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On the characteristic rank of vector bundles over oriented Grassmannians

by

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Abstract. We study the cohomology algebra of the Grassmann manifold $\widetilde{G}_{k,n}$ of oriented k-dimensional subspaces in $\mathbb{R}^{n+k}$ via the characteristic rank of the canonical vector bundle $\widetilde{\gamma}_{k,n}$ over $\widetilde{G}_{k,n}$ (denoted by $\operatorname{charrank}(\widetilde{\gamma}_{k,n})$). Using Gröbner bases for the ideals determining the cohomology algebras of the "unoriented" Grassmannians $G_{k,n}$ we prove that $\operatorname{charrank}(\widetilde{\gamma}_{k,n})$ increases with k. In addition, we calculate the exact value of $\operatorname{charrank}(\widetilde{\gamma}_{4,n})$, and for $k \geq 5$ we improve a general lower bound for $\operatorname{charrank}(\widetilde{\gamma}_{k,n})$ obtained by Korbaš. Some corollaries concerning the cup-length of $\widetilde{G}_{4,n}$ are also given.

1. Introduction. Let $G_{k,n}$ be the Grassmann manifold of k-dimensional subspaces in $\mathbb{R}^{n+k}$. According to Borel's description of the cohomology algebra $H^*(G_{k,n};\mathbb{Z}_2)$, every cohomology class in $H^*(G_{k,n};\mathbb{Z}_2)$ is a polynomial in the Stiefel–Whitney classes of the canonical vector bundle $\gamma_{k,n}$ over $G_{k,n}$. On the other hand, the mod 2 cohomology algebra of the Grassmann manifold $\widetilde{G}_{k,n}$ of oriented k-dimensional subspaces in $\mathbb{R}^{n+k}$ is more complicated and much less understood. Besides the polynomials in the Stiefel–Whitney classes of the canonical bundle $\widetilde{\gamma}_{k,n}$, it contains the so-called "anomalous" classes—the ones that are not expressible by the Stiefel–Whitney classes of $\widetilde{\gamma}_{k,n}$. There has been some significant interest lately in finding the minimal r such that an "anomalous" class occurs in $H^r(\widetilde{G}_{k,n};\mathbb{Z}_2)$ (see [4, 6, 9, 10]). This is actually the task of determining the characteristic rank of $\widetilde{\gamma}_{k,n}$.

If α is a real vector bundle over a d-dimensional CW-complex X, then the *characteristic rank* of α, denoted by $\operatorname{charrank}(\alpha)$, is defined as the maximal integer $q \in \{0,1,\ldots,d\}$ such that every cohomology class in $H^j(X;\mathbb{Z}_2)$ for

2010 *Mathematics Subject Classification*: Primary 57R20, 13P10; Secondary 55R25.
Key words and phrases: characteristic rank, Stiefel–Whitney class, Grassmann manifold, Gröbner basis.
Received 20 October 2017; revised 12 March 2018.
Published online *.

$0 \leq j \leq q$ is expressible as a polynomial in the Stiefel–Whitney classes $w_i(\alpha)$, $i \geq 1$.

In this paper we improve many existing results concerning $\mathrm{charrank}(\widetilde{\gamma}_{k,n})$ and offer some new (algebraic) methods for studying it. We prove that $\mathrm{charrank}(\widetilde{\gamma}_{k,n}) \leq \mathrm{charrank}(\widetilde{\gamma}_{k+1,n})$ for all positive integers k and n (Theorem 3.1), and thus improve [6, Proposition 3.4(2)]. The key ingredient in the proof is the observation that every relation in $H^*(G_{k+1,n};\mathbb{Z}_2)$ produces a relation in $H^*(G_{k,n};\mathbb{Z}_2)$ in an interesting way (Theorem 2.11). This observation heavily relies on Gröbner bases (obtained in [8]) for the ideals determining $H^*(G_{k,n};\mathbb{Z}_2)$ in Borel's description (in Section 2.2 we give a brief background on the theory of Gröbner bases, sufficient for the applications in this paper).

