On the $\mathbb{Z}_2$-cohomology cup-length of some real flag manifolds

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Abstract. In this paper we discuss two different techniques for calculating the $\mathbb{Z}_2$-cohomology cup-length – one based on fiberings and a result of Horanska and Korbaš, and the other based on Gröbner bases. We use these techniques to obtain $\mathbb{Z}_2$-cohomology cup-length or bounds for the $\mathbb{Z}_2$-cohomology cup-length of some of the real flag manifolds $F(1,\ldots,1,2\ldots,2,n)$.

1. Introduction

The $\mathbb{Z}_2$-cohomology cup-length of a path connected space X, denoted by $\mathrm{cup}(X)$, is the supremum of all positive integers m such that there exist classes $a_1, a_2, \ldots, a_m \in \widetilde{H}^*(X;\mathbb{Z}_2)$ with nonzero cup product, i.e., $a_1 a_2 \cdots a_m \neq 0$. The problem of finding the number $\mathrm{cup}(M)$ is extensively studied in the literature, particularly since it is known that $\mathrm{cup}(M)$ provides a lower bound for the Lyusternik-Shnirel'man category of M. Recall that the Lyusternik-Shnirel'man category of M, denoted by $\mathrm{cat}(M)$, is the minimum number of open subsets of M covering M, each of which is contractible in M. Specifically, one has $1+\dim(M) \geq \mathrm{cat}(M) \geq 1+\mathrm{cup}(M)$. Since we work only with cohomologies with $\mathbb{Z}_2$-coefficients, in the remainder of this paper we will write "cup-length" instead of "$\mathbb{Z}_2$-cohomology cup-length".

In this paper we are concerned with obtaining $\mathrm{cup}(M)$, when M is a real flag manifold of a specified type. For positive integers $n_1,\ldots,n_r$, $r\geq 2$, the real flag manifold $F(n_1,\ldots,n_r)$ is the set of flags of type $(n_1,\ldots,n_r)$ (r–tuples $(V_1,\ldots,V_r)$ of mutually orthogonal subspaces in $\mathbb{R}^m$, where $m = n_1+\ldots+n_r$, $\dim(V_i) = n_i$, $i=\overline{1,r}$) with the manifold structure coming from the identification $F(n_1,\ldots,n_r) = O(m)/O(n_1)\times\cdots\times O(n_r)$. This identification makes $F(n_1,\ldots,n_r)$ into a closed manifold of dimension $\delta(F(n_1,\ldots,n_r)) = \sum_{1\leq i<j\leq r} n_i n_j$. For a general real flag manifold the cup-length is not known; it is not known even in the special case of Grassmann manifolds (that is the real flag manifolds $F(k,n)$). In [4] and [12] the cup-length of some Grassmann manifolds is obtained, namely $F(2,n)$, $F(3,n)$ and $F(4,n)$; in [6] and [7] some bounds for the cup-length of oriented Grassmann manifolds are obtained; in [9] the cup-length of some of the real flag manifolds $F(1^{\cdot\cdot j}, 2^{\cdot\cdot d}, n)$ is obtained (we are using the notation from [9]: $F(1^{\cdot\cdot j}, 2^{\cdot\cdot d}, n)$ stands for the flag manifold $F(1,\ldots,1,2,\ldots,2,n)$, with j ones and d twos).

In this paper we continue research in this area. We use two different techniques in order to obtain our results. The first one is the method of fiberings, which is presented in Section 3. The second one is the method of Gröbner bases, which is presented in Sections 4 and 5 (in Section 4 we restate some results from [11] on Gröbner bases for the real flag manifolds $F(1^{\cdot\cdot j}, 2^{\cdot\cdot d}, n)$), and in Section 5 we use these results to obtain the cup-length of some manifolds of this type). Finally, in Section 6 we give a brief comparison of these two techniques.

2. The cup-length of the real flag manifolds $F(1^{\cdot\cdot j}, 2^{\cdot\cdot d}, n)$

Throughout this paper $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Also, we assume that $j,d\in\mathbb{N}_0$, and $n \geq \min\{2, d+1\}$.

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As mentioned in the Introduction, an obvious upper bound for the cup-length of a manifold is its dimension. In our case we have

$$\operatorname{cup}(F(1^{\cdots j}, 2^{\cdots d}, n)) \leq nj + 2nd + \binom{j}{2} + 4\binom{d}{2} + 2jd.$$

Berstein detected all Grassmann manifolds for which the cup-length is equal to the dimension.

Theorem 2.1 ([2]). *One has:* $\operatorname{cup}(F(k, n-k)) = \delta(F(k, n-k))$ *if and only if* $k = 1$*, or* $k = 2$ *and* $n = 2^t + 1$*, for* $t \in \mathbb{N}$.

Finding all flag manifolds for which the cup-length is equal to the dimension is still an open problem. In [9] the authors obtained an infinite family of the real flag manifolds $F(1^{\cdots j}, 2^{\cdots d}, n)$ for which the cup-length is equal to the dimension, and asked if these are the only manifolds of this type for which this is true ([9, Remark (a)]).

In [9] (and more generally [6]) the authors offered an algorithm for computing the cup-length of flag manifolds. Although this algorithm is very hard to apply for a general flag manifold, it gives an idea of how one can find the cup-length of some specific flag manifold. The starting point of this algorithm is connected with the following result.

Lemma 2.2 ([9]). *For a flag manifold* $F(n_1, n_2, \ldots, n_q)$ *of dimension* δ*, let* $\operatorname{ht}(i)$*,* $i = \overline{1, q-1}$*, be the heights of the first Stiefel-Whitney classes of the canonical vector bundles, and* $S = \operatorname{ht}(1) + \cdots + \operatorname{ht}(q-1)$*. If* $S \leq \delta$*, then*

$$\operatorname{cup}(F(n_1, n_2, \ldots, n_q)) \leq S + \left\lfloor \frac{\delta - S}{2} \right\rfloor.$$

Let $\beta(F(n_1, n_2, \ldots, n_q))$ denote the upper bound from this lemma.

As suggested by the previous lemma, the heights of the first Stiefel-Whitney classes are often needed for calculating the cup-length. For (all) flag manifolds these are known due to Korbaš and Lörinc (see [9]). The following proposition is a special case of their result.

Proposition 2.3 ([9]). *For the flag manifold* $F(1^{\cdots j}, 2^{\cdots d}, n)$ *we have* $\operatorname{ht}(x_i) = n + j + 2d - 1$*,* $i = \overline{1, j}$*. If* $d \geq 1$*, and* $s \geq 3$ *is the unique integer such that* $2^{s-1} < n + j + 2d \leq 2^s$*, then* $\operatorname{ht}(y_{i,1}) = 2^s - 2$*,* $i = \overline{1, d}$*.*

Note that by this proposition, if $\beta(F(1^{\cdots j}, 2^{\cdots d}, n))$ is well-defined, then we have

$$\begin{aligned}\beta(F(1^{\cdots j}, 2^{\cdots d}, n)) &= j(n + j + 2d - 1) + d(2^s - 2) + \left\lfloor \frac{2nd + 4\binom{d}{2} - \binom{j}{2} - d(2^s - 2)}{2} \right\rfloor \\ &= j(n + j + 2d - 1) + d(n + d + 2^{s-1} - 2) + \left\lfloor -\frac{j(j-1)}{4} \right\rfloor.\end{aligned}$$

3. Application of fiberings for cup-length calculations

In this section we use the method of fiberings to obtain the cup-length of some of the real flag manifolds $F(1^{\cdots j}, 2^{\cdots d}, n)$. The main idea of this method is to build an appropriate sequence of fiber bundles, and then use the following result by Horanska and Korbaš ([5]).

