# On endomorphisms of real Grassmannians that commute with Steenrod squares

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**Abstract**

For $k \in \{2,3\}$ we classify endomorphisms of $H^*(G_{k,n};\mathbb{Z}_2)$ that commute with Steenrod squares (here, $G_{k,n}$ denote the Grassmann manifold of $k$-dimensional subspaces in $\mathbb{R}^{n+k}$). Additionally, for all positive integers $k$ and $n$ we classify endomorphisms of $H^*(G_{k,n};\mathbb{Z}_2)$ that commute with Steenrod squares and such that the image of each class is a polynomial in the (nonzero) class of $H^1(G_{k,n};\mathbb{Z}_2)$.

## 1 Introduction

Let $G_{k,n}$ denote the Grassmann manifold of $k$-dimensional subspaces in $\mathbb{R}^{n+k}$. To study properties of a continuous map $f : G_{\ell,m} \to G_{\ell',m'}$ one may consider the map $f^* : H^*(G_{\ell',m'};\mathbb{Z}_2) \to H^*(G_{\ell,m};\mathbb{Z}_2)$. Then:

(1) $f^*$ is a homomorphism;

(2) $f^*$ commutes with Steenrod squares.

It turned out that for various $(\ell,\ell',m,m')$ conditions (1) and (2) were enough to obtain a classification of maps $f^*$ that was sufficient to conclude that all continuous map $f : G_{\ell,m} \to G_{\ell',m'}$ have some specific properties. For example, in [15] this classification was obtained in the case $\ell = \ell' = 2$ and $m = m' \equiv 0, 1 \pmod 4$, and used to prove that then $G_{2,m}$ has the fixed point property; in [12] the classification was obtained in the case $m = m'$, $1 \leqslant \ell' < \ell \leqslant m$ and $m \geqslant 2\ell' - 1$, and used to prove that then there does not exist an equivariant map between $\widetilde{G}_{\ell,m}$ and $\widetilde{G}_{\ell',m'}$ (here, $\widetilde{G}_{k,m}$ denotes the oriented Grassmann manifold of oriented $k$-planes in $\mathbb{R}^{n+k}$); later, in [14] this classification was used to prove that any continuous map $f : G_{\ell,m} \to G_{\ell',m'}$ has an invariant point. We note that similar results were obtained for related manifolds in [2, 3, 5, 6, 7, 9, 10, 13]. These results suggest that it is a natural question to consider homomorphisms that satisfy (1) and (2).

For the $\mathbb{Z}$-cohomology of complex Grassmannians it was conjectured by Glover and Homer (see [5]) that all endomorphisms have a very simple form, i.e. that they are the so called

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Adams' maps (it can be said that these are just "scaled" identity maps). So, it is a natural question to ask if a similar conjecture holds for the $\mathbb{Z}_2$-cohomology of real Grassmannians. Already in [5] it was shown that there are endomorphisms of $H^*(G_{2,n};\mathbb{Z}_2)$ (for certain $n$) that are not Adams' maps (for any $\mathbb{Z}_2$-cohomology the only Adams' maps are the identity map and the map that is zero in positive dimensions). But all these examples are, what we call *projective* endomorphisms (for them the image of each class is a polynomial in the (nonzero) class of $H^1(G_{k,n};\mathbb{Z}_2)$); analogous maps appeared in similar classifications (see [6, 9, 12, 13]). So, one might hope that any endomorphism of $H^*(G_{k,n};\mathbb{Z}_2)$ is either Adams' map or projective (there is a similar conjecture, see [6], for the endomorphisms of complex flag manifolds). We will prove that this is, unfortunately, not the case.

In this paper, for $k \in \{2,3\}$, we classify endomorphisms of $H^*(G_{k,n};\mathbb{Z}_2)$ that commute with Steenrod squares. For $k=2$ this completes the work done by O'Neill in [15]. Although the classification for $k=2$ is rather simple, already the case $k=3$ proved to be much more demanding. Indeed, for $k=3$ we have endomorphisms that are neither the identity map nor *projective* (this notion is defined in Section 6). These results suggest that for general $k$ it should be very difficult to obtain the full classification of endomorphisms of $H^*(G_{k,n};\mathbb{Z}_2)$ that satisfy (2). Finally, in Section 6 we classify, for every $k$ and $n$, projective endomorphisms of $H^*(G_{k,n};\mathbb{Z}_2)$ that commute with Steenrod squares.

## 2 Cohomology of Grassmann manifolds

In this paper, all cohomology groups are assumed to have coefficients in $\mathbb{Z}_2$.

Let $\gamma_{k,n}$ be the tautological $k$-dimensional vector bundle over $G_{k,n}$, and let $w_i$ be the $i$-th Stiefel-Whitney class of $\gamma_{k,n}$. Then, by Borel's description (see [1]), the algebra $H^*(G_{k,n})$ is given by:

$$H^*(G_{k,n}) \cong \mathbb{Z}_2[w_1, w_2, \dots, w_k]/I_{k,n},$$

where $I_{k,n} = \langle \overline{w}_{n+1}, \overline{w}_{n+2}, \dots, \overline{w}_{n+k}\rangle$. Here, $\overline{w}_t$, for $t \geqslant 0$, denote the dual classes, which are defined with $(1+w_1+\cdots+w_k)(1+\overline{w}_1+\overline{w}_2+\cdots)=1$, or, equivalently, by the following recurrence relation:

$$\overline{w}_{t+k} = w_1\overline{w}_{t+k-1} + w_2\overline{w}_{t+k-2} + \cdots + w_k\overline{w}_t \quad \text{for } t \geqslant 0. \tag{2.1}$$

These leads to the following identity:

$$\overline{w}_t = \sum_{a_1+2a_2+\cdots+ka_k=t} [a_1, a_2, \dots, a_k] w_1^{a_1} w_2^{a_2}\cdots w_k^{a_k}, \tag{2.2}$$

where $[a_1, a_2, \dots, a_k] := \binom{a_1+a_2+\cdots+a_k}{a_1}\binom{a_2+\cdots+a_k}{a_2}\cdots\binom{a_{k-1}+a_k}{a_{k-1}}$ denotes the multinomial coefficient.

**Remark 2.1** *We abuse the notation and use $w_i$ (and $\overline{w}_j$) to denote both the class of the cohomology algebra $H^*(G_{k,n})$ and the element of the polynomial ring $\mathbb{Z}_2[w_1, w_2, \dots, w_k]$.*

The following theorem gives an additive basis for $H^*(G_{k,n})$ in terms of Stiefel-Whitney classes (see [11]).

**Theorem 2.2** *The set $D_{k,n} = \{w_1^{\alpha_1}w_2^{\alpha_2}\cdots w_k^{\alpha_k} \mid (\alpha_1,\alpha_2,\dots,\alpha_k) \in \mathbb{N}_0^k,\ \alpha_1+\alpha_2+\cdots+\alpha_k \leqslant n\}$ is an additive basis for $H^*(G_{k,n})$.*

The *height* of a class $c \in \widetilde{H}^*(X)$, denoted by $\mathrm{ht}(c)$, is the largest $m \in \mathbb{N}$ such that $c^m \neq 0$. Height of $w_1 \in H^*(G_{k,n})$ is obtained by Stong in [16]: if $s$ is the unique non-negative integer such that $2^s < n+k \leqslant 2^{s+1}$, then

$$\mathrm{ht}(w_1) = \begin{cases} 2^{s+1}-2, & \text{if } k=2 \text{ or } (k,n)=(3,2^s-2), \\ 2^{s+1}-1, & \text{otherwise.} \end{cases} \tag{2.3}$$

Since we work with $\mathbb{Z}_2$ coefficients, the following result will be used throughout the paper.