Up to now, the exact value of $\mathrm{charrank}(\widetilde{\gamma}_{k,n})$, where $n \geq k$, has been known only in the cases $k = 1, 2, 3$. For $k = 1$ we have $\widetilde{G}_{1,n} = S^n$, and $\mathrm{charrank}(\widetilde{\gamma}_{1,n}) = n - 1$. When $k = 2$, from [6, Theorem 3.6] we know that $\mathrm{charrank}(\widetilde{\gamma}_{2,n}) = n - 1$ if n is even, and $\mathrm{charrank}(\widetilde{\gamma}_{2,n}) = n$ if n is odd. For $k = 3$ we know that $\mathrm{charrank}(\widetilde{\gamma}_{3,n}) = \min\{3n - 2^t + 7, 2^t - 5\}$, where t is the integer such that $2^{t-1} \leq n + 3 < 2^t$. In full generality, this result is proved in [9], but many partial results were obtained in [4, 6, 10]. It turns out that the methods used in [9] for $k = 3$ work equally well in the case $k = 4$, and in this paper we calculate the exact value of $\mathrm{charrank}(\widetilde{\gamma}_{4,n})$ for all $n \geq 4$ (Theorem 6.6), improving [4, Theorem 2.1(2)], where several results are obtained for some n "close" to a power of 2. As a consequence, new upper bounds for the cup-length of $\widetilde{G}_{4,n}$ appear. Along with some new lower bounds, these are presented in Proposition 6.11.

For $k \geq 5$, the best known lower bound for $\mathrm{charrank}(\widetilde{\gamma}_{k,n})$ is $n + 1$ [4, Theorem 2.1(3)]. In Section 4 we use the method of Korbaš and Rusin [6] to improve this bound (Theorem 4.4). However, it is worth mentioning that the monotonicity of $\mathrm{charrank}(\widetilde{\gamma}_{k,n})$ (established in Theorem 3.1) implies that another lower bound for $\mathrm{charrank}(\widetilde{\gamma}_{k,n})$, $k \geq 5$, is $\mathrm{charrank}(\widetilde{\gamma}_{4,n})$ (determined in Theorem 6.6), and this bound quite often exceeds the one obtained in Theorem 4.4 (see Remark 4.5).

In Section 5 we show that the problem of finding $\mathrm{charrank}(\widetilde{\gamma}_{k,n})$ can be reduced to testing linear independence of certain polynomials closely related to the elements of the Gröbner bases obtained in [8].

The characteristic rank of a smooth closed connected manifold is defined as the characteristic rank of its tangent bundle. If $n+k$ is even (and $k \geq 2$), then $\mathrm{charrank}(\widetilde{G}_{k,n}) = 1$ [5, p. 73]. When $n + k$ is odd, it is known that the Stiefel–Whitney classes of the Grassmannian $\widetilde{G}_{k,n}$ are expressible as polynomials in the Stiefel–Whitney classes of $\widetilde{\gamma}_{k,n}$ and vice versa [5, p. 72], and so $\mathrm{charrank}(\widetilde{G}_{k,n}) = \mathrm{charrank}(\widetilde{\gamma}_{k,n})$ in this case. Therefore, for $n + k$

odd, all our results about $\operatorname{charrank}(\widetilde{\gamma}_{k,n})$ are valid for $\operatorname{charrank}(\widetilde{G}_{k,n})$ as well.

2. Some facts concerning the cohomology algebra $H^*(G_{k,n};\mathbb{Z}_2)$. Throughout this paper all cohomology groups are assumed to have coefficients in $\mathbb{Z}_2$. The set of all nonnegative integers is denoted by $\mathbb{N}_0$.

2.1. Borel's description. Let k and n be positive integers and $G_{k,n}$ the Grassmann manifold of (unoriented) k-dimensional linear subspaces in $\mathbb{R}^{n+k}$. By Borel's [2] classical result,

$$H^*(G_{k,n}) \cong \mathbb{Z}_2[w_1,\dots,w_k]/J_{k,n},$$

where $J_{k,n}$ is the ideal generated by certain polynomials $\overline{w}_{n+1},\dots,\overline{w}_{n+k}$. In this isomorphism, the classes of the variables $w_1,\dots,w_k$ (on the right-hand side) correspond to the Stiefel–Whitney classes of the canonical vector bundle $\gamma_{k,n}$ over $G_{k,n}$ (on the left-hand side). (In what follows, by abuse of notation, these Stiefel–Whitney classes will also be denoted by $w_1,\dots,w_k$.) The explicit formula for the polynomials $\overline{w}_r$ ($r\ge 0$) is

$$\overline{w}_r = \sum_{a_1+2a_2+\dots+ka_k=r} [a_1,a_2,\dots,a_k]\, w_1^{a_1}w_2^{a_2}\cdots w_k^{a_k}, \tag{2.1}$$

where $[a_1,a_2,\dots,a_k] := \binom{a_1+a_2+\dots+a_k}{a_1}\binom{a_2+\dots+a_k}{a_2}\cdots\binom{a_{k-1}+a_k}{a_{k-1}}$ is the multinomial coefficient (considered modulo 2).

