GRÖBNER BASIS (PARTIAL) FOR FLAG MANIFOLDS

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Abstract. Gröbner bases for ideals determining cohomology of (partial) flag manifolds in the Borel picture are obtained. In addition to that, corresponding Gröbner bases for isotropic flag manifolds are also given.

1. Introduction

Let $n_1, n_2, \ldots, n_r, m \in \mathbb{N}$. The complex flag manifold $F = F^{\mathbb{C}}(n_1, \ldots, n_r, m)$ is isomorphic to the homogeneous space

$$U(n_1 + n_2 + \cdots + n_r + m)/U(n_1) \times U(n_2) \times \cdots \times U(n_r) \times U(m)$$

(here $U(k)$ denotes the group of $k \times k$ unitary matrices). To avoid any confusion, let us note that some authors use the term *partial flag manifolds* for these manifolds, and the term *flag manifolds* for the special case when $n_1 = n_2 = \ldots = n_r = m = 1$ (in this paper these are called *complete flag manifolds*).

There are several ways to describe cohomology algebra $H^*(F;\mathbb{Z})$. The most notable are the ones using Schubert calculus and the Borel description via Chern classes of canonical vector bundles over F. For Schubert calculus a natural additive basis Σ_F of $H^*(F;\mathbb{Z})$ is given by Schubert classes, and furthermore, some multiplication rules for elements in Σ_F are known. Another convenient additive basis B_F is given by monomials in Chern classes (see Theorem 3.1), so it is natural to relate it to Borel's description.

Our main motivation comes from topology, where one naturally works with Chern or Stiefel–Whitney classes, since problems in topology are often connected to vector bundles and maps between various classifying spaces – for example, immersion and embedding dimension of manifolds, Lusternik–Schnirelmann category, cup-length, the existence of a non-zero section of a vector bundle etc. (see, e.g., [4, 9, 11, Doe (1996), 13, 22]). Therefore, it is of great interest to better understand multiplicative structure in the additive basis B_F. Let us note that we cannot use basis Σ_F for this purpose, since the transition matrix between Σ_F and B_F is hard to compute already in the case of Grassmannians (Kostka numbers appear there; see [15]).

2010 *Mathematics Subject Classification.* 13P10, 14M15, 57N65.

The first author was partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia Project #174032. The second author was partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia Project #174034. The third author was partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia Project #174008.

Cohomology algebra $H^*(F;\mathbb{Z})$ is described in the Borel picture as a quotient of a polynomial algebra in Chern classes by certain ideal I_F. Our aim is to make this description "finer", such that it supplies us with multiplication rules for B_F. In order to do so, we construct a suitable Gröbner basis for the ideal I_F. This basis will provide formulae for multiplication of elements of B_F by Chern classes, which may be understood as Pieri-type formulae for B_F. These kind of formulae certainly do not follow directly from Borel's description.

Gröbner bases for complete flags are well-known (see, e.g., [5, 23, 24]), but there are only few results for other flags (see [6, 14, 16, 17, 18, 20]). All of these results are either for Grassmannians or for flags for which every step (i.e., every n_i) is at most 2. It turned out that members of Gröbner bases, which were obtained in our previous papers, gave important identities related to multiplication in question. This proved to be rather useful for various applications.

In this paper we complete this project by determining (in closed form) Gröbner bases for all flag manifolds. As in [18], we deal with both complex and real flag manifolds. In fact, we do much more – Gröbner bases that we obtain are such that they produce the (minimal) set of identities that completely determine multiplication in cohomology in terms of additive bases B_F (see the discussion after the proof of Theorem 3.1 and Remark 2). What may be also significant, especially from the computational point of view, is that we give recurrence relations which may be used to find these identities.

In [8] the authors discuss Gröbner bases for a class of ideals motivated by computations in the Chow ring $A(X)$ (for a nonsingular variety X) related to flag bundles over X. They described the method that can be used to obtain them, but they explicitly determined only the leading terms of these bases.

Another standard way of dealing with computations in the cohomology of flag manifolds is to embed it in the cohomology of complete flags. We do not find it useful for our purpose. One look at the formula from Remark 2 and a brief glance at the elements of our Gröbner bases, should convince a reader that no computation with symmetric functions would easily give such formulae. So, we prefer to work with native classes of our flag manifolds.

The presentation of the results in this paper is as follows. Sections 2 and 3 are dedicated mainly to establish notation and recollect some necessary results about cohomology of flag manifolds in general, together with some technical results. Section 4 is dedicated to the determination of Gröbner bases for complex flag manifolds $F^{\mathbb{C}}(n_1,\ldots,n_r,m)$, where n_i and m are arbitrary positive integers. Finally, Section 5 just describes Gröbner bases for real flag manifolds which follows from previous considerations for complex ones.

2. Notation

In this paper we use the same notation as in [18]. The set of all positive integers is denoted by $\mathbb{N}$ and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For $n\in\mathbb{N}$, in $\mathbb{Z}^n$ we have the vectors $e_0=(0,0,\ldots,0)$, $e_1=(1,0,\ldots,0)$, $e_2=(0,1,0,\ldots,0),\ldots$, $e_n=(0,\ldots,0,1)$. For an n-tuple $\alpha=(a_1,a_2,\ldots,a_n)\in\mathbb{Z}^n$ and $0\le i\le j\le n$:

- $\alpha^i:=\alpha+e_i$ and $\alpha_i:=\alpha-e_i$;
- $\alpha^{i,j}:=\alpha+e_i+e_j$ and $\alpha_{i,j}:=\alpha-e_i-e_j$;

- $|\alpha| := \sum_{j=1}^{n} a_j$ and $\|\alpha\| := \sum_{j=1}^{n} j a_j$.

If $\mu = (m_2, \dots, m_n) \in \mathbb{Z}^{n-1}$, let:

- $[\alpha, \mu]_t := \binom{\sum_{j=t-1}^{n} a_j - \sum_{j=t}^{n} m_j}{a_{t-1}}, \quad 2 \le t \le n;$
- $[\alpha, \mu] := \prod_{t=2}^{n} [\alpha, \mu]_t.$

The number $[\alpha, \mu]_t$ is defined for all integers $a_1, a_2, \dots, a_n$, since we are assuming that $\binom{a}{b}$ is zero if $b < 0$ ($a, b \in \mathbb{Z}$).

It is clear that $[\alpha, \mathbf{0}] = [a_1, a_2, \dots, a_n]$, where $\mathbf{0} = (0, 0, \dots, 0) \in \mathbb{Z}^{n-1}$ and

$$[a_1, a_2, \dots, a_n] = \binom{a_1 + \cdots + a_n}{a_1} \binom{a_2 + \cdots + a_n}{a_2} \cdots \binom{a_{n-1} + a_n}{a_{n-1}}$$

is the multinomial coefficient. Also, in the case $n = 1$ note that μ must be $\emptyset$, $|\emptyset| = \|\emptyset\| = 0$, and it is understood that $[\alpha, \emptyset] = 1$ for all $\alpha = a_1 \in \mathbb{Z}$.

Let us also note that for $\mu = (m_2, \dots, m_n) \in \mathbb{Z}^{n-1}$, by our definition, $\|\mu\| = m_2 + 2m_3 + \cdots + (n-1)m_n$ (Proposition 3.2 explains why we are indexing $(n-1)$-tuples by the integers $2, 3, \dots, n$).

Let $n \in \mathbb{N}$. The following lemma is proved in [18, p. 68].

Lemma 2.1. *If $\alpha = (a_1, \dots, a_n) \in \mathbb{N}_0^n$ and $\mu = (m_2, \dots, m_n) \in \mathbb{N}_0^{n-1}$ are such that $[\alpha, \mu] \ne 0$, then $|\alpha| < |\mu|$ or else $\sum_{j=t}^{n} a_j \ge \sum_{j=t}^{n} m_j$ for all t such that $2 \le t \le n$.*

3. Cohomology of flag manifolds

Let $r, n_1, n_2, \dots, n_r, m \in \mathbb{N}$ be fixed integers and $n = m + n_1 + n_2 + \cdots + n_r$. The flag manifold $F = F^{\mathbb{C}}(n_1, n_2, \dots, n_r, m)$ consists of complex flags in $\mathbb{C}^n$ of type $(n_1, n_2, \dots, n_r, m)$, that is, $(r+1)$-tuples $(V_1, V_2, \dots, V_{r+1})$ of mutually orthogonal complex vector subspaces of $\mathbb{C}^n$ with $\dim_{\mathbb{C}}(V_i) = n_i$, $1 \le i \le r$, and $\dim_{\mathbb{C}}(V_{r+1}) = m$. For $r = 1$, the flag manifold $F = F^{\mathbb{C}}(n_1, m)$ is actually the Grassmann manifold $G_{n_1,m}(\mathbb{C}) = G_{n_1}(\mathbb{C}^n)$.

There are $r + 1$ canonical complex vector bundles over flag manifold F; let us denote them by $\gamma_1, \gamma_2, \dots, \gamma_r, \gamma_{r+1}$ ($\dim_{\mathbb{C}}(\gamma_i) = n_i$, $1 \le i \le r$, $\dim_{\mathbb{C}}(\gamma_{r+1}) = m$). Let $c_{i,j} \in H^{2j}(F; \mathbb{Z})$, for $1 \le i \le r$ and $1 \le j \le n_i$, be the j-th Chern class of the bundle γ_i (observe that we now consider only the first r of the canonical bundles, the last one is omitted).

For $1 \le i \le r$ and an n_i-tuple $\alpha = (a_1, a_2, \dots, a_{n_i}) \in \mathbb{N}_0^{n_i}$ we use the notation C_i^{α} for the monomial $c_{i,1}^{a_1} c_{i,2}^{a_2} \cdots c_{i,n_i}^{a_{n_i}}$. Also, let $S_0 := m$ and $S_i := m + n_1 + n_2 + \cdots + n_i$, $1 \le i \le r$.

The following theorem is probably known, but we were not able to find a reference, so we prove it here. It establishes one of our main objects of study here – an additive basis of a flag manifold.

Theorem 3.1. *The set*

$$B_F := \left\{ C_1^{\alpha(1)} C_2^{\alpha(2)} \cdots C_r^{\alpha(r)} \ : \ |\alpha(i)| \le S_{i-1}, 1 \le i \le r \right\}$$

is an additive basis for $H^*(F; \mathbb{Z})$.

