Characteristic rank of canonical vector bundles over oriented Grassmann manifolds $\widetilde{G}_{3,n}$

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Abstract

We determine the characteristic rank of the canonical oriented vector bundle over $\widetilde{G}_{3,n}$ for all $n \geq 3$, and as a consequence, we obtain the affirmative answer to a conjecture of Korbaš and Rusin. As an application of this result, we calculate the $\mathbb{Z}_2$-cup-length for a new infinite family of manifolds $\widetilde{G}_{3,n}$. This result confirms the corresponding claim of Fukaya's conjecture.

1 Introduction

The characteristic rank of a real vector bundle α over a d-dimensional CW-complex X, denoted by $\operatorname{charrank}(\alpha)$, is defined in [9] as the maximal integer $q \in \{0, 1, \ldots, d\}$ such that all cohomology classes in $H^j(X; \mathbb{Z}_2)$ for $0 \leq j \leq q$ are polynomials in Stiefel–Whitney classes $w_1(\alpha), w_2(\alpha), \ldots$ of the bundle α.

There has been much work done recently in studying the characteristic rank of various vector bundles (see [5, 6, 9]), and especially the canonical vector bundle $\widetilde{\gamma}_{k,n}$ over Grassmann manifold $\widetilde{G}_{k,n}$ $(k \leq n)$ of oriented k-dimensional subspaces in $\mathbb{R}^{n+k}$ (see [4, 7, 8, 12]). The majority of the obtained results pertains to the case $k = 3$. As the main result of this paper, we determine the exact value of $\operatorname{charrank}(\widetilde{\gamma}_{3,n})$ for all $n \geq 3$. This is stated in the following theorem, which is proven in Section 3. (In the rest of the paper we assume that $n \geq 3$.)

Theorem 1.1 *If $t \geq 3$ is the unique integer such that $2^{t-1} \leq n + 3 < 2^t$, then*

$$\operatorname{charrank}(\widetilde{\gamma}_{3,n}) = \min\{3n - 2^t + 7, 2^t - 5\}.$$

The technique of the proof combines the method of Korbaš and Rusin used in [7, 12] (for obtaining the lower bound for $\operatorname{charrank}(\widetilde{\gamma}_{3,n})$) with Gröbner bases for "unoriented" Grassmann manifolds $G_{3,n}$ constructed in [11] (for obtaining the upper bound for $\operatorname{charrank}(\widetilde{\gamma}_{3,n})$).

An immediate corollary of Theorem 1.1 is the positive answer to Conjecture 3.3 from [7] (see Remark 3.5).

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The $\mathbb{Z}_2$-cup-length of a path connected space X, denoted by $\operatorname{cup}(X)$, is defined as the maximal r such that there exist classes $x_1, x_2, \ldots, x_r \in \widetilde{H}^*(X;\mathbb{Z}_2)$ with nontrivial cup product ($x_1 x_2 \cdots x_r \neq 0$). In [2] Fukaya studied the $\mathbb{Z}_2$-cup-length of Grassmanians $\widetilde{G}_{3,n}$ and conjectured [2, Conjecture 1.2] that for n in approximately the first half of the interval $[2^{t-1}-4, 2^t-4)$ (where $t \geq 4$) this cup-length is equal to $2^{t-1}-3$. In that paper he obtains Gröbner bases for certain ideals related to Grassmanians $\widetilde{G}_{3,2^{t-1}-4}$ and uses them to verify this conjecture for $n = 2^{t-1}-4$. Later, in [4, 7] Korbaš and Rusin verified it for $n \in \{2^{t-1}-3, 2^{t-1}-2, 2^{t-1}-1, 2^{t-1}\}$ ($t \geq 5$ in the case $n = 2^{t-1}$). We prove that the conjecture is true for all n in the first third of the interval $[2^{t-1}-4, 2^t-4)$. This result is stated in the following theorem, the proof of which occupies Section 4 of the paper.

