big)\\
&= \Big(g_1, z_{n+2,2}, \dots, \boxed{z_{n+j+2d-2,2}}, \boxed{z_{n+j+2d-1,2}}, z_{n+j+2d,3}\Big)\\
&\quad \dots\\
&= \Big(g_1, \dots, g_{j-1}, z_{n+j,j}, \dots, z_{n+j+2d-2,j}, \boxed{z_{n+j+2d-1,j}}, \boxed{z_{n+j+2d,j}}\Big)\\
&= \Big(g_1, \dots, g_{j-1}, z_{n+j,j}, \dots, \boxed{z_{n+j+2d-2,j}}, \boxed{z_{n+j+2d-1,j}}, z'_{n+D}\Big)\\
&\quad \dots\\
&= (g_1, \dots, g_j, z'_{n+j+1}, \dots, z'_{n+j+2d}). \qquad (3.5)
\end{aligned}$$

To continue our proof we need the following lemma which can be easily extracted from [12]. Since it is not stated there in a form suitable for us, we include its proof.

Lemma 3.2. *Let $1 \le m \le d$, $-2 \le N \le n+j+2m-1$, $r \ge 0$ and $s \ge 1$. The following identities hold:*

a) $\mathrm{LT}(g^{(N)}_{m,r}) = y^{N+1-r}_{m,1} y^r_{m,2}$, *for* $r \le N+1$;
b) $\overline{y}_{m,r+2} = y_{m,1}\overline{y}_{m,r+1} + y_{m,2}\overline{y}_{m,r}$;
c) $g^{(N)}_{m,r+2} = y_{m,1} g^{(N)}_{m,r+1} + y_{m,2} g^{(N)}_{m,r}$;
d) $y^s_{m,2} g^{(N)}_{m,r} + y^s_{m,1} g^{(N)}_{m,r+s} = \sum_{i=0}^{s-1} y^i_{m,1} y^{s-1-i}_{m,2} g^{(N)}_{m,r+2+i}$.
e) $g^{(N)}_{m,N+2} = 0$, *and* $g^{(N)}_{m,r} = g^{(r-2)}_{m,N+2}$ *for* $r \ge N+3$.

Proof. a) By Lemma 3.1, if $b < r$ and $a+b \ge r$ we have $\binom{a+b-r}{a} = 0$, and therefore every nonzero term of

$$g^{(N)}_{m,r} = \sum_{a+2b=N+1+r} \binom{a+b-r}{a} y^a_{m,1} y^b_{m,2}$$

satisfies $b \ge r$ or $a+b < r$. So, $a+b \le \max\{N+1, r-1\} = N+1$, and therefore, $\mathrm{LT}(g^{(N)}_{m,r}) = y^{N+1-r}_{m,1} y^r_{m,2}$.

b) We have (modulo 2)

$$\begin{aligned}
y_{m,1}\overline{y}_{m,r+1}+y_{m,2}\overline{y}_{m,r} &= \\
&= \sum_{a+2b=r+1}\binom{a+b}{a}y_{m,1}^{a+1}y_{m,2}^{b}+\sum_{a+2b=r}\binom{a+b}{a}y_{m,1}^{a}y_{m,2}^{b+1}\\
&= \sum_{a+2b=r+2}\binom{a+b-1}{a-1}y_{m,1}^{a}y_{m,2}^{b}+\sum_{a+2b=r+2}\binom{a+b-1}{a}y_{m,1}^{a}y_{m,2}^{b}\\
&= \sum_{a+2b=r+2}\binom{a+b}{a}y_{m,1}^{a}y_{m,2}^{b}.
\end{aligned}$$

The change of variable $a \mapsto a-1$ (resp. $b \mapsto b-1$) does not affect the requirement that $a \geq 0$ (resp. $b \geq 0$), since for $a = 0$ (resp. $b = 0$) the binomial coefficient $\binom{a+b-1}{a-1}$ (resp. $\binom{a+b-1}{a} = \binom{r+1}{r+2}$) is equal to 0. So, the last sum is equal to $\overline{y}_{m,r+2}$.

c) We have (modulo 2)

$$\begin{aligned}
y_{m,1}g_{m,r+1}^{(N)}+y_{m,2}g_{m,r}^{(N)} &= \\
&= \sum_{a+2b=N+r+2}\binom{a+b-r-1}{a}y_{m,1}^{a+1}y_{m,2}^{b}+\sum_{a+2b=N+r+1}\binom{a+b-r}{a}y_{m,1}^{a}y_{m,2}^{b+1}\\
&= \sum_{a+2b=N+r+3}\binom{a+b-r-2}{a-1}y_{m,1}^{a}y_{m,2}^{b}+\sum_{a+2b=N+r+3}\binom{a+b-r-1}{a}y_{m,1}^{a}y_{m,2}^{b}\\
&= \sum_{a+2b=N+r+3}\binom{a+b-r-2}{a}y_{m,1}^{a}y_{m,2}^{b}.
\end{aligned}$$

Note that, similarly as in part b), the change of variable $a \mapsto a-1$ (resp. $b \mapsto b-1$) does not affect the requirement that $a \geq 0$ (resp. $b \geq 0$). So, the last sum is equal to $g_{m,r+2}^{(N)}$.