**Theorem 2.3 (Lucas' theorem)** *Let $a$ and $b$ be positive integers, and $a = \sum_{i=0}^{\infty} a_i 2^i$, $b = \sum_{i=0}^{\infty} b_i 2^i$, $a_i, b_i \in \{0,1\}$, their binary representations. Then*

$$\binom{a}{b} \equiv \prod_{i=0}^{\infty} \binom{a_i}{b_i} \pmod 2.$$

# 3 Steenrod squares

In this paper we are interested in finding endomorphisms $\phi$ such that the following diagram is commutative:

$$\begin{array}{ccc} H^*(G_{k,n}) & \xrightarrow{\phi} & H^*(G_{k,n}) \\ \downarrow{\scriptstyle \mathrm{Sq}^i} & & \downarrow{\scriptstyle \mathrm{Sq}^i} \\ H^*(G_{k,n}) & \xrightarrow{\phi} & H^*(G_{k,n}) \end{array}$$

Using the notation from the previous section, we have $H^*(G_{k,n}) \cong \mathbb{Z}_2[w_1,\dots,w_k]/I_{k,n}$, so the above diagram is commutative if and only if

$$\phi(\mathrm{Sq}^i(w_j)) = \mathrm{Sq}^i(\phi(w_j)) \quad \text{for all } 1 \leqslant i < j \leqslant k \tag{3.1}$$

(indeed, since $\mathrm{Sq}^0$ is the identical transformation we may assume $i \geqslant 1$, since $\mathrm{Sq}^a(w_b) = 0$ for $a > b$ we may assume $i \leqslant j$, and since (3.1) is satisfied for $i = j$ we may assume $i \neq j$). By Wu's formula from [17], we can calculate the action of the Steenrod squares on the Stiefel-Whitney classes:

$$\mathrm{Sq}^i(w_j) = \sum_{t=0}^{i} \binom{j-i+t-1}{t} w_{i-t} w_{j+t}. \tag{3.2}$$

Since in this paper we mostly consider the real Grassmannian $G_{k,n}$ for $k \in \{2,3\}$, this formula will be particulary useful for $i \in \{1,2\}$. So, let us write down these two special cases:

$$\mathrm{Sq}^1(w_j) = w_1 w_j + (j-1) w_{j+1}, \tag{3.3}$$

$$\mathrm{Sq}^2(w_j) = w_2 w_j + (j-2) w_1 w_{j+1} + \binom{j-1}{2} w_{j+2} \tag{3.4}$$

(here, we take $w_\ell = 0$ for $\ell > k$). Of course, to obtain the action of $\mathrm{Sq}^i$ on $H^*(G_{k,n})$, we combine these formulas with:

$$\mathrm{Sq}^i(xy) = \sum_{t=0}^{i} \mathrm{Sq}^t(x) \mathrm{Sq}^{i-t}(y). \tag{3.5}$$

Let us observe $\mathbb{Z}_2[w_1, \ldots, w_k]$ as a graded algebra (where $\deg w_i = i$, $1 \leqslant i \leqslant k$). Then for any endomorphism $\phi : H^*(G_{k,n}) \to H^*(G_{k,n})$ one has an endomorphism $\varphi : \mathbb{Z}_2[w_1, \ldots, w_k] \to \mathbb{Z}_2[w_1, \ldots, w_k]$ defined with $\varphi(w_i) = \phi(w_i)$ that further satisfies $\varphi(I_{k,n}) \subseteq I_{k,n}$. Of course, the converse is also true, i.e. any such endomorphism $\varphi$ induces an endomorphism $\phi$ of $H^*(G_{k,n})$. In fact, in this paper we find it more convenient to work with endomorphisms $\varphi$ of $\mathbb{Z}_2[w_1, \ldots, w_k]$, and classify all that satisfy $\varphi(I_{k,n}) \subseteq I_{k,n}$ and such that the corresponding endomorphism $\phi$ commutes with Steenrod squares.

# 4 The case $k = 2$

Let $s$ be the unique positive integer such that $2^s < n + 2 \leqslant 2^{s+1}$.

Let $\varphi : \mathbb{Z}_2[w_1, w_2] \to \mathbb{Z}_2[w_1, w_2]$ be an endomorphism that satisfies $\varphi(I_{2,n}) \subseteq I_{2,n}$. So, let $\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2, \lambda_{1,1} \in \mathbb{Z}_2$ be such that

- $\varphi(w_1) = \alpha w_1$ and $\varphi(w_2) = \beta_1 w_2 + \beta_2 w_1^2$,
- $\varphi(\overline{w}_{n+1}) = \lambda_1 \overline{w}_{n+1}$ and $\varphi(\overline{w}_{n+2}) = \lambda_2 \overline{w}_{n+2} + \lambda_{1,1} w_1 \overline{w}_{n+1}$.

Also, let $\phi : H^*(G_{2,n}) \to H^*(G_{2,n})$ be the endomorphism induced by $\varphi$. Suppose that $\phi$ commutes with Steenrod squares. Then the condition (3.1) should be checked only for $i = 1$ and $j = 2$, which, by (3.3), is equivalent with

$$\alpha w_1(\beta_1 w_2 + \beta_2 w_1^2) = \beta_1 w_1 w_2.$$

This implies (by Theorem 2.2):

$$\alpha\beta_1 = \beta_1, \qquad \alpha\beta_2 = 0.$$

So, we have the following four possibilities: 1° $\alpha = 1$, $\beta_1 = 0$, $\beta_2 = 0$; 2° $\alpha = 1$, $\beta_1 = 1$, $\beta_2 = 0$; 3° $\alpha = 0$, $\beta_1 = 0$, $\beta_2 = 1$; 4° $\alpha = 0$, $\beta_1 = 0$, $\beta_2 = 0$. In what follows we check, case by case, if the corresponding endomorphism $\varphi$ induces an endomorphism $\phi$ of $H^*(G_{k,n})$ (note: if it does, then $\phi$ commutes with Steenrod squares). We consider each of them separately.

*Case* 1° $\alpha = 1$, $\beta_1 = 0$, $\beta_2 = 0$, i.e. $\varphi(w_1) = w_1$ and $\varphi(w_2) = 0$. Then

$$\varphi(\overline{w}_{n+1}) = \varphi\left(\sum_{a+2b=n+1} \binom{a+b}{a} w_1^a w_2^b\right) = \sum_{a+2b=n+1} \binom{a+b}{b} \varphi(w_1)^a \varphi(w_2)^b = w_1^{n+1}.$$

It follows that $\lambda_1 = 1$ and $\overline{w}_{n+1} = w_1^{n+1}$. By formula (2.2):

$$\overline{w}_{n+1} = \sum_{a+2b=n+1} \binom{a+b}{b} w_1^a w_2^b. \tag{4.1}$$

We can deduce that $n + 1$ must be odd, since otherwise $w_2^{(n+1)/2}$ is a monomial of $\overline{w}_{n+1}$. Also, by (2.3), we have $\mathrm{ht}(w_1) = 2^{s+1} - 2$. Since $w_1^{n+1} = \overline{w}_{n+1} \in I_{2,n}$, we conclude that $n + 1 > 2^{s+1} - 2$, and therefore $n = 2^{s+1} - 2$. Similarly as above, we have $\varphi(\overline{w}_{n+2}) = w_1^{n+2}$.

Hence, to have $\varphi$ induce a desired endomorphism we must have $w_1^{n+1} = \overline{w}_{n+1}$. It was proven above that this identity can hold only for $n = 2^{s+1} - 2$. On the other hand, if $n = 2^{s+1} - 2$, then $\mathrm{ht}(w_1) = n$, and hence $\varphi(\overline{w}_{n+1}) = w_1^{n+1} \in I_{2,n}$ and $\varphi(\overline{w}_{n+2}) = w_1^{n+2} \in I_{2,n}$.

This shows that $\phi(w_1) = w_1$ and $\phi(w_2) = 0$ define an endomorphism $\phi : H^*(G_{2,n}) \to H^*(G_{2,n})$ if and only if $n = 2^{s+1} - 2$ for some $s \geqslant 1$.

*Case* $2^\circ$ $\alpha = 1$, $\beta_1 = 1$, $\beta_2 = 0$. Clearly, in this case $\varphi$ induces the identity map $\phi$.

*Case* $3^\circ$ $\alpha = 0$, $\beta_1 = 0$, $\beta_2 = 1$, i.e. $\varphi(w_1) = 0$ and $\varphi(w_2) = w_1^2$. Then

$$\varphi(\overline{w}_{n+1}) = \sum_{a+2b=n+1} \binom{a+b}{b} \varphi(w_1)^a \varphi(w_2)^b = \begin{cases} 0, & 2 \nmid n+1, \\ w_1^{n+1}, & 2 \mid n+1. \end{cases}$$

If $n+1$ is even, then $w_1^{n+1} = \lambda_1 \overline{w}_{n+1} = 0$ in $H^*(G_{2,n})$. On the other hand, by (2.3), we have $\mathrm{ht}(w_1) = 2^{s+1} - 2 \geqslant n+1$, a contradiction.