Theorem 3.1 ([5]). *Let* $p : E \to B$ *be a smooth fiber bundle with connected base* B *and connected fiber* F*. Suppose that the fiber inclusion induces an epimorphism in* $\mathbb{Z}_2$*-cohomology. Then* $\operatorname{cup}(E) \geq \operatorname{cup}(F) + \operatorname{cup}(B)$*.*

To put this result into context, let us observe the following fiber bundle (see [9]), with total space supposed to satisfy the condition $S \leq \delta$ (from Lemma 2.2),

$$\begin{tikzcd} F(n_{t+1},\dots,n_q) \arrow[r, hook] & F(n_1,\dots,n_q) \arrow[d] \\ & F(n_1,\dots,n_t,n_{t+1}+\cdots+n_q). \end{tikzcd}$$

Since the inclusion $i : F(n_{t+1},\dots,n_q) \to F(n_1,\dots,n_q)$ induces an epimorphism in $\mathbb{Z}_2$-cohomology (see [9]), by Theorem 3.1 if

(1) $\mathrm{cup}(F(n_{t+1},\dots,n_q)) \geq d_1$ and

(2) $\mathrm{cup}(F(n_1,\dots,n_t,n_{t+1}+\cdots+n_q)) \geq d_2$,

then we have

$$\min\{\delta(F(n_1,\dots,n_q)),\beta(F(n_1,\dots,n_q))\} \geq \mathrm{cup}(F(n_1,\dots,n_q)) \geq d_1+d_2. \tag{1}$$

This will be used throughout this section.

Note that one also has

$$\delta(F(n_{t+1},\dots,n_q)) + \delta(F(n_1,\dots,n_t,n_{t+1}+\cdots+n_q)) = \delta(F(n_1,\dots,n_q)). \tag{2}$$

The cup-length of the real flag manifolds $F(1^{\cdots j},n)$ is known due to Korbaš and Lörinc (their proof is based on the method of fiberings).

Proposition 3.2 ([9]). *The cup length of* $F(1^{\cdots j},n)$ *is* $\binom{j}{2}+nj$.

As mentioned in the Introduction, the cup-length of Grassmann manifold $F(2,n)$ is known due to Hiller.

Proposition 3.3 ([4]). *Let* $2^{s-1} < n+2 \leq 2^s$. *Then* $\mathrm{cup}(F(2,n)) = n+2^{s-1}-1$.

Note that $\beta(F(2,n)) = 2^s-2+\left\lfloor \dfrac{2n-2^s+2}{2} \right\rfloor = n+2^{s-1}-1$, i.e., $\mathrm{cup}(F(2,n))$ is equal to the upper bound obtained in Lemma 2.2.

For the remainder of the paper let $s\in\mathbb{N}$ be the unique integer such that $2^{s-1} < n+j+2d \leq 2^s$. Moreover, for $1\leq m\leq d$, let $s(m)$ be the unique positive integer such that $2^{s(m)-1} < n+j+2m \leq 2^{s(m)}$.

First, we investigate the case $j=0$, and extend the result from Proposition 3.3.

Proposition 3.4. *If* δ *is the dimension of* $F(2^{\cdots d},n)$, *then* $\mathrm{cup}(F(2^{\cdots d},n)) \geq \delta - \sum_{t=1}^{d}(n+2t-1-2^{s(t)-1})$. *In particular, if* $2^{s-1} < n+2 \leq n+2d \leq 2^s$, *then* $\mathrm{cup}(F(2^{\cdots d},n)) = d(n+d+2^{s-1}-2)$.

Proof. We proceed by induction on d. For $d=1$ the proof follows from Proposition 3.3. So, let us assume that the proposition holds for $d-1\geq 1$ and prove it for d.

Let us observe the following fiber bundle

$$\begin{tikzcd} F(2^{\cdots d-1},n) \arrow[r, hook] & F(2^{\cdots d},n) \arrow[d] \\ & F(2,n+2(d-1)). \end{tikzcd}$$

By Proposition 3.3 we have $\mathrm{cup}(F(2, n + 2(d - 1))) = n + 2d - 3 + 2^{s-1}$, and by the inductive hypothesis

$$\mathrm{cup}(F(2^{\cdot\cdot d-1}, n)) \geq \delta(F(2^{\cdot\cdot d-1}, n)) - \sum_{t=1}^{d-1}(n + 2t - 1 - 2^{s(t)-1}) = \delta - n - 2d + 3 - 2^{s-1} - \sum_{t=1}^{d}(n + 2t - 1 - 2^{s(t)-1}).$$

So, by Theorem 3.1,

$$\mathrm{cup}(F(2^{\cdot\cdot d}, n)) \geq \mathrm{cup}(F(2^{\cdot\cdot d-1}, n)) + \mathrm{cup}(F(2, n + 2(d - 1)) = \delta - \sum_{t=1}^{d}(n + 2t - 1 - 2^{s(t)-1}),$$

which completes the proof of the first part of the proposition.

Now, let $2^{s-1} < n + 2 \leq n + 2d \leq 2^s$. Then $s(i) = s$, for $i = \overline{1, d}$, and $\delta = 2nd + 2d(d - 1)$, so, by the inequality obtained in the first part of the proof, we have

$$\mathrm{cup}(F(2^{\cdot\cdot d}, n)) \geq d(n + d + 2^{s-1} - 2).$$

On the other hand,

$$\beta(F(2^{\cdot\cdot d}, n)) = (2^s - 2)d + \left\lfloor \frac{2d(d - 1) + 2dn - (2^s - 2)d}{2} \right\rfloor = d(n + d + 2^{s-1} - 2),$$

and therefore, by Lemma 2.2, $\mathrm{cup}(F(2^{\cdot\cdot d}, n)) = d(n + d + 2^{s-1} - 2)$. □

In the following proposition we extend the previous result.

Proposition 3.5. *If δ is the dimension of $F(1^{\cdot\cdot j}, 2^{\cdot\cdot d}, n)$, then*

$$\mathrm{cup}(F(1^{\cdot\cdot j}, 2^{\cdot\cdot d}, n)) \geq \delta - \sum_{t=1}^{d}(n + j + 2t - 1 - 2^{s(t)-1}).$$

Proof. Obviously, $F(1^{\cdot\cdot j}, 2^{\cdot\cdot d}, n)$ is homeomorphic to $F(2^{\cdot\cdot d}, 1^{\cdot\cdot j}, n)$. So, let us observe the following fiber bundle

$$\begin{array}{ccc} F(1^{\cdot\cdot j}, n) & \hookrightarrow & F(2^{\cdot\cdot d}, 1^{\cdot\cdot j}, n) \\ & & \downarrow \\ & & F(2^{\cdot\cdot d}, n + j). \end{array}$$

By Proposition 3.2 we have $\mathrm{cup}(F(1^{\cdot\cdot j}, n)) = \delta(F(1^{\cdot\cdot j}, n))$, and by Proposition 3.4

$$\mathrm{cup}(F(2^{\cdot\cdot d}, n + j)) \geq \delta(F(2^{\cdot\cdot d}, n + j)) - \sum_{t=1}^{d}(n + j + 2t - 1 - 2^{s(t)-1}).$$

Therefore the desired inequality follows from Theorem 3.1. □

In the following proposition we extend Proposition 3.2.4 from [9].