Proof. The proof is by induction on r. It is a known fact that the theorem is true for $r = 1$ (see e.g. [12, Proposition 2.5] or [18, Proposition 2]). Proceeding to the induction step, for $r \geq 2$ observe the map $p : F \to F^{\mathbb{C}}(n_2, \dots, n_r, n_1 + m)$,

$$p(V_1, V_2, \dots, V_{r+1}) = (V_2, \dots, V_r, V_1 \oplus V_{r+1}).$$

The map p is a fiber bundle with fiber $F^{\mathbb{C}}(n_1, m)$.

$$\begin{tikzcd} F^{\mathbb{C}}(n_1, m) \arrow[r, hook, "l"] & F^{\mathbb{C}}(n_1, n_2, \dots, n_r, m) = F \arrow[d, "p"] \\ & F^{\mathbb{C}}(n_2, \dots, n_r, n_1 + m) \end{tikzcd} \tag{3.1}$$

Moreover, if $l : F^{\mathbb{C}}(n_1, m) \hookrightarrow F$ is an inclusion of the fiber, then the pullback vector bundle $l^*(\gamma_1)$ is identified with the canonical vector bundle over the Grassmannian $F^{\mathbb{C}}(n_1, m)$ (cf. [Doe (1996), p. 146]), and so the class $l^*(c_{1,j}) \in H^{2j}(F^{\mathbb{C}}(n_1, m); \mathbb{Z})$ is equal to the j-th Chern class of this bundle ($1 \leq j \leq n_1$). Therefore (see the remark at the beginning of the proof), $H^*(F^{\mathbb{C}}(n_1, m); \mathbb{Z})$ is a free $\mathbb{Z}$-module with basis

$$\{l^*(c_{1,1})^{a_1} \cdots l^*(c_{1,n_1})^{a_{n_1}} \,:\, a_1 + \cdots + a_{n_1} \leq m\} = \{l^*(C_1^{\alpha}) \,:\, |\alpha| \leq S_0\}.$$

This means that we can apply the Leray–Hirsch theorem to the fiber bundle (3.1). By the induction hypothesis, an additive basis for $H^*(F^{\mathbb{C}}(n_2, \dots, n_r, n_1 + m); \mathbb{Z})$ is the set $\{C_2^{\alpha(2)} \cdots C_r^{\alpha(r)} \,:\, |\alpha(i)| \leq S_{i-1}, 2 \leq i \leq r\}$, where we use the same notation for the appropriate monomials in $H^*(F^{\mathbb{C}}(n_2, \dots, n_r, n_1 + m); \mathbb{Z})$ as in $H^*(F; \mathbb{Z})$. We conclude that $H^*(F; \mathbb{Z})$ is free with basis

$$\left\{C_1^{\alpha(1)} p^*\big(C_2^{\alpha(2)} \cdots C_r^{\alpha(r)}\big) \,:\, |\alpha(i)| \leq S_{i-1}, 1 \leq i \leq r\right\},$$

so it remains to prove that $p^*\big(C_2^{\alpha(2)} \cdots C_r^{\alpha(r)}\big) = C_2^{\alpha(2)} \cdots C_r^{\alpha(r)}$. However, this is an immediate consequence of the fact that the pullbacks of the first $r-1$ canonical vector bundles over $F^{\mathbb{C}}(n_2, \dots, n_r, n_1 + m)$ via p are easily identified with the canonical bundles $\gamma_2, \dots, \gamma_r$ over F. □

Simple counting arguments show that there are $[n_1, \dots, n_r, m]$ elements in this basis. Of course, this agrees with the result one may obtain from the appropriate Poincaré polynomial (see, e.g., [9, Corollary 9.5.15]).

Observe that a monomial $C_1^{\alpha(1)} C_2^{\alpha(2)} \cdots C_r^{\alpha(r)}$ is an element of the additive basis B_F if and only if it is not divisible by any of the monomials C_i^{α}, $1 \leq i \leq r$, $|\alpha| = S_{i-1}+1$. This leads us to the idea of finding a Gröbner basis for the ideal determining the cohomology algebra $H^*(F; \mathbb{Z})$ with the property that the set of all its leading monomials is exactly the set

$$B_F^+ = \{C_i^{\alpha} \,:\, 1 \leq i \leq r,\, |\alpha| = S_{i-1} + 1\}.$$

In fact, in this paper we obtain much more. Every element of the Gröbner basis that we construct will contain only one monomial not belonging to the additive basis B_F – its leading monomial. Therefore, this bases gives the representation of each element of B_F^+ in the additive basis B_F. Thus, it produces the (minimal) set of rules for the multiplication in B_F, which is similar to some classical rules of this type (for example, Pieri's formula). This is what we do in the next section.

For $1 \leq i \leq r$ the dual classes $\overline{c}_{i,j}$, $j \geq 1$, are defined by the equation

$$(1 + c_{i,1} + c_{i,2} + \cdots + c_{i,n_i})(1 + \overline{c}_{i,1} + \overline{c}_{i,2} + \cdots) = 1.$$

We also define $\overline{c}_{i,0} := 1$ and $\overline{c}_{i,j} := 0$ for $j < 0$. From the equation one easily obtains that, for all integers s,

$$\overline{c}_{i,s} = -\sum_{j=1}^{n_i} c_{i,j}\overline{c}_{i,s-j}, \tag{3.2}$$

and

$$\begin{aligned}\overline{c}_{i,s} &= \sum_{a_1+2a_2+\cdots+n_i a_{n_i}=s} (-1)^{a_1+a_2+\cdots+a_{n_i}}[a_1, a_2, \ldots, a_{n_i}]c_{i,1}^{a_1}c_{i,2}^{a_2}\cdots c_{i,n_i}^{a_{n_i}} \\ &= \sum_{\|\alpha\|=s} (-1)^{|\alpha|}[\alpha, \mathbf{0}]C_i^{\alpha}, \end{aligned}\tag{3.3}$$

where the sum is over all $\alpha = (a_1, a_2, \ldots, a_{n_i}) \in \mathbb{N}_0^{n_i}$ such that $\|\alpha\| = s$.

By Borel's description ([3]), the cohomology algebra $H^*(F;\mathbb{Z})$ is isomorphic to the quotient algebra

$$\mathbb{Z}[c_{1,1}, \ldots, c_{1,n_1}, c_{2,1}, \ldots, c_{2,n_2}, \ldots, c_{r,1}, \ldots, c_{r,n_r}]/I_F,$$

where I_F is the ideal generated by the classes (polynomials) $\overline{c}_{m+1}, \overline{c}_{m+2}, \ldots, \overline{c}_n$, which are obtained from the equation

$$(1 + \overline{c}_1 + \overline{c}_2 + \cdots) \cdot \prod_{i=1}^{r}(1 + c_{i,1} + c_{i,2} + \cdots + c_{i,n_i}) = 1,$$

that is,

$$1 + \overline{c}_1 + \overline{c}_2 + \cdots = \prod_{i=1}^{r}(1 + c_{i,1} + c_{i,2} + \cdots + c_{i,n_i})^{-1} = \prod_{i=1}^{r}(1 + \overline{c}_{i,1} + \overline{c}_{i,2} + \cdots).$$

Now we have that

$$\overline{c}_s = \sum_{j_1+\cdots+j_r=s} \overline{c}_{1,j_1}\cdots\overline{c}_{r,j_r}, \qquad s \in \mathbb{N}, \tag{3.4}$$

where the sum is taken over all r-tuples of nonnegative integers $(j_1, \ldots, j_r)$ such that $j_1 + \cdots + j_r = s$.

We are going to use the following abbreviation:

$$\mathbb{Z}[C_1, C_2, \ldots, C_r] := \mathbb{Z}[c_{1,1}, \ldots, c_{1,n_1}, c_{2,1}, \ldots, c_{2,n_2}, \ldots, c_{r,1}, \ldots, c_{r,n_r}].$$

The polynomials $\overline{c}_{i,s}$ ($1 \leq i \leq r$, $s \in \mathbb{Z}$) and $\overline{c}_s$ ($s \in \mathbb{N}$) are elements of this polynomial algebra. Let us now define one important set of polynomials in $\mathbb{Z}[C_1, C_2, \ldots, C_r]$. For $1 \leq k \leq r$, an (n_k-1)-tuple of nonnegative integers $\mu = (m_2, \ldots, m_{n_k})$ and an integer N, let

$$g_{k,\mu}^{(N)} = \sum_{\|\alpha\|=N+1+\|\mu\|} (-1)^{N+1+|\alpha|}[\alpha, \mu]C_k^{\alpha}, \tag{3.5}$$

where the sum is taken over all $\alpha \in \mathbb{N}_0^{n_k}$ that satisfy $\|\alpha\| = N + 1 + \|\mu\|$. So, $g_{k,\mu}^{(N)}$ is a polynomial in variables $c_{k,1}, c_{k,2}, \ldots, c_{k,n_k}$ only. Note the similarity of these polynomials with those which form Gröbner bases in [18]; consequently, they share some of the properties. Observe that

$$g_{k,\mathbf{0}}^{(N)} = \sum_{\|\alpha\|=N+1} (-1)^{N+1+|\alpha|}[\alpha,\mathbf{0}]C_k^\alpha = (-1)^{N+1}\overline{c}_{k,N+1}.$$

Note also that $g_{k,\mu}^{(N)} = 0$ if $N < -1 - \|\mu\|$.

Example 1. Let $F := F(2,3,3)$. Then for $k=1$ and $\mu=(2)$ we have

$$g_{1,(2)}^{(N)} = \sum_{a+2b=N+3} (-1)^{N+1+a+b}\binom{a+b-2}{a} c_{1,1}^a c_{1,2}^b,$$

where the sum is over all $(a,b)\in\mathbb{N}_0^2$ that satisfy $a+2b=N+3$. In particular, for $-2\le N\le 3$ the polynomial $g_{1,(2)}^{(N)}$ is given in the following table:

N	-2	-1	0	1	2	3
$g_{1,(2)}^{(N)}$	$-c_{1,1}$	$-c_{1,2}$	0	$c_{1,2}^2$	$c_{1,1}c_{1,2}^2$	$c_{1,1}^2c_{1,2}^2 - c_{1,2}^3$

As usual, for a polynomial $f = \alpha_1\mathfrak{C}_1 + \cdots + \alpha_l\mathfrak{C}_l \in \mathbb{Z}[C_1, C_2, \ldots, C_r]$, where $\alpha_1,\ldots,\alpha_l \in \mathbb{Z}\setminus\{0\}$ and $\mathfrak{C}_1,\ldots,\mathfrak{C}_l$ are pairwise different monomials in variables $c_{i,j}$ ($1\le i\le r$, $1\le j\le n_i$), we define $M(f) := \{\mathfrak{C}_1,\mathfrak{C}_2,\ldots,\mathfrak{C}_l\}$ (it is understood that $M(0)=\emptyset$).