So, $n+1$ must be odd. Then, as above we conclude that $\varphi(\overline{w}_{n+2}) = w_1^{n+2} = \lambda_2 \overline{w}_{n+2} + \lambda_{1,1} w_1 \overline{w}_{n+1}$, which is 0 in $H^*(G_{2,n})$. This implies $n+2 > \mathrm{ht}(w_1) = 2^{s+1} - 2$ and hence $n = 2^{s+1} - 2$. Finally, for $n = 2^{s+1} - 2$ the endomorphism $\phi : H^*(G_{2,n}) \to H^*(G_{2,n})$, induced by $\varphi$ is well defined, since $\varphi(\overline{w}_{n+1}), \varphi(\overline{w}_{n+2}) \in I_{2,n}$ (indeed, $\varphi(\overline{w}_{n+1}) = 0$ and $\varphi(\overline{w}_{n+2}) = w_1^{n+2}$ which is zero in $H^*(G_{2,n})$). This shows that $\phi(w_1) = 0$ and $\phi(w_2) = w_1^2$ defines an endomorphism $\phi : H^*(G_{2,n}) \to H^*(G_{2,n})$ if and only if $n = 2^{s+1} - 2$ for some $s \geqslant 1$.

*Case* $4^\circ$ $\alpha = 0$, $\beta_1 = 0$, $\beta_2 = 0$. In this case $\varphi$ induces the map $\phi$ that vanishes in positive dimensions.

We collect the results obtained in this section in the following theorem.

**Theorem 4.1** *Let $\phi : H^*(G_{2,n}) \to H^*(G_{2,n})$ be an endomorphism that commutes with Steenrod squares. If $n \neq 2^{s+1} - 2$ for $s \in \mathbb{N}$, then $\phi$ is the identity map or it vanishes in positive dimensions. If $n = 2^{s+1} - 2$ for some $s \in \mathbb{N}$, then we have the following two additional possibilities:*

*(1) $\phi$ is defined with $\phi(w_1) = w_1$ and $\phi(w_2) = 0$,*

*(2) $\phi$ is defined with $\phi(w_1) = 0$ and $\phi(w_2) = w_1^2$.*

## 5 The case $k = 3$

Let $s \geqslant 2$ be the unique positive integer such that $2^s < n+3 \leqslant 2^{s+1}$ (we assume that $n \geqslant 3$). For $k = 3$ the equation (2.2) becomes:

$$\overline{w}_t = \sum_{a+2b+3c=t} \binom{a+b+c}{a} \binom{b+c}{b} w_1^a w_2^b w_3^c, \tag{5.1}$$

while (2.3) gives:

$$\mathrm{ht}(w_1) = \begin{cases} 2^{s+1} - 2, & \text{if } n = 2^s - 2, \\ 2^{s+1} - 1, & \text{otherwise.} \end{cases} \tag{5.2}$$

Let $\varphi : \mathbb{Z}_2[w_1, w_2, w_3] \to \mathbb{Z}_2[w_1, w_2, w_3]$ be an endomorphism that induces an endomorphism $\phi : H^*(G_{3,n}) \to H^*(G_{3,n})$ which commutes with Steenrod squares. The mapping $\varphi$ satisfies relation $\varphi(I_{3,n}) \subseteq I_{3,n}$. Therefore we have

$$\begin{aligned} \varphi(w_1) &= \alpha_1 w_1, \\ \varphi(w_2) &= \beta_1 w_2 + \beta_2 w_1^2, \\ \varphi(w_3) &= \gamma_1 w_3 + \gamma_2 w_1 w_2 + \gamma_3 w_1^3, \end{aligned}$$

for some $\alpha_1,\beta_1,\beta_2,\gamma_1,\gamma_2,\gamma_3\in\mathbb{Z}_2$, and

$$\varphi(\overline{w}_{n+1})=\lambda_1\overline{w}_{n+1}, \tag{5.3}$$
$$\varphi(\overline{w}_{n+2})=\lambda_2\overline{w}_{n+2}+\lambda_{1,1}w_1\overline{w}_{n+1}, \tag{5.4}$$
$$\varphi(\overline{w}_{n+3})=\lambda_3\overline{w}_{n+3}+\lambda_{1,2}w_1\overline{w}_{n+2}+\lambda_{1,1,1}w_1^2\overline{w}_{n+1}+\lambda_{2,1}w_2\overline{w}_{n+1}, \tag{5.5}$$

for some $\lambda_1,\lambda_2,\lambda_3,\lambda_{1,1},\lambda_{1,2},\lambda_{2,1},\lambda_{1,1,1}\in\mathbb{Z}_2$.

In this case we have the condition (3.1) for $(i,j)\in\{(1,2),(1,3),(2,3)\}$. For $(i,j)=(1,2)$, formula (3.3) leads to:

$$(\alpha_1\beta_1+\gamma_2)w_1w_2+(\alpha_1\beta_2+\gamma_3)w_1^3+\gamma_1w_3=\beta_1w_1w_2+\beta_1w_3,$$

which, by Theorem 2.2, yields:

$$\beta_1=\gamma_1,\quad \beta_1=\alpha_1\beta_1+\gamma_2,\quad 0=\alpha_1\beta_2+\gamma_3. \tag{5.6}$$

In a similar way we can write (3.1) for $(i,j)=(1,3)$ and $(i,j)=(2,3)$. It turns out that $(i,j)=(1,3)$ does not lead to any new relations, while $(i,j)=(2,3)$ gives us the following additional relation:

$$\beta_1(1+\alpha_1+\beta_2)=0 \tag{5.7}$$

(for calculating $\mathrm{Sq}^2$ we use (3.3), (3.4) and (3.5)).

We now divide our proof into several cases. Two main are: $1^\circ$ $\alpha_1=0$ and $2^\circ$ $\alpha_1=1$.

*Case* $1^\circ$ $\alpha_1=0$. Then (5.6) and (5.7) simplify to $\beta_1=\gamma_1=\gamma_2$, $\gamma_3=0$ and $\beta_1(1+\beta_2)=0$. We now divide our proof into two subcases: $1.1^\circ$ $\beta_1=0$ and $1.2^\circ$ $\beta_1=1$.

*Subcase* $1.1^\circ$ $\beta_1=0$. Then $\gamma_1=\gamma_2=\gamma_3=0$, which amounts to:

$$\varphi(w_1)=0,\quad \varphi(w_2)=\beta_2w_1^2,\quad \varphi(w_3)=0.$$

If $\beta_2=0$, then $\varphi$ vanishes in positive dimensions. So, assume that $\beta_2=1$. Then

$$\varphi(w_1)=0,\quad \varphi(w_2)=w_1^2,\quad \varphi(w_3)=0,$$

which together with (5.1) gives:

$$\begin{aligned}\varphi(\overline{w}_{n+1})&=\sum_{a+2b+3c=n+1}\binom{a+b+c}{a}\binom{b+c}{b}\varphi(w_1)^a\varphi(w_2)^b\varphi(w_3)^c\\&=\sum_{2b=n+1}\varphi(w_2)^b=\begin{cases}0, & 2\nmid n+1,\\ w_1^{n+1}, & 2\mid n+1.\end{cases}\end{aligned}$$

Suppose that $2\mid n+1$. Then the monomial $w_2^{(n+1)/2}$ appears in $\overline{w}_{n+1}$ (by (5.1)), so from (5.3) we conclude that $\lambda_1=0$ and hence $\varphi(\overline{w}_{n+1})=w_1^{n+1}=0$, a contradiction.