d) We proceed by induction on s. For $s = 1$, we need to prove that $g_{m,r+2}^{(N)} = y_{m,2}g_{m,r}^{(N)} + y_{m,1}g_{m,r+1}^{(N)}$, which follows from part c). For the induction step, using part c) we obtain

$$\begin{aligned}
& y_{m,2}^s g_{m,r}^{(N)} + y_{m,1}^s g_{m,r+s}^{(N)} \\
&\quad = y_{m,2}^s g_{m,r}^{(N)} + y_{m,2} y_{m,1}^{s-1} g_{m,r+s-1}^{(N)} + y_{m,2} y_{m,1}^{s-1} g_{m,r+s-1}^{(N)} + y_{m,1}^s g_{m,r+s}^{(N)} \\
&\quad = y_{m,2} \left(y_{m,2}^{s-1} g_{m,r}^{(N)} + y_{m,1}^{s-1} g_{m,r+s-1}^{(N)} \right) + y_{m,1}^{s-1} \left(y_{m,2} g_{m,r+s-1}^{(N)} + y_{m,1} g_{m,r+s}^{(N)} \right) \\
&\quad = y_{m,2} \sum_{i=0}^{s-2} y_{m,1}^i y_{m,2}^{s-2-i} g_{m,r+2+i}^{(N)} + y_{m,1}^{s-1} g_{m,r+s+1}^{(N)} \\
&\quad = \sum_{i=0}^{s-1} y_{m,1}^i y_{m,2}^{s-1-i} g_{m,r+2+i}^{(N)}.
\end{aligned}$$

e) First, let $r = N+2$. If $a + 2b = N + 1 + r = 2N + 3$, for $a, b \geq 0$, then $2a + 2b \geq 2N + 3$, i.e., $a + b \geq N + 2$. At the same time, $2b \leq 2N + 3$, i.e., $b < N + 2$. So, $0 \leq a + b - r < a$, and therefore $\binom{a+b-r}{a} = 0$, i.e., $g_{m,N+2}^{(N)} = 0$.

For the other identity, let $r \geq N + 3$, and $a, b \geq 0$ be such that $a + 2b = N + 1 + r$. Then $r - b - 1 = a + b - N - 2$ and

$$\binom{a+b-r}{a} = (-1)^a \binom{r-b-1}{a} = \binom{a+b-N-2}{a}.$$

Therefore,

$$\begin{aligned}
g_{m,r}^{(N)} &= \sum_{a+2b=N+1+r} \binom{a+b-r}{a} y_{m,1}^a y_{m,2}^b \\
&= \sum_{a+2b=N+1+r} \binom{a+b-N-2}{a} y_{m,1}^a y_{m,2}^b \\
&= g_{m,N+2}^{(r-2)},
\end{aligned}$$

which completes our proof. □

Let

$$z''_{n+j+m} = \sum_{R(2)=n+j+m} \overline{Y}^{R_2}, \ m = \overline{1, 2d},$$

where the sum is taken over all d-tuples $R = (r_1, \ldots, r_d)$ of nonnegative integers, such that $R(2) = n + j + m$.

Note that if we define $\overline{y}_{m,-1}$ to be 0, then part b) of the previous lemma also holds for $r = -1$. So, for a d-tuple of nonnegative integers $R = (r_1, r_2, \ldots, r_d)$

we have

$$\begin{aligned}
z'_{n+j+m+2} &= \sum_{R(1)=n+j+m+2} \overline{Y}^{R_1} \\
&= \sum_{\substack{R(1)=n+j+m+2\\ r_1\geq 1}} \overline{Y}^{R_1} + \sum_{R(2)=n+j+m+2} \overline{Y}^{R_2} \\
&= \sum_{\substack{R(1)=n+j+m+2\\ r_1\geq 1}} \big(y_{1,1}\overline{y}_{1,r_1-1} + y_{1,2}\overline{y}_{1,r_1-2}\big)\,\overline{Y}^{R_2} + \sum_{R(2)=n+j+m+2} \overline{Y}^{R_2} \\
&= \sum_{R(1)=n+j+m+1} y_{1,1}\overline{Y}^{R_1} + \sum_{R(1)=n+j+m} y_{1,2}\overline{Y}^{R_1} + z''_{n+j+m+2} \\
&= y_{1,1}z'_{n+j+m+1} + y_{1,2}z'_{n+j+m} + z''_{n+j+m+2}. \qquad (3.6)
\end{aligned}$$

Using the previous identities, as for (3.5) we obtain

$$\begin{aligned}
I_{j,d,n} &= (g_1,\dots,g_j,z'_{n+j+1},z'_{n+j+2},z'_{n+j+3},\dots,z'_{n+j+2d}) \\
&= (g_1,\dots,g_j,z'_{n+j+1},z'_{n+j+2},z''_{n+j+3},\dots,z''_{n+j+2d}). \qquad (3.7)
\end{aligned}$$

Next, we prove the following lemma.

Lemma 3.3. $(z'_{n+j+1}, z'_{n+j+2}) = (g_{1,0}, g_{1,1}, \dots, g_{1,n+j+1})$.

Proof. Note that $g_{1,0}^{(r_1-1)} = \overline{y}_{1,r_1}$, for $r_1 = \overline{-1, n+j+1}$, and therefore

$$g_{1,0} = \sum_{R(1)=n+j+1} \overline{Y}^{R_1} = z'_{n+j+1}.$$

Since $g_{1,1}^{(-2)} = 1$ and $g_{1,1}^{(-1)} = 0$, we have

$$\begin{aligned}
g_{1,1} &= \sum_{\substack{R(1)=n+j+1\\ r_1\geq -1}} g_{1,1}^{(r_1-1)}\overline{Y}^{R_2} \\
&= \sum_{\substack{R(1)=n+j+1\\ r_1\geq -1}} \sum_{a+2b=r_1+1} \binom{a+b-1}{a} y_{1,1}^a y_{1,2}^b \overline{Y}^{R_2} \\
&= \sum_{\substack{R(1)=n+j+1\\ r_1\geq 1}} \sum_{a+2b=r_1-1} \binom{a+b}{a} y_{1,1}^a y_{1,2}^{b+1} \overline{Y}^{R_2} + \sum_{R(2)=n+j+2} \overline{Y}^{R_2} \\
&= y_{1,2} \sum_{R(1)=n+j+1} \overline{y}_{1,r_1-1}\overline{Y}^{R_2} + \sum_{R(2)=n+j+2} \overline{Y}^{R_2} \\
&= y_{1,2} \sum_{R(1)=n+j} \overline{Y}^{R_1} + \sum_{R(2)=n+j+2} \overline{Y}^{R_2},
\end{aligned}$$

and therefore, by part b) of Lemma 3.2,

$$\begin{aligned}
z'_{n+j+2} &= \sum_{R(1)=n+j+2} \overline{Y}^{R(1)} \\
&= \sum_{\substack{R(1)=n+j+2\\ r_1\geq 1}} (y_{1,1}\overline{y}_{1,r_1-1} + y_{1,2}\overline{y}_{1,r_1-2})\overline{Y}^{R(2)} + \sum_{R(2)=n+j+2} \overline{Y}^{R(2)} \\
&= y_{1,1}z'_{n+j+1} + y_{1,2}\sum_{R(1)=n+j} \overline{Y}^{R(1)} + \sum_{R(2)=n+j+2} \overline{Y}^{R(2)} \\
&= y_{1,1}g_{1,0} + g_{1,1}.
\end{aligned}$$

So, $(z'_{n+j+1}, z'_{n+j+2}) \subseteq (g_{1,0}, g_{1,1}, \ldots, g_{1,n+j+1})$.