To simplify notation, for $\alpha=(a_1,\dots,a_k)\in\mathbb{N}_0^k$ we define

$$W_k^\alpha := w_1^{a_1}\cdots w_k^{a_k}.$$

Also, if we write W_k^α, then it is understood that $\alpha\in\mathbb{N}_0^k$.

For $\alpha=(a_1,\dots,a_k)\in\mathbb{N}_0^k$, let

$$|\alpha| = a_1+\dots+a_k \quad\text{and}\quad \|\alpha\| = a_1+2a_2+\dots+ka_k.$$

Note that $|\alpha|$ is the degree, and $\|\alpha\|$ is the (cohomological) dimension of the monomial W_k^α.

The following theorem is well known (see e.g. [3]).

Theorem 2.1. *The set $D_{k,n}=\{W_k^\alpha : |\alpha|\le n\}$ is a vector space basis for $H^*(G_{k,n})$.*

Following [6], we say that an element W_k^α of $D_{k,n}$ is *regular* if $|\alpha|<n$, and *singular* if $|\alpha|=n$.

2.2. Gröbner bases. Let $\mathbb{F}$ be a field and $\mathbb{F}[x_1,\dots,x_k]$ the polynomial algebra over $\mathbb{F}$ in k variables. The set of all monomials in $\mathbb{F}[x_1,\dots,x_k]$ will be denoted by M. Also, let $\preceq$ be a well ordering of M (a total ordering such that every nonempty subset of M has a least element) with the property that $m_1\preceq m_2$ implies $mm_1\preceq mm_2$ for all $m,m_1,m_2\in M$.

For $f = \sum_{i=1}^{r} \alpha_i m_i \in \mathbb{F}[x_1, \dots, x_k]$, where $m_i \in M$ are pairwise different and $\alpha_i \in \mathbb{F} \setminus \{0\}$, let $M(f) := \{m_i : 1 \le i \le r\}$. We define the *leading monomial* of f, denoted by $\mathrm{LM}(f)$, as $\max M(f)$ with respect to $\preceq$. The *leading coefficient* of f, denoted by $\mathrm{LC}(f)$, is the coefficient of $\mathrm{LM}(f)$ in f, and the *leading term* of f is $\mathrm{LT}(f) := \mathrm{LC}(f) \cdot \mathrm{LM}(f)$.

Definition 2.2. Let $\mathcal{G} \subset \mathbb{F}[x_1, \dots, x_k]$ be a finite set of nonzero polynomials and $I_{\mathcal{G}} = (\mathcal{G})$ the ideal in $\mathbb{F}[x_1, \dots, x_k]$ generated by $\mathcal{G}$. We say that $\mathcal{G}$ is a *Gröbner basis* for $I_{\mathcal{G}}$ (with respect to $\preceq$) if for each $f \in I_{\mathcal{G}} \setminus \{0\}$ there exists $g \in \mathcal{G}$ such that $\mathrm{LM}(g) \mid \mathrm{LM}(f)$.

The crucial property of Gröbner bases is the following: if $\mathcal{G}$ is such a basis, then every polynomial from the ideal $I_{\mathcal{G}}$ reduces to zero modulo $\mathcal{G}$. For polynomials $f, h \in \mathbb{F}[x_1, \dots, x_k]$ we say that f *reduces to* h *modulo* $\mathcal{G}$ if there exist $n \ge 1$ and polynomials $f_1, \dots, f_n \in \mathbb{F}[x_1, \dots, x_k]$ such that $f_1 = f$, $f_n = h$ and for every $i \in \{1, \dots, n-1\}$ one has

$$f_{i+1} = f_i - \frac{\mathrm{LC}(f_i)}{\mathrm{LC}(g_i)} \cdot m_i \cdot g_i$$

for some $g_i \in \mathcal{G}$ and $m_i \in M$ such that $m_i \cdot \mathrm{LM}(g_i) = \mathrm{LM}(f_i)$ (note that $\mathrm{LT}(f_i)$ cancels out on the right-hand side, and so $\mathrm{LM}(f_{i+1}) \prec \mathrm{LM}(f_i)$).