Theorem 1.2 *Let $t \geq 3$ be the unique integer such that $2^{t-1}-4 \leq n < 2^t-4$. If $n \leq 2^{t-1}-4+\frac{2^{t-1}}{3}$, then:*

(a) $\operatorname{cup}(\widetilde{G}_{3,n}) = 2^{t-1}-3$;

(b) $\operatorname{ht}(w_2(\widetilde{\gamma}_{3,n})) = 2^{t-1}-4$.

In the theorem, $\operatorname{ht}(w_2(\widetilde{\gamma}_{3,n}))$ is, as usual, the height of the class $w_2(\widetilde{\gamma}_{3,n})$, that is, the maximal m such that $w_2(\widetilde{\gamma}_{3,n})^m \neq 0$.

2 Cohomology of Grassmann manifolds

In this paper, all cohomology groups are assumed to have coefficients in $\mathbb{Z}_2$.

Let $G_{3,n}$ be the Grassmann manifold of unoriented three-dimensional subspaces in $\mathbb{R}^{n+3}$. By the Borel description [1], the cohomology algebra $H^*(G_{3,n})$ is isomorphic to the quotient

$$\mathbb{Z}_2[w_1, w_2, w_3]/J_{3,n},$$

where $J_{3,n} = (\overline{w}_{n+1}, \overline{w}_{n+2}, \overline{w}_{n+3})$ is the ideal generated by dual classes. Hence, these dual classes are polynomials in variables w_1, w_2 and w_3, and they are obtained from the relation

$$(1 + w_1 + w_2 + w_3)(1 + \overline{w}_1 + \overline{w}_2 + \cdots) = 1.$$

In the above isomorphism the classes of the variables w_1, w_2 and w_3 correspond to the Stiefel–Whitney classes of the canonical vector bundle $\gamma_{3,n}$ over $G_{3,n}$ (which, by an abuse of notation, we also denote by w_1, w_2 and w_3).

The proof of the following theorem can be found in [3].

Theorem 2.1 *The set $D_{3,n} = \{w_1^a w_2^b w_3^c : a+b+c \leq n\}$ is a vector space basis for $H^*(G_{3,n})$.*

In [11] a Gröbner basis for the ideal $J_{3,n}$ is obtained. It consists of the polynomials $\widetilde{g}_{m,l}$, indexed by the pairs of nonnegative integers (m,l) such that $m+l \leq n+1$. They are defined by

$$\widetilde{g}_{m,l} := \sum_{a+2b+3c=n+1+m+2l} \binom{a+b+c-m-l}{a}\binom{b+c-l}{b} w_1^a w_2^b w_3^c, \tag{1}$$

where the sum is taken over all triples (a,b,c) of nonnegative integers such that $a+2b+3c = n+1+m+2l$ ([11, p. 80]; see also [10, p. 3]). It is clear that the monomial $w_1^{n+1-m-l} w_2^m w_3^l$ appears in $\widetilde{g}_{m,l}$ with coefficient 1, and it is a fact ([11, Proposition 5]; see also [10, Proposition 2.5]) that all other monomials of $\widetilde{g}_{m,l}$ have degree (sum of the exponents) at most n. Since $\widetilde{g}_{m,l}=0$ in $H^*(G_{3,n})$, this means that (1) gives us the representation of the monomial $w_1^{n+1-m-l}w_2^m w_3^l$ in the additive basis $D_{3,n}$:

$$w_1^{n+1-m-l}w_2^m w_3^l=\sum_{\substack{a+2b+3c=n+1+m+2l\\(a,b,c)\neq(n+1-m-l,m,l)}}\binom{a+b+c-m-l}{a}\binom{b+c-l}{b}w_1^a w_2^b w_3^c. \tag{2}$$

This is an equality in $H^{n+1+m+2l}(G_{3,n})$, and in the rest of the paper, we say that an element of $H^j(G_{3,n})$ (or $H^j(\widetilde{G}_{3,n})$) has dimension j.