Therefore, $n+1$ must be odd. Then, similarly as for $\varphi(\overline{w}_{n+1})$ we conclude that $\varphi(\overline{w}_{n+2})=w_1^{n+2}\in I_{3,n}$. So, $w_1^{n+2}=0$ in $H^*(G_{3,n})$, and hence $\mathrm{ht}(w_1)\leqslant n+1$. If $n=2^s-2$, then, by (5.2), $\mathrm{ht}(w_1)=2^{s+1}-2>2^s-1=n+1$, a contradiction; if $n>2^s-2$, then, again by (5.2), $\mathrm{ht}(w_1)=2^{s+1}-1\geqslant n+2$, a contradiction.

In conclusion, there is no endomorphism $\phi:H^*(G_{3,n})\to H^*(G_{3,n})$ which commutes with Steenrod squares and such that $\phi(w_1)=0$, $\phi(w_2)=w_1^2$, $\phi(w_3)=0$.

*Subcase* 1.2° $\beta_1 = 1$. Then $\beta_2 = 1$, $\gamma_1 = \gamma_2 = 1$ and $\gamma_3 = 0$, i.e.

$$\varphi(w_1) = 0, \quad \varphi(w_2) = w_2 + w_1^2, \quad \varphi(w_3) = w_3 + w_1 w_2. \tag{5.8}$$

Then the equations (5.1) and (5.3) yield

$$\varphi(\overline{w}_{n+1}) = \sum_{2b+3c=n+1} \binom{b+c}{b} (w_2 + w_1^2)^b (w_3 + w_1 w_2)^c = \lambda_1 \overline{w}_{n+1}. \tag{5.9}$$

First, let us assume that $n$ is even. Then the coefficient of $w_1^{n+1}$ on the left-hand side of (5.9) is 0 (monomial $w_1^{n+1}$ does not appear on the left-hand side), while it is equal to $\lambda_1$ on the right-hand side. So, $\lambda_1 = 0$.

Let us now observe the monomial $w_2^b w_3^c$ for $b, c \geqslant 0$ such that $2b+3c = n+1$ in the identity (5.9). The coefficient of this monomial on the left-hand side of this identity is $\binom{b+c}{b}$, which implies that $\binom{b+c}{b} \equiv 0 \pmod 2$ for all non-negative integers $b$ and $c$ such that $2b+3c = n+1$. By Lemma 7.3, this implies $n = 2^{s+1} - 4$.

Finally, we prove that for $n = 2^{s+1} - 4$ , we have $\varphi(\overline{w}_{n+i}) \in I_{3,n}$ for $i \in \{1, 2, 3\}$ and consequently that $\varphi$ induces desired endomorphism $\phi$. In fact, we prove that the following holds:

$$\varphi(\overline{w}_{n+1}) = 0, \quad \varphi(\overline{w}_{n+2}) = \overline{w}_{n+2}, \quad \varphi(\overline{w}_{n+3}) = \overline{w}_{n+3} + w_1 \overline{w}_{n+2}.$$

Our previous arguments prove that the first identity holds. We prove the remaining two by induction on $s \geqslant 2$. For $s = 2$ this can be checked directly. So, we assume that they are correct for $s - 1 \geqslant 2$ and prove them for $s$.

By Lemma 7.2 and inductional hypothesis we have

$$\begin{aligned}
\varphi(\overline{w}_{2^{s+1}-2}) &= \varphi(\overline{w}_{2^s-1}^2 + w_2 \overline{w}_{2^s-2}^2) = \varphi(\overline{w}_{2^s-1})^2 + \varphi(w_2)\varphi(\overline{w}_{2^s-2})^2 \\
&= (\overline{w}_{2^s-1} + w_1 \overline{w}_{2^s-2})^2 + (w_2 + w_1^2)\overline{w}_{2^s-2}^2 \\
&= \overline{w}_{2^s-1}^2 + w_2 \overline{w}_{2^s-2}^2 = \overline{w}_{2^{s+1}-2}
\end{aligned}$$

and

$$\begin{aligned}
\varphi(\overline{w}_{2^{s+1}-1}) &= \varphi(w_1 \overline{w}_{2^s-1}^2 + w_3 \overline{w}_{2^s-2}^2) = \varphi(w_1)\varphi(\overline{w}_{2^s-1})^2 + \varphi(w_3)\varphi(\overline{w}_{2^s-2})^2 \\
&= (w_3 + w_1 w_2)\overline{w}_{2^s-2}^2 = w_1 \overline{w}_{2^s-1}^2 + w_3 \overline{w}_{2^s-2}^2 + w_1(\overline{w}_{2^s-1}^2 + w_2 \overline{w}_{2^s-2}^2) \\
&= \overline{w}_{2^{s+1}-1} + w_1 \overline{w}_{2^{s+1}-2}.
\end{aligned}$$

Next, let us consider the case when $n$ is odd. Then $2 \mid n+1$. Hence, the coefficient of $w_2^{(n+1)/2}$ on the left-hand side of (5.9) is equal to 1 (for $b = \frac{n+1}{2}$ and $c = 0$), while on the right-hand side it is $\lambda_1$. So, $\lambda_1 = 1$, i.e. $\varphi(\overline{w}_{n+1}) = \overline{w}_{n+1}$. Let us now observe the monomial $w_1^{n-1} w_2$ in the same identity. The coefficient of this monomial on the right-hand side is $\binom{n}{n-1} \equiv 1 \pmod 2$. On the other hand, its coefficient on the left-hand side is $\binom{(n+1)/2}{1}$ (indeed, since $2 \mid n+1$, we have $2 \mid c$, and hence the only summand that contains this monomial is $(w_2 + w_1^2)^{(n+1)/2}$). We conclude that $n = 4\ell + 1$ for some $\ell \in \mathbb{Z}$.

Now, the identities (5.1) and (5.4) yield

$$\varphi(\overline{w}_{n+2}) = \sum_{2b+3c=4\ell+3} \binom{b+c}{b} (w_2 + w_1^2)^b (w_3 + w_1 w_2)^c = \lambda_2 \overline{w}_{n+2} + \lambda_{1,1} w_1 \overline{w}_{n+1}. \tag{5.10}$$

In a similar way as above, by considering the coefficient of $w_2^{2\ell}w_3$ in this identity we obtain: $\lambda_2 = \binom{2\ell+1}{1} = 1$, and by considering the coefficient of $w_1w_2^{2\ell+1}$: $\binom{2\ell+2}{1}\lambda_2 + \lambda_{1,1} = \binom{2\ell+1}{1}$, i.e. $\lambda_{1,1} = 1$. We conclude that $\varphi(\overline{w}_{n+2}) = \overline{w}_{n+2} + w_1\overline{w}_{n+1}$.

Finally, we use identities (5.1) and (5.5) to obtain

$$\begin{aligned}\varphi(\overline{w}_{n+3}) &= \sum_{2b+3c=4\ell+4} \binom{b+c}{b}(w_2+w_1^2)^b(w_3+w_1w_2)^c \\ &= \lambda_3\overline{w}_{n+3} + \lambda_{1,2}w_1\overline{w}_{n+2} + \lambda_{1,1,1}w_1^2\overline{w}_{n+1} + \lambda_{2,1}w_2\overline{w}_{n+1}. \qquad (5.11)\end{aligned}$$

Now, by considering the coefficient of $w_1^{4\ell+4}$ in this identity we obtain:

$$\lambda_3 + \lambda_{1,2} + \lambda_{1,1,1} = 1;$$

by considering the coefficient of $w_2^{2\ell+2}$:

$$\lambda_3 + \lambda_{2,1} = 1;$$

by considering the coefficient of $w_1^{4\ell+1}w_3$:

$$\binom{4\ell+2}{1}\lambda_3 + \binom{4\ell+1}{1}\lambda_{1,2} + \binom{4\ell}{1}\lambda_{1,1,1} = 0,$$

i.e. $\lambda_{1,2} = 0$; by considering the coefficient of $w_1^{4\ell+2}w_2$:

$$\binom{4\ell+3}{1}\lambda_3 + \binom{4\ell+2}{1}\lambda_{1,2} + \binom{4\ell+1}{1}\lambda_{1,1,1} + \lambda_{2,1} = \binom{2\ell+2}{1},$$

i.e. $\lambda_3 + \lambda_{1,1,1} + \lambda_{2,1} = 0$. We conclude that $\lambda_3 = \lambda_{1,2} = 0$, $\lambda_{1,1,1} = \lambda_{2,1} = 1$, and hence $\varphi(\overline{w}_{n+3}) = (w_2 + w_1^2)\overline{w}_{n+1}$.