Proposition 3.6. *i) If $2^{s-1} < n + 2$, then* $\mathrm{cup}(F(1, 2^{\cdot\cdot d}, n)) = n + d(n + d + 2^{s-1})$.

ii) If $2^{s-1} < n + 2$, then $\mathrm{cup}(F(1, 1, 2^{\cdot\cdot d}, n)) = 2n + 1 + d(n + d + 2^{s-1} + 2)$.

Proof. i) Let us observe the following fiber bundle

$$\begin{tikzcd} F(2^{\cdot\cdot d}, n) \arrow[r, hook] & F(1, 2^{\cdot\cdot d}, n) \arrow[d] \\ & F(1, n+2d). \end{tikzcd}$$

By Proposition 3.2, we have $\operatorname{cup}(F(1, n+2d)) = n+2d$, and by Proposition 3.4, $\operatorname{cup}(F(2^{\cdot\cdot d}, n)) = d(n+d+2^{s-1}-2)$. On the other hand,

$$\beta(F(1, 2^{\cdot\cdot d}, n)) = n + 2d + d(2^s - 2) + \left\lfloor \frac{2nd + 2d(d-1) - d(2^s - 2)}{2} \right\rfloor = n + d(n + d + 2^{s-1}),$$

and therefore, by (1), we have $\operatorname{cup}(F(1, 2^{\cdot\cdot d}, n)) = n + d(n + d + 2^{s-1})$.

ii) Let us observe the following fiber bundle

$$\begin{tikzcd} F(2^{\cdot\cdot d}, n) \arrow[r, hook] & F(1, 1, 2^{\cdot\cdot d}, n) \arrow[d] \\ & F(1, 1, n+2d). \end{tikzcd}$$

By Proposition 3.2, we have $\operatorname{cup}(F(1, 1, n + 2d)) = 2n + 4d + 1$, and by Proposition 3.4, $\operatorname{cup}(F(2^{\cdot\cdot d}, n)) = d(n + d + 2^{s-1} - 2)$. On the other hand,

$$\begin{aligned} \beta(F(1, 1, 2^{\cdot\cdot d}, n)) &= 2(n + 2d + 1) + d(2^s - 2) + \left\lfloor \frac{2nd + 2d(d-1) - d(2^s - 2) - 1}{2} \right\rfloor \\ &= 2n + 1 + d(n + d + 2^{s-1} + 2), \end{aligned}$$

and therefore, by (1), we have $\operatorname{cup}(F(1, 2^{\cdot\cdot d}, n)) = 2n + 1 + d(n + d + 2^{s-1} + 2)$. ◻

Remark 3.7. *In Proposition 5.6 we will show that, if $n + 2 = 2^{s-1}$, then $\operatorname{cup}(F(1, 2^{\cdot\cdot d}, n)) \neq n + d(n + d + 2^{s-1})$, i.e., the inequality $2^{s-1} < n + 2$ in part i) of the previous proposition is, in a sense, the best possible.*

The bound obtained in Proposition 3.5 is often not very tight. In the following proposition we improve this bound for some of the real flag manifolds $F(1^{\cdot\cdot\cdot j}, 2^{\cdot\cdot d}, n)$.

Proposition 3.8. *If $n + 1 \leq 2^{s-1} \leq n + j + 1$, then*

$$\operatorname{cup}(F(1^{\cdot\cdot\cdot j}, 2^{\cdot\cdot d}, n)) \geq \delta(F(1^{\cdot\cdot\cdot j}, 2^{\cdot\cdot d}, n)) - d(d-1).$$

Proof. Let $t = 2^{s-1} - n - 1$. Obviously, $F(1^{\cdot\cdot\cdot j}, 2^{\cdot\cdot d}, n)$ is homeomorphic to $F(1^{\cdot\cdot\cdot j-t}, 2^{\cdot\cdot d}, 1^{\cdot\cdot\cdot t}, n)$. So, let us observe the following sequence of fiber bundles

$$\begin{array}{ccc}
F(1^{\dots t}, n) & \hookrightarrow & F(1^{\dots j-t}, 2^{\dots d}, 1^{\dots t}, n) \\
& & \downarrow \\
F(2, 2^{s-1}-1) & \hookrightarrow & F(1^{\dots j-t}, 2^{\dots d}, 2^{s-1}-1) \\
& & \downarrow \\
F(2, 2^{s-1}+1) & \hookrightarrow & F(1^{\dots j-t}, 2^{\dots d-1}, 2^{s-1}+1) \\
& & \downarrow \\
& & \vdots \\
& & \downarrow \\
F(2, 2^{s-1}+2d-3) & \hookrightarrow & F(1^{\dots j-t}, 2, 2^{s-1}+2d-3) \\
& & \downarrow \\
& & F(1^{\dots j-t}, 2^{s-1}+2d-1).
\end{array}$$

Going through these sequence of fiber bundles, and using Theorem 3.1, we obtain

$$\operatorname{cup}(F(1^{\dots j}, 2^{\dots d}, n)) \geq \operatorname{cup}(F(1^{\dots t}, n)) + \sum_{i=0}^{d-1} \operatorname{cup}(F(2, 2^{s-1}-1+2i)) + \operatorname{cup}(F(1^{\dots j-t}, 2^{s-1}+2d-1)). \tag{3}$$

By Proposition 3.2, $\operatorname{cup}(F(1^{\dots t}, n)) = \delta(F(1^{\dots t}, n))$, and since $2^{s-1} < 2^{s-1}+2i+1 \leq n+j+2d \leq 2^s$, for $i = \overline{0, d-1}$, by Proposition 3.3 we have $\operatorname{cup}(F(2, 2^{s-1}-1+2i)) = \delta(F(2, 2^{s-1}-1+2i)) - 2i$, for $i = \overline{0, d-1}$. Therefore, using (2), the inequality (3) becomes

$$\begin{aligned}
\operatorname{cup}(F(1^{\dots j}, 2^{\dots d}, n)) &\geq \delta(F(1^{\dots t}, n)) + \sum_{i=0}^{d-1} \delta(F(2, 2^{s-1}-1+2i)) + \delta(F(1^{\dots j-t}, 2^{s-1}+2d-3)) - d(d-1) \\
&= \delta(F(1^{\dots j}, 2^{\dots d}, n)) - d(d-1),
\end{aligned}$$

which completes our proof. □

Remark 3.9. *Note that for $d = 1$ and $n+1 \leq 2^{s-1} \leq n+j+1$ we have $\operatorname{cup}(F(1^{\dots j}, 2, n)) = \delta(F(1^{\dots j}, 2, n))$, which is a special case of Theorem 3.1.3 from [9].*