The obvious map $p:\widetilde{G}_{3,n}\to G_{3,n}$ (which forgets the orientation of a three-dimensional subspace in $\mathbb{R}^{n+3}$) is a two-sheeted covering map, and it is well known that the associated Gysin exact sequence is of the form

$$\cdots\xrightarrow{w_1}H^j(G_{3,n})\xrightarrow{p^*}H^j(\widetilde{G}_{3,n})\to H^j(G_{3,n})\xrightarrow{w_1}H^{j+1}(G_{3,n})\xrightarrow{p^*}\cdots, \tag{3}$$

where $H^j(G_{3,n})\xrightarrow{w_1}H^{j+1}(G_{3,n})$, $j\geq 0$, is the homomorphism given with $\sigma\mapsto w_1\sigma$, $\sigma\in H^j(G_{3,n})$.

The canonical bundle $\gamma_{3,n}$ over $G_{3,n}$ pulls back via p to the canonical bundle $\widetilde{\gamma}_{3,n}$ over $\widetilde{G}_{3,n}$, and therefore, $w_i(\widetilde{\gamma}_{3,n})=p^*w_i$, $i=1,2,3$. Since every cohomology class in $H^*(G_{3,n})$ is a polynomial in w_1, w_2 and w_3, we have that $\operatorname{charrank}(\widetilde{\gamma}_{3,n})\geq q$ if and only if $p^*:H^j(G_{3,n})\to H^j(\widetilde{G}_{3,n})$ is onto for all $j\in\{0,1,\ldots,q\}$ (where $0\leq q\leq 3n=\dim\widetilde{G}_{3,n}$). The following equivalence is now straightforward from the exactness of sequence (3):

$$\begin{aligned}\operatorname{charrank}(\widetilde{\gamma}_{3,n})\geq q\quad\iff\quad & H^j(G_{3,n})\xrightarrow{w_1}H^{j+1}(G_{3,n})\text{ is a monomorphism}\\ & \text{for all } j\in\{0,1,\ldots,q\}.\end{aligned} \tag{4}$$

Using this equivalence and Theorem 2.1, it is easy to see that

$$\operatorname{charrank}(\widetilde{\gamma}_{3,n})\geq n-1. \tag{5}$$

3 Proof of Theorem 1.1

We divide the proof of our main result into two parts. In the first we show that $\operatorname{charrank}(\widetilde{\gamma}_{3,n})\leq\min\{3n-2^t+7,2^t-5\}$, and in the second that $\operatorname{charrank}(\widetilde{\gamma}_{3,n})\geq\min\{3n-2^t+7,2^t-5\}$.

3.1 Upper bound

The following lemma is proved in [2, Proposition 3.2].

Lemma 3.1 *Let $t\geq 3$ be an integer. Then for all nonnegative integers b and c such that $2b+3c=2^{t-1}-3$ the number $\binom{b+c}{b}$ is divisible by 2.*

Proposition 3.2 *If $t\geq 3$ is the unique integer such that $2^{t-1}\leq n+3<2^t$, then*

$$\operatorname{charrank}(\widetilde{\gamma}_{3,n})\leq\min\{3n-2^t+7,2^t-5\}.$$

PROOF — Suppose first that $3n - 2^t + 7 \le 2^t - 5$, i.e., $n < 2^{t-1} - 4 + \frac{2^{t-1}}{3}$. We need to show that $\operatorname{charrank}(\widetilde{\gamma}_{3,n}) \le 3n - 2^t + 7$.

Let $l = n - 2^{t-1} + 4$. Note that $1 \le l \le n$. Namely, since $n + 3 \ge 2^{t-1}$, it holds $l = n + 4 - 2^{t-1} \ge 1$; also, since $t \ge 3$, we have that $2^{t-1} \ge 4$, so $l = n + 4 - 2^{t-1} \le n$. Equation (2) for this l and $m = 0$ gives us that

$$w_1^{n+1-l} w_3^l = \sum_{\substack{a+2b+3c=n+1+2l \\ (a,b,c)\neq(n+1-l,0,l)}} \binom{a+b+c-l}{a} \binom{b+c-l}{b} w_1^a w_2^b w_3^c. \tag{6}$$

Note that if $w_1^a w_2^b w_3^c$ is a monomial with nonzero coefficient in this sum (i.e., if the integer $\binom{a+b+c-l}{a}\binom{b+c-l}{b}$ is odd), then $c \ge l$ and $a \ge 1$.