Proposition 3.2. *Let $1\le k\le r$ and $\mu=(m_2,\ldots,m_{n_k})\in\mathbb{N}_0^{n_k-1}$ be such that $|\mu|\le S_{k-1}+1$.*

(a) *If $N<S_{k-1}$ and $\alpha\in\mathbb{N}_0^{n_k}$ such that $C_k^\alpha\in M(g_{k,\mu}^{(N)})$, then $|\alpha|\le S_{k-1}$.*

(b) *For $\overline{\mu} := (S_{k-1}+1-|\mu|, m_2,\ldots,m_{n_k})$ we have that $C_k^{\overline{\mu}}\in M(g_{k,\mu}^{(S_{k-1})})$ and the coefficient of $C_k^{\overline{\mu}}$ in $g_{k,\mu}^{(S_{k-1})}$ is 1. Moreover, if $C_k^\alpha\in M(g_{k,\mu}^{(S_{k-1})})\setminus\{C_k^{\overline{\mu}}\}$ for some $\alpha\in\mathbb{N}_0^{n_k}$, then $|\alpha|\le S_{k-1}$.*

Proof. (a) Since $C_k^\alpha\in M(g_{k,\mu}^{(N)})$, $[\alpha,\mu]\ne 0$ (by definition (3.5)). By Lemma 2.1, we have that either $|\alpha|<|\mu|\le S_{k-1}+1$ (in which case the proof is completed) or

$$\sum_{j=t}^{n_k} a_j \ge \sum_{j=t}^{n_k} m_j, \quad 2\le t\le n_k.$$

Summing up these inequalities we obtain $\sum_{j=2}^{n_k}(j-1)a_j \ge \sum_{j=2}^{n_k}(j-1)m_j$. However, $\|\alpha\| = N+1+\|\mu\|$ (since $C_k^\alpha\in M(g_{k,\mu}^{(N)})$), and so

$$\|\alpha\|-|\alpha| = \sum_{j=2}^{n_k}(j-1)a_j \ge \sum_{j=2}^{n_k}(j-1)m_j = \|\mu\| = \|\alpha\|-N-1 > \|\alpha\|-S_{k-1}-1,$$

i.e., $|\alpha|<S_{k-1}+1$.

(b) The proof of this part of the proposition is exactly the same as the one of Proposition 5 in [18]. □

As for the part (b) of Proposition 3.2, the proofs of the following results are identical with the corresponding proofs in [18, Proposition 7 and Lemma 9].

Proposition 3.3. *Let $1 \leq k \leq r$, $\mu \in \mathbb{N}_0^{n_k-1}$ and $1 \leq i \leq j \leq n_k - 1$. Then for all integers N the following identity holds*

$$g^{(N)}_{k,\mu^{i,j}} = c_{k,i} g^{(N)}_{k,\mu^j} - c_{k,j+1} g^{(N)}_{k,\mu^{i-1}} + g^{(N)}_{k,\mu^{i-1,j+1}},$$

where it is understood that $g^{(N)}_{k,\mu^{i-1,j+1}}$ is equal to zero if $j = n_k - 1$.

Lemma 3.1. *Let $1 \leq k \leq r$, $s \geq 0$ and $\mathbf{s} = (s, 0, \ldots, 0) \in \mathbb{N}_0^{n_k-1}$ (if $n_k = 1$, then $\mathbf{s} = \emptyset$, and we allow the case $s = 0$ only). Then for all integers N the following identity holds*

$$g^{(N)}_{k,\mathbf{s}} = (-1)^{N+1} \sum_{i=0}^{s} \binom{s}{i} c_{k,1}^{s-i} \cdot \overline{c}_{k,N+1+i}.$$

4. Gröbner basis for the ideal I_F

For $1 \leq k \leq r$ and $s \in \mathbb{Z}$, let

$$\overline{c}^{(k)}_s = \sum_{d_k + \cdots + d_r = s} \overline{c}_{k,d_k} \cdots \overline{c}_{r,d_r}, \tag{4.1}$$

where the sum is over all $(r-k+1)$-tuples $(d_k, d_{k+1}, \ldots, d_r)$ of nonnegative integers, such that $d_k + d_{k+1} + \cdots + d_r = s$. (Note that the classes of these polynomials in $\mathbb{Z}[C_1, \ldots, C_r]/I_F \cong H^*(F;\mathbb{Z})$ are actually Chern classes of the bundle $(\gamma_k \oplus \cdots \oplus \gamma_r)^\perp = \gamma_1 \oplus \cdots \oplus \gamma_{k-1} \oplus \gamma_{r+1}$, which is an S_{k-1}-bundle.) By looking at (3.4) and (4.1) we see that $\overline{c}^{(1)}_s = \overline{c}_s$ for all $s \in \mathbb{N}$. Also, it is obvious that $\overline{c}^{(k)}_0 = 1$, and that $\overline{c}^{(k)}_s = 0$ for $s < 0$ $(1 \leq k \leq r)$.

Example 2. Let $F := F(2,3,3)$. Then, for $k = 2$ we have

$$\overline{c}^{(2)}_s = \overline{c}_{2,s} = \sum_{a+2b+3c=s} (-1)^{a+b+c} \binom{a+b+c}{a} \binom{b+c}{b} c_{2,1}^a c_{2,2}^b c_{2,3}^c,$$

where the sum is over all $(a,b,c) \in \mathbb{N}_0^3$ that satisfy $a + b + c = s$. In particular, for $0 \leq s \leq 4$ the polynomial $\overline{c}^{(2)}_s$ is given in the following table:

s	0	1	2	3	4
$\overline{c}^{(2)}_s$	1	$-c_{2,1}$	$c_{2,1}^2 - c_{2,2}$	$-c_{2,1}^3 + 2c_{2,1}c_{2,2} - c_{2,3}$	$c_{2,1}^4 - 3c_{2,1}^2 c_{2,2} + 2c_{2,1}c_{2,3} + c_{2,2}^2$

.

We shall need the following extension of definition (4.1) to the case $k = r+1$: $\overline{c}^{(r+1)}_s := 0$ for $s \neq 0$, and $\overline{c}^{(r+1)}_0 := 1$.

Proposition 4.1. *For $1 \leq k \leq r$ and all integers s the following identity holds*

$$\overline{c}^{(k)}_s = -\sum_{j=1}^{n_k} c_{k,j} \overline{c}^{(k)}_{s-j} + \overline{c}^{(k+1)}_s.$$

Proof. The equality is obviously true for $s \leq 0$. For $s \geq 1$, by definition (4.1), identity (3.2) and the fact that $\overline{c}_{i,j} = 0$ for $j < 0$, one has

$$\begin{aligned}
\overline{c}_s^{(k)} &= \sum_{\substack{d_k+\cdots+d_r=s\\ d_k\geq 1}} \overline{c}_{k,d_k}\cdots\overline{c}_{r,d_r} + \sum_{d_{k+1}+\cdots+d_r=s} \overline{c}_{k+1,d_{k+1}}\cdots\overline{c}_{r,d_r}\\
&= -\sum_{\substack{d_k+\cdots+d_r=s\\ d_k\geq 1}} \left(\sum_{j=1}^{n_k} c_{k,j}\overline{c}_{k,d_k-j}\right)\overline{c}_{k+1,d_{k+1}}\cdots\overline{c}_{r,d_r} + \overline{c}_s^{(k+1)}\\
&= -\sum_{j=1}^{n_k} c_{k,j} \sum_{\substack{(d_k-j)+d_{k+1}+\cdots+d_r=s-j\\ d_k-j\geq 0}} \overline{c}_{k,d_k-j}\overline{c}_{k+1,d_{k+1}}\cdots\overline{c}_{r,d_r} + \overline{c}_s^{(k+1)}\\
&= -\sum_{j=1}^{n_k} c_{k,j}\overline{c}_{s-j}^{(k)} + \overline{c}_s^{(k+1)},
\end{aligned}$$

which completes our proof. □

The ideal $I_F \lhd \mathbb{Z}[C_1, C_2, \ldots, C_r]$, which determines $H^*(F;\mathbb{Z})$, was defined as the ideal generated by the classes $\overline{c}_{m+1}, \overline{c}_{m+2}, \ldots, \overline{c}_n$. We are now able to present a more suitable set of generators for this ideal (cf. [6]).

Corollary 4.1.1. *The ideal I_F is generated by the set*

$$G_1 := \{\overline{c}_s^{(k)} \,:\, 1 \leq k \leq r,\ S_{k-1}+1 \leq s \leq S_k\}.$$

Proof. For $1 \leq k \leq r$ observe the set

$$G_1^{(k)} := \{\overline{c}_s^{(i)} \,:\, 1 \leq i \leq k-1,\ S_{i-1}+1 \leq s \leq S_i\} \cup \{\overline{c}_s^{(k)} \,:\, S_{k-1}+1 \leq s \leq S_r\}.$$

Since $S_0 = m$, $S_r = n$ and $\overline{c}_s^{(1)} = \overline{c}_s$ for all $s \in \mathbb{N}$, we have that

$$G_1^{(1)} = \{\overline{c}_{m+1}, \overline{c}_{m+2}, \ldots, \overline{c}_n\},$$

that is, $I_F = (G_1^{(1)})$. Since obviously $G_1^{(r)} = G_1$, it suffices to prove that $(G_1^{(k)}) = (G_1^{(k+1)})$ for all $k \in \{1, 2, \ldots, r-1\}$.

By definition, $G_1^{(k+1)} \setminus G_1^{(k)} = \{\overline{c}_s^{(k+1)} \,:\, S_k+1 \leq s \leq S_r\}$. However, for $S_k+1 \leq s \leq S_r$, by Proposition 4.1 we have that

$$\overline{c}_s^{(k+1)} = \overline{c}_s^{(k)} + \sum_{j=1}^{n_k} c_{k,j}\overline{c}_{s-j}^{(k)} = \sum_{j=0}^{n_k} c_{k,j}\overline{c}_{s-j}^{(k)} \in (G_1^{(k)}),$$

since if $0 \leq j \leq n_k$, then $S_{k-1}+1 = S_k - n_k + 1 \leq s-j \leq S_r$, and so $\overline{c}_{s-j}^{(k)} \in G_1^{(k)}$. Therefore, $(G_1^{(k+1)}) \subseteq (G_1^{(k)})$.