4. A Gröbner basis for the cohomology algebra $H^*(F(1^{\dots j}, 2^{\dots d}, n); \mathbb{Z}_2)$

Let $n \geq 3$. By Borel's description (see [3]), the cohomology algebra $H^*(F(1^{\dots j}, 2^{\dots d}, n); \mathbb{Z}_2)$ is isomorphic to a quotient algebra $\mathbb{Z}_2[x_1, \dots, x_j, y_{1,1}, y_{1,2}, \dots, y_{d,1}, y_{d,2}]/I_{j,d,n}$. Here $x_i \in H^1(F(1^{\dots j}, 2^{\dots d}, n); \mathbb{Z}_2)$, $i = \overline{1, j}$, are the Stiefel-Whitney classes of the canonical line bundles over $F(1^{\dots j}, 2^{\dots d}, n)$; $y_{i,l} \in H^l(F(1^{\dots j}, 2^{\dots d}, n); \mathbb{Z}_2)$, $i = \overline{1, d}$, $l = \overline{1, 2}$, are the Stiefel-Whitney classes of the canonical two-dimensional vector bundles over $F(1^{\dots j}, 2^{\dots d}, n)$; $I_{j,d,n}$ is the ideal of $\mathbb{Z}_2[x_1, \dots, x_j, y_{1,1}, y_{1,2}, \dots, y_{d,1}, y_{d,2}]$ generated by the dual classes z_{n+1}, $z_{n+2}, \dots, z_{n+j+2d}$. The following identity holds:

$$1 + z_1 + z_2 + \cdots = \prod_{i=1}^{j} (1+x_i)^{-1} \prod_{i=1}^{d} (1+y_{i,1}+y_{i,2})^{-1}.$$

Although this description is simple enough, concrete calculations in the cohomology is often very difficult to perform (see for example [8, 12]). In this paper we show that having a Gröbner basis for the ideal $I_{j,d,n}$ can be very useful for cup-length calculations (see also [10]).

Recently, Gröbner bases for the ideals $I_{j,d,n}$ were obtained in [11]. In this section we restate those results from [11] that are going to be used in this paper. We note that in order to understand and use these results one does not need to be familiar with the theory of Gröbner bases.

By [11] the ideal $I_{j,d,n}$ is generated by $G = G_1 \cup G_2$, where the sets G_1 and G_2 are defined as follows.

Let $G_1 = \{g_m \mid 1 \le m \le j\}$, where

$$g_m = \sum_{l+r_1+\cdots+r_d=n+m} h_l(x_m, \ldots, x_j)\overline{y}_{1,r_1} \cdots \overline{y}_{d,r_d},$$

and the sum is taken over all $(d+1)$-tuples $R = (l, r_1, \ldots, r_d)$ of nonnegative integers, such that $l+r_1+\cdots+r_d = n+m$. Also, for $1 \le m \le j$, $h_l(x_m, \ldots, x_j)$ denotes the complete homogenous symmetric polynomial of degree l in the variables $x_m, \ldots, x_j$.

Let $G_2 = \{g_{m,r} \mid 1 \le m \le d,\ 0 \le r \le n + j + 2m - 1\}$, where

$$g_{m,r} = \sum_{r_m+\cdots+r_d=n+j+2m-1} g^{(r_m-1)}_{m,r}\overline{y}_{m+1,r_{m+1}} \cdots \overline{y}_{d,r_d},$$

and the sum is taken over all $(d - m + 1)$-tuples $(r_m, \ldots, r_d)$ such that $r_m \ge -1$, $r_i \ge 0$, for $i = \overline{m+1, d}$, and $r_m + \cdots + r_d = n + j + 2m - 1$.

Here, for $1 \le m \le d$, $-2 \le N \le n + j + 2m - 2$, and $r \ge 0$:

- $\overline{y}_{m,r} = \displaystyle\sum_{a+2b=r} \binom{a+b}{a} y^a_{m,1}y^b_{m,2};$

- $g^{(N)}_{m,r} = \displaystyle\sum_{a+2b=N+1+r} \binom{a+b-r}{a} y^a_{m,1}y^b_{m,2};$

where the sum is taken over all $(a, b) \in \mathbb{N}_0^2$ that satisfy the appropriate equality.

In this paper we mostly use elements from the set G_1 (elements from the set G_2 are only used in Proposition 5.6). Obviously, in $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$ one has $g_m = 0$, for $1 \le m \le j$, and $g_{m,r} = 0$, for $1 \le m \le d, 0 \le r \le n + j + 2m - 1$.

In the remainder of this paper $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$ will simply be denoted by H^*. Also, for a polynomial $p \in \mathbb{Z}_2[x_1, \ldots, x_j, y_{1,1}, y_{2,1}, \ldots, y_{d,1}, y_{d,2}]$, we will denote the class of p in H^* by the same letter.

Proposition 4.1 ([11]). *The set*

$$B_{j,d,n} = \left\{ \prod_{i=1}^{j} x_i^{a_i} \prod_{i=1}^{d} y^{b'_i}_{i,1}y^{b''_i}_{i,2} \ :\ a_i \le n + i - 1,\ i = \overline{1, j},\ b'_i + b''_i \le n + j + 2i - 2,\ i = \overline{1, d} \right\}$$

is a vector space basis for H^*.

For a polynomial $p \in \mathbb{Z}_2[x_1, \ldots, x_j, y_{1,1}, y_{1,2}, \ldots, y_{d,1}, y_{d,2}]$, and $1 \le i \le j$, we denote $p \prec x_i$ if p does not contain the variables $x_1, \ldots, x_i$. For a polynomial $p \in \mathbb{Z}_2[x_1, \ldots, x_j, y_{1,1}, y_{1,2}, \ldots, y_{d,1}, y_{d,2}]$, and $1 \le i \le d$, we denote $p \prec y_{i,2}$ if p does not contain the variables $x_1, \ldots, x_j, y_{1,1}, y_{1,2}, \ldots, y_{i,1}, y_{i,2}$.

Proposition 4.2 ([11]). (1) *Let $1 \le i \le j$, $0 \le a_1 < a_2 < \cdots < a_k \le n + i - 1$, and $p_1, p_2, \ldots, p_k$ be polynomials such that $p_l \prec x_i$, for $l = \overline{1, k}$. Then $\displaystyle\sum_{l=1}^{k} x_i^{a_l} p_l = 0$ in H^* if and only if $p_l = 0$ in H^*, for $l = \overline{1, k}$.*

(2) *Let* $1 \le i \le d$, $(b_1, c_1), \ldots, (b_k, c_k)$ *be distinct pairs of nonnegative integers such that* $b_l + c_l \le n + j + 2l - 2$, *for* $l = \overline{1,k}$, *and* $p_1, \ldots, p_k$ *be polynomials such that* $p_l < y_{i,2}$, *for* $l = \overline{1,k}$. *Then* $\sum_{l=1}^{k} y_{i,1}^{b_l} y_{i,2}^{c_l} p_l = 0$ *in* H^* *if and only if* $p_l = 0$ *in* H^*, *for* $l = \overline{1,k}$.

Proposition 4.3 ([11]). *Let* $a_l \ge 0$, $l = \overline{1,j}$, *and* $b_l, c_l \ge 0$, $l = \overline{1,d}$. *If*

i) $\sum_{l=i+1}^{j} a_l + \sum_{l=1}^{d} (b_l + 2c_l) > \sum_{l=i+1}^{j} (n + l - 1) + \sum_{l=1}^{d} (2n + 2j + 4l - 4)$, *for some* $0 \le i \le j$, *or*

ii) $\sum_{l=i+1}^{d} (b_l + 2c_l) > \sum_{l=i+1}^{d} (2n + 2j + 4l - 4)$, *for some* $0 \le i \le d$,

then $\prod_{l=1}^{j} x_l^{a_l} \prod_{l=1}^{d} y_{l,1}^{b_l} y_{l,2}^{c_l} = 0$ *in* H^* .