Indeed, if $c < l$, then since $\binom{b+c-l}{b} \neq 0$, we have that $b + c - l < 0$, and now, since $\binom{a+b+c-l}{a} \neq 0$, it must be $a+b+c-l < 0$. But then $n+1+2l = a+2b+3c \le 3(a+b+c) < 3l$, which leads to the contradiction $l > n + 1$.

Assume now that $a = 0$. Since then $2b + 3(c - l) = n + 1 + 2l - 3l = n + 1 - l = 2^{t-1} - 3$, we can apply Lemma 3.1 to (nonnegative) integers b and $c - l$. Thus we obtain that $\binom{b+c-l}{b}$ is even, which is a contradiction.

So, all monomials with nonzero coefficient in (6) are divisible by w_1. Hence, in $H^{n+1+2l}(G_{3,n})$ we have the equality

$$w_1 \left(w_1^{n-l} w_3^l + \sum_{\substack{a+2b+3c=n+1+2l \\ (a,b,c)\neq(n+1-l,0,l)}} \binom{a+b+c-l}{a} \binom{b+c-l}{b} w_1^{a-1} w_2^b w_3^c \right) = 0.$$

Since the expression in the brackets is a nontrivial linear combination of elements of the set $D_{3,n}$, by Theorem 2.1 it is a nonzero element in the kernel of $H^{n+2l}(G_{3,n}) \xrightarrow{w_1} H^{n+1+2l}(G_{3,n})$. By (4),

$$\operatorname{charrank}(\widetilde{\gamma}_{3,n}) \le n + 2l - 1 = 3n - 2^t + 7.$$

Suppose now that $3n - 2^t + 7 \ge 2^t - 5$, i.e., $n > 2^{t-1} - 4 + \frac{2^{t-1}}{3}$ $(> 2^{t-1} - 3)$. We now want to prove that $\operatorname{charrank}(\widetilde{\gamma}_{3,n}) \le 2^t - 5$.

Let $m = 2^t - 4 - n$. It is obvious that $m \ge 0$ (since $n \le 2^t - 4$), and we also have that $m \le n$ (since this is equivalent to $n \ge 2^{t-1} - 2$). Therefore, $0 \le m \le n$, and so, we can use equation (2) for this m and $l = 0$:

$$w_1^{n+1-m} w_2^m = \sum_{\substack{a+2b+3c=n+1+m \\ (a,b,c)\neq(n+1-m,m,0)}} \binom{a+b+c-m}{a} \binom{b+c}{b} w_1^a w_2^b w_3^c.$$

Since $n + 1 + m = 2^t - 3$, by Lemma 3.1, for every summand with $a = 0$ in this sum, the coefficient $\binom{b+c}{b}$ is even, and so, such a summand vanishes. Therefore, all monomials that appear in the sum are divisible by w_1. Similarly as in the first part of the proof, in $H^{n+1+m}(G_{3,n})$ we now have the relation

$$w_1 \left(w_1^{n-m} w_2^m + \sum_{\substack{a+2b+3c=n+1+m \\ (a,b,c)\neq(n+1-m,m,0)}} \binom{a+b+c-m}{a} \binom{b+c}{b} w_1^{a-1} w_2^b w_3^c \right) = 0,$$

which, by Theorem 2.1, leads to a nontrivial element in the kernel of the homomorphism $H^{n+m}(G_{3,n}) \xrightarrow{w_1} H^{n+1+m}(G_{3,n})$. By (4) this implies that $\operatorname{charrank}(\widetilde{\gamma}_{3,n}) \le n+m-1 = 2^t-5$. □

3.2 Lower bound

Let us recall some notation from [4, 7, 8, 12]. For $i \geq 1$, let $g_i \in \mathbb{Z}_2[w_2, w_3] \cong \mathbb{Z}_2[w_1, w_2, w_3]/(w_1)$ denote the reduction of the polynomial (dual class) $\overline{w}_i$ modulo w_1. The corresponding polynomial in Stiefel–Whitney classes w_2 and w_3 in $H^i(G_{3,n})$ is again denoted by the same symbol. The main result of this subsection is based on [7, Proposition 2.4]. We are stating only a part of that proposition, and only for $k = 3$.