On the other hand, $G_1^{(k)} \setminus G_1^{(k+1)} = \{\overline{c}_s^{(k)} \,:\, S_k+1 \leq s \leq S_r\}$. We know that $\overline{c}_s^{(k)} \in G_1^{(k+1)}$ for $S_{k-1}+1 \leq s \leq S_k$, and using induction and Proposition 4.1, for $S_k+1 \leq s \leq S_r$ we obtain that

$$\overline{c}_s^{(k)} = -\sum_{j=1}^{n_k} c_{k,j}\overline{c}_{s-j}^{(k)} + \overline{c}_s^{(k+1)} \in (G_1^{(k+1)}),$$

since $s-j \geq s-n_k \geq S_k+1-n_k = S_{k-1}+1$. Hence, $(G_1^{(k)}) \subseteq (G_1^{(k+1)})$. □

Corollary 4.1.2. *For $1 \leq k \leq r$ and every $s \geq S_{k-1}+1$ one has $\overline{c}_s^{(k)} \in I_F$.*

Proof. The proof is by reverse induction on $k \in \{1,2,\ldots,r\}$. For $k=r$, by Corollary 4.1.1 $\overline{c}_s^{(r)} \in I_F$ if $S_{r-1}+1 \leq s \leq S_r$. For $s \geq S_r+1$, the result now follows by induction on s, since by Proposition 4.1

$$\overline{c}_s^{(r)} = -\sum_{j=1}^{n_r} c_{r,j}\overline{c}_{s-j}^{(r)}$$

and $s-j \geq S_r+1-n_r = S_{r-1}+1$.

Let us now prove the result for an integer $k \in \{1,2,\ldots,r-1\}$ assuming that it holds for $k+1$. According again to Corollary 4.1.1, $\overline{c}_s^{(k)} \in I_F$ if $S_{k-1}+1 \leq s \leq S_k$. For $s \geq S_k+1$, we know that $\overline{c}_s^{(k+1)} \in I_F$ (by induction hypothesis), and so, by Proposition 4.1 we have that

$$\overline{c}_s^{(k)} = -\sum_{j=1}^{n_k} c_{k,j}\overline{c}_{s-j}^{(k)} + \overline{c}_s^{(k+1)} \in I_F,$$

using induction on s and the fact that $s-j \geq S_k+1-n_k = S_{k-1}+1$. □

In order to construct a Gröbner basis for the ideal I_F, we now define a monomial ordering $\prec$ in $\mathbb{Z}[C_1,C_2,\ldots,C_r]$. Let us fix an integer $k \in \{1,2,\ldots,r\}$ for a moment, and consider the monomials in variables $c_{k,1}$, $c_{k,2}$,..., c_{k,n_k} only. Recall that the graded lexicographical ordering $\prec_{grlex}$ on these monomials is defined as follows. For $\alpha,\beta \in \mathbb{N}_0^{n_k}$ ($\alpha = (a_1,\ldots,a_{n_k})$, $\beta = (b_1,\ldots,b_{n_k})$), $C_k^\alpha \prec_{grlex} C_k^\beta$ if and only if $|\alpha| < |\beta|$ or else $|\alpha| = |\beta|$ and $a_s < b_s$, where $s = \min\{j : a_j \neq b_j\}$.

Returning to monomials in $\mathbb{Z}[C_1,C_2,\ldots,C_r]$, for $\alpha(i),\beta(i) \in \mathbb{N}_0^{n_i}$, $1 \leq i \leq r$, we define:

$$C_1^{\alpha(1)}C_2^{\alpha(2)}\cdots C_r^{\alpha(r)} \prec C_1^{\beta(1)}C_2^{\beta(2)}\cdots C_r^{\beta(r)}$$

if and only if $C_k^{\alpha(k)} \prec_{grlex} C_k^{\beta(k)}$, where $k = \min\{i : \alpha(i) \neq \beta(i)\}$.

For example, if $1 \leq k < l \leq r$, then for arbitrary $\alpha \in \mathbb{N}_0^{n_k} \setminus \{\mathbf{0}\}$ and $\beta \in \mathbb{N}_0^{n_l}$ one has $C_k^\alpha \succ C_l^\beta$. Also, if $1 \leq k \leq r$, note that the assertion that a monomial $\mathfrak{C}$ does not contain the variables $c_{i,j}$ for $i \leq k$ can be stated as $\mathfrak{C} \prec c_{k,n_k}$.

Now that we have specified the monomial ordering $\prec$, we recall the basic notions in the theory of Gröbner bases. For a nonzero polynomial $f \in \mathbb{Z}[C_1,C_2,\ldots,C_r]$, *the leading monomial* of f is $\mathrm{LM}(f) = \max M(f)$ with respect to $\prec$; *the leading coefficient* of f, denoted by $\mathrm{LC}(f)$, is the coefficient of $\mathrm{LM}(f)$ in f; and *the leading term* of f is $\mathrm{LT}(f) = \mathrm{LC}(f)\cdot\mathrm{LM}(f)$.

Now, if $1 \leq k \leq r$, then the assertion that a nonzero polynomial f does not contain the variables $c_{i,j}$ for $i \leq k$, i.e., that polynomial f contains only variables $c_{k+1,1},\ldots,c_{k+1,n_{k+1}},\ldots,c_{r,1},\ldots,c_{r,n_r}$, is equivalent to $\mathrm{LM}(f) \prec c_{k,n_k}$. For example, $\mathrm{LM}(\overline{c}_s^{(k+1)}) \prec c_{k,n_k}$ ($s \geq 0$, $1 \leq k \leq r$).

Let us now define a set of polynomials in $\mathbb{Z}[C_1,C_2,\ldots,C_r]$, which will turn out to be a strong Gröbner basis for I_F.

Definition 4.1. For $1 \le k \le r$ and $\mu \in \mathbb{N}_0^{n_k-1}$ let

$$g_{k,\mu} := \sum_{N=-n_k}^{S_{k-1}} (-1)^{S_{k-1}-N} g_{k,\mu}^{(N)} \cdot \overline{c}_{S_{k-1}-N}^{(k+1)}.$$

Moreover, let

$$G^{\mathbb{C}} := \{g_{k,\mu} \,:\, 1 \le k \le r,\, |\mu| \le S_{k-1}+1\}.$$

Note that $g_{r,\mu} = g_{r,\mu}^{(S_{r-1})}$ since $\overline{c}_{S_{r-1}-N}^{(r+1)}$ is nonzero only for $N = S_{r-1}$. Therefore, when $r = 1$, i.e., when F is the Grassmannian, the set $G^{\mathbb{C}}$ simplifies to the Gröbner basis obtained in [18, Definition 2].

It is evident that the polynomial $g_{k,\mu}$ does not contain the variables $c_{i,j}$ for $i \le k-1$, so every monomial in $g_{k,\mu}$ is of the form $C_k^{\alpha(k)} \cdots C_r^{\alpha(r)}$ for some $\alpha(i) \in \mathbb{N}_0^{n_i}$, $k \le i \le r$.

Example 3. Let $F := F(2,3,3)$. Then for $k = 1$ and $\mu = (2)$ we have $n_1 = 2$, $S_0 = 3$ and

$$g_{1,(2)} = \sum_{N=-2}^{3} (-1)^{3-N} g_{1,(2)}^{(N)} \cdot \overline{c}_{3-N}^{(2)}.$$

So, using the calculations done in Examples 1 and 2 we have

$$\begin{aligned} g_{1,(2)} = {}& c_{1,1}c_{2,1}^5 - 4c_{1,1}c_{2,1}^3c_{2,2} + 3c_{1,1}c_{2,1}^2c_{2,3} + 3c_{1,1}c_{2,1}c_{2,2}^2 - 2c_{1,1}c_{2,2}c_{2,3} - c_{1,2}c_{2,1}^4 \\ & + 3c_{1,2}c_{2,1}^2c_{2,2} - 2c_{1,2}c_{2,1}c_{2,3} - c_{1,2}c_{2,2}^2 + c_{1,2}^2c_{2,1}^2 - c_{1,2}^2c_{2,2} - c_{1,1}c_{1,2}^2c_{2,1} + c_{1,1}^2c_{1,2}^2 - c_{1,2}^3. \end{aligned}$$

Proposition 4.2. *Let $1 \le k \le r$ and $\mu = (m_2, \dots, m_{n_k}) \in \mathbb{N}_0^{n_k-1}$ be such that $|\mu| \le S_{k-1}+1$ (i.e., such that $g_{k,\mu} \in G^{\mathbb{C}}$). Then $g_{k,\mu} \ne 0$ and $\mathrm{LT}(g_{k,\mu}) = C_k^{\overline{\mu}}$, where $\overline{\mu} = (S_{k-1}+1-|\mu|, m_2, \dots, m_{n_k})$. Moreover, if $C_k^{\alpha(k)} \cdots C_r^{\alpha(r)} \in M(g_{k,\mu}) \setminus \{C_k^{\overline{\mu}}\}$, then $|\alpha(i)| \le S_{i-1}$ for all $i \in \{k, k+1, \dots, n\}$.*

Proof. The summand in $g_{k,\mu}$ for $N = S_{k-1}$ (see Definition 4.1) is $g_{k,\mu}^{(S_{k-1})}$, and $C_k^{\overline{\mu}}$ is a monomial in $g_{k,\mu}^{(S_{k-1})}$ with coefficient 1 (by Proposition 3.2). Also, all other monomials $C_k^{\alpha} \in M(g_{k,\mu}^{(N)})$, $-n_k \le N \le S_{k-1}$, have the property $|\alpha| \le S_{k-1}$ (again by Proposition 3.2). Since $|\overline{\mu}| = S_{k-1}+1$ and since the polynomials $\overline{c}_{S_{k-1}-N}^{(k+1)}$, $-n_k \le N \le S_{k-1}$, contain none of the variables $c_{k,1}, c_{k,2}, \dots, c_{k,n_k}$, this proves that $g_{k,\mu} \ne 0$, $\mathrm{LT}(g_{k,\mu}) = C_k^{\overline{\mu}}$ and that $|\alpha(k)| \le S_{k-1}$ for every $C_k^{\alpha(k)} \cdots C_r^{\alpha(r)} \in M(g_{k,\mu}) \setminus \{C_k^{\overline{\mu}}\}$.

If $C_k^{\alpha(k)} \cdots C_r^{\alpha(r)} \in M(g_{k,\mu}) \setminus \{C_k^{\overline{\mu}}\}$ and $k+1 \le i \le r$, then $C_{k+1}^{\alpha(k+1)} \cdots C_r^{\alpha(r)}$ is a monomial in $\overline{c}_{S_{k-1}-N}^{(k+1)}$ for some $N \in \{-n_k, \dots, S_{k-1}\}$ (since $g_{k,\mu}^{(N)}$ may contain variables $c_{k,1}, c_{k,2}, \dots, c_{k,n_k}$ only). By (4.1) and (3.3), $\sum_{l=k+1}^{r} \|\alpha(l)\| = S_{k-1} - N$, and thus

$$|\alpha(i)| \le \|\alpha(i)\| \le \sum_{l=k+1}^{r} \|\alpha(l)\| = S_{k-1} - N \le S_{k-1} + n_k = S_k \le S_{i-1}.$$

$\square$

Remark 1. In [8] the authors used the following monomial ordering on the polynomial algebra $\mathbb{Z}[C_1, C_2, \dots, C_r]$. In this paper this monomial ordering will be called *block graded reverse lexicographical ordering*, and denoted with $\prec_r$.