Proposition 4.4 ([11]). *Let* $1 \le m \le d$ *and* $M = n + j + 2m - 2$. *If* p *and* q *are polynomials in the variables* $y_{m,1}$ *and* $y_{m,2}$, *such that the corresponding classes are equal in* $H^*(F(2, M); \mathbb{Z}_2)$, *then* $p = q + r$ *in* $H^*(F(1^{\cdots j}, 2^{\cdots d}, n); \mathbb{Z}_2)$, *where* r *is a polynomial in the variables* $y_{m,1}, y_{m,2}, \ldots, y_{d,1}, y_{d,2}$, *such that each monomial of* r *has at least one of the variables* $y_{m+1,1}, y_{m+1,2}, \ldots, y_{d,1}, y_{d,2}$ *in positive degree.*

5. Applications of Gröbner bases for cup-length calculation

We start this section with the case $d = 0$, by reproving Proposition 3.2.

Proof of Proposition 3.2. Note that $\delta(F(1^{\cdots j}, n)) = \binom{j}{2} + nj$, so $\mathrm{cup}(F(1^{\cdots j}, n)) \le \binom{j}{2} + nj$. Additionally, by Proposition 4.1, $x_1^n x_2^{n+1} \cdots x_j^{n+j-1} \neq 0$ in H^*, and therefore the other inequality also holds. $\square$

So, in the remainder of this section we may assume that $d \ge 1$.

The following lemma is a direct consequence of Proposition 4.4 and Proposition 5 from [4].

Lemma 5.1. *Let* $1 \le m \le d$. *Then in* H^* *the following hold*

i) $y_{m,1}^{2^{s(m)}-2} y_{m,2}^{n+j+2m-1-2^{s(m)-1}} = y_{m,2}^{n+j+2m-2} + p$;

ii) for $l \ge 2^{s(m)} - 1$, $y_{m,1}^{l} = q$;

where p *and* q *are polynomials in the variables* $y_{m,1}, y_{m,2}, \ldots, y_{d,1}, y_{d,2}$, *such that each monomial of* p *and* q *is in* $B_{j,d,n}$ *and the degree of at least one of the variables* $y_{m+1,1}, y_{m+1,2}, \ldots, y_{d,1}, y_{d,2}$ *in each monomial is positive.*

As a consequence of Lemma 5.1 we have the following lemma, which will be used throughout this section.

Lemma 5.2. *In* H^* *the following identities hold for* $1 \le i \le d$.

i) $\prod_{t=i}^{d} y_{t,1}^{2^{s(t)}-2} y_{t,2}^{n+j+2t-1-2^{s(t)-1}} = \prod_{t=i}^{d} y_{t,2}^{n+j+2t-2}$.

ii) If $l \ge 2^{s(i)} - 1$, *then* $y_{i,1}^{l} \prod_{t=i+1}^{d} y_{t,1}^{2^{s(t)}-2} y_{t,2}^{n+j+2t-1-2^{s(t)-1}} = 0$.

Proof. i) We proceed by reverse induction on i. For $i = d$, the lemma follows from Lemma 5.1. So, let us assume that it holds for $i \geq 2$ and prove it for $i-1$.

By Lemma 5.1, $y_{i-1,1}^{2^{s(i-1)}-2} y_{i-1,2}^{n+j+2i-3-2^{s(i-1)-1}} = y_{i-1,2}^{n+j+2i-4} + p$, where p is a polynomial in the variables $y_{i-1,1}$, $y_{i-1,2}, \ldots, y_{d,1}, y_{d,2}$, such that each monomial of p is in $B_{j,d,n}$ and the degree of at least one of the variables $y_{i,1}$, $y_{i,2}, \ldots, y_{d,1}, y_{d,2}$ in each monomial is positive. Therefore, by Proposition 4.3, $p \prod_{t=i}^{d} y_{t,1}^{2^{s(t)}-2} y_{t,2}^{n+j+2t-1-2^{s(t)-1}} = 0$, i.e., by inductive hypothesis, $\prod_{t=i-1}^{d} y_{t,1}^{2^{s(t)}-2} y_{t,2}^{n+j+2t-1-2^{s(t)-1}} = \prod_{t=i-1}^{d} y_{t,2}^{n+j+2t-2}$.

ii) By Lemma 5.1, $y_{i-1,1}^{l} = q$, such that each monomial of q is in $B_{j,d,n}$ and the degree of at least one of the variables $y_{i,1}, y_{i,2}, \ldots, y_{d,1}, y_{d,2}$ in each monomial is positive. Therefore, by Proposition 4.3 $q \prod_{t=i}^{d} y_{t,1}^{2^{s(t)-2}} y_{t,2}^{n+j+2t-1-2^{s(t)-1}} = 0$, which completes our proof. □

As a consequence of the previous lemma, we obtain new proofs of Propositions 3.4 and 3.5.

Proof of Proposition 3.4. Since $s(i) = s$, for $1 \leq i \leq d$, by Lemma 5.2,

$$\operatorname{cup}(F(2^{\ldots d}, n)) \geq (2^s - 2)d + \sum_{i=1}^{d} (n + 2i - 1 - 2^{s-1}) = d(n + d + 2^{s-1} - 2).$$

On the other hand, since $\operatorname{ht}(y_{i,1}) = 2^s - 2$, for $1 \leq i \leq d$, by Lemma 2.2,

$$\operatorname{cup}(F(2^{\ldots d}, n)) \leq (2^s - 2)d + \left\lfloor \frac{2nd + 4\binom{d}{2} - (2^s - 2)d}{2} \right\rfloor = d(n + d + 2^{s-1} - 2),$$

which completes our proof. □

Proof of Proposition 3.5. By Lemma 5.2 and Proposition 4.1 we have

$$\prod_{t=1}^{j} x_t^{n+t-1} \prod_{t=1}^{d} y_{t,1}^{2^{s(t)}-2} y_{t,2}^{n+j+2t-1-2^{s(t)-1}} \neq 0.$$

It is easy to check that the degree of this monomial is equal to the expression on the right hand side of the desired inequality. □

For $1 \leq k \leq d$, let $e_k = \sum_{1 \leq i_1 < \cdots < i_k \leq d} y_{i_1,2} \cdots y_{i_k,2}$ be the k-th elementary symmetric polynomial in the variables $y_{i,2}$, for $i = \overline{1,d}$ ($e_0 = 1$ and $e_k = 0$ for $k > d$), and h_k the complete symmetric homogenous polynomial in variables $y_{i,2}$, for $i = \overline{1,d}$ ($h_0 = 1$). Note that mod 2, for $k \geq 0$, the following identity holds

$$\sum_{i=0}^{\min\{d,k\}} e_i h_{k-i} = 0. \tag{4}$$

Lemma 5.3. *Let $1 \leq i \leq j$, $m = \lfloor \frac{n+i}{2} \rfloor$, and $\alpha = n + i - 2m$. In H^* the following identities hold for $0 \leq t \leq m-1$ and $\beta \in \{0, 1\}$*

$$x_i^{n+i+2t+\beta} = x_i^{\beta} \sum_{r=1}^{m-t} x_i^{n+i-2r} \sum_{l=\max\{0,r+t-d\}}^{r-1} h_l e_{r+t-l} + x_i^{\alpha+\beta} \sum_{r=0}^{t-1} x_i^{2r} \sum_{l=0}^{\min\{r,d+r-t\}} h_{m-l} e_{t+l-r} + p,$$

where p is a polynomial such that for each monomial of p at least one of the variables $x_{i+1}, \ldots, x_j, y_{1,1}, y_{2,1}, \ldots, y_{d,1}$ has a positive degree.