Proposition 3.3 *For an integer $x \geq 0$ observe the following set of polynomials in $H^{n+1+x}(G_{3,n})$:*

$$N_x(G_{3,n}) = \bigcup_{i=0}^{2} \{w_2^b w_3^c g_{n+1+i} \,:\, 2b + 3c = x - i\}.$$

If $x \leq n - 1$ and the set $N_x(G_{3,n})$ is linearly independent, then

$$H^{n+x}(G_{3,n}) \xrightarrow{w_1} H^{n+1+x}(G_{3,n})$$

is a monomorphism.

In the polynomial algebra $\mathbb{Z}_2[w_2, w_3]$, for all $i \geq 4$ one has the following recurrence relation (see [12, (2.3)])

$$g_i = w_2 g_{i-2} + w_3 g_{i-3},$$

which can also be written in the matrix form (see [12, p. 55]):

$$\begin{pmatrix} g_i \\ g_{i-1} \\ g_{i-2} \end{pmatrix} = \begin{pmatrix} 0 & w_2 & w_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} g_{i-1} \\ g_{i-2} \\ g_{i-3} \end{pmatrix}.$$

This identity implies that for all integers $r > 0$ and $i \geq r + 3$, one has

$$\begin{pmatrix} g_i \\ g_{i-1} \\ g_{i-2} \end{pmatrix} = \begin{pmatrix} 0 & w_2 & w_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^r \begin{pmatrix} g_{i-r} \\ g_{i-r-1} \\ g_{i-r-2} \end{pmatrix}. \tag{7}$$

In the remainder of this section we use the following notation:

$$A = \begin{pmatrix} 0 & w_2 & w_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & w_3 & 0 \\ 0 & 0 & w_3 \\ 1 & 0 & w_2 \end{pmatrix}.$$

Note that $AB = w_3 I = BA$, where I is the identity matrix, and hence $A^r B^r = w_3^r I$, for all $r > 0$.

We will also need the following facts, which hold in $\mathbb{Z}_2[w_2, w_3]$:

- $g_i = 0$ if and only if $i = 2^{t-1} - 3$ for some $t \geq 3$ ([4, Lemma 2.3(i)]);
- for all $t \geq 3$, $g_{2^{t-1}-2}$ and $g_{2^{t-1}-1}$ are coprime ([12, Lemma 2.5]).

Proposition 3.4 *If $t \geq 3$ is the unique integer such that $2^{t-1} \leq n + 3 < 2^t$, then*

$$\operatorname{charrank}(\widetilde{\gamma}_{3,n}) \geq \min\{3n - 2^t + 7, 2^t - 5\}.$$

PROOF — Let $\delta = \min\{3n - 2^t + 7, 2^t - 5\}$. By (5) and (4) it suffices to prove that

$$H^{n+x}(G_{3,n}) \xrightarrow{w_1} H^{n+1+x}(G_{3,n})$$

is a monomorphism for all $x \in \{0, 1, \ldots, \delta - n\}$. Note that $x \le \delta - n$ implies $x \le n - 1$. Indeed, if $n = 2^{t-1} - 3$, then $x \le \delta - n \le 3n - 2^t + 7 - n = 1 < n - 1$; and if $n \ge 2^{t-1} - 2$, then $x \le \delta - n \le 2^t - 5 - n \le n - 1$. So, by Proposition 3.3, it is now enough to show that $N_x(G_{3,n})$ is linearly independent for all $x \in \{0, 1, \ldots, \delta - n\}$.