First, for an integer $k \in \{1, 2, \dots, r\}$, the *graded reverse lexicographical ordering* $\prec_{rlex}$ on monomials in variables $c_{k,1}, c_{k,2}, \dots, c_{k,n_k}$ is defined as follows. For $\alpha, \beta \in \mathbb{N}_0^{n_k}$ ($\alpha = (a_1, \dots, a_{n_k})$, $\beta = (b_1, \dots, b_{n_k})$), $C_k^\alpha \prec_{rlex} C_k^\beta$ if and only if $\|\alpha\| < \|\beta\|$ or else $\|\alpha\| = \|\beta\|$ and $a_s > b_s$, where $s = \max\{j : a_j \neq b_j\}$.

Returning to monomials in $\mathbb{Z}[C_1, C_2, \dots, C_r]$, for $\alpha(i), \beta(i) \in \mathbb{N}_0^{n_i}$, $1 \le i \le r$, one has

$$C_1^{\alpha(1)} C_2^{\alpha(2)} \cdots C_r^{\alpha(r)} \prec_r C_1^{\beta(1)} C_2^{\beta(2)} \cdots C_r^{\beta(r)}$$

if and only if $C_k^{\alpha(k)} \prec_{rlex} C_k^{\beta(k)}$, where $k = \min\{i : \alpha(i) \neq \beta(i)\}$.

Interestingly enough, for a polynomial $g \in G^{\mathbb{C}}$ its leading terms with respect to monomial orderings $\prec$ and $\prec_r$ are the same.

To prove this, let us consider a polynomial $g_{k,\mu}$, where $1 \le k \le r$ and $|\mu| \le S_{k-1} + 1$. By the definition of $\prec_r$, the leading term of $g_{k,\mu}$ with respect to this ordering is the same as the leading term of $g_{k,\mu}^{(S_{k-1})}$ with respect to the graded reverse lexicographical ordering. So, let us assume to the contrary that for this leading term $C_k^\alpha := c_{k,1}^{a_1} c_{k,2}^{a_2} \dots c_{k,n_k}^{a_{n_k}}$, one has $C_k^{\overline{\mu}} \prec_{rlex} C_k^\alpha$, and let $l = \max\{l : a_l \neq m_l\}$ (note that this set is nonempty; otherwise, since $\|\alpha\| = \|\overline{\mu}\| = S_{k-1} + 1 + \|\mu\|$, the equalities $a_l = m_l$, $2 \le l \le n_k$, would imply that $\alpha = \overline{\mu}$). Let us consider the coefficient $[\alpha, \mu]$. Since, $[\alpha, \mu] \neq 0$, one has $[\alpha, \mu]_t \neq 0$, for $2 \le t \le n_k$. In particular,

$$[\alpha, \mu]_{l-1} = \binom{a_{l-1} + a_l - m_l}{a_{l-1}} \neq 0.$$

By assumption $a_l < m_l$, and therefore $a_{l-1} + a_l < m_l$. Now, using a reverse induction on $t < l$ (and the fact $[\alpha, \mu] \neq 0$), one proves that $a_t + a_{t+1} + \cdots + a_l < m_{t+1} + \cdots + m_l$, $1 \le t \le l - 1$. Finally, adding these inequalities gives

$$a_1 + 2a_2 + \cdots + la_l < m_2 + 2m_3 + \cdots + (l-1)m_l,$$

and therefore

$$\|\alpha\| - \|\mu\| < m_{l+1} + m_{l+2} + \cdots + m_{n_k} \le |\mu| \le S_{k-1} + 1,$$

which contradicts the fact $\|\alpha\| = S_{k-1} + 1 + \|\mu\|$.

The following result is obtained by multiplying the equality from Proposition 3.3 with $(-1)^{S_{k-1}-N} \cdot \overline{c}_{S_{k-1}-N}^{(k+1)}$, and then summing up for $-n_k \le N \le S_{k-1}$.

Proposition 4.3. *Let $1 \le k \le r$, $\mu \in \mathbb{N}_0^{n_k - 1}$ and $1 \le i \le j \le n_k - 1$. Then*

$$g_{k,\mu^{i,j}} = c_{k,i} g_{k,\mu^j} - c_{k,j+1} g_{k,\mu^{i-1}} + g_{k,\mu^{i-1,j+1}},$$

where it is understood that $g_{k,\mu^{i-1,j+1}}$ is equal to zero for $j = n_k - 1$.

Using this proposition, in the same way as it was done in the proof of Corollary 8 in [18], one can verify that $g_{k,\mu} \in (G^{\mathbb{C}})$ for all $k \in \{1, 2, \dots, r\}$ and all $\mu \in \mathbb{N}_0^{n_k - 1}$ (not just for those with the property $|\mu| \le S_{k-1} + 1$).

Lemma 4.1. *Let $1 \le k \le r$, $0 \le s \le n_k - 1$ and $\mathbf{s} = (s, 0, \dots, 0) \in \mathbb{N}_0^{n_k-1}$. Then*

$$g_{k,\mathbf{s}} = (-1)^{S_{k-1}+1} \sum_{i=0}^{s} \binom{s}{i} c_{k,1}^{s-i} \cdot \overline{c}^{(k)}_{S_{k-1}+1+i}. \tag{4.2}$$

Moreover, for any $s \ge 0$ one has $g_{k,\mathbf{s}} \in I_F$.

Proof. For every $i \ge 0$,

$$\begin{aligned}
\overline{c}^{(k)}_{S_{k-1}+1+i} &= \sum_{d_k+\cdots+d_r=S_{k-1}+1+i} \overline{c}_{k,d_k} \cdots \overline{c}_{r,d_r} \\
&= \sum_{d_k=0}^{S_{k-1}+1+i} \overline{c}_{k,d_k} \sum_{d_{k+1}+\cdots+d_r=S_{k-1}+1+i-d_k} \overline{c}_{k+1,d_{k+1}} \cdots \overline{c}_{r,d_r} \\
&= \sum_{N=-i-1}^{S_{k-1}} \overline{c}_{k,N+1+i} \sum_{d_{k+1}+\cdots+d_r=S_{k-1}-N} \overline{c}_{k+1,d_{k+1}} \cdots \overline{c}_{r,d_r} \\
&= \sum_{N=-i-1}^{S_{k-1}} \overline{c}_{k,N+1+i} \cdot \overline{c}^{(k+1)}_{S_{k-1}-N}.
\end{aligned} \tag{4.3}$$

By Definition 4.1 and Lemma 3.1, for all $s \ge 0$ we have

$$\begin{aligned}
g_{k,\mathbf{s}} &= \sum_{N=-n_k}^{S_{k-1}} (-1)^{S_{k-1}-N} g^{(N)}_{k,\mathbf{s}} \cdot \overline{c}^{(k+1)}_{S_{k-1}-N} \\
&= \sum_{N=-n_k}^{S_{k-1}} (-1)^{S_{k-1}-N} \left((-1)^{N+1} \sum_{i=0}^{s} \binom{s}{i} c_{k,1}^{s-i} \overline{c}_{k,N+1+i} \right) \overline{c}^{(k+1)}_{S_{k-1}-N} \\
&= (-1)^{S_{k-1}+1} \sum_{i=0}^{s} \binom{s}{i} c_{k,1}^{s-i} \sum_{N=-n_k}^{S_{k-1}} \overline{c}_{k,N+1+i} \cdot \overline{c}^{(k+1)}_{S_{k-1}-N}.
\end{aligned} \tag{4.4}$$

If $s \le n_k - 1$, then $-n_k \le -s-1 \le -i-1$, and $\overline{c}_{k,N+1+i} = 0$ for $N < -i-1$, so identities (4.4) and (4.3) imply (4.2).

Proceeding to the second part of the lemma, if $0 \le s \le n_k - 1$, then $g_{k,\mathbf{s}} \in I_F$ by (4.2) and Corollary 4.1.2. For the case $s \ge n_k$, let us observe the identity (4.4). For $i \le n_k - 1$, as in the previous case we have

$$\sum_{N=-n_k}^{S_{k-1}} \overline{c}_{k,N+1+i} \cdot \overline{c}^{(k+1)}_{S_{k-1}-N} = \sum_{N=-i-1}^{S_{k-1}} \overline{c}_{k,N+1+i} \cdot \overline{c}^{(k+1)}_{S_{k-1}-N} = \overline{c}^{(k)}_{S_{k-1}+1+i} \in I_F,$$

by (4.3) and Corollary 4.1.2. Now, let $i \ge n_k$. In this case there are summands of (4.3) not appearing in (4.4) – these are exactly the ones for which $N \le -n_k - 1$. So

$$\sum_{N=-n_k}^{S_{k-1}} \overline{c}_{k,N+1+i} \cdot \overline{c}^{(k+1)}_{S_{k-1}-N} = \overline{c}^{(k)}_{S_{k-1}+1+i} - \sum_{N=-i-1}^{-n_k-1} \overline{c}_{k,N+1+i} \cdot \overline{c}^{(k+1)}_{S_{k-1}-N} \in I_F,$$

again by Corollary 4.1.2, since $S_{k-1} - N \ge S_{k-1} + n_k + 1 = S_k + 1$. This completes the proof of the lemma. $\square$

In Corollary 4.1.1 we introduced the set G_1 – another generating set for the ideal I_F. We now prove that $G^{\mathbb{C}}$ generates I_F as well, and this turns out to be the hardest part of the proof that $G^{\mathbb{C}}$ is a Gröbner basis for this ideal.

Proposition 4.4. $I_F = (G^{\mathbb{C}})$.

Proof. Let $k \in \{1, 2, \dots, r\}$ be a fixed integer. We are going to prove that $g_{k,\mu} \in I_F$ for all $\mu \in \mathbb{N}_0^{n_k-1}$, by induction on the lexicographical right ordering on the set $\mathbb{N}_0^{n_k-1}$ ($\mu \prec_{lexr} \mu'$ if and only if $m_t < m'_t$ where $t = \max\{i : m_i \neq m'_i\}$), and then the inclusion $(G^{\mathbb{C}}) \subseteq I_F$ follows immediately.