Next, we apply $\varphi$ on the identity (2.1) for $t = n$. We get:

$$\varphi(w_3)\varphi(\overline{w}_n) = \varphi(\overline{w}_{n+3}) + \varphi(w_1)\varphi(\overline{w}_{n+2}) + \varphi(w_2)\varphi(\overline{w}_{n+1}) = 0,$$

and hence $\varphi(\overline{w}_n) = 0$. Now, as in the case when $n$ is even, it can be shown that $\varphi(\overline{w}_n) = 0$ implies $n = 2^{s+1} - 3$, and that for $n = 2^{s+1} - 3$, $\varphi$ defined with (5.8) induces a desired endomorphism $\phi$.

*Case* 2° $\alpha_1 = 1$. Then (5.6) and (5.7) imply $\beta_1 = \gamma_1$, $\gamma_2 = 0$, $\beta_2 = \gamma_3$ and $\beta_1\beta_2 = 0$.

If $\beta_1 = 1$, then $\beta_2 = 0$, and hence $\varphi$ induces the identity map. So, let $\beta_1 = 0$. Then

$$\varphi(w_1) = w_1, \quad \varphi(w_2) = \beta_2w_1^2, \quad \varphi(w_3) = \beta_2w_1^3. \qquad (5.12)$$

*Subcase* 2.1° $\beta_2 = 0$. Then (5.12) yields

$$\varphi(w_1) = w_1, \quad \varphi(w_2) = 0, \quad \varphi(w_3) = 0.$$

Using (5.1), similarly as in the previous cases we conclude $\varphi(\overline{w}_{n+1}) = w_1^{n+1} \in I_{3,n}$, and hence $w_1^{n+1} = 0$ in $H^*(G_{3,n})$. This implies $\mathrm{ht}(w_1) \leqslant n$, which contradicts (5.2) (since $\mathrm{ht}(w_1) \geqslant 2^{s+1} - 2 \geqslant n+1$). Therefore, in this case $\varphi$ does not induce a desired endomorphism.

*Subcase* 2.2° $\beta_2 = 1$. Then (5.12) yields

$$\varphi(w_1) = w_1, \quad \varphi(w_2) = w_1^2, \quad \varphi(w_3) = w_1^3.$$

For a polynomial $p \in \mathbb{Z}_2[w_1, w_2, w_3]$ let $n(p)$ denote the number of its (nonzero) monomials modulo 2.

Then (5.1) implies $\varphi(\overline{w}_m) = n(\overline{w}_m)w_1^m$, for all $m \geqslant 0$. In particular, $\varphi(\overline{w}_{n+i}) = n(\overline{w}_{n+i})w_1^{n+i} \in I_{3,n}$ for $i \in \{1,2,3\}$. If $n(\overline{w}_{n+i}) = 1$ for some $i \in \{1,2\}$, then $w_1^{n+i} = 0$ in $H^*(G_{3,n})$. This implies $\mathrm{ht}(w_1) \leqslant n+i-1 \leqslant n+1$, which leads to a contradiction (as in 1.1°). So, $n(\overline{w}_{n+1}) = n(\overline{w}_{n+2}) = 0$, and hence, by Lemma 7.1, $n(\overline{w}_{n+3}) = 1$ and $n \equiv 1 \pmod 4$. As before, $n(\overline{w}_{n+3}) = 1$ implies $w_1^{n+3} = 0$ in $H^*(G_{3,n})$, and hence $\mathrm{ht}(w_1) \leqslant n+2$. On the other hand, since $n \neq 2^s - 2$ ($n$ is odd), we have $\mathrm{ht}(w_1) = 2^{s+1} - 1 \geqslant n+2$, and hence $n = 2^{s+1} - 3$.

For $n = 2^{s+1} - 3$ we have $\varphi(\overline{w}_{n+1}) = 0$, $\varphi(\overline{w}_{n+2}) = 0$ and $\varphi(\overline{w}_{n+3}) = w_1^{n+3} \in I_{3,n}$, so $\varphi$ induces an endomorphism that commutes with Steenrod squares.

We summarize results obtained in this section in the following theorem.

**Theorem 5.1** *Let $\phi : H^*(G_{3,n}) \to H^*(G_{3,n})$ be an endomorphism that commutes with Steenrod squares and which is not the identity map nor it vanishes in positive dimensions. Then*

*(1) $n = 2^{s+1} - 4$, for some $s \geqslant 2$, and $\phi$ is defined with $\phi(w_1) = 0$, $\phi(w_2) = w_2 + w_1^2$ and $\phi(w_3) = w_3 + w_1w_2$, or*

*(2) $n = 2^{s+1} - 3$, for some $s \geqslant 2$, and $\phi$ is defined with $\phi(w_1) = 0$, $\phi(w_2) = w_2 + w_1^2$ and $\phi(w_3) = w_3 + w_1w_2$, or*

*(3) $n = 2^{s+1} - 3$, for some $s \geqslant 2$, and $\phi$ is defined with $\phi(w_1) = w_1$, $\phi(w_2) = w_1^2$ and $\phi(w_3) = w_1^3$.*

**Remark 5.2** *As in [15], using the previous result and the Lefschetz fixed point theorem, it can be proven that for $n \equiv 0,2,4 \pmod 8$, $n \neq 2^{s+1}-4$, the Grassmannian $G_{3,n}$ has the fixed point property. However, this is only a special case of [5, Theorem 6].*

# 6 Projective endomorphisms

We say that an endomorphism $\phi : H^*(G_{k,n}) \to H^*(G_{k,n})$ is *projective* if its image is in $\mathbb{Z}[w_1]$ (we have borrowed this name from [6], where similar notion was defined for endomorphisms of complex flag manifolds).

In this section we classify projective endomorphisms of real Grassmaniann s for $k \geqslant 4$ and prove that each of them commutes with Steenrod squares. Throughout, let $s$ be the unique integer such that $2^s < n+k \leqslant 2^{s+1}$.

Let $\varphi : \mathbb{Z}_2[w_1, w_2, \ldots, w_k] \to \mathbb{Z}_2[w_1, w_2, \ldots, w_k]$ be an endomorphism that induces a projective endomorphism $\phi : H^*(G_{k,n}) \to H^*(G_{k,n})$. Then there are $\alpha_i \in \mathbb{Z}_2$, $1 \leqslant i \leqslant k$, such that

$$\varphi(w_i) = \alpha_i w_1^i.$$

By applying $\varphi$ on the identity (2.2) we conclude that there are $c_i \in \mathbb{Z}_2$ for $i \geqslant 0$, such that

$$\varphi(\overline{w}_m) = c_m w_1^m.$$

Let us consider the identity $\varphi(\overline{w}_{n+i}) = c_{n+i}w_1^{n+i}$ for $1 \leqslant i \leqslant k-1$. By (2.3), $\mathrm{ht}(w_1) = 2^{s+1}-1 \geqslant n+k-1$, and hence $w_1^{n+i} \notin I_{k,n}$ for every $1 \leqslant i \leqslant k-1$. Since $\varphi(\overline{w}_{n+i}) \in I_{k,n}$, this implies $c_{n+i} = 0$ for $1 \leqslant i \leqslant k-1$, i.e.

$$\varphi(\overline{w}_{n+1}) = \varphi(\overline{w}_{n+2}) = \cdots = \varphi(\overline{w}_{n+k-1}) = 0. \tag{6.1}$$

Suppose now that $\phi$ does not vanish in positive dimensions. Then $\varphi(w_i) \neq 0$ for some $1 \leqslant i \leqslant k$; let $j$ be the largest such $j$. We prove that $\varphi(\overline{w}_{n+k}) = w_1^{n+k}$. Suppose that this is not the case. Then $\varphi(\overline{w}_{n+k}) = 0$. Let us now apply $\varphi$ on (2.1). We get:

$$\varphi(w_j)\varphi(\overline{w}_m) = \varphi(w_{j-1})\varphi(\overline{w}_{m+1}) + \cdots + \varphi(w_1)\varphi(\overline{w}_{m+j-1}) + \varphi(\overline{w}_{m+j}).$$

Now, an easy reverse induction on $t$, $t \leqslant n+k$, shows that $\varphi(\overline{w}_t) = 0$ for all $0 \leqslant t \leqslant n+k$, which is not possible (since $\varphi(\overline{w}_0) = \varphi(1) = 1$).