2.3. Gröbner bases for Grassmannians. Let $k, n \ge 1$ be fixed integers. We now present a Gröbner basis for the ideal $J_{k,n}$ in $\mathbb{Z}_2[w_1, \dots, w_k]$, which was obtained in [8]. That is a Gröbner basis with respect to the grlex ordering $\preccurlyeq$ on the monomials (for $\alpha = (a_1, \dots, a_k), \beta = (b_1, \dots, b_k) \in \mathbb{N}_0^k$, $W_k^\alpha \prec W_k^\beta$ if and only if either $|\alpha| < |\beta|$, or $|\alpha| = |\beta|$ and $a_s < b_s$, where $s = \min\{i \mid a_i \ne b_i\}$).

For $\alpha = (a_1, \dots, a_k) \in \mathbb{N}_0^k$ and $\mu = (m_2, \dots, m_k) \in \mathbb{N}_0^{k-1}$ we define an integer $[\alpha, \mu]$ as the following product of binomial coefficients:

$$[\alpha, \mu] := \binom{a_1 + a_2 + \cdots + a_k - m_2 - m_3 - \cdots - m_k}{a_1} \binom{a_2 + \cdots + a_k - m_3 - \cdots - m_k}{a_2} \cdots \binom{a_{k-1} + a_k - m_k}{a_{k-1}}.$$

Remark 2.3. The $(k-1)$-tuple $\mu = (m_2, \dots, m_k)$ is indexed by integers from 2 to k (not from 1 to $k-1$) for the reason which will become obvious in Proposition 2.6. Nevertheless, we should remark in this regard that, by definition, the value of $\|\mu\|$ is $m_2 + 2m_3 + \cdots + (k-1)m_k$ (and not $2m_2 + 3m_3 + \cdots + km_k$).

For $k = 1$ it is understood that $\mu = \emptyset$, $|\mu| = \|\mu\| = 0$, and $[\alpha, \mu] = 1$ for all $\alpha = a_1 \in \mathbb{N}_0$.

For $\mu \in \mathbb{N}_0^{k-1}$, let $g_\mu \in \mathbb{Z}_2[w_1, \dots, w_k]$ be defined as

$$g_\mu := \sum_{\|\alpha\| = n+1+\|\mu\|} [\alpha, \mu] W_k^\alpha, \tag{2.2}$$

where the sum is taken over all $\alpha \in \mathbb{N}_0^k$ such that $\|\alpha\| = n + 1 + \|\mu\|$ (and the integers $[\alpha, \mu]$ are considered modulo 2).

Example 2.4. For $\alpha = (a_1, a_2, a_3, a_4) \in \mathbb{N}_0^4$ and $\mu = (1, 0, 2) \in \mathbb{N}_0^3$ we have $[\alpha, \mu] = \binom{a_1+a_2+a_3+a_4-3}{a_1}\binom{a_2+a_3+a_4-2}{a_2}\binom{a_3+a_4-2}{a_3}$. So, for $k = 4$ and $n = 6$,

$$\begin{aligned} g_{(1,0,2)} &= \sum_{a_1+2a_2+3a_3+4a_4=14} [(a_1, a_2, a_3, a_4), (1, 0, 2)] w_1^{a_1} w_2^{a_2} w_3^{a_3} w_4^{a_4} \\ &= w_1^4 w_2 w_4^2 + w_1^3 w_3 w_4^2 + w_1^2 w_2^2 w_4^2 + w_1^2 w_4^3 + w_2^3 w_4^2 + w_3^2 w_4^2. \end{aligned}$$

Now we can formulate a result about Gröbner bases.

Theorem 2.5 ([8, Theorem 14]). *If $S_{k,n} = \{\mu \in \mathbb{N}_0^{k-1} : |\mu| \leq n + 1\}$, then the set*

$$\mathcal{G}_{k,n} := \{g_\mu : \mu \in S_{k,n}\}$$

is a Gröbner basis for the ideal $J_{k,n}$ with respect to the grlex ordering $\preccurlyeq$.