From Lemma 4.1 we know that $g_{k,\mathbf{s}} \in I_F$ for all (n_k-1)-tuples of the form $\mathbf{s} = (s, 0, \dots, 0)$, $s \geq 0$. Now, let $\mu = (m_2, \dots, m_{t+1}, 0, \dots, 0) \in \mathbb{N}_0^{n_k-1}$ be such that $m_{t+1} > 0$, where $t \geq 2$, and suppose that $g_{k,\nu} \in I_F$ for all ν such that $\nu \prec_{lexr} \mu$. Then, by Proposition 4.3 applied to $\mu_t = (m_2, \dots, m_{t+1}-1, 0, \dots, 0) \in \mathbb{N}_0^{n_k-1}$, $i = 1$ and $j = t-1$, we obtain that

$$g_{k,\mu} = g_{k,\mu_t^{1,t-1}} - c_{k,1} g_{k,\mu_t^{t-1}} + c_{k,t} g_{k,\mu_t} \in I_F,$$

since $\mu_t \prec_{lexr} \mu_t^{t-1} \prec_{lexr} \mu_t^{1,t-1} \prec_{lexr} \mu$.

In order to establish the inclusion $I_F \subseteq (G^{\mathbb{C}})$, by Corollary 4.1.1 it is enough to prove that $\overline{c}_t^{(k)} \in (G^{\mathbb{C}})$ for $1 \leq k \leq r$ and $S_{k-1}+1 \leq t \leq S_k$. This will be done by induction on t. According to Lemma 4.1 (for $s = 0$), we know that $\overline{c}^{(k)}_{S_{k-1}+1} = (-1)^{S_{k-1}+1} g_{k,\mathbf{0}} \in (G^{\mathbb{C}})$. Proceeding to the induction step, for $S_{k-1}+1 < t \leq S_k$ we apply Lemma 4.1 in the case $s = t - S_{k-1} - 1$ (then $0 < s \leq S_k - S_{k-1} - 1 = n_k - 1$), and obtain

$$\overline{c}_t^{(k)} = \overline{c}^{(k)}_{S_{k-1}+1+s} = (-1)^{S_{k-1}+1} g_{k,\mathbf{s}} - \sum_{i=0}^{s-1} \binom{s}{i} c_{k,1}^{s-i} \cdot \overline{c}^{(k)}_{S_{k-1}+1+i} \in (G^{\mathbb{C}})$$

by induction hypothesis. □

Let us now recall the definition of strong Gröbner basis (see [1, p. 251] and [2, p. 455]; in [2] the authors use the phrase D-Gröbner basis). The finite set G of nonzero polynomials is a *strong Gröbner basis* for an ideal I in a polynomial algebra if $(G) = I$ and for every $f \in I \setminus \{0\}$ there exists $g \in G$ such that $\mathrm{LT}(g) \mid \mathrm{LT}(f)$. The strong Gröbner basis G is *minimal* if $\mathrm{LT}(g_1) \nmid \mathrm{LT}(g_2)$ for all $g_1, g_2 \in G$ such that $g_1 \neq g_2$.

Theorem 4.5. *The set $G^{\mathbb{C}}$ is a minimal strong Gröbner basis for the ideal I_F with respect to the monomial ordering $\prec$.*

Proof. It is evident from Definition 4.1 that $G^{\mathbb{C}}$ is finite. By Proposition 4.2, $0 \notin G^{\mathbb{C}}$, and by Proposition 4.4, $G^{\mathbb{C}}$ is a generating set for I_F.

We are left to prove that for each $f \in I_F \setminus \{0\}$ there is a polynomial $g \in G^{\mathbb{C}}$ with the property $\mathrm{LT}(g) \mid \mathrm{LT}(f)$. Note that, according to Proposition 3.2,

$$\{\mathrm{LT}(g) : g \in G^{\mathbb{C}}\} = \{C_k^{\alpha} \;:\; 1 \leq k \leq r,\, |\alpha| = S_{k-1}+1\}. \tag{4.5}$$

If there was a polynomial $f \in I_F \setminus \{0\}$ such that $\mathrm{LT}(g) \nmid \mathrm{LT}(f)$ for all $g \in G^{\mathbb{C}}$, then it would mean that for any $C_1^{\alpha(1)} C_2^{\alpha(2)} \cdots C_r^{\alpha(r)} \in M(f)$ the inequality $|\alpha(k)| \leq S_{k-1}$ holds for all $k \in \{1, 2, \dots, r\}$. However, by Theorem 3.1, then we would have that the class of f in $\mathbb{Z}[C_1, C_2, \dots, C_r]/I_F \cong H^*(F;\mathbb{Z})$ is nonzero, which is a contradiction since $f \in I_F$. So, $G^{\mathbb{C}}$ is a strong Gröbner basis for I_F.

The minimality of $G^{\mathbb{C}}$ is straightforward from (4.5) and the fact that distinct polynomials from $G^{\mathbb{C}}$ have distinct leading terms (which is easily seen from Proposition 4.2). □

By Remark 1, the set $G^{\mathbb{C}}$ is also a minimal strong Gröbner basis for the ideal I_F with respect to the monomial ordering $\prec_r$ (see [2, Exercise 10.5] or [8, Lemma 1.3]). Nevertheless, we feel that it is more natural to use the monomial ordering $\prec$ having in mind Theorem 3.1 and Proposition 4.2. This will be explained further in the following remark.

Remark 2. For $1 \le k \le r$ and $\lambda = (l_1, l_2, \dots, l_{n_k}) \in \mathbb{N}_0^{n_k}$, we define $\underline{\lambda} := (l_2, \dots, l_{n_k}) \in \mathbb{N}_0^{n_k-1}$.

Now, let $\lambda = (l_1, l_2, \dots, l_{n_k}) \in \mathbb{N}_0^{n_k}$ be such that $|\lambda| = S_{k-1} + 1$. By Proposition 4.4, $g_{k,\mu} = 0$ (in $H^*(F;\mathbb{Z})$), which gives rise to the following identity

$$C_k^{\lambda} = \sum_{\substack{\|\alpha\| = S_{k-1}+1+\|\underline{\lambda}\| \\ \alpha \neq \lambda}} (-1)^{S_{k-1}+|\alpha|}[\alpha, \underline{\lambda}] C_k^{\alpha} + \sum_{N=-n_k}^{S_{k-1}-1} (-1)^{S_{k-1}-N+1} g_{k,\underline{\lambda}}^{(N)} \cdot \overline{c}_{S_{k-1}-N}^{(k+1)}. \tag{4.6}$$

By Proposition 4.2, all monomials on the right-hand side of this identity are in B_F, so it actually gives the unique representation of C_k^{λ} in the additive basis B_F. Therefore, formula (4.6) can be understood as a Pieri-type formula for the elements of the basis B_F.

More precisely, for $1 \le k \le r$, $1 \le i \le n_k$ and $C_k^{\lambda} \in B_F$, if $|\lambda| < S_{k-1}$, then $c_{k,i} \cdot C_k^{\lambda} \in B_F$, and if $|\lambda| = S_{k-1}$, then

$$c_{k,i} \cdot C_k^{\lambda} = C_k^{\lambda^i} = \sum_{\substack{\|\alpha\| = S_{k-1}+1+\|\underline{\lambda^i}\| \\ \alpha \neq \lambda^i}} (-1)^{S_{k-1}+|\alpha|}[\alpha, \underline{\lambda}^i] C_k^{\alpha} + \sum_{N=-n_k}^{S_{k-1}-1} (-1)^{S_{k-1}-N+1} g_{k,\underline{\lambda}^i}^{(N)} \cdot \overline{c}_{S_{k-1}-N}^{(k+1)}. \tag{4.7}$$

In our previous papers we obtained special cases of formula (4.6) for flag manifolds that we studied. These proved very useful in several applications: in [17] to give a lower bound on the embedding and immersion dimension of the manifolds $F(1,2,n)$; in [18] to provide new formulas for Kostka numbers; in [21] to obtain cup-length of some flag manifolds of the form $F(1,\dots,1,2,\dots,2,n)$; in [7] to estimate topological complexity of $F(1,\dots,1,m)$; in [19] to give new results on the characteristic rank of oriented Grassmanians.

Example 4. Let $F := F(2,3,m)$ and $0 \le l \le m+1$. Then $S_0 = m$, $S_1 = m+2$ and $S_2 = m+5$. For $k = 1$ and $\lambda = (m+1-l, l)$, formula (4.6) gives

$$\begin{aligned} c_{1,1}^{m+1-l} c_{1,2}^{l} = & \sum_{\substack{a+2b=m+1+l \\ (a,b)\neq(m+1-l,l)}} (-1)^{m+a+b} \binom{a+b-l}{a} c_{1,1}^{a} c_{1,2}^{b} \\ & + \sum_{N=-2}^{m-1} \sum_{a+2b=N+1+l} (-1)^{m+a+b} \binom{a+b-l}{a} c_{1,1}^{a} c_{1,2}^{b} \sum_{u+v+w=m-N} (-1)^{u+v+w} [u,v,w] c_{2,1}^{u} c_{2,2}^{v} c_{2,3}^{w}. \end{aligned}$$

Remark 3. Gröbner basis obtained in Theorem 4.5, i.e., the set $G^{\mathbb{C}}$, has

$$\binom{m+n_1}{n_1-1}+\binom{m+n_1+n_2}{n_2-1}+\cdots+\binom{m+n_1+\cdots+n_r}{n_r-1}$$

elements. Since the Gröbner bases that can be obtained using method from [8] are such that their elements have the same leading monomials as ours (see Remark 1), they naturally have the same number of elements (see [8, Remark 3.11 and Proposition 2.2]).

Remark 4. We have obtained closed forms of elements of the set $G^{\mathbb{C}}$. Nevertheless, in practice, these elements can be calculated using recurrence formulas given in Proposition 4.3 and Lemma 4.1.

Let us now recall the notion of reduction in a polynomial algebra over integers (with a fixed monomial ordering), which is closely related to Gröbner bases (see [2, p. 453], where the phrase *D-reduction* is used). For a polynomial $f=\alpha_1\mathfrak{C}_1+\cdots+\alpha_l\mathfrak{C}_l$, where $\alpha_1,\ldots,\alpha_l\in\mathbb{Z}\setminus\{0\}$ and $\mathfrak{C}_1,\ldots,\mathfrak{C}_l$ are pairwise different monomials, we define the set of terms of f, $T(f):=\{\alpha_1\mathfrak{C}_1,\alpha_2\mathfrak{C}_2,\ldots,\alpha_l\mathfrak{C}_l\}$ (it is understood that $T(0)=\emptyset$). If f and g are nonzero polynomials such that $\mathrm{LT}(g)\mid t$ for some $t\in T(f)$, then we say that the polynomial $h:=f-\frac{t}{\mathrm{LT}(g)}\cdot g$ is a *reduction of f by g*, and we write

$$f\longrightarrow_g h.$$

For a finite set G of nonzero polynomials, we say that a polynomial f *can be reduced modulo G* if $\mathrm{LT}(g)\mid t$ for some $g\in G$ and $t\in T(f)$. Moreover, for a polynomial h we say that f *reduces to h modulo G* , and write

$$f\overset{*}{\longrightarrow}_G h,$$

if there exists $l\in\mathbb{N}_0$, $g_1,g_2,\ldots,g_l\in G$ and polynomials $h_1,\ldots,h_{l-1}$ such that

$$f\longrightarrow_{g_1} h_1\longrightarrow_{g_2} h_2\longrightarrow_{g_3}\cdots\longrightarrow_{g_{l-1}} h_{l-1}\longrightarrow_{g_l} h.$$

It is understood that $f\overset{*}{\longrightarrow}_G f$ (this corresponds to the case $l=0$). Note that $f\overset{*}{\longrightarrow}_G h$ implies $\mathrm{LM}(h)\preccurlyeq\mathrm{LM}(f)$ and $f-h\in(G)$.