To prove the other inclusion, first note that $g_{1,0} = z'_{n+j+1}$ and $g_{1,1} = z'_{n+j+2} + y_{1,1}z'_{n+j+1} \in (z'_{n+j+1}, z'_{n+j+2})$. Now, by simple induction on r and part c) of Lemma 3.2, we have

$$g_{1,r+2} = y_{1,1}g_{1,r+1} + y_{1,2}g_{1,r} \in (z'_{n+j+1}, z'_{n+j+2}),\ r = \overline{0, n+j-1},$$

which completes our proof. □

By the previous lemma

$$I_{j,d,n} = (g_1, \ldots, g_j, g_{1,0}, \ldots, g_{1,n+j+1}, z''_{n+j+3}, \ldots, z''_{n+j+2d}).$$

Note that the polynomials z''_{n+j+i}, $i = \overline{3, 2d}$, have the same form as polynomials z'_{n+j+i}, $i = \overline{1, 2d}$, and are in variables $y_{2,1}, y_{2,2}, \ldots, y_{d,1}, y_{d,2}$. Therefore, we can continue as before, and obtain the desired result, i.e.,

$$I_{j,d,n} = (G). \tag{3.8}$$

To prove that G is a Gröbner basis for the ideal $I_{j,d,n}$, it is convenient to extend the definition of $g_{m,r}$ to $r = n+j+2m$ as follows

$$g_{m,n+j+2m} = \sum_{R(m)=n+j+2m-1} g^{(r_m-1)}_{m,n+j+2m}\overline{Y}^{R_{m+1}},\ m = \overline{1, d},$$

where the sum is taken over all d-tuples $R = (r_1, \ldots, r_d)$ such that $r_m \geq -1$, $r_i \geq 0$, for $i = \overline{m+1, d}$, and $R(m) = n+j+2m-1$.

Note that by part e) of Lemma 3.2

$$\begin{aligned}
\mathrm{LT}(g_{m,n+j+2m}) &= \max_{-2\leq N\leq n+j+2m-2} \left\{\mathrm{LT}\left(g^{(N)}_{m,n+j+2m}\overline{y}_{m+1,n+j+2m-N-2}\right)\right\} \\
&= \max_{-2\leq N\leq n+j+2m-2} \left\{\mathrm{LT}\left(g^{(n+j+2m-2)}_{m,N+2}\overline{y}_{m+1,n+j+2m-N-2}\right)\right\} \\
&= \max_{-2\leq N\leq n+j+2m-2} \left\{y_{m,1}^{n+j+2m-N-3}y_{m,2}^{N}\overline{y}_{m+1,n+j+2m-N-2}\right\} \\
&= y_{m,1}^{n+j+2m-1}y_{m+1,1}^{n+j+2m}.
\end{aligned}$$

Moreover, the following holds.

Lemma 3.4. *For $1 \le m \le d$,*

$$\sum_{r+s=n+j+2m} g_{m,r} \sum_{R(m+1)=s} \overline{Y}^{R_{m+1}} = 0,$$

where the double sum is taken over all pairs of nonnegative integers (r, s), and all d-tuples of nonnegative integers $R = (r_1, r_2, \dots, r_d)$, such that $r+s = n+j+2m$ and $R(m+1) = s$.

Proof. Let A denote the double sum in the lemma. By the definition $g_{m,r} = \sum_{R'(m)=n+j+2m-1} g_{m,r}^{(r'_m-1)} \overline{Y}^{R'_{m+1}}$. Plugging this in A we obtain

$$\begin{aligned} A &= \sum_{r+s=n+j+2m} \sum_{R'(m)=n+j+2m-1} g_{m,r}^{(r'_m-1)} \overline{Y}^{R'_{m+1}} \sum_{R(m+1)=s} \overline{Y}^{R_{m+1}} \\ &= \sum_{0 \le r, r' \le n+j+2m} g_{m,r}^{(r'-2)} \sum_{R(m+1)=n+j+2m-r} \sum_{R'(m+1)=n+j+2m-r'} \overline{Y}^{R_{m+1}} \overline{Y}^{R'_{m+1}}. \end{aligned}$$

By part e) of Lemma 3.2 $g_{m,r}^{(r'-2)} = g_{m,r'}^{(r-2)}$ and $g_{m,r}^{(r-2)} = 0$, and so $A = 0$. □

From the previous lemma

$$g_{m,n+j+2m} = \sum_{\substack{r+s=n+j+2m \\ s \ge 1}} g_{m,r} \sum_{R(m+1)=s} \overline{Y}^{R_{m+1}}. \tag{3.9}$$

Note that each $g_{m,r}$ that appears on the right hand side of (3.9) is in G_2.

We are ready to prove the main theorem of the paper.

Theorem 3.1. *The set G is the reduced Gröbner basis for the ideal $I_{j,d,n}$ with respect to the ordering $\preccurlyeq$.*

Proof. To prove that G is a Gröbner basis it is enough to prove that G satisfies part (ii) of Theorem 2.1. Let $g', g'' \in G$, $g' \ne g''$.