We will make use of the following proposition from [8] as well (it is actually the modulo 2 version of [8, Proposition 5]).

Proposition 2.6. *If $\mu = (m_2, \ldots, m_k) \in S_{k,n}$, then $g_\mu \neq 0$ and $\mathrm{LT}(g_\mu) = W_k^{\overline{\mu}}$, where $\overline{\mu} = (n+1-|\mu|, m_2, \ldots, m_k)$. Moreover, if $W_k^\alpha \in M(g_\mu) \setminus \{W_k^{\overline{\mu}}\}$ for some $\alpha \in \mathbb{N}_0^k$, then $|\alpha| \leq n$.*

2.4. A relation in $H^*(G_{k,n})$ yields a relation in $H^*(G_{k-1,n})$. In this subsection, for $k \geq 2$ we construct a function

$$F : \mathbb{Z}_2[w_1, \ldots, w_{k-1}, w_k] \to \mathbb{Z}_2[w_1, \ldots, w_{k-1}]$$

with the following properties:

- if $p \neq 0$, then $F(p) \neq 0$;
- if (for some positive integer n) $p \in J_{k,n}$, then $F(p) \in J_{k-1,n}$.

In other words, given any (nontrivial) relation $p = 0$ in $H^*(G_{k,n})$, we shall have a (nontrivial) relation $F(p) = 0$ in $H^*(G_{k-1,n})$.

Let us first define a function $f : \mathbb{N}_0^k \to \mathbb{N}_0^{k-1}$ by

$$f(a_1, \ldots, a_{k-2}, a_{k-1}, a_k) := (a_1, \ldots, a_{k-2}, a_{k-1} + a_k).$$

Note that

$$|f(\alpha)| = |\alpha| \quad \text{and} \quad \|f(\alpha)\| = \|\alpha\| - a_k \tag{2.3}$$

for all $\alpha = (a_1, \ldots, a_k) \in \mathbb{N}_0^k$.

We use this function f to define F on the monomials in $\mathbb{Z}_2[w_1, \ldots, w_k]$:

$$F(W_k^\alpha) := W_{k-1}^{f(\alpha)},$$

that is,

$$F(w_1^{a_1} \cdots w_{k-2}^{a_{k-2}} w_{k-1}^{a_{k-1}} w_k^{a_k}) := w_1^{a_1} \cdots w_{k-2}^{a_{k-2}} w_{k-1}^{a_{k-1}+a_k}.$$

Now we extend F to $\mathbb{Z}_2[w_1,\dots,w_k]$, but not linearly. Namely, we do define $F(0) := 0$, but for a given nonzero polynomial (sum of monomials) we take into account only monomials with minimal exponent of the variable w_k, ignoring all others. The precise definition goes as follows. For $p \in \mathbb{Z}_2[w_1,\dots,w_k]$, $p \neq 0$, let $M_p := \{\alpha \in \mathbb{N}_0^k : W_k^\alpha \in M(p)\}$. So, M_p is a finite nonempty set and

$$p = \sum_{\alpha \in M_p} W_k^\alpha.$$

If $s(p) := \min\{a_k : \alpha \in M_p\}$, then

$$F(p) := \sum_{\substack{\alpha \in M_p \\ a_k = s(p)}} F(W_k^\alpha) = \sum_{\substack{\alpha \in M_p \\ a_k = s(p)}} W_{k-1}^{f(\alpha)}.$$

Example 2.7. For $k = 4$, let $p = w_1^3 w_2^2 w_3 w_4^2 + w_1 w_2 w_3 w_4^3 + w_2^2 w_3^2 w_4^2 + w_2 w_4^4 \in \mathbb{Z}_2[w_1,w_2,w_3,w_4]$. Then $s(p) = 2$ and

$$F(p) = F(w_1^3 w_2^2 w_3 w_4^2) + F(w_2^2 w_3^2 w_4^2) = w_1^3 w_2^2 w_3^3 + w_2^2 w_3^4 \in \mathbb{Z}_2[w_1,w_2,w_3].$$

Now we list some basic properties of the function F. The first one is a direct consequence of (2.3):

Proposition 2.8. *If p is homogeneous of (cohomological) dimension d (that is, $\|\alpha\| = d$ for all $\alpha \in M_p$), then $F(p)$ is homogeneous of dimension $d - s(p)$.*