Every polynomial f has a *normal form* modulo G, that is, a polynomial h which cannot be reduced modulo G, and such that $f\overset{*}{\longrightarrow}_G h$ (see [2, Lemma 10.2(ii)]).

It is a well-known fact that the following two assertions are equivalent for a finite set G of nonzero polynomials:

(i) G is a strong Gröbner basis for (G);
(ii) for every polynomial f, $f\in(G)$ if and only if $f\overset{*}{\longrightarrow}_G 0$.

Returning to the polynomial algebra $\mathbb{Z}[C_1,C_2,\ldots,C_r]$, for a polynomial $f\in\mathbb{Z}[C_1,C_2,\ldots,C_r]$ the class of f in $\mathbb{Z}[C_1,C_2,\ldots,C_r]/I_F\cong H^*(F;\mathbb{Z})$ will be denoted by the same letter (as usual).

Since $\{\mathrm{LT}(g):g\in G^{\mathbb{C}}\}=\{C_k^\alpha\ :\ 1\le k\le r,\ |\alpha|=S_{k-1}+1\}$, note that the claim that $f\in\mathbb{Z}[C_1,C_2,\ldots,C_r]$ cannot be reduced modulo $G^{\mathbb{C}}$ is equivalent to $M(f)\subseteq B_F$ (see Theorem 3.1).

From the previous discussion we obtain the following consequence of Theorems 4.5 and 3.1.

Corollary 4.5.1. *For every $f \in \mathbb{Z}[C_1, C_2, \dots, C_r]$ there is a unique polynomial p such that $p = f$ in $H^*(F;\mathbb{Z})$, $\mathrm{LM}(p) \preccurlyeq \mathrm{LM}(f)$ and $M(p) \subseteq B_F$.*

The case $p = 0$ in Corollary 4.5.1 is not forbidden – the relation $\mathrm{LM}(0) \preccurlyeq \mathrm{LM}(f)$ for all polynomials f is understood.

Note that a base for reducing a polynomial f to the polynomial p (with the property $M(p) \subseteq B_F$) is provided by Proposition 4.2 and Remark 2. Namely, if $M(f) \nsubseteq B_F$, then f contains a monomial that is divisible by $C_k^{\overline{\mu}} = \mathrm{LT}(g_{k,\mu})$ for some $g_{k,\mu} \in G^{\mathbb{C}}$, and Remark 2 gives us the representation of $C_k^{\overline{\mu}}$ as a linear combination of elements of B_F.

Let us now prove another corollary, which might be helpful in concrete calculations in $H^*(F;\mathbb{Z})$.

Corollary 4.5.2. *Let $1 \le k \le r$, $l \in \mathbb{N}$ and let $\alpha^{(1)}, \dots, \alpha^{(l)}$ be distinct n_k-tuples of nonnegative integers such that $|\alpha^{(j)}| \le S_{k-1}$ for $1 \le j \le l$. If $f_1, \dots, f_l \in \mathbb{Z}[C_1, C_2, \dots, C_r]$ are such that $\mathrm{LM}(f_j) \prec c_{k,n_k}$, $1 \le j \le l$, then the following equivalence holds:*

$$\sum_{j=1}^{l} C_k^{\alpha^{(j)}} f_j = 0 \ \text{in } H^*(F;\mathbb{Z}) \iff f_j = 0 \ \text{in } H^*(F;\mathbb{Z}) \text{ for all } j \in \{1, \dots, l\}.$$

Proof. Assume that $\sum_{j=1}^{l} C_k^{\alpha^{(j)}} f_j = 0$ in $H^*(F;\mathbb{Z})$. By Corollary 4.5.1, there are polynomials $p_1, \dots, p_l$ such that $p_j = f_j$ in $H^*(F;\mathbb{Z})$, $\mathrm{LM}(p_j) \preccurlyeq \mathrm{LM}(f_j)$ and $M(p_j) \subseteq B_F$ for all $j \in \{1, 2, \dots, l\}$. Note that then all monomials of $C_k^{\alpha^{(j)}} p_j$ are in B_F (since $|\alpha^{(j)}| \le S_{k-1}$ and $\mathrm{LM}(p_j) \preccurlyeq \mathrm{LM}(f_j) \prec c_{k,n_k}$). If some of $p_1, \dots, p_l \in \mathbb{Z}[C_1, C_2, \dots, C_r]$ were nonzero, then the sum $\sum_{j=1}^{l} C_k^{\alpha^{(j)}} p_j$ would be a nontrivial linear combination of elements of B_F. However, this contradicts Theorem 3.1, since $\sum_{j=1}^{l} C_k^{\alpha^{(j)}} p_j = \sum_{j=1}^{l} C_k^{\alpha^{(j)}} f_j = 0$ in $H^*(F;\mathbb{Z})$. So, $f_j = p_j = 0$ in $H^*(F;\mathbb{Z})$ for all $j \in \{1, 2, \dots, l\}$.

The opposite implication is obvious. □

5. Gröbner bases for real flag manifolds

The description of the mod 2 cohomology algebra of the real flag manifolds is essentially the same as the one of the integral cohomology in the complex case. Therefore, the completely analogous consideration to the one of the preceding two sections is valid in the real case. In short, Chern classes are substituted with Stiefel–Whitney classes and all the coefficients are considered mod 2 (in particular, the signs are ignored). We just point out the crucial definitions and facts.

Let $r, n_1, \dots, n_r, m \in \mathbb{N}$ be fixed integers, $n = m + n_1 + \cdots + n_r$ and $\widetilde{F} = F^{\mathbb{R}}(n_1, \dots, n_r, m)$ the real flag manifold (which consists of the real flags in $\mathbb{R}^n$ of type $(n_1, \dots, n_r, m)$). For $1 \le i \le r$ and $1 \le j \le n_i$, let $w_{i,j} \in H^j(\widetilde{F};\mathbb{Z}_2)$ be the j-th Stiefel–Whitney class of the canonical n_i-dimensional vector bundle $\widetilde{\gamma}_i$ over $\widetilde{F}$. If $\alpha = (a_1, a_2, \dots, a_{n_i}) \in \mathbb{N}_0^{n_i}$, then the monomial $w_{i,1}^{a_1} w_{i,2}^{a_2} \cdots w_{i,n_i}^{a_{n_i}}$ will be abbreviated to W_i^{α}. As before, $S_0 = m$ and $S_i = m + n_1 + \cdots + n_i$, $1 \le i \le r$.

Theorem 5.1. *The set*

$$D_{\widetilde{F}} := \left\{ W_1^{\alpha(1)} W_2^{\alpha(2)} \cdots W_r^{\alpha(r)} \,:\, |\alpha(i)| \le S_{i-1}, 1 \le i \le r \right\}$$

is an additive basis for $H^*(\widetilde{F};\mathbb{Z}_2)$.

Let

$$\mathbb{Z}_2[W_1,W_2,\ldots,W_r] := \mathbb{Z}_2[w_{1,1},\ldots,w_{1,n_1},w_{2,1},\ldots,w_{2,n_2},\ldots,w_{r,1},\ldots,w_{r,n_r}].$$

By Borel's description, the cohomology algebra $H^*(\widetilde{F};\mathbb{Z}_2)$ is isomorphic to the quotient of the polynomial algebra $\mathbb{Z}_2[W_1,W_2,\ldots,W_r]$ by the ideal $J_{\widetilde{F}}$ generated by the classes (polynomials) $\overline{w}_{m+1}$, $\overline{w}_{m+2}$,..., $\overline{w}_n$. The explicit formula for these polynomials is

$$\overline{w}_s = \sum_{j_1+\cdots+j_r=s} \overline{w}_{1,j_1}\cdots\overline{w}_{r,j_r}, \qquad s\in\mathbb{N},$$

where

$$\overline{w}_{i,j} = \sum_{\|\alpha\|=j}[\alpha,\mathbf{0}]W_i^{\alpha}, \quad 1\le i\le r, \quad j\ge 0.$$

Now, for $1\le k\le r$, $\mu\in\mathbb{N}_0^{n_k-1}$ and integer N, let

$$\widetilde{g}_{k,\mu}^{(N)} = \sum_{\|\alpha\|=N+1+\|\mu\|}[\alpha,\mu]W_k^{\alpha},$$

where the sum is taken over all $\alpha\in\mathbb{N}_0^{n_k}$ that satisfy $\|\alpha\|=N+1+\|\mu\|$. Also, for $1\le k\le r$, let

$$\overline{w}_s^{(k)} = \sum_{d_k+\cdots+d_r=s}\overline{w}_{k,d_k}\cdots\overline{w}_{r,d_r}, \quad s\ge 0,$$

and $\overline{w}_s^{(k)}=0$ for $s<0$. As in the complex case, this definition extends to: $\overline{w}_s^{(r+1)}=0$ for $s\ne 0$, $\overline{w}_0^{(r+1)}=1$.

Definition 5.1. For $1\le k\le r$ and $\mu\in\mathbb{N}_0^{n_k-1}$, let

$$\widetilde{g}_{k,\mu} := \sum_{N=-n_k}^{S_{k-1}}\widetilde{g}_{k,\mu}^{(N)}\cdot\overline{w}_{S_{k-1}-N}^{(k+1)}.$$

Moreover, let

$$G^{\mathbb{R}} := \{\widetilde{g}_{k,\mu} \,:\, 1\le k\le r,\, |\mu|\le S_{k-1}+1\}.$$

In the theory of Gröbner bases over a field, the notion of Gröbner bases coincides with the notion of strong Gröbner bases from the theory over a commutative domain. Besides that, the Gröbner basis G is called *reduced* if for all $g_1,g_2\in G$, $g_1\ne g_2$, one has that $\mathrm{LM}(g_1)\nmid\mathfrak{m}$ for any $\mathfrak{m}\in M(g_2)$ (note that this condition is stronger than minimality). The reduced Gröbner basis has the important feature of being unique (for a fixed monomial ordering).

Theorem 5.2. *The set* $G^{\mathbb{R}}$ *is the reduced Gröbner basis for the ideal* $J_{\widetilde{F}}$ *with respect to the monomial ordering* $\prec$.