Note that for $1 \le m \le j$, $\mathrm{LT}(g_m) = x_m^{n+m}$. By part a) of Lemma 3.2, for $1 \le m \le d$ and $0 \le r \le n+j+2m-1$,

$$\mathrm{LT}(g_{m,r}) = \mathrm{LT}(g_{m,r}^{(n+j+2(m-1))}) = y_{m,1}^{n+j+2m-1-r} y_{m,2}^{r}.$$

So, if $g' \in G_1$, $g'' \in G_2$, or $g', g'' \in G_1$, or $g' = g_{m',r'}$, $g'' = g_{m'',r''}$, $m' \ne m''$, by Lemma 2.1 we have $S(g', g'') \to_G^* 0$.

So, we may assume that $g' = g_{m,r'}$ and $g'' = g_{m,r''}$, for some $1 \le m \le d$ and $0 \le r' < r'' \le n+j+2m-1$. Then

$$\mathrm{lcm}(\mathrm{LT}(g_{m,r'}), \mathrm{LT}(g_{m,r''})) = y_{m,1}^{n+j+2m-1-r'} y_{m,2}^{r''},$$

and therefore

$$\begin{aligned}S(g_{m,r'},g_{m,r''}) &= y_{m,2}^{r''-r'}g_{m,r'}+y_{m,1}^{r''-r'}g_{m,r''}\\ &= \sum_{R(m)=n+j+2m-1}\left(y_{m,2}^{r''-r'}g_{m,r'}^{(r_m-1)}+y_{m,1}^{r''-r'}g_{m,r''}^{(r_m-1)}\right)\overline{Y}^{R_{m+1}}\end{aligned}$$

Let $\delta = r''-r'-1$. By part d) of Lemma 3.2 we have

$$\begin{aligned}S(g_{m,r'},g_{m,r''}) &= \sum_{R(m)=n+j+2m-1}\sum_{i=0}^{\delta} y_{m,1}^{i}y_{m,2}^{\delta-i}g_{m,r'+2+i}^{(r_m-1)}\overline{Y}^{R_{m+1}}\\ &= \sum_{i=0}^{\delta} y_{m,1}^{i}y_{m,2}^{\delta-i}\sum_{R(m)=n+j+2m-1} g_{m,r'+2+i}^{(r_m-1)}\overline{Y}^{R_{m+1}}\\ &= \sum_{i=0}^{\delta} y_{m,1}^{i}y_{m,2}^{\delta-i}g_{m,r'+2+i}. \qquad (3.10)\end{aligned}$$

Note that if $r'+2+i\le n+j+2m-1$ then

$$\mathrm{LT}(y_{m,1}^{i}y_{m,2}^{\delta-i}g_{m,r'+2+i}) = y_{m,1}^{n+j+2m-3-r'}y_{m,2}^{r''+1}\prec y_{m,1}^{n+j+2m-1-r'}y_{m,2}^{r''}.$$

So, if $r''<n+j+2m-1$, then (3.10) is the representation of $S(g_{m,r'},g_{m,r''})$ that satisfies part (ii) of Theorem 2.1. Let $r''=n+j+2m-1$. By (3.9) and (3.10),

$$\begin{aligned}S(g_{m,r'},g_{m,r''}) &= \sum_{i=0}^{\delta-1} y_{m,1}^{i}y_{m,2}^{\delta-i}g_{m,r'+2+i}\\ &\quad + y_{m,1}^{\delta}\sum_{\substack{r+s=n+j+2m\\ s\ge 1}} g_{m,r}\sum_{R(m+1)=s}\overline{Y}^{R(m+1)}. \qquad (3.11)\end{aligned}$$

Note that for every $h\in\mathbb{Z}_2[y_{m+1,1},y_{m+1,2},\dots,y_{d,1},y_{d,2}]$ and $0\le r\le n+j+2m-1$, we have

$$\mathrm{LT}\left(y_{m,1}^{\delta}g_{m,r}h\right) = y_{m,1}^{\delta+n+j+2m-1-r}y_{m,2}^{r}\mathrm{LT}(h)\prec y_{m,1}^{n+j+2m-1-r'}y_{m,2}^{n+j+2m-1}.$$

So, (3.11) is the representation of $S(g_{m,r'},g_{m,r''})$ that satisfies part (ii) of Theorem 2.1.

To prove that G is a reduced Gröbner basis, let us assume to the contrary that $\mathrm{LT}(g')$ divides a term of g'', for some $g',g''\in G$, $g'\ne g''$. If $g'=g_{m'}$, for some $1\le m'\le j$, then, from the definition of G, we have $g''=g_{m''}$, for some $1\le m''<m'$. But the degree of $g_{m''}$ is less than the degree of $\mathrm{LT}(g_{m'})$, which is a contradiction. The case $g'=g_{m,r}$ is dealt with similarly. □

Remark 2. Using the results from [14], in a similar way as in this paper, one can obtain Gröbner bases for flag manifolds of type $F(1^{\cdots j}, 2^{\cdots d}, k, n)$, for $k \in \mathbb{N}$. Since this will need more complicated notations, we decided not to include this proof.

Remark 3. Using the method of this paper, it should be difficult to obtain a result for a general flag manifold. The main problem should be obtaining the result similar to Lemma 3.4. Note that this was very complicated even for $F(3, n)$ (see [13, Proposition 2.8])

For a polynomial $p \in \mathbb{Z}_2[x_1, \ldots, x_j, y_{1,1}, y_{2,1}, \ldots, y_{d,1}, y_{d,2}]$, we will denote the class of p in $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$ by the same letter. By Theorem 2.1 (implication (i)⇒(iii)) we have the following corollary.

Corollary 3.1.1. *The set*

$$\left\{\prod_{i=1}^{j} x_i^{a_i} \prod_{i=1}^{d} y_{i,1}^{b_i'} y_{i,2}^{b_i''} : a_i \leq n+i-1,\ i=\overline{1,j},\ b_i' + b_i'' \leq n+j+2i-2,\ i=\overline{1,d}\right\}$$

is a vector space basis for $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$.

Additive basis obtained in the previous corollary will be denoted by $B_{j,d,n}$. Note that the Gröbner basis detected in Theorem 3.1 gives us more. By the definition of the reduction, if $p \to_f q$, then $\mathrm{LT}(q) \preccurlyeq \mathrm{LT}(p)$, and therefore we have the following corollary.