Proof. First, note that $\overline{y}_{i,k} = y_{i,2}^{k/2} + p_k$, if k is even, and $\overline{y}_{i,k} = p_k$, if k is odd, where p_k is a polynomial in the variables $y_{i,1}$ and $y_{i,2}$, such that $y_{i,1}$ is in positive degree in each monomial of p_k (see [11, Remark 1]). Therefore, in H^* the following holds

$$0 = g_i = x_i^{n+i} + \sum_{r=1}^{\lfloor\frac{n+i}{2}\rfloor} x_i^{n+i-2r} \sum_{r_1+\cdots+r_d=r} y_{1,2}^{r_1}\cdots y_{d,2}^{r_d} + \widetilde{p} = x_i^{n+i} + \sum_{r=1}^{\lfloor\frac{n+i}{2}\rfloor} x_i^{n+i-2r} h_r + \widetilde{p}, \tag{5}$$

where $\widetilde{p}$ is such that for each monomial of $\widetilde{p}$ at least one of the variables $x_{i+1},\dots,x_j,y_{1,1},y_{2,1},\dots,y_{d,1}$ has a positive degree. Multiplying the last identity by x_i^{β}, and using (4), we obtain the identity from the lemma for $t=0$.

We proceed by induction on $t \geq 1$. By the inductive hypothesis, identities (4) and (5), and since $e_{t+1} = 0$ for $t \geq d$, in H^* the following holds

$$\begin{aligned}
x_i^{n+i+2t+2+\beta} &= x_i^{\beta} \sum_{r=1}^{m-t} x_i^{n+i-2r+2} \sum_{l=\max\{0,r+t-d\}}^{r-1} h_l e_{r+t-l} + x_i^{\alpha+\beta} \sum_{r=0}^{t-1} x_i^{2r+2} \sum_{l=0}^{\min\{r,d+r-t\}} h_{m-l} e_{t+l-r} + x_i^2 p \\
&= x_i^{\beta} x_i^{n+i} e_{t+1} + x_i^{\beta} \sum_{r=1}^{m-t-1} x_i^{n+i-2r} \sum_{l=\max\{0,r+1+t-d\}}^{r} h_l e_{r+1+t-l} + x_i^{\alpha+\beta} \sum_{r=1}^{t} x_i^{2r} \sum_{l=0}^{\min\{r-1,d+r-1-t\}} h_{m-l} e_{t+l-(r-1)} + x_i^2 p \\
&= x_i^{\beta} \sum_{r=1}^{m-t-1} x_i^{n+i-2r} h_r e_{t+1} + x_i^{\beta} \sum_{r=m-t}^{m} x_i^{2(m-r)+\alpha} h_r e_{t+1} + x_i^{\beta} e_{t+1}\widetilde{p} \\
&\quad + x_i^{\beta} \sum_{r=1}^{m-t-1} x_i^{n+i-2r} \sum_{l=\max\{0,r+1+t-d\}}^{r} h_l e_{r+1+t-l} + x_i^{\alpha+\beta} \sum_{r=1}^{t} x_i^{2r} \sum_{l=0}^{\min\{r-1,d+r-1-t\}} h_{m-l} e_{t+1+l-r} + x_i^2 p \\
&= x_i^{\beta} \sum_{r=1}^{m-t-1} x_i^{n+i-2r} h_r e_{t+1} + x_i^{\beta} \sum_{r=1}^{m-t-1} x_i^{n+i-2r} \sum_{l=\max\{0,r+1+t-d\}}^{r} h_l e_{r+1+t-l} \\
&\quad + x_i^{\beta} \sum_{r=0}^{t} x_i^{\alpha+2r} h_{m-r} e_{t+1} + x_i^{\alpha+\beta} \sum_{r=1}^{t} x_i^{2r} \sum_{l=0}^{\min\{r-1,d+r-1-t\}} h_{m-l} e_{t+1+l-r} + x_i^{\beta} e_{t+1}\widetilde{p} + x_i^2 p \\
&= x_i^{\beta} \sum_{r=1}^{m-t-1} x_i^{n+i-2r} \sum_{l=\max\{0,r+1+t-d\}}^{r-1} h_l e_{r+1+t-l} + x_i^{\alpha+\beta} \sum_{r=0}^{t} x_i^{2r} \sum_{l=0}^{\min\{r,d+r-1-t\}} h_{m-l} e_{t+1+l-r} + x_i^{\beta} e_{t+1}\widetilde{p} + x_i^2 p,
\end{aligned}$$

where p and $\widetilde{p}$ are polynomials such that for each monomial of p and $\widetilde{p}$ at least one of the variables $x_{i+1},\dots,x_j,y_{1,1},y_{2,1},\dots,y_{d,1}$ has a positive degree. This also holds for $x_i^2 p + x_i^{\beta} e_{t+1}\widetilde{p}$, so the proof is completed. $\square$

Remark 5.4. *Note that for $t \geq d$ the first double sum in the equality for $x_i^{n+i+2t+\beta}$ is equal to zero, since it is empty. Also, looking at the proof, it is easy to conclude that the lemma holds for every $t \geq 0$. Since the purpose of this lemma is to represent $x^{n+i+2t+\beta}$ in the additive basis $B_{j,d,n}$, we omitted the case $t \geq m$.*

Proof of Proposition 3.6. i) Let $m = \lfloor\frac{n+1}{2}\rfloor$ and $\alpha = n+1-2m$. Note that $2d \leq 2^s - (n+1) \leq n+1$, and therefore $d-1 \leq m-1$. Now, by Lemma 5.3

$$\begin{aligned}
x_1^{n+2d} &= \sum_{r=1}^{m-d+1} x_1^{n+2-2r} \sum_{l=\max\{0,r-1\}}^{r-1} h_l e_{r+d-1-l} + x_1^{\alpha} \sum_{r=0}^{d-2} x_1^{2r+1} \sum_{l=0}^{\min\{r,r+1\}} h_{m-l} e_{d-1+l-r} + p \\
&= \sum_{r=1}^{m-d+1} x_1^{n+2-2r} h_{r-1} e_d + x_i^{\alpha} \sum_{r=0}^{d-2} x_1^{2r+1} \sum_{l=0}^{r} h_{m-l} e_{d-1+l-r} + p,
\end{aligned}$$

where p is a polynomial such that for each monomial of p at least one of the variables $y_{1,1}, y_{2,1}, \dots, y_{d,1}$ has a positive degree. Note that $2(d-2)+1+\alpha < n$, so in the second sum the degree of x_1 is less than n. Therefore, by Proposition 2.3, Corollary 4.3 and Lemma 5.3,

$$x_1^{n+2d}\prod_{i=1}^{d} y_{i,1}^{2^s-2}y_{i,2}^{n+2i-2^{s-1}-1} = x_1^n e_d \prod_{i=1}^{d} y_{i,1}^{2^s-2}y_{i,2}^{n+2i-2^{s-1}-1} = x_1^n \prod_{i=1}^{d} y_{i,1}^{2^s-2}y_{i,2}^{n+2i-2^{s-1}},$$

which is nonzero by Lemma 5.2. Note that this monomial is in the maximal dimension, and degrees of the first Stiefel-Whitney classes of this monomial are equal to their heights. So, by Lemma 2.2, this monomial gives the cup-length, i.e., $\mathrm{cup}(F(1, 2^{\cdot\cdot d}, n)) = n + d(n + d + 2^{s-1})$.