The monomial ordering $\prec$ is the analogue of the one defined in Section 4:

$$W_1^{\alpha(1)}W_2^{\alpha(2)}\cdots W_r^{\alpha(r)} \prec W_1^{\beta(1)}W_2^{\beta(2)}\cdots W_r^{\beta(r)}$$

if and only if $W_k^{\alpha(k)}\prec_{grlex}W_k^{\beta(k)}$, where $k=\min\{i\,:\,\alpha(i)\ne\beta(i)\}$. According to (the mod 2 variant of) Proposition 4.2, for $1\le k\le r$ and $\mu=(m_2,\ldots,m_{n_k})\in\mathbb{N}_0^{n_k-1}$ such that $\widetilde{g}_{k,\mu}\in G^{\mathbb{R}}$, one has that $\mathrm{LT}(\widetilde{g}_{k,\mu})=\mathrm{LM}(\widetilde{g}_{k,\mu})=W_k^{\overline{\mu}}$, where $\overline{\mu}=(S_{k-1}+$

$1-|\mu|, m_2, \ldots, m_{n_k}) \in \mathbb{N}_0^{n_k}$, and that $|\alpha(i)| \leq S_{i-1}$, $k \leq i \leq r$, for all others $W_k^{\alpha(k)} \cdots W_r^{\alpha(r)} \in M(\widetilde{g}_{k,\mu})$. Therefore, it is easily seen that the Gröbner basis $G^{\mathbb{R}}$ is really the reduced one.

Remark 5. As in the complex case, Gröbner basis $G^{\mathbb{R}}$ gives rise to a Pieri-type formula for the additive basis $D_{\widetilde{F}}$. Indeed, if $\lambda = (l_1, l_2, \ldots, l_{n_k}) \in \mathbb{N}_0^{n_k}$ is such that $|\lambda| = S_{k-1}+1$, then

$$W_k^{\lambda} = \sum_{\substack{\|\alpha\| = S_{k-1}+1+\|\underline{\lambda}\| \\ \alpha \neq \lambda}} [\alpha, \underline{\lambda}] W_k^{\alpha} + \sum_{N=-n_k}^{S_{k-1}-1} \widetilde{g}_{k,\underline{\lambda}}^{(N)} \cdot \overline{w}_{S_{k-1}-N}^{(k+1)}. \tag{5.1}$$

Also, it is easy to see that (mod 2) variants of Corollaries 4.5.1 and 4.5.2 hold in $H^*(\widetilde{F}; \mathbb{Z}_2)$.

Example 5. Let $s \geq 2$. In what follows we prove that the cup-length of the flag manifold $\widetilde{F} := F^{\mathbb{R}}(2, 3, 2^s - 1)$ is $9 \cdot 2^{s-1} - 1$ (recall that the $\mathbb{Z}_2$-*cup-length* (or simply *cup-length*) of a path connected space X, denoted by $\operatorname{cup} X$, is the supremum of all integers d such that there exist classes $a_1, a_2, \ldots, a_d \in \widetilde{H}^*(X; \mathbb{Z}_2)$ with nonzero cup product $a_1 a_2 \ldots a_d$).

First, let us prove that $\operatorname{cup} \widetilde{F} \leq 9 \cdot 2^{s-1} - 1$. Let $t = w_{1,1}^a w_{1,2}^b w_{2,1}^c w_{2,2}^d w_{2,3}^e \neq 0$ be the class that achieves the cup-length of $\widetilde{F}$. Then, by Poincaré duality, $t \in H^{\dim \widetilde{F}}(\widetilde{F}; \mathbb{Z}_2)$, and hence $a + 2b + c + 2d + 3e = \dim \widetilde{F} = 5 \cdot 2^s + 1$. Since $\operatorname{ht}(w_{1,1}) = 2^{s+1} - 2$ and $\operatorname{ht}(w_{2,1}) = 2^{s+1} - 1$ (see [Doe (1996), p. 147]), we have $a \leq 2^{s+1} - 2$ and $c \leq 2^{s+1} - 1$ (recall that the *height* of a class $y \in \widetilde{H}^*(X; \mathbb{Z}_2)$, denoted by $\operatorname{ht}(y)$, is the largest positive integer n such that $y^n \neq 0$). This implies

$$2 \cdot \operatorname{cup} \widetilde{F} = 2a + 2b + 2c + 2d + 2e \leq 2^{s+1} - 2 + 2^{s+1} - 1 + a + 2b + c + 2d + 3e = 9 \cdot 2^s - 2$$

as desired.

So, to finish the proof it is enough to show that the class $x := w_{1,1}^{2^{s+1}-2} w_{1,2}^3 w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}-1}$ is nonzero (in $H(\widetilde{F}; \mathbb{Z}_2)$).

Let us first prove that $w_{1,1}^{2^{s+1}-2} w_{1,2}^3 = w_{1,2}^{2^s+2}$. Since $\widetilde{F} \approx F^{\mathbb{R}}(3, 2, 2^s - 1)$, we observe the fiber bundle $p : F^{\mathbb{R}}(3, 2, 2^s - 1) \to F^{\mathbb{R}}(2, 2^s + 2)$ defined as in Theorem 3.1. By [10, Proposition 5], in $H^*(F^{\mathbb{R}}(2, 2^s + 2); \mathbb{Z}_2)$ one has $w_{1,1}^{2^{s+1}-2} w_{1,2}^3 = w_{1,2}^{2^s+2}$, and hence $w_{1,1}^{2^{s+1}-2} w_{1,2}^3 = p^*(w_{1,1}^{2^{s+1}-2} w_{1,2}^3) = p^*(w_{1,2}^{2^s+2}) = w_{1,2}^{2^s+2}$ in $H^*(F^{\mathbb{R}}(3, 2, 2^s - 1); \mathbb{Z}_2)$. This implies $x = w_{1,2}^{2^s+2} w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}-1}$.

Next, we prove that $w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}} = 0$. Suppose that this is not the case. Observe the fiber bundle $p : F^{\mathbb{R}}(2, 3, 2^s - 1) \to F^{\mathbb{R}}(3, 2^s + 1)$ defined as in Theorem 3.1. Then $p^*(w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}}) = w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}}$, and hence $w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}}$ is nonzero in $H^*(F^{\mathbb{R}}(3, 2^s + 1); \mathbb{Z}_2)$. So, by Poincaré duality, there exists a class $z \in H^4(F^{\mathbb{R}}(3, 2^s + 1); \mathbb{Z}_2)$, such that $w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}} z \neq 0$. Since z is a polynomial in classes $w_{2,1}$, $w_{2,2}$ and $w_{2,3}$, and $\operatorname{ht}(w_{2,1}) = 2^{s+1} - 1$ (in $H^*(F^{\mathbb{R}}(3, 2^s + 1); \mathbb{Z}_2)$), we have $z = w_{2,2}^2$. Hence, $w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}+2} \neq 0$, and therefore $\operatorname{cup} F^{\mathbb{R}}(3, 2^s + 1) \geq 5 \cdot 2^{s-1} + 1$. But $\operatorname{cup} F^{\mathbb{R}}(3, 2^s + 1) = 5 \cdot 2^{s-1}$ (see [22, p. 104]), a contradiction.

Now, we use formula (5.1) to represent $w_{1,2}^{2^s+2}$ in the additive basis $D_{\widetilde{F}}$. In fact, since $w_{2,1}^{2^{s+1}} = 0$ and $w_{2,1}^{2^{s+1}-1}w_{2,2}^{2^{s-1}} = 0$, in order to prove $x \neq 0$, it is enough to represent $w_{1,2}^{2^s+2}$ modulo $w_{2,1}$ and modulo $w_{2,2}$. Additionally, by Corollary 4.5.1 (i.e., its variant for real flag manifolds), every nonzero monomial in variables $w_{2,1}$, $w_{2,2}$ and $w_{2,3}$ has cohomological dimension at most $3(2^s+1)$. Hence, if a monomial t is such that $tw_{2,1}^{2^{s+1}-1}w_{2,2}^{2^{s-1}-1} \neq 0$, then the degree of $w_{2,3}$ in t is at most 2. So, in what follows, for $t', t'' \in H^*(\widetilde{F};\mathbb{Z}_2)$, we write $t' \sim_d t''$ if and only if t' and t'' are the same modulo $w_{2,1}$ and modulo $w_{2,2}$, and the degree of $w_{2,3}$ in each monomial of t'' is at most d.

By formula (5.1), applied to $\lambda = (3, 2^s - 3)$ and $\lambda = (0, 2^s)$ (and $k = 1$), we have

$$
\begin{aligned}
&w_{1,1}^3 w_{1,2}^{2^s-3} \sim_0 0\\
&\qquad w_{1,2}^{2^s} \sim_1 w_{1,1}w_{1,2}^{2^s-2}w_{2,3}\\
&\qquad w_{1,2}^{2^s} \sim_2 w_{1,1}w_{1,2}^{2^s-2}w_{2,3} + (w_{1,1}^4 w_{1,2}^{2^s-5} + w_{1,1}^2 w_{1,2}^{2^s-4} + w_{1,2}^{2^s-3})w_{2,3}^2,
\end{aligned}
$$

and hence

$$
\begin{aligned}
x &= w_{1,2}^2 w_{1,2}^{2^s} w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}-1}\\
&= w_{1,2}^2 (w_{1,1}w_{1,2}^{2^s-2}w_{2,3} + (w_{1,1}^4 w_{1,2}^{2^s-5} + w_{1,1}^2 w_{1,2}^{2^s-4} + w_{1,2}^{2^s-3})w_{2,3}^2) w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}-1}\\
&= w_{1,1}w_{1,2}^{2^s} w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}-1} w_{2,3} + (w_{1,1}^4 w_{1,2}^{2^s-3} + w_{1,1}^2 w_{1,2}^{2^s-2} + w_{1,2}^{2^s-1}) w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}-1} w_{2,3}^2\\
&= w_{1,1}^2 w_{1,2}^{2^s-2} w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}-1} w_{2,3}^2 + (0 + w_{1,1}^2 w_{1,2}^{2^s-2} + w_{1,2}^{2^s-1}) w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}-1} w_{2,3}^2\\
&= w_{1,2}^{2^s-1} w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}-1} w_{2,3}^2.
\end{aligned}
$$

By Corollary 4.5.2 (i.e., its variant for real flag manifolds), to prove that $x \neq 0$ it is enough to prove that $w_{2,1}^{2^{s+1}-1} w_{2,2}^{2^{s-1}-1} w_{2,3}^2 \neq 0$. This follows from [22, p. 112], using the fiber bundle $p : F^{\mathbb{R}}(2, 3, 2^s - 1) \to F^{\mathbb{R}}(3, 2^s + 1)$ defined as in Theorem 3.1.

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