So, $\varphi(\overline{w}_{n+k}) = w_1^{n+k}$. Since $\varphi(\overline{w}_{n+k}) = w_1^{n+k} \in I_{k,n}$, we have $n+k > \mathrm{ht}(w_1) = 2^{s+1}-1$, which implies $n+k = 2^{s+1}$.

Now, by applying $\varphi$ on (2.1) for $t = n$ we get

$$w_1^{n+k} = \varphi(\overline{w}_{n+k}) = \varphi(w_1)\varphi(\overline{w}_{n+k-1}) + \varphi(w_2)\varphi(\overline{w}_{n+k-2}) + \cdots + \varphi(w_k)\varphi(\overline{w}_n) = \varphi(w_k)\varphi(\overline{w}_n),$$

which implies $\varphi(w_k) \neq 0$ and $\varphi(\overline{w}_n) \neq 0$, i.e. $\varphi(w_k) = w_1^k$ ($\alpha_k = 1$) and $\varphi(\overline{w}_n) = w_1^n$ ($c_n = 1$).

Now we prove that $c_{n+k+i} = c_i$ for $i \geqslant 0$, i.e. that the sequence $\{c_i\}$ is periodic with period $n+k = 2^{s+1}$. We do this by induction on $i$. The case $i = 0$ is already proven above. So, we assume $c_{n+k+j} = c_j$ for $0 \leqslant j \leqslant i-1$ and prove $c_{n+k+i} = c_i$. If $i \leqslant k$, then this follows by applying $\varphi$ on (2.1) for $t = n+i$:

$$\begin{aligned} c_{n+k+i} &= \alpha_1 c_{n+k+i-1} + \cdots + \alpha_i c_{n+k} + \alpha_{i+1}c_{n+k-1} + \cdots + \alpha_k c_{n+i}, \\ &= \alpha_1 c_{i-1} + \cdots + \alpha_i c_0 = c_i. \end{aligned}$$

If $i > k$, then we again applying $\varphi$ on (2.1) for $t = n+i$:

$$\begin{aligned} c_{n+k+i} &= \alpha_1 c_{n+k+i-1} + \alpha_2 c_{n+k+i-2} + \cdots + \alpha_k c_{n+i}, \\ &= \alpha_1 c_{i-1} + \alpha_2 c_{i-2} + \cdots + \alpha_k c_{i-k} = c_i. \end{aligned}$$

As mentioned in Section 2, the identity (2.1) can be written as

$$(1 + w_1 + w_2 + \cdots + w_k)\sum_{t\geqslant 0} \overline{w}_t = 1. \tag{6.2}$$

We will apply $\varphi$ on this identity, but before we do that let us note that the periodicity of $\{c_i\}$ implies

$$\sum_{t\geqslant 0} \varphi(\overline{w}_t) = (c_0 + c_1 w_1 + c_2 w_1^2 + \cdots + c_{n+k-1}w_1^{n+k-1})\sum_{t\geqslant 0} w_1^{(n+k)t}.$$

So, if we denote $P(w_1) = \alpha_0 + \alpha_1 w_1 + \cdots + \alpha_k w_1^k$ and $Q(w_1) = c_0 + c_1 w_1 + \cdots + c_{n+k-1}w_1^{n+k-1} = c_0 + c_1 w_1 + \cdots + c_n w_1^n$, then by applying $\varphi$ on (6.2) we get:

$$1 = P(w_1)Q(w_1)\sum_{t\geqslant 0} w_1^{(n+k)t} = P(w_1)Q(w_1) + w_1^{n+k}P(w_1)Q(w_1)\sum_{t\geqslant 0} w_1^{(n+k)t}. \tag{6.3}$$

Since $\alpha_k = 1$ and $c_n = 1$, the degree of $P$ is $k$ and the degree of $Q$ is $n$, and hence the degree of $P(w_1)Q(w_1)$ is $n+k$. Since the coefficient of $w_1^{n+k}$ in $P(w_1)Q(w_1)$ is 1, (6.3) immediately implies

$$P(w_1)Q(w_1) = 1 + w_1^{n+k} = 1 + w_1^{2^{s+1}} = (1+w_1)^{2^{s+1}},$$

and hence $P(w_1) = (1+w_1)^k$ and $Q(w_1) = (1+w_1)^n$. So, $\alpha_i = \binom{k}{i}$ for $0 \leqslant i \leqslant k$.

Next, we prove that for $n+k = 2^{s+1}$, $\varphi$ defined with $\varphi(w_i) = \binom{k}{i} w_1^i$, for $1 \leqslant i \leqslant k$, induces an endomorphism of $H^*(G_{k,n})$. Note that the identity obtained by applying $\varphi$ on (6.2) uniquely defines $\varphi(\overline{w}_i)$ for $i \geqslant 0$. So,

$$P(w_1)Q(w_1)\sum_{t\geqslant 0} w_1^{(n+k)t} = (1+w_1)^{n+k}(1+w_1)^{-n-k} = 1,$$

implies

$$\sum_{i\geqslant 0}\varphi(\overline{w}_i) = Q(w_1)\sum_{t\geqslant 0} w_1^{(n+k)t},$$

and hence $\varphi(\overline{w}_{n+1}) = \varphi(\overline{w}_{n+2}) = \cdots = \varphi(\overline{w}_{n+k-1}) = 0$ and $\varphi(\overline{w}_{n+k}) = w_1^{n+k} = w_1^{2^{s+1}}$. Since $\mathrm{ht}(w_1) = 2^{s+1}-1$ (by (2.3)), we have $\varphi(\overline{w}_{n+k}) \in I_{k,n}$, so $\varphi$ defined in this way induces a projective endomorphism $\phi$ of $H^*(G_{k,n})$.

Finally, we prove that $\phi$ commutes with Steenrod squares. By (3.2), we have

$$\phi\left(\mathrm{Sq}^i(w_j)\right) = \phi\left(\sum_{t=0}^{i}\binom{j-i+t-1}{t} w_{i-t}w_{j+t}\right) = \sum_{t=0}^{i}\binom{j-i+t-1}{t}\binom{k}{i-t}\binom{k}{j+t} w_1^{i+j},$$

while

$$\mathrm{Sq}^i\left(\phi(w_j)\right) = \mathrm{Sq}^i\left(\binom{k}{j} w_1^j\right) = \binom{k}{j}\mathrm{Sq}^i(w_1^j).$$

Using (3.5), an easy induction on $j$ shows that $\mathrm{Sq}^i(w_1^j) = \binom{j}{i} w_1^{i+j}$, so, by (3.1), it is enough to prove

$$\sum_{t=0}^{i}\binom{j-i+t-1}{t}\binom{k}{i-t}\binom{k}{j+t} \equiv \binom{k}{j}\binom{j}{i} \pmod 2. \tag{6.4}$$

This follows from Lemma 7.4.

**Theorem 6.1** *Let $k \geqslant 4$. An endomorphism $\phi : H^*(G_{k,n}) \to H^*(G_{k,n})$, that does not vanish in positive dimensions, is projective if and only if $n+k = 2^{s+1}$ for some $s \geqslant 2$ and*

$$\phi(w_i) = \binom{k}{i} w_1^i \quad \text{for all } 1 \leqslant i \leqslant k.$$

*Further, in this case $\phi$ commutes with Steenrod squares.*

# 7 Proofs of some auxiliary results

In this section we collect and prove auxiliary results used in the proof of our main theorems. First, we prove two lemmas that hold in $\mathbb{Z}_2[w_1, w_2, w_3]$.