Corollary 3.1.2. *For* $f \in \mathbb{Z}_2[x_1, \ldots, x_j, y_{1,1}, y_{1,2}, \ldots, y_{d,1}, y_{d,2}]$ *there is a polynomial* p*, such that* $f = p$ *in* $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$*,* $\mathrm{LT}(p) \preccurlyeq \mathrm{LT}(f)$*, and all monomials of* p *are in* $B_{j,d,n}$*.*

The previous corollary can be restated as follows. If a polynomial p does not contain variables $x_1, x_2, \ldots, x_i$, and f is a sum of elements of $B_{j,d,n}$ such that $f = p$ in $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$, then f does not contain variables $x_1, x_2, \ldots, x_i$. Similarly, if a polynomial q does not contain variables $x_1, \ldots, x_j, y_{1,1}, y_{1,2}, \ldots, y_{i,1}, y_{i,2}$, and g is a sum of elements of $B_{j,d,n}$ such that in $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$ one has $g = q$, then g does not contain variables $x_1, \ldots, x_j, y_{1,1}, y_{1,2}, \ldots, y_{i,1}, y_{i,2}$. Therefore, we have the following corollary.

Corollary 3.1.3. *(1) Let* $1 \leq i \leq j$*,* $0 \leq a_1 < a_2 < \cdots < a_k \leq n+i-1$*, and* $p_1, p_2, \ldots, p_k$ *polynomials such that* $\mathrm{LT}(p_l) \prec x_i$*, for* $l = \overline{1,k}$*. Then*

$$\sum_{l=1}^{k} x_i^{a_l} p_l = 0$$

in $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$ *if and only if* $p_l = 0$ *in* $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$*, for all* $1 \leq l \leq k$*.*

(2) Let $1 \le i \le d$, $(b_1, c_1), \dots, (b_k, c_k)$ distinct pairs of nonnegative integers such that $b_l + c_l \le n + j + 2l - 2$, for $1 \le l \le k$, and $p_1, \dots, p_k$ polynomials such that $\mathrm{LT}(p_l) \prec y_{i,2}$, for $l = \overline{1,k}$. Then

$$\sum_{l=1}^{k} y_{i,1}^{b_l} y_{i,2}^{c_l} p_l = 0$$

in $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$ if and only if $p_l = 0$ in $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$, for all $1 \le l \le k$.

By Corollary 3.1.1, if $p \in B_{j,d,n}$ does not contain variables $x_1, x_2, \dots, x_i$, then the maximum degree of p is $\sum_{l=i+1}^{j}(n + l - 1) + \sum_{l=1}^{d}(n + j + 2i - 2)$, and the maximum dimension of p is $\sum_{l=i+1}^{j}(n+l-1)+\sum_{l=1}^{d}(2n+2j+4i-4)$. Similarly, if $q \in B_{j,d,n}$ does not contain variables $x_1, \dots, x_j, y_{1,1}, y_{1,2}, \dots, y_{i,1}, y_{i,2}$, then the maximum degree of q is $\sum_{l=i+1}^{d}(n+j+2i-2)$, and the maximum dimension of q is $\sum_{l=i+1}^{d}(2n+2j+4i-4)$. Therefore, by Corollary 3.1.2 we have the following.

Corollary 3.1.4. *Let $a_l \ge 0$, $l = \overline{1,j}$, and $b_l, c_l \ge 0$, $l = \overline{1,d}$. If*

(1) $\displaystyle\sum_{l=i+1}^{j} a_l + \sum_{l=1}^{d}(b_l + 2c_l) > \sum_{l=i+1}^{j} (n + l - 1) + \sum_{l=1}^{d}(2n + 2j + 4l - 4)$, *for some $0 \le i \le j$, or*

(2) $\displaystyle\sum_{l=i+1}^{d} (b_l + 2c_l) > \sum_{l=i+1}^{d} (2n + 2j + 4l - 4)$, *for some $0 \le i \le d$,*

then in $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$

$$\prod_{l=1}^{j} x_l^{a_l} \prod_{l=1}^{d} y_{l,1}^{b_l} y_{l,2}^{c_l} = 0.$$

Let $1 \le m \le d$, $M = n + j + 2m - 2$, $G_{j,m,n} = \{g_{m,0}, g_{m,1}, \dots, g_{m,M+1}\}$, and $G'_{j,m,n} = \{g^{(M)}_{m,0}, g^{(M)}_{m,1}, \dots, g^{(M)}_{m,M+1}\}$. By [12], $G'_{m,j,n}$ is a Gröbner basis for the ideal $(G'_{j,m,n})$, and

$$\mathbb{Z}_2[y_{m,1}, y_{m,2}]/(G'_{j,m,n}) \cong H^*(F(2, M); \mathbb{Z}_2). \tag{3.12}$$

Via this isomorphism the classes $y_{m,1}$ and $y_{m,2}$ correspond to the Stiefel-Whitney classes w_1 and w_2 of the canonical bundle γ_2 over the Grassmannian $F(2, M)$. Also, if p and q are polynomials in variables $y_{m,1}$ and $y_{m,2}$ such that

$$p \longrightarrow^*_{G'_{j,m,n}} q,$$

replacing every $g^{(M)}_{m,i}$ that appears in this reduction with $g_{m,i}$, we obtain

$$p \longrightarrow^*_{G_{j,m,n}} q + r,$$

where r is a polynomial in variables $y_{m,1}, y_{m,2}, \dots, y_{d,1}, y_{d,2}$, such that each monomial of r has at least one of the variables $y_{m+1,1}, y_{m+1,2}, \dots, y_{d,1}, y_{d,2}$ in positive degree. This observation, together with (3.12), gives us the following result.