Let $0 \le x \le \delta - n$. If some linear combination of elements of the set $N_x(G_{3,n})$ vanishes, then in $H^{n+1+x}(G_{3,n})$ one has the equality

$$q_{x-2}g_{n+3} + q_{x-1}g_{n+2} + q_x g_{n+1} = 0, \tag{8}$$

where q_{x-2}, q_{x-1} and q_x are some polynomials in Stiefel–Whitney classes w_2 and w_3, and the dimension of q_{x-i}, $i = 0, 1, 2$, is equal to $x - i$ (of course, $q_{x-i} = 0$ if $x < i$). In order to finish the proof, as in [12, p. 55], we are left to prove that $q_{x-2} = q_{x-1} = q_x = 0$, where q_{x-2}, q_{x-1} and q_x are interpreted as elements of the polynomial algebra $\mathbb{Z}_2[w_2, w_3]$ (since that will mean that all coefficients in the starting linear combination vanish).

So, from now on, $q_{x-2}, q_{x-1}, q_x \in \mathbb{Z}_2[w_2, w_3]$. Note that (8) holds in $\mathbb{Z}_2[w_2, w_3]$ as well. This is due to Theorem 2.1, since for a monomial $w_2^b w_3^c$ of the left-hand side in (8) one has $2(b + c) \le 2b + 3c = n + 1 + x \le 2n$ (since $x \le n - 1$).

Let $s = n + 3 - 2^{t-1}$ (i.e., $n = 2^{t-1} - 3 + s$) and

$$\begin{pmatrix} p_{x+s-1} & p_{x+s} & p_{x+s+1} \end{pmatrix} = \begin{pmatrix} q_{x-2} & q_{x-1} & q_x \end{pmatrix} A^{s+1}. \tag{9}$$

Note that $s \ge 0$ and that the (cohomological) dimensions of polynomials $p_{x+s-1}, p_{x+s}, p_{x+s+1} \in \mathbb{Z}_2[w_2, w_3]$ are $x + s - 1, x + s, x + s + 1$ respectively. Multiplying equality (9) with the column $(g_{2^{t-1}-1} \quad g_{2^{t-1}-2} \quad g_{2^{t-1}-3})^T$, by (7) (for $r = s+1$ and $i = n+3$) and (8), we obtain that

$$p_{x+s-1}g_{2^{t-1}-1} + p_{x+s}g_{2^{t-1}-2} + p_{x+s+1}g_{2^{t-1}-3} = 0 \tag{10}$$

in $\mathbb{Z}_2[w_2, w_3]$. Since $g_{2^{t-1}-3} = 0$ and the polynomials $g_{2^{t-1}-2}$ and $g_{2^{t-1}-1}$ are coprime (and nonzero), we conclude that $g_{2^{t-1}-2} \mid p_{x+s-1}$. If $p_{x+s-1} \ne 0$, then by comparing dimensions of $g_{2^{t-1}-2}$ and p_{x+s-1}, we have $2^{t-1} - 2 \le x + s - 1 \le 2^{t-1} - 2 - s + s - 1 = 2^{t-1} - 3$ (since $x \le \delta - n \le 2^t - 5 - n = 2^t - 5 - (2^{t-1} - 3 + s) = 2^{t-1} - 2 - s$). This contradiction proves that $p_{x+s-1} = 0$, and then from (10) it follows that $p_{x+s} = 0$ also (since $g_{2^{t-1}-2} \ne 0$).

Now, if we multiply identity (9) with the matrix B^{s+1}, we obtain that

$$\begin{pmatrix} 0 & 0 & p_{x+s+1} \end{pmatrix} B^{s+1} = \begin{pmatrix} w_3^{s+1}q_{x-2} & w_3^{s+1}q_{x-1} & w_3^{s+1}q_x \end{pmatrix}, \tag{11}$$

since $A^{s+1}B^{s+1} = w_3^{s+1}I$. For a matrix C over the ring $\mathbb{Z}_2[w_2, w_3]$, let the matrix $\overline{C}$ be defined as the reduction of C modulo w_3 (that is, each entry of $\overline{C}$ is the reduction of the corresponding entry of C modulo w_3). Then, it is easy to check that

$$\overline{B}^r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ w_2^{r-1} & 0 & w_2^r \end{pmatrix}$$

for all $r \ge 1$. The reduction $\mathbb{Z}_2[w_2, w_3] \to \mathbb{Z}_2[w_2, w_3]/(w_3) \cong \mathbb{Z}_2[w_2]$ is a ring homomorphism, and so, $\overline{B^r} = \overline{B}^r$, $r \ge 1$. This means that the low-right entry of the matrix B^{s+1} is equal to