**Lemma 7.1** *In $\mathbb{Z}_2[w_1, w_2, w_3]$ we have:*

$$n(\overline{w}_m) = \begin{cases} 1, & \text{if } m \equiv 0, 1 \pmod 4 \\ 0, & \text{otherwise.} \end{cases}$$

PROOF — By (2.1) we have

$$n(\overline{w}_{m+3}) = n(\overline{w}_{m+2}) + n(\overline{w}_{m+1}) + n(\overline{w}_m) \quad \text{for all } m \geqslant 0.$$

Since $n(\overline{w}_0) = 1$, $n(\overline{w}_1) = 1$ and $n(\overline{w}_2) = 0$, the result is obtained by induction on $m$ using the previous recurrence relation. □

**Lemma 7.2** *In $\mathbb{Z}_2[w_1, w_2, w_3]$ the following identities hold for any positive integer $t$:*

$$\overline{w}_{2^{t+1}-2} = \overline{w}_{2^t-1}^2 + w_2\overline{w}_{2^t-2}^2,$$
$$\overline{w}_{2^{t+1}-1} = w_1\overline{w}_{2^t-1}^2 + w_3\overline{w}_{2^t-2}^2.$$

PROOF — Let us prove the first identity. Let $A = \{(a, b, c) \,:\, a, b, c \geqslant 0,\ a+2b+3c = 2^{t+1}-2\}$. Clearly, if $(a, b, c) \in A$, then $a + 3c$ is even, and hence $a$ and $c$ have the same parity. Next, we prove that if $(a, b, c) \in A$ and $a$ and $c$ are odd, then $\binom{a+b+c}{a}\binom{b+c}{b}$ is even. Suppose that this is not the case. Then $\binom{b+c}{c}$ is odd, and hence, by Lucas' theorem, $b$ is even. So, $a + b + c$ is even and $a$ odd, and again by Lucas' theorem $\binom{a+b+c}{a}$ is even, a contradiction.

So, if we set

$$A_1' = \{(a, b, c) \,:\, a, b, c \geqslant 0,\ a + 2b + 3c = 2^{t+1} - 2,\ 2 \mid a, b, c\},$$
$$A_1'' = \{(a, b, c) \,:\, a, b, c \geqslant 0,\ a + 2b + 3c = 2^{t+1} - 2,\ 2 \mid a, c,\ 2 \nmid b\},$$

then by (5.1):

$$\overline{w}_{2^{t+1}-2} = \sum_{(a,b,c)\in A} \binom{a+b+c}{a}\binom{b+c}{b} w_1^a w_2^b w_3^c = \sum_{(a,b,c)\in A_1'\cup A_1''} \binom{a+b+c}{a}\binom{b+c}{b} w_1^a w_2^b w_3^c.$$

Note that for each $(a, b, c) \in A_1'$ we have $a = 2a'$, $b = 2b'$, $c = 2c'$, where $a', b', c'$ are non-negative integers such that $a' + 2b' + 3c' = 2^t - 1$ and $\binom{a+b+c}{a}\binom{b+c}{b} = \binom{2(a'+b'+c')}{2a'}\binom{2(b'+c')}{2b'} = \binom{a'+b'+c'}{a'}\binom{b'+c'}{b'}$ (by Lucas' theorem); similarly, for each $(a, b, c) \in A_1''$ we have $a = 2a'$, $b = 2b' + 1$, $c = 2c'$, where $a', b', c'$ are non-negative integers such that $a' + 2b' + 3c' = 2^t - 2$ and $\binom{a+b+c}{a}\binom{b+c}{b} = \binom{2(a'+b'+c')+1}{2a'}\binom{2(b'+c')+1}{2b'+1} = \binom{a'+b'+c'}{a'}\binom{b'+c'}{b'}$ (by Lucas' theorem). So,

$$\begin{aligned}
\overline{w}_{2^{t+1}-2} &= \sum_{a'+2b'+3c'=2^t-1} \binom{a'+b'+c'}{a'}\binom{b'+c'}{b'} w_1^{2a'} w_2^{2b'} w_3^{2c'} \\
&\qquad + \sum_{a'+2b'+3c'=2^t-2} \binom{a'+b'+c'}{a'}\binom{b'+c'}{b'} w_1^{2a'} w_2^{2b'+1} w_3^{2c'} \\
&= \left(\sum_{a'+2b'+3c'=2^t-1} \binom{a'+b'+c'}{a'}\binom{b'+c'}{b'} w_1^{a'} w_2^{b'} w_3^{c'}\right)^2 \\
&\qquad + w_2\left(\sum_{a'+2b'+3c'=2^t-2} \binom{a'+b'+c'}{a'}\binom{b'+c'}{b'} w_1^{a'} w_2^{b'} w_3^{c'}\right)^2 \\
&= \overline{w}_{2^t-1}^2 + w_2\overline{w}_{2^t-2}^2.
\end{aligned}$$

We prove the second identity in a similar way as the first one. Let $B = \{(a,b,c) : a,b,c \geqslant 0,\ a+2b+3c = 2^{t+1}-1\}$. Clearly, if $(a,b,c) \in B$, then $a+3c$ is odd, and hence $a$ and $c$ have different parity. Next, let $(a,b,c) \in B$ be such that $\binom{a+b+c}{a}\binom{b+c}{b}$ is odd. We prove that $b$ is even. Suppose that this is not the case. Since $\binom{b+c}{b}$ is odd, then by Lucas' theorem $c$ must be even, and hence $a$ is odd. This implies $2 \mid a+b+c$, and hence $\binom{a+b+c}{a}$ is even (by Lucas' theorem), a contradiction.

So, if we set

$$\begin{aligned} B_1' &= \{(a,b,c) : a,b,c \geqslant 0,\ a+2b+3c = 2^{t+1}-1,\ 2 \mid a,b,\ 2 \nmid c\}, \\ B_1'' &= \{(a,b,c) : a,b,c \geqslant 0,\ a+2b+3c = 2^{t+1}-1,\ 2 \mid b,c,\ 2 \nmid a\}, \end{aligned}$$

then by (5.1):

$$\overline{w}_{2^{t+1}-1} = \sum_{(a,b,c)\in B} \binom{a+b+c}{a}\binom{b+c}{b} w_1^a w_2^b w_3^c = \sum_{(a,b,c)\in B_1'\cup B_1''} \binom{a+b+c}{a}\binom{b+c}{b} w_1^a w_2^b w_3^c.$$

Note that for each $(a,b,c) \in B_1'$ we have $a = 2a'$, $b = 2b'$, $c = 2c'+1$, where $a', b', c'$ are non-negative integers such that $a'+2b'+3c' = 2^t-2$ and $\binom{a+b+c}{a}\binom{b+c}{b} = \binom{2(a'+b'+c')+1}{2a'}\binom{2(b'+c')+1}{2b'} = \binom{a'+b'+c'}{a'}\binom{b'+c'}{b'}$ (by Lucas' theorem); similarly, for each $(a,b,c) \in B_1''$ we have $a = 2a'+1$, $b = 2b'$, $c = 2c'$, where $a', b', c'$ are non-negative integers such that $a'+2b'+3c' = 2^t-1$ and $\binom{a+b+c}{a}\binom{b+c}{b} = \binom{2(a'+b'+c')+1}{2a'+1}\binom{2(b'+c')}{2b'} = \binom{a'+b'+c'}{a'}\binom{b'+c'}{b'}$ (by Lucas' theorem). So,

$$\begin{aligned} \overline{w}_{2^{t+1}-1} &= \sum_{a'+2b'+3c'=2^t-2} \binom{a'+b'+c'}{a'}\binom{b'+c'}{b'} w_1^{2a'} w_2^{2b'} w_3^{2c'+1} \\ &\qquad + \sum_{a'+2b'+3c'=2^t-1} \binom{a'+b'+c'}{a'}\binom{b'+c'}{b'} w_1^{2a'+1} w_2^{2b'} w_3^{2c'} \\ &= w_3 \left( \sum_{a'+2b'+3c'=2^t-2} \binom{a'+b'+c'}{a'}\binom{b'+c'}{b'} w_1^{a'} w_2^{b'} w_3^{c'} \right)^2 \\ &\qquad + w_1 \left( \sum_{a'+2b'+3c'=2^t-1} \binom{a'+b'+c'}{a'}\binom{b'+c'}{b'} w_1^{a'} w_2^{b'} w_3^{c'} \right)^2 \\ &= w_3 \overline{w}_{2^t-2}^2 + w_1 \overline{w}_{2^t-1}^2. \end{aligned}$$

This completes the proof of the lemma. □

In the remainder of this section, we prove two arithmetic lemma.