Corollary 3.1.5. *Let $1 \le m \le d$ and $M = n + j + 2m - 2$. If $p, q \in \mathbb{Z}_2[y_{m,1}, y_{m,2}]$ are such that $p = q$ in $H^*(F(2, M); \mathbb{Z}_2)$, then $p = q + r$ in $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$, where r is a polynomial in variables $y_{m,1}, y_{m,2}, \dots, y_{d,1}, y_{d,2}$, such that each monomial of r has at least one of the variables $y_{m+1,1}, y_{m+1,2}, \dots, y_{d,1}, y_{d,2}$ in positive degree.*

Using this result in the case $m = d$, we conclude that the heights of the classes $y_{d,1}$ and $y_{d,2}$, and therefore, by symmetry, of the classes $y_{i,1}, y_{i,2}$, for $i = \overline{1, d}$, are equal to the heights of classes w_1 and w_2, which are well-known ([16]). Thus, we obtained a special case of the corresponding result by Korbaš and Lörinc (see [5, p. 147]).

Corollary 3.1.6. *Let $d \ge 1$, $n \ge 2$, and let $y_{i,1}, y_{i,2} \in H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$, $i = \overline{1, d}$, be the Stiefel-Whitney classes of the canonical two-dimensional vector bundle over $F(1^{\cdots j}, 2^{\cdots d}, n)$. Then $\operatorname{ht}(y_{i,2}) = n + j + 2d - 1$, for $i = \overline{1, d}$, and if $s \ge 3$ is the integer such that $2^{s-1} < n + j + 2d \le 2^s$, then $\operatorname{ht}(y_{i,1}) = 2^s - 2$.*

Let us calculate a few elements of the Gröbner basis G. For $m = \overline{1, d}$, let $n + j + 2m - 2 = M$. First, from the definition (3.2) one can obtain (see [12, p. 118], or proof of Lemma 3.2), $g^{(M)}_{m,M+1} = y^{M+1}_{m,2}$, $g^{(M-1)}_{m,M} = y^{M}_{m,2}$, $g^{(M-1)}_{m,M-1} = y_{m,1} y^{M-1}_{m,2}$, $g^{(M)}_{m,M} = y_{m,1} y^{M}_{m,2}$, $g^{(M-2)}_{m,M-1} = y^{M-1}_{m,2}$, $g^{(M)}_{m,M-1} = y^2_{m,1} y^{M-1}_{m,2} + y^{M}_{m,2}$. Now, from part e) of Lemma 3.2 we have

$$\begin{aligned} g_{m,M+1} &= g^{(M)}_{m,M+1} + g^{(M-1)}_{m,M+1}\sigma^{(m+1)}_1 + g^{(M-2)}_{m,M+1}\sigma^{(m+1)}_2 + g^{(M-3)}_{m,M+1}\sigma^{(m+1)}_3 + p, \\ &= y^{M+1}_{m,2} + 0 + g^{(M-1)}_{m,M}\sigma^{(m+1)}_2 + g^{(M-1)}_{m,M-1}\sigma^{(m+1)}_3 + p \\ &= y^{M+1}_{m,2} + y^{M}_{m,2}\sigma^{(m+1)}_2 + y_{m,1} y^{M-1}_{m,2}\sigma^{(m+1)}_3 + p, \end{aligned} \tag{3.13}$$

$$\begin{aligned} g_{m,M} &= g^{(M)}_{m,M} + g^{(M-1)}_{m,M}\sigma^{(m+1)}_1 + g^{(M-2)}_{m,M}\sigma^{(m+1)}_2 + g^{(M-3)}_{m,M}\sigma^{(m+1)}_3 + q \\ &= y_{m,1} y^{M}_{m,2} + y^{M}_{m,2}\sigma^{(m+1)}_1 + 0 + g^{(M-2)}_{m,M-1}\sigma^{(m+1)}_3 + q \\ &= y_{m,1} y^{M}_{m,2} + y^{M}_{m,2}\sigma^{(m+1)}_1 + y^{M-1}_{m,2}\sigma^{(m+1)}_3 + q \end{aligned} \tag{3.14}$$

$$\begin{aligned} g_{m,M-1} &= g^{(M)}_{m,M-1} + g^{(M-1)}_{m,M-1}\sigma^{(m+1)}_1 + g^{(M-2)}_{m,M-1}\sigma^{(m+1)}_2 + g^{(M-3)}_{m,M-1}\sigma^{(m+1)}_3 + r \\ &= y^2_{m,1} y^{M-1}_{m,2} + y^{M}_{m,2} + y_{m,1} y^{M-1}_{m,2}\sigma^{(m+1)}_1 + y^{M-1}_{m,2}\sigma^{(m+1)}_2 + r, \end{aligned} \tag{3.15}$$

where

$$\sigma^{(m+1)}_k = \sum_{r_{m+1}+\cdots+r_d = k} \overline{y}_{m+1,r_{m+1}} \cdots \overline{y}_{d,r_d}, \; k = \overline{1, 4},$$

and p, q, r are polynomials in variables $y_{i,1}, y_{i,2}$, $i = \overline{m, d}$, such that the total dimension of $y_{m,1}$ and $y_{m,2}$ in each monomial of these polynomials is at most $2M - 2$.

As the conclusion, let us illustrate the use of Gröbner bases by the following examples.

Example 1. Let us consider the flag manifold $F(1^{\cdots j}, n)$. By Corollary 3.1.1 the monomial

$$x_1^n x_2^{n+1} \dots x_j^{n+j-1}$$

is nonzero in $H^*(F(1^{\cdots j}, n); \mathbb{Z}_2)$. Note that its degree is $nj + \binom{j}{2}$, which is equal to the dimension of the manifold $F(1^{\cdots j}, n)$.