If $p \in \mathbb{Z}_2[w_1,\dots,w_k]$ and $\alpha = (a_1,\dots,a_k)$, $\beta = (b_1,\dots,b_k) \in M_p$ are such that $\alpha \neq \beta$ and $a_k = b_k = s(p)$, then $a_i \neq b_i$ for some $i \in \{1,\dots,k-1\}$, and so $f(\alpha) \neq f(\beta)$. This means that

$$p \neq 0 \;\Rightarrow\; F(p) \neq 0. \tag{2.4}$$

If $p, q \in \mathbb{Z}_2[w_1,\dots,w_k]$ are nonzero, then $s(p\cdot q) = s(p)+s(q)$, and monomials with minimal exponent of w_k in $p\cdot q$ are precisely the products of monomials in p and monomials of q with minimal exponent of w_k. Therefore,

$$F(p\cdot q) = F(p)\cdot F(q). \tag{2.5}$$

(This equality is obvious if either $p = 0$ or $q = 0$, and so it holds for all $p, q \in \mathbb{Z}_2[w_1,\dots,w_k]$.)

On the other hand, it is evident that

$$F(p+q) = \begin{cases} F(p), & s(p) < s(q), \\ F(q), & s(p) > s(q), \\ F(p)+F(q), & s(p) = s(q) = s(p+q). \end{cases} \tag{2.6}$$

The only case that is not covered by (2.6) is $s(p) = s(q) < s(p+q)$, that is, when p and q have exactly the same monomials with minimal exponent of w_k, and they cancel out in $p+q$. This case, however, will not appear in our considerations.

Until the end of the section, let n be a fixed positive integer.

LEMMA 2.9. *If* $\mu=(m_2,\dots,m_k)\in S_{k,n}$, *then* $s(g_\mu)=m_k$.

Proof. By Proposition 2.6, $\overline{\mu}=(n+1-|\mu|,m_2,\dots,m_k)\in M_{g_\mu}$, and so $s(g_\mu)\le m_k$.

Suppose that $s(g_\mu)<m_k$, i.e., $a_k<m_k$ for some $\alpha=(a_1,\dots,a_k)\in M_{g_\mu}$. Since then $[\alpha,\mu]\ne 0$, first we have $\binom{a_{k-1}+a_k-m_k}{a_{k-1}}\ne 0$, which implies $a_{k-1}+a_k-m_k<0$ (this is because $a_k<m_k$, and, for $a,b\in\mathbb{Z}$, $\binom{a}{b}\ne 0$ implies that either $a\ge b$ or $a<0$). Now, from this inequality and the facts $m_{k-1}\ge 0$ and $\binom{a_{k-2}+a_{k-1}+a_k-m_{k-1}-m_k}{a_{k-2}}\ne 0$, we conclude that $a_{k-2}+a_{k-1}+a_k-m_{k-1}-m_k<0$. Continuing in the same manner, we obtain the following $k-1$ inequalities:

$$\begin{aligned} a_{k-1}+a_k&<m_k,\\ a_{k-2}+a_{k-1}+a_k&<m_{k-1}+m_k,\\ &\ \vdots\\ a_1+a_2+\dots+a_k&<m_2+m_3+\dots+m_k. \end{aligned}$$

Summing up, we get $\|\mu\|>\|\alpha\|-a_k=n+1+\|\mu\|-a_k$ (the equality is due to the fact $\alpha\in M_{g_\mu}$), that is, $a_k>n+1$. But this leads to

$$n+1<a_k<m_k\le m_2+\dots+m_k=|\mu|\le n+1.$$

This contradiction proves that $s(g_\mu)=m_k$. ∎

Note that if $\mu=(m_2,\dots,m_{k-2},m_{k-1},m_k)\in S_{k,n}$, then (2.3) implies that $f(\mu)=(m_2,\dots,m_{k-2},m_{k-1}+m_k)\in S_{k-1,n}$, i.e., $g_{f(\mu)}\in\mathcal{G}_{k-1,n}$ (for $k=2$, it is understood that $f(\mu)=\emptyset$, and the relation $g_{f(\mu)}\in\mathcal{G}_{1,n}$ reduces to $g_\emptyset=w_1^{n+1}\in\mathcal{G}_{1,n}$).

PROPOSITION 2.10. *If* $\mu\in S_{k,n}$, *then* $F(g_\mu)=g_{f(\mu)}$.