**Lemma 7.3** *Let $m$ be a positive integer. Then*

$$\binom{b+c}{b} \equiv 0 \pmod 2$$

*for all non-negative integers $b$ and $c$ such that $2b+3c = m$ if and only if $m = 2^t-3$ for some $t \geqslant 2$.*

PROOF — The "if" part of this lemma is proven in [4, Proposition 3.2].

To prove the "only if" part, we prove that for every $t \geqslant 3$ and every $m$ which satisfies $2^{t-1} < m+3 < 2^t$ there are $b, c \geqslant 0$ such that $2b + 3c = m$ and $\binom{b+c}{b} \equiv 1 \pmod 2$. Our proof is by induction on $t \geqslant 3$. For $t = 3$ this is checked directly. So, suppose that it is true for $t \geqslant 3$ and prove it for $t+1$. Let $m$ be such that $2^t < m+3 < 2^{t+1}$. If $m$ is even, then for $b = m/2$ and $c = 0$ we have $\binom{b+c}{b} \equiv 1 \pmod 2$. So, let $m$ be odd and define $m' = \frac{m-3}{2}$. Then $2^{t-1} < m'+3 < 2^t$ and hence, by inductive hypothesis, there are $b', c' \geqslant 0$ such that $2b' + 3c' = m'$ and $\binom{b'+c'}{b'} \equiv 1 \pmod 2$. So, for $b = 2b'$ and $c = 2c'+1$ we have $2b + 3c = 2m' + 3 = m$ and $\binom{b+c}{b} = \binom{2(b'+c')+1}{2b'} \equiv \binom{b'+c'}{b'} \equiv 1 \pmod 2$, which completes our proof. □

**Lemma 7.4** *For all non-negative integers $k$, $i$ and $j$:*

$$\sum_{t=0}^{i} \binom{j-i+t-1}{t}\binom{k}{i-t}\binom{k}{j+t} \equiv \binom{k}{j}\binom{j}{i} \pmod 2. \tag{7.1}$$

PROOF — Our proof is by induction on $k$. Base case $k = 0$ is trivial. So, we assume that (7.1) holds for all $k' < k$ and prove it for $k$. We distinguish between two cases. In both cases we denote the left-hand side of (7.1) with $L$.

*Case* 1° $k = 2k'$ is even ($k' \in \mathbb{N}$). Let us observe an odd summand of $L$. Then, by Lucas' theorem, $i - t$ and $j + t$ are even, so $i, j, t$ are of the same parity. So, $j - i$ is even, and, by Lucas' theorem, $\binom{j-i+t-1}{t} \equiv 1 \pmod 2$ implies that $t$ is even (and hence $i$ and $j$ are also even). So, if $i$ or $j$ is odd, then $L$ is even, and so is the right-hand side (if $j$ is odd, then $\binom{k}{j}$ is even; if $j$ is even, and $i$ odd, then $\binom{j}{i}$ is even). Hence, we may assume that $i = 2i'$, $j = 2j'$ and $t = 2t'$, for some $i', j', t' \in \mathbb{N}_0$, $t' \leqslant i'$. Hence, By Lucas' theorem,

$$\begin{aligned} L &\equiv \sum_{t'=0}^{i'} \binom{2(j'-i'+t'-1)+1}{2t'}\binom{2k'}{2(i'-t')}\binom{2k'}{2(j'+t')} \\ &\equiv \sum_{t'=0}^{i'} \binom{j'-i'+t'-1}{t'}\binom{k'}{i'-t'}\binom{k'}{j'+t'} \pmod 2, \end{aligned}$$

and

$$\binom{k}{j}\binom{j}{i} = \binom{2k'}{2j'}\binom{2j'}{2i'} \equiv \binom{k'}{j'}\binom{j'}{i'} \pmod 2,$$

so (7.1) follows by inductive hypothesis.

*Case* 2° $k = 2k'+1$ is odd ($k' \in \mathbb{N}_0$). Let us first assume that $i$ and $j$ have the same parity, i.e. $i = 2i' + \delta$ and $j = 2j' + \delta$ for some $i', j' \in \mathbb{N}_0$ and $\delta \in \{0, 1\}$. We now proceed similarly as in Case 1. Let us consider an odd summand of $L$. Since $\binom{j-i+t-1}{t} = \binom{2(j'-i')+t-1}{t}$ is odd, by Lucas' theorem $t$ is even, and hence $t = 2t'$ for some $0 \leqslant t' \leqslant i'$. Now, again by Lucas' theorem

$$\begin{aligned} L &\equiv \sum_{t'=0}^{i'} \binom{2(j'-i'+t'-1)+1}{2t'}\binom{2k'+1}{2(i'-t')+\delta}\binom{2k'+1}{2(j'+t')+\delta} \\ &\equiv \sum_{t'=0}^{i'} \binom{j'-i'+t'-1}{t'}\binom{k'}{i'-t'}\binom{k'}{j'+t'} \pmod 2, \end{aligned}$$

and

$$\binom{k}{j}\binom{j}{i} = \binom{2k'+1}{2j'+\delta}\binom{2j'+\delta}{2i'+\delta} \equiv \binom{k'}{j'}\binom{j'}{i'} \pmod 2.$$

Hence (7.1) follows by inductive hypothesis.

So, we may assume that exactly one of $i$ and $j$ is odd. First, let us assume that $i = 2i'+1$ and $j = 2j'$ for some $i', j' \in \mathbb{N}_0$. Then, by Lucas' theorem,

$$\binom{k}{j}\binom{j}{i} = \binom{2k'+1}{2j'}\binom{2j'}{2i'+1} \equiv 0 \pmod 2.$$

To obtain $L$ modulo 2, we divide the sum into two parts, for $t$ even and $t$ odd. Using Lucas' theorem we have:

$$\begin{aligned} L &= \sum_{t'=0}^{i'} \binom{2(j'-i'+t'-1)}{2t'}\binom{2k'+1}{2(i'-t')+1}\binom{2k'+1}{2(j'+t')} \\ &\quad + \sum_{t'=0}^{i'} \binom{2(j'-i'+t'-1)+1}{2t'+1}\binom{2k'+1}{2(i'-t')}\binom{2k'+1}{2(j'+t')+1} \\ &\equiv 2\sum_{t'=0}^{i'} \binom{j'-i'+t'-1}{t'}\binom{k'}{i'-t'}\binom{k'}{j'+t'} \equiv 0 \pmod 2. \end{aligned}$$

Finally, we consider the case when $i = 2i'$ and $j = 2j'+1$ for some $i', j' \in \mathbb{N}_0$. Then, by Lucas' theorem,

$$\binom{k}{j}\binom{j}{i} = \binom{2k'+1}{2j'+1}\binom{2j'+1}{2i'} \equiv \binom{k'}{j'}\binom{j'}{i'} \pmod 2.$$

To obtain $L$ modulo 2, we again divide the sum into two parts, for $t$ even and $t$ odd. Using Lucas' theorem

$$\begin{aligned} L &= \sum_{t'=0}^{i'} \binom{2(j'-i'+t')}{2t'}\binom{2k'+1}{2(i'-t')}\binom{2k'+1}{2(j'+t')+1} \\ &\quad + \sum_{t'=0}^{i'} \binom{2(j'-i'+t')+1}{2t'+1}\binom{2k'+1}{2(i'-t'-1)+1}\binom{2k'+1}{2(j'+t'+1)} \\ &\equiv \sum_{t'=0}^{i'} \binom{j'-i'+t'}{t'}\binom{k'}{i'-t'}\binom{k'}{j'+t'} + \sum_{t'=0}^{i'} \binom{j'-i'+t'}{t'}\binom{k'}{i'-t'-1}\binom{k'}{j'+t'+1} \\ &\equiv \sum_{t'=0}^{i'} \left(\binom{j'-i'+t'}{t'} + \binom{j'-i'+t'-1}{t'-1}\right)\binom{k'}{i'-t'}\binom{k'}{j'+t'} \\ &\equiv \sum_{t'=0}^{i'} \binom{j'-i'+t'-1}{t'}\binom{k'}{i'-t'}\binom{k'}{j'+t'} \pmod 2, \end{aligned}$$

and hence (7.1) follows by inductive hypothesis. □

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