Example 2. Let $F(1^{\cdots j}, 2^{\cdots d}, n)$ be the real flag manifold with $d \geq 2$. By Corollary 3.1.1, in $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$ we have $y_{d-1,2}^{n+j+2d-4} y_{d,2}^{n+j+2d-2} \neq 0$, and by symmetry $y_{d,2}^{n+j+2d-4} y_{d-1,2}^{n+j+2d-2} \neq 0$. Moreover, by Corollary 3.1.3,

$$y_{d-1,2}^{n+j+2d-4} y_{d,2}^{n+j+2d-2} = y_{d,2}^{n+j+2d-4} y_{d-1,2}^{n+j+2d-2}.$$

On the other hand, we will show that the monomial $y_{d-1,2}^{n+j+2d-3} y_{d,2}^{n+j+2d-3}$, which is in the same dimension as the previous two, is zero. By formula (3.13)

$$0 = g_{d-1,n+j+2d-3} = y_{d-1,2}^{n+j+2d-3} + y_{d-1,2}^{n+j+2d-4}(y_{d,1}^2 + y_{d,2}) + \widetilde{p},$$

where $\widetilde{p}$ is a polynomial in $y_{d-1,1}, y_{d-1,2}, y_{d,1}, y_{d,2}$, such that the total dimension of $y_{d-1,1}$ and $y_{d-1,2}$ in each monomial of $\widetilde{p}$ is at most $2n + 2j + 4d - 9$. Also, by formula (3.15)

$$0 = g_{d,n+j+2d-3} = y_{d,1}^2 y_{d,2}^{n+j+2d-3} + y_{d,2}^{n+j+2d-2},$$

and therefore

$$\begin{aligned} y_{d-1,2}^{n+j+2d-3} & y_{d,2}^{n+j+2d-3} \\ &= \left(y_{d-1,2}^{n+j+2d-4}(y_{d,1}^2 + y_{d,2}) + \widetilde{p}\right) y_{d,2}^{n+j+2d-3} \\ &= y_{d-1,2}^{n+j+2d-4} y_{d,1}^2 y_{d,2}^{n+j+2d-3} + y_{d-1,2}^{n+j+2d-4} y_{d,2}^{n+j+2d-2} + \widetilde{p} y_{d,2}^{n+j+2d-3} \\ &= \widetilde{p} y_{d,2}^{n+j+2d-3} = 0, \end{aligned}$$

where the last equality follows from Corollary 3.1.3.

4. Non-embeddings and non-immersions of some flag manifolds of type $F(1^{\cdots j}, 2^{\cdots d}, n)$

In this section we use results from the previous section to obtain some non-embeddings and non-immersions of flag manifolds. Let

$$\begin{aligned} \mathrm{em}(F(1^{\cdots j}, 2^{\cdots d}, n)) &= \min\{m \mid F(1^{\cdots j}, 2^{\cdots d}, n) \text{ embeds into } \mathbb{R}^m\} \\ \mathrm{imm}(F(1^{\cdots j}, 2^{\cdots d}, n)) &= \min\{m \mid F(1^{\cdots j}, 2^{\cdots d}, n) \text{ immerses into } \mathbb{R}^m\}. \end{aligned}$$

It is well known (see [8, p. 120 and 49]) that if $w_t(\nu)$ is nontrivial, where ν is the stable normal bundle of $F(1^{\cdots j}, 2^{\cdots d}, n)$, then

$$\begin{aligned} \mathrm{em}(F(1^{\cdots j}, 2^{\cdots d}, n)) &\geq \delta(F(1^{\cdots j}, 2^{\cdots d}, n)) + t + 1 \\ &= jn + 2dn + 2jd + \binom{j}{2} + 4\binom{d}{2} + t + 1 \qquad (4.1) \\ \mathrm{imm}(F(1^{\cdots j}, 2^{\cdots d}, n)) &\geq \delta(F(1^{\cdots j}, 2^{\cdots d}, n)) + t \\ &= jn + 2dn + 2jd + \binom{j}{2} + 4\binom{d}{2} + t. \qquad (4.2) \end{aligned}$$

These inequalities will be used to obtain lower bounds for $\mathrm{em}(F(1^{\cdots j}, 2^{\cdots d}, n))$ and $\mathrm{imm}(F(1^{\cdots j}, 2^{\cdots d}, n))$, for some j, d, n.

Let γ_i, $i = \overline{1,j}$, γ_i', $i = \overline{1,d}$, and γ'', be the canonical vector bundles over $F(1^{\cdots j}, 2^{\cdots d}, n)$ ($\dim(\gamma_i) = 1$, $i = \overline{1,j}$, $\dim(\gamma_i') = 2$, $i = \overline{1,d}$, $\dim(\gamma'') = n$). By Lam's formula ([6]), for the tangent bundle τ over $F(1^{\cdots j}, 2^{\cdots d}, n)$, we have

$$\begin{aligned} \tau \cong & \bigoplus_{1\leq l<k\leq j} (\gamma_l \otimes \gamma_k) \oplus \bigoplus_{1\leq l<k\leq d} (\gamma_l' \otimes \gamma_k') \oplus \bigoplus_{\substack{1\leq l\leq j \\ 1\leq k\leq d}} (\gamma_l \otimes \gamma_k') \\ & \oplus \bigoplus_{1\leq l\leq j} (\gamma_l \otimes \gamma'') \oplus \bigoplus_{1\leq l\leq d} (\gamma_l' \otimes \gamma''). \end{aligned}$$

Adding $\bigoplus_{1\leq l\leq k\leq j} (\gamma_l \otimes \gamma_k) \oplus \bigoplus_{1\leq l\leq k\leq d} (\gamma_l' \otimes \gamma_k') \oplus \bigoplus_{\substack{1\leq l\leq j \\ 1\leq k\leq d}} (\gamma_l \otimes \gamma_k')$ to both sides of the previous isomorphism, and using the fact that $\bigoplus_{1\leq l\leq j} \gamma_l \oplus \bigoplus_{1\leq l\leq d} \gamma_l' \oplus \gamma''$ is a trivial $(n + j + 2d)$-dimensional bundle, we obtain

$$\begin{aligned} \tau \oplus & \bigoplus_{1\leq l\leq k\leq j} (\gamma_l \otimes \gamma_k) \oplus \bigoplus_{1\leq l\leq k\leq d} (\gamma_l' \otimes \gamma_k') \oplus \bigoplus_{\substack{1\leq l\leq j \\ 1\leq k\leq d}} (\gamma_l \otimes \gamma_k') \\ & \cong \bigoplus_{1\leq l\leq j} (n + j + 2d)\gamma_l \oplus \bigoplus_{1\leq l\leq d} (n + j + 2d)\gamma_l', \end{aligned}$$

and therefore

$$
\begin{aligned}
w(\tau)\cdot \prod_{1\le l<k\le j} w(\gamma_l\otimes\gamma_k) \prod_{\substack{1\le l\le j\\ 1\le k\le d}} w(\gamma_l\otimes\gamma_k') \prod_{1\le l\le k\le d} w(\gamma_l'\otimes\gamma_k')\\
=\prod_{l=1}^{j}(1+x_i)^{n+j+2d}\prod_{l=1}^{d}(1+y_{l,1}+y_{l,2})^{n+j+2d}. \qquad (4.3)
\end{aligned}
$$