ii) Similarly as in the first part of the proof we will prove that the class

$$x_1^{n+2d}x_2^{n+1+2d}\prod_{i=1}^{d} y_{i,1}^{2^s-2}y_{i,2}^{n+2i-1-2^{s-1}} \tag{6}$$

is nonzero. Let $m' = \lfloor \frac{n+1}{2} \rfloor$, $m'' = \lfloor \frac{n+2}{2} \rfloor$, $\alpha' = n+1-2m'$ and $\alpha'' = n+1-2m''$. Note that $2d \le 2^s - (n+2) \le n$, i.e., $d-1 \le m'-1 \le m''-1$. Therefore, as in part i), we have

$$x_1^{n+2d} = \sum_{r=1}^{m'-d+1} x_1^{n+2-2r}h_{r-1}e_d + x_i^{\alpha'}\sum_{r=0}^{d-2} x_1^{2r+1}\sum_{l=0}^{r} h_{m-l}e_{d-1+l-r} + p',$$

where p' is a polynomial such that for each monomial of p' at least one of the variables $x_2, y_{1,1}, y_{2,1}, \dots, y_{d,1}$ has a positive degree. Note that $2(d-2)+1+\alpha' < n$, so in the second sum the degree of x_1 is less than n. So, by Corollary 4.3 and Proposition 2.3, the class

$$\left(x_i^{\alpha'}\sum_{r=0}^{d-2} x_1^{2r+1}\sum_{l=0}^{r} h_{m-l}e_{d-1+l-r} + p'\right)x_2^{n+1+2d}\prod_{i=1}^{d} y_{i,1}^{2^s-2}y_{i,2}^{n+2i-1-2^{s-1}}$$

iz zero, and therefore the class in (6) is equal to

$$x_1^n e_d x_2^{n+1+2d}\prod_{i=1}^{d} y_{i,1}^{2^s-2}y_{i,2}^{n+2i-1-2^{s-1}} = x_1^n x_2^{n+1+2d}\prod_{i=1}^{d} y_{i,1}^{2^s-2}y_{i,2}^{n+2i-2^{s-1}}.$$

Similarly as in the previous part of the proof, this term is by Lemma 5.3, Proposition 2.3 and Corollary 4.3 equal to

$$x_1^n x_2^{n+1} e_d \prod_{i=1}^{d} y_{i,1}^{2^s-2}y_{i,2}^{n+2i-2^{s-1}} = x_1^n x_2^{n+1}\prod_{i=1}^{d} y_{i,1}^{2^s-2}y_{i,2}^{n+2i+1-2^{s-1}},$$

which is by Lemma 5.2 nonzero. Now, the total degree of the class in (6) is $2n+1+d(n+d+2^{s-1}+1)$, which is equal to $\beta(F(1, 1, 2^{\cdot\cdot d}, n))$. □

Proof of Proposition 3.8. Let $g = n + j + 1 - 2^{s-1}$. It is enough to prove that in H^*

$$\prod_{i=1}^{j-g} x_i^{n+i-1}\prod_{i=j-g+1}^{j} x_i^{n+i+2d-1}\prod_{i=1}^{d} y_{i,1}^{2^s-2}\prod_{i=1}^{d} y_{i,2}^{2i-2} \neq 0,$$

since the degree of this class is equal to $\delta(F(1^{\cdot\cdot j}, 2^{\cdot\cdot d}, n)) - d(d-1)$. This can be done similarly as in the proof of the previous proposition. We omit the detailed proof since it is too technical. □

In the following proposition we present an infinite family of real flag manifolds $F(1^{\cdot\cdot j}, 2^{\cdot\cdot d}, n)$ for which $\beta(F(1^{\cdot\cdot j}, 2^{\cdot\cdot d}, n))$ is well-defined and the cup-length is smaller than $\min\{\delta, \beta\}$, where $\delta = \delta(F(1^{\cdot\cdot j}, 2^{\cdot\cdot d}, n))$ and $\beta = \beta(F(1^{\cdot\cdot j}, 2^{\cdot\cdot d}, n))$.

It is easy to check that if $\binom{j}{2} \le 2d(n+d-2^{s-1})$, then $\beta(F(1^{\cdot\cdot j}, 2^{\cdot\cdot d}, n))$ is well-defined and one has $\beta \le \delta$.

Proposition 5.5. *Let $j, d, n \in \mathbb{N}$ be such that $\binom{j}{2}$ is even, $6 \le \binom{j}{2} \le 2d(n+d-2^{s-1})$, $\left\lfloor \frac{j}{2} \right\rfloor + d \le \left\lfloor \frac{n+1}{2} \right\rfloor$, and let $\delta = \delta(F(1^{\cdots j}, 2^{\cdot\cdot d}, n))$ and $\beta = \beta(F(1^{\cdots j}, 2^{\cdot\cdot d}, n))$. Then* $\operatorname{cup}(F(1^{\cdots j}, 2^{\cdot\cdot d}, n)) < \beta \le \delta$.

Proof. Let $\prod_{i=1}^{j} x_i^{a_i} \prod_{i=1}^{d} y_{i,1}^{b_i} y_{i,2}^{c_i}$ be a nonzero monomial for which $\sum_{i=1}^{j} a_i + \sum_{i=1}^{d}(b_i + c_i) = D$ is maximum. Let us assume that $D \ge \beta$, i.e., by Lemma 2.2, that $\operatorname{cup}(F(1^{\cdots j}, 2^{\cdot\cdot d}, n)) = \beta$. By Lemma 2.3, $a_i \le n + j + 2d - 1$, $i = \overline{1, j}$, $b_i \le 2^s - 2$, $i = \overline{1, d}$, and by Corollary 4.3, $\sum_{i=1}^{j} a_i + \sum_{i=1}^{d}(b_i + 2c_i) \le \delta$. If $S = j(n + j + 2d - 1) + d(2^s - 2)$, then by adding the previously obtained inequalities we have

$$2S + 2\left\lfloor \frac{\delta - S}{2} \right\rfloor = 2\beta = 2\sum_{i=1}^{j} a_i + 2\sum_{i=1}^{d}(b_i + c_i) \le S + \delta. \tag{7}$$

Since δ and S are even, (7) is in fact equality, and therefore $a_i = n + j + 2d - 1$, $i = \overline{1, j}$, and $b_i = 2^s - 2$, $i = \overline{1, d}$. So,

$$\prod_{i=1}^{j} x_i^{n+j+2d-1} \prod_{i=1}^{d} y_{i,1}^{2^s-2} y_{i,2}^{c_i} \ne 0. \tag{8}$$

Let $m = \lfloor \frac{n+1}{2} \rfloor$, $\alpha = n + 1 - 2m$, $t = \lfloor \frac{j}{2} \rfloor + d - 1$, $\beta = j + 2d - 2 - 2t$. Since $t \le m - 1$, we can use Lemma 5.3 to represent $x_1^{n+j+2d-1}$. Note that the first double sum in this representation is zero (since the inner sum is empty), and therefore the class in (8) is equal to

$$\left(x_1^{\alpha+\beta} \sum_{r=0}^{t-1} x_1^{2r} \sum_{l=0}^{d+r-t} h_{m-l} e_{d+l-r} + p \right) \prod_{i=2}^{j} x_i^{n+j+2d-1} \prod_{i=2}^{d} y_{i,1}^{2^s-2} y_{i,2}^{b_i},$$

where p is such that for each monomial of p at least one of the variables $x_2, \ldots, x_j, y_{1,1}, y_{2,1}, \ldots, y_{d,1}$ has a positive degree. Since, by Proposition 2.3,

$$p \prod_{i=2}^{j} x_i^{n+j+2d-1} \prod_{i=1}^{d} y_{i,1}^{2^s-2} y_{i,2}^{b_i} = 0,$$

and, by Proposition 4.3,

$$x_1^{2r+\alpha} \prod_{i=1}^{j} x_i^{n+j+2d-1} \prod_{i=2}^{d} y_{i,1}^{2^s-2} y_{i,2}^{b_i} = 0,$$

for $r < \frac{n-1}{2}$, the class in (8) is equal to zero, which is a contradiction. $\square$

We conclude this section with the following proposition, which extends Example 3.3.3 from [9].