Proof. By Lemma 2.9, $s(g_\mu)=m_k$, and so

$$\begin{aligned} F(g_\mu)&=\sum_{\substack{\|\alpha\|=n+1+\|\mu\|\\ a_k=m_k}}[\alpha,\mu]W_{k-1}^{f(\alpha)}=\sum_{\substack{\|\alpha\|-a_k=n+1+\|\mu\|-m_k\\ a_k=m_k}}[f(\alpha),f(\mu)]W_{k-1}^{f(\alpha)}\\ &=\sum_{\substack{\|f(\alpha)\|=n+1+\|f(\mu)\|\\ a_k=m_k}}[f(\alpha),f(\mu)]W_{k-1}^{f(\alpha)}. \end{aligned}$$

The second equality holds because the last factor in the product $[\alpha,\mu]$ is $\binom{a_{k-1}+a_k-m_k}{a_{k-1}}=\binom{a_{k-1}}{a_{k-1}}=1$ (since $a_k=m_k$ for every summand), and the remaining factors are easily seen to give $[f(\alpha),f(\mu)]$. The third equality comes from (2.3).

On the other hand,

$$g_{f(\mu)} = \sum_{\|\beta\|=n+1+\|f(\mu)\|} [\beta, f(\mu)] W_{k-1}^{\beta},$$

where the sum is taken over all $\beta = (b_1, \ldots, b_{k-1}) \in \mathbb{N}_0^{k-1}$ such that $\|\beta\| = n+1+\|f(\mu)\|$. However, the last two sums are equal. Namely, in the first sum (for $F(g_\mu)$) the summand corresponding to $\alpha = (a_1, \ldots, a_{k-2}, a_{k-1}, m_k)$ is equal to the one in the second sum (for $g_{f(\mu)}$) corresponding to $\beta := f(\alpha) = (a_1, \ldots, a_{k-2}, a_{k-1} + m_k)$; and vice versa, for $\beta = (b_1, \ldots, b_{k-2}, b_{k-1})$ in the second sum, we have (unique) $\alpha := (b_1, \ldots, b_{k-2}, b_{k-1} - m_k, m_k)$ in the first (the summands with $b_{k-1} < m_k$ vanish in the second sum, since then $b_{k-1} < m_k \le m_{k-1} + m_k = s(g_{f(\mu)})$ by Lemma 2.9). ∎

Now we come to the main result of this section.

Theorem 2.11. *Let $J_{k,n} \lhd \mathbb{Z}_2[w_1, \ldots, w_k]$ and $J_{k-1,n} \lhd \mathbb{Z}_2[w_1, \ldots, w_{k-1}]$ be the ideals determining $H^*(G_{k,n})$ and $H^*(G_{k-1,n})$ respectively. If $F : \mathbb{Z}_2[w_1, \ldots, w_k] \to \mathbb{Z}_2[w_1, \ldots, w_{k-1}]$ is the function constructed above, then*

$$F(J_{k,n}) \subseteq J_{k-1,n}.$$

Proof. We know that $F(0) = 0$. Let $p \in J_{k,n} \setminus \{0\}$. We want to prove that $F(p) \in J_{k-1,n}$.

Since $\mathcal{G}_{k,n}$ is a Gröbner basis for $J_{k,n}$ (Theorem 2.5), the polynomial p can be reduced to zero modulo $\mathcal{G}_{k,n}$. Therefore, we have some polynomials $p_1, \ldots, p_r \in \mathbb{Z}_2[w_1, \ldots, w_k]$ $(r \ge 2)$ such that $p_1 = p$, $p_r = 0$ and

$$p_i - p_{i+1} = W_k^{\beta_i} \cdot g_{\mu_i}, \quad 1 \le i \le r-1,$$

for some $g_{\mu_i} \in \mathcal{G}_{k,n}$ and monomials $W_k^{\beta_i}$ with $W_k^{\beta_i} \cdot \mathrm{LM}(g_{\mu_i}) = \mathrm{LM}(p_i)$. Summing up these equalities, we obtain

$$p = \sum_{i=1}^{r-1} W_k^{\beta_i} \cdot g_{\mu_i}. \tag{2.7}$$

Furthermore,

$$\mathrm{LM}(W_k^{\beta_i} \cdot g_{\mu_i}) = W_k^{\beta_i} \cdot \mathrm{LM}(g_{\mu_i}) = \mathrm{LM}(p_i) \succ \mathrm{LM}(p_{i+1}) = \mathrm{LM}(W_k^{\beta_{i+1}} \cdot g_{\mu_{i+1}})$$

(in the grlex ordering) for all $i \in \{1, \ldots, r-2\}$, which means that all polynomials $W_k^{\beta_i} \cdot g_{\mu_i}$, $1 \le i \le r-1$, have distinct leading monomials.