Using the method described in [8, Problem 7-C] we obtain

$$w(\gamma_l\otimes\gamma_k) = 1+x_l+x_k,\ 1\le l\le k\le j \qquad (4.4)$$

$$w(\gamma_l\otimes\gamma_k') = 1+y_{k,1}+x_l^2+x_ly_{k,1}+y_{k,2},\ 1\le l\le j,\, 1\le k\le d \qquad (4.5)$$

$$
\begin{aligned}
w(\gamma_l'\otimes\gamma_k') =&1+y_{l,1}^2+y_{k,1}^2+y_{l,1}y_{k,1}+y_{l,1}^2y_{k,2}+y_{k,1}^2y_{l,2}+y_{l,2}^2+y_{k,2}^2\\
&+y_{l,1}y_{k,1}(y_{l,1}+y_{k,1})+y_{l,1}y_{k,1}(y_{l,2}+y_{k,2}),\ 1\le l\le k\le d. \qquad (4.6)
\end{aligned}
$$

We are ready to prove the main theorem of this section.

Theorem 4.1. *If $2^{s-1}<n<n+j+2d\le 2^s$, then $w_t(\nu)\neq 0$ for*

$$t=(j+2d)(2^s-n-j)-2d^2+\binom{j}{2}.$$

Proof. By formula (4.3), we have

$$
\begin{aligned}
w(\nu)=&\prod_{1\le l<k\le j} w(\gamma_l\otimes\gamma_k) \prod_{\substack{1\le l\le j\\ 1\le k\le d}} w(\gamma_l\otimes\gamma_k') \prod_{1\le l\le k\le d} w(\gamma_l'\otimes\gamma_k')\\
&\times\prod_{l=1}^{j}(1+x_l)^{-n-j-2d}\prod_{l=1}^{d}(1+y_{l,1}+y_{l,2})^{-n-j-2d}.
\end{aligned}
$$

Since, $(1+x_l)^{2^s}=1+x_l^{2^s}$, and the height of x_l is $n+j+2d-1$ (see [5, p. 147]), we have $x_l^{2^s}=0$, i.e., $(1+x_l)^{2^s}=1$. Similarly, by Corollary 3.1.6, $(1+y_{l,1}+y_{l,2})^{2^s}=1$, and therefore

$$
\begin{aligned}
w(\nu)=&\prod_{1\le l<k\le j} w(\gamma_l\otimes\gamma_k) \prod_{\substack{1\le l\le j\\ 1\le k\le d}} w(\gamma_l\otimes\gamma_k') \prod_{1\le l\le k\le d} w(\gamma_l'\otimes\gamma_k')\\
&\times\prod_{l=1}^{j}(1+x_l)^{2^s-n-j-2d}\prod_{l=1}^{d}(1+y_{l,1}+y_{l,2})^{2^s-n-j-2d}. \qquad (4.7)
\end{aligned}
$$

Using formulas (4.4)–(4.6), we conclude that the top class in (4.7) is in dimension t and

$$w_t(\nu) = \prod_{l=1}^{j} x_l^{2^s-n-j-2d} \prod_{l=1}^{d} y_{l,2}^{2^s-n-j-2d} \prod_{1\le l<k\le j} (x_l+x_k) \prod_{\substack{1\le i\le j\\ 1\le k\le d}} (x_l^2+x_l y_{k,1}+y_{k,2})$$
$$\times \prod_{l=1}^{d} y_{l,1}^2 \prod_{1\le l<k\le d} (y_{l,1}y_{k,1}(y_{l,2}+y_{k,2})+y_{l,1}^2 y_{k,2}+y_{k,1}^2 y_{l,2}+y_{l,2}^2+y_{k,2}^2). \tag{4.8}$$

To prove that $w_t(\nu) \neq 0$, let us examine one monomial m of $w_t(\nu)$. The degree of x_l in m is at most $(2^s-n-j-2d)+(j-1)+2d = 2^s-n-1$, for $1 \le l \le j$. The sum of degrees of $y_{l,1}$ and $y_{l,2}$ in m is at most $(2^s-n-j-2d)+j+2+2(d-1) = 2^s-n$, for $1 \le l \le j$. Since $2^s-n-1 \le n$, by Corollary 3.1.1, after multiplication in (4.8) $w_t(\nu)$ is represented as a sum of elements of $B_{j,d,n}$. In this sum

$$\prod_{l=1}^{j} x_l^{2^s-n-l} \prod_{l=1}^{d} y_{l,1}^2 y_{l,2}^{2^s-n-j-2l}$$

appears only once, so $w_t(\nu) \neq 0$ (this term is obtained by always choosing x_l from x_l+x_k, x_l^2 from $x_l^2+x_l y_{k,1}+y_{k,2}$, and $y_{l,2}^2$ from $y_{l,1}y_{k,1}(y_{l,2}+y_{k,2}) + y_{l,1}^2 y_{k,2}+y_{k,1}^2 y_{l,2}+y_{l,2}^2+y_{k,2}^2$ in (4.8)). □

By the previous theorem and inequalities (4.1)-(4.2), we have the following corollary.

Corollary 4.1.1. *If* $2^{s-1} < n < n+j+2d \le 2^s$, *then*

$$\operatorname{em}(F(1^{\cdots j}, 2^{\cdots d}, n)) \ge (j+2d)(2^s-1)+1;$$
$$\operatorname{imm}(F(1^{\cdots j}, 2^{\cdots d}, n)) \ge (j+2d)(2^s-1).$$

Note that this result extends Theorem 1.1.(a) from [11] and, in part, Corollary 1.1. and Corollary 1.2. from [15].

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