Proposition 5.6. *For $s \ge 3$ we have* $\operatorname{cup}(F(1, 2, 2, 2^{s-1} - 2)) = \delta(F(1, 2, 2, 2^{s-1} - 2)) - 1$.

Proof. Let $N = 2^{s-1} - 1$. First, let us represent the class $t = y_{2,1}^{2N} y_{2,2}$ in the additive basis $B = B_{1,2,N-1}$ from Proposition 4.1. Note that the dimension of t is $2N + 2$. Therefore, by Proposition 4.2, and since the only classes of B in the variables $y_{2,1}$, $y_{2,2}$ and dimension $2N+2$ are $y_{2,2}^{N+1}$ and $y_{2,1}^2 y_{2,2}^N$, we have $t = \alpha y_{2,2}^{N+1} + \beta y_{2,1}^2 y_{2,2}^N$, for some $\alpha, \beta \in \{0, 1\}$. Note that the height of $y_{2,1}$ is equal to $2N$, and therefore $ty_{2,1} = 0 = \alpha y_{2,1}y_{2,2}^{N+1} + \beta y_{2,1}^3 y_{2,2}^N$. At the same time,

$$0 = g_{2,N} = \sum_{a+2b=2N+3} y_{2,1}^a y_{2,2}^b = \binom{3}{3} y_{2,1}^3 y_{2,2}^N + \binom{2}{1} y_{2,1} y_{2,2}^{N+1} = y_{2,1}^3 y_{2,2}^N,$$

and therefore, since $y_{2,1}y_{2,2}^N \in B$, we have $\alpha = 0$. Finally, since $t \neq 0$, we conclude that $\beta = 1$, i.e., $y_{2,1}^{2N} y_{2,2} = y_{2,1}^2 y_{2,2}^N$. By symmetry $y_{1,1}^{2N} y_{1,2} = y_{1,1}^2 y_{1,2}^N$.

Now, let us prove that $\mathrm{cup}(F(1, 2, 2, N-1)) \neq \delta(F(1, 2, 2, 2^{s-1} - 2))$. Since $\delta(F(1, 2, 2, N-1)) = 5N + 3$, $\mathrm{ht}(x_1) = N + 3$, $\mathrm{ht}(y_{1,1}) = \mathrm{ht}(y_{2,1}) = 2N$, this is equivalent to proving that the class $u = x_1^{N+3} y_{1,1}^{2N} y_{2,1}^{2N}$ iz zero. By Lemma 5.3, we have

$$x_1^{N+3} = x_1^{N-1} y_{1,2} y_{2,2} + p',$$

where p' is a polynomial such that each monomial of p' contains $y_{1,1}$ or $y_{2,1}$, or the degree of x_1 in it is less than $N - 1$. So, by Propositions 2.3 and 4.3, $p' y_{1,1}^{2N} y_{2,1}^{2N} = 0$, and therefore

$$u = x_1^{N-1} y_{1,1}^{2N} y_{1,2} y_{2,1}^{2N} y_{2,2} = x_1^{N-1} y_{1,1}^2 y_{1,2}^{N+1} y_{2,1}^2 y_{2,2}^{N+1}.$$

Now, $0 = g_{1,N+1} = y_{1,1}^2 y_{1,2}^N + y_{1,2}^{N+1} + y_{1,2}^N y_{2,2} + p''$, where p'' is a polynomial such that each monomial of p'' contains $y_{2,1}$, or the degree of $y_{2,2}$ in it is at least 2. So, by Propositions 2.3 and 4.3, we have $p'' y_{2,1}^{2N} y_{2,2} = 0$, and therefore $u = x_1^{N-1}(y_{1,2}^{N+1} + y_{1,2}^N y_{2,2}) y_{2,1}^2 y_{2,2}^{N+1}$. Finally, $0 = g_{1,N+3} = y_{1,2}^{N+1} + y_{1,2}^N y_{2,2} + p'''$, where p''' is a polynomial such that each monomial of p''' contains $y_{2,1}$, or the degree of $y_{2,2}$ in it is at least 2. Again, by Propositions 2.3 and 4.3, we have $p''' y_{2,1}^{2N} y_{2,2} = 0$, and therefore $u = 0$.

To finish the proof it is enough to prove that the class $v = x_1^{N+1} y_{1,1}^{2N} y_{2,1}^{2N} y_{2,2}$ is nonzero. By Lemma 5.3, we have

$$x_1^{N+1} = x_1^{N-1}(y_{1,2} + y_{2,2}) + q',$$

where q' is a polynomial such that each monomial of q' contains $y_{1,1}$ or $y_{2,1}$, or the degree of x_1 is less than $N - 1$. By Propositions 2.3 and 4.3, $q' y_{1,1}^{2N} y_{2,1}^{2N} y_{2,2} = 0$, and therefore

$$v = x_1^{N-1}(y_{1,2} + y_{2,2}) y_{1,1}^{2N} y_{2,1}^{2N} y_{2,2} = x_1^{N-1} y_{1,1}^{2N} y_{1,2} y_{2,1}^{2N} y_{2,2} + x_1^{N-1} y_{1,1}^{2N} y_{2,1}^{2N} y_{2,2}^2.$$

As in the first part of the proof one has $x_1^{N-1} y_{1,1}^{2N} y_{1,2} y_{2,1}^{2N} y_{2,2} = 0$. Additionally, by Lemma 5.1, $x_1^{N-1} y_{1,1}^{2N} y_{2,1}^{2N} y_{2,2}^2 = x_1^{N-1} y_{1,2}^N y_{2,2}^{N+2} \neq 0$, and therefore $v \neq 0$, which completes our proof. $\square$

6. Concluding remarks

In this paper we presented two techniques for cup-length calculations, and provided a number of examples that show that both of them can be very useful for these calculations. Of course, both of them have their advantages and shortcomings, so depending on the situation in question, one technique can be more efficient than the other.

The method of fiberings is very effective and elegant, and often requires very little calculation. The "only" deficiency of this method is that it only provides us with a lower bound for the cup-length, and often with a "bad" one. On the other hand, using Gröbner basis one can obtain both upper and lower bounds for the cup-length, but in order to do so, often a lot of calculation is required. Unfortunately, when dealing with a general flag manifold of a given type, these calculations can become too involved to give us any result. Of course, having a Gröbner basis for a concrete flag manifold means that all calculations can be done using well-known algorithms (see [1]), i.e., using a computer, and therefore the cup-length can also be computed.

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