$w_2^{s+1} + w_3\widetilde{p}$ for some $\widetilde{p} \in \mathbb{Z}_2[w_2, w_3]$. By (11), it follows that $w_3^{s+1}q_x = p_{x+s+1}(w_2^{s+1} + w_3\widetilde{p})$, and so, $w_3^{s+1} \mid p_{x+s+1}$. If $p_{x+s+1} \neq 0$, then by comparing dimensions of w_3^{s+1} and p_{x+s+1}, we obtain that $3(s+1) \leq x+s+1 \leq 2s+1+s+1 = 3s+2$ (since $x \leq \delta - n \leq 3n - 2^t + 7 - n = 2n - 2^t + 7 = 2^t - 6 + 2s - 2^t + 7 = 2s+1$). This contradiction implies that $p_{x+s+1} = 0$. Finally, by (11) we have that

$$q_{x-2} = q_{x-1} = q_x = 0,$$

and the proof is completed. $\square$

Remark 3.5 *In [7] Korbaš and Rusin proved that if* $1 \leq s \leq 6$ *and* $2^{t-1} + \lfloor \frac{s-1}{2} \rfloor + 1 \leq n+3 \leq 2^t - s - 3$, *then* $\operatorname{charrank}(\widetilde{\gamma}_{3,n}) \geq n+s+1$. *They also conjectured [7, Conjecture 3.3] that this is true for all* $s \geq 1$. *Proposition 3.4 (Theorem 1.1) confirms this conjecture. Indeed, since* $\lfloor \frac{s-1}{2} \rfloor \geq \frac{s-2}{2}$, *we have that* $2(n+3) \geq 2^t + 2 \cdot \lfloor \frac{s-1}{2} \rfloor + 2 \geq 2^t + 2 \cdot \frac{s-2}{2} + 2 = 2^t + s$, *which implies* $3n - 2^t + 7 \geq n+s+1$. *Also,* $n+3 \leq 2^t - s - 3$ *implies* $2^t - 5 \geq n+s+1$, *and therefore, we have that*

$$\operatorname{charrank}(\widetilde{\gamma}_{3,n}) \geq \min\{3n - 2^t + 7, 2^t - 5\} \geq n+s+1.$$

Remark 3.6 *The characteristic rank of a smooth connected manifold was introduced in [5], and it is actually the characteristic rank of the tangent bundle over the manifold. If* n *is even, then it is a known fact that all Stiefel–Whitney classes of (the tangent bundle over the) Grassmannian* $\widetilde{G}_{3,n}$ *are polynomials in Stiefel–Whitney classes of the canonical bundle* $\widetilde{\gamma}_{3,n}$, *and vice versa (see [5, p. 72]). Therefore, by Theorem 1.1, for even* n *and* $t \geq 3$ *such that* $2^{t-1} \leq n+3 < 2^t$ *we have that*

$$\operatorname{charrank}(\widetilde{G}_{3,n}) = \operatorname{charrank}(\widetilde{\gamma}_{3,n}) = \min\{3n - 2^t + 7, 2^t - 5\}.$$

4 Proof of Theorem 1.2

The following theorem of Naolekar and Thakur gives an upper bound for the $\mathbb{Z}_2$-cup-length in terms of characteristic rank.

Theorem 4.1 ([9]) *Let* M *be a connected closed smooth* d*-dimensional manifold. Let* α *be a real vector bundle over* M *and let* $j \leq \operatorname{charrank}(\alpha)$ *be an integer such that every monomial* $w_{i_1}(\alpha) \cdots w_{i_s}(\alpha)$, $1 \leq i_1 \leq \cdots \leq i_s \leq j$, *in dimension* d *vanishes. Then*

$$\operatorname{cup}(M) \leq 1 + \frac{d-j-1}{r},$$

where r *is the smallest positive integer such that* $H^r(M) \neq 0$.

In the case $M = \widetilde{G}_{3,n}$, we have that $d = 3n$ and $r = 2$. It is also a well known (and easily seen) fact that the nonzero class in $H^{3n}(\widetilde{G}_{3,n}) \cong \mathbb{Z}_2$ is not a polynomial in Stiefel–Whitney classes of the canonical bundle $\widetilde{\gamma}_{3,n}$ (this can be seen, for instance, from the Gysin sequence (3): obviously, the map $H^{3n}(G_{3,n}) \xrightarrow{w_1} H^{3n+1}(G_{3,n}) = 0$ is not a monomorphism, and so, $p^* : H^{3n}(G_{3,n}) \to H^{3n}(\widetilde{G}_{3n})$ is not onto). Therefore, for the bundle $\alpha := \widetilde{\gamma}_{3,n}$ we can take $j := \operatorname{charrank}(\widetilde{\gamma}_{3,n})$, and then Theorem 4.1 gives us the inequality

$$\operatorname{cup}(\widetilde{G}_{3,n}) \leq 1 + \frac{3n - \operatorname{charrank}(\widetilde{\gamma}_{3,n}) - 1}{2}. \tag{12}$$

The equality $\mathrm{cup}(\widetilde{G}_{3,2^{t-1}-4}) = 2^{t-1}-3$, $t \geq 4$, was proved in [2] and, independently, in [5].

If $2^{t-1}-3 \leq n \leq 2^{t-1}-4+\frac{2^{t-1}}{3}$ (for some $t \geq 4$), then by multiplying the second inequality with 3 we obtain that $3n-2^t+7 \leq 2^t-5$. Hence, by Theorem 1.1, $\mathrm{charrank}(\widetilde{\gamma}_{3,n}) = 3n-2^t+7$, and from (12) it follows that

$$\mathrm{cup}(\widetilde{G}_{3,n}) \leq 1 + \frac{3n-(3n-2^t+7)-1}{2} = 2^{t-1}-3.$$

The opposite inequality $\mathrm{cup}(\widetilde{G}_{3,n}) \geq 2^{t-1}-3$ holds by [7, (13)]. So, we have proved part (a) of Theorem 1.2.

For part (b), let $2^{t-1}-4 \leq n \leq 2^{t-1}-4+\frac{2^{t-1}}{3}$ (for some $t \geq 4$). We know that $w_2(\widetilde{\gamma}_{3,2^{t-1}-4})^{2^{t-1}-4} \neq 0$ (by [2, Corollary 4.12]), and for $n \geq 2^{t-1}-3$, the fact $w_2(\widetilde{\gamma}_{3,n})^{2^{t-1}-4} \neq 0$ was proved in [7, p. 83].

Suppose that the class $w_2(\widetilde{\gamma}_{3,n})^{2^{t-1}-3}$ is nonzero. Then, by the Poincaré duality there exists a class $\sigma \in H^{3n-2^t+6}(\widetilde{G}_{3,n})$ such that $\sigma \cdot w_2(\widetilde{\gamma}_{3,n})^{2^{t-1}-3} \neq 0$, and, since $2^t-6 \leq 2n+2 < 3n$, we have that $\mathrm{cup}(\widetilde{G}_{3,n}) \geq 2^{t-1}-2$, which contradicts part (a) of the theorem. Therefore, $w_2(\widetilde{\gamma}_{3,n})^{2^{t-1}-3} = 0$.

We conclude that $\mathrm{ht}(w_2(\widetilde{\gamma}_{3,n})) = 2^{t-1}-4$, which finishes the proof of Theorem 1.2.

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