Let $s := \min\{s(W_k^{\beta_i} \cdot g_{\mu_i}) : 1 \le i \le r-1\}$. By Lemma 2.9 and Proposition 2.6, the minimal exponent of w_k, $s(W_k^{\beta_i} \cdot g_{\mu_i}) = s(W_k^{\beta_i}) + s(g_{\mu_i})$, is reached by the leading monomial $\mathrm{LM}(W_k^{\beta_i} \cdot g_{\mu_i}) = W_k^{\beta_i} \cdot \mathrm{LM}(g_{\mu_i})$. Now, all these leading monomials are different, and so they do not cancel out in (2.7). Namely, if $I := \{i \in \{1, \ldots, r-1\} : s(W_k^{\beta_i} \cdot g_{\mu_i}) = s\}$, then the maximal (with respect to $\preccurlyeq$) of all leading monomials $\mathrm{LM}(W_k^{\beta_i} \cdot g_{\mu_i})$ with $i \in I$ appears exactly once in (2.7). Therefore, $s(p) = s$.

Finally, by (2.5)–(2.7), and Proposition 2.10,

$$F(p) = \sum_{i\in I} F(W_k^{\beta_i} \cdot g_{\mu_i}) = \sum_{i\in I} F(W_k^{\beta_i}) \cdot F(g_{\mu_i}) = \sum_{i\in I} W_{k-1}^{f(\beta_i)} \cdot g_{f(\mu_i)} \in J_{k-1,n},$$

since $g_{f(\mu_i)} \in \mathcal{G}_{k-1,n} \subset J_{k-1,n}$ for all $i \in I$. ∎

3. Monotonicity of $\mathrm{charrank}(\widetilde{\gamma}_{k,n})$**.** Let k and n be positive integers, $\widetilde{G}_{k,n}$ the Grassmann manifold of oriented k-dimensional linear subspaces in $\mathbb{R}^{n+k}$, and $\widetilde{\gamma}_{k,n}$ the canonical vector bundle over $\widetilde{G}_{k,n}$. The standard approach to studying $\mathrm{charrank}(\widetilde{\gamma}_{k,n})$ uses the Gysin sequence of the two-fold covering map $p : \widetilde{G}_{k,n} \to G_{k,n}$ (which forgets the orientation of a k-dimensional subspace):

$$(3.1) \quad \cdots \xrightarrow{w_1} H^j(G_{k,n}) \xrightarrow{p^*} H^j(\widetilde{G}_{k,n}) \to H^j(G_{k,n}) \xrightarrow{w_1} H^{j+1}(G_{k,n}) \xrightarrow{p^*} \cdots$$

($H^j(G_{k,n}) \xrightarrow{w_1} H^{j+1}(G_{k,n})$ is multiplication by $w_1 = w_1(\gamma_{k,n})$). By the Borel description every cohomology class in $H^*(G_{k,n})$ is a polynomial in the Stiefel–Whitney classes of $\gamma_{k,n}$ (i.e., $\mathrm{charrank}(\gamma_{k,n}) = \dim G_{k,n} = kn$), and since $p^*(\gamma_{k,n}) = \widetilde{\gamma}_{k,n}$, for every nonnegative integer $q \le kn$ we have $\mathrm{charrank}(\widetilde{\gamma}_{k,n}) \ge q$ if and only if $p^* : H^j(G_{k,n}) \to H^j(\widetilde{G}_{k,n})$ is onto for all $j \in \{0, 1, \ldots, q\}$. Now, from the exactness of (3.1) we conclude that

$$(3.2) \quad \mathrm{charrank}(\widetilde{\gamma}_{k,n}) \ge q \iff H^j(G_{k,n}) \xrightarrow{w_1} H^{j+1}(G_{k,n}) \text{ is a monomorphism}$$
$$\text{for all } j \in \{0, 1, \ldots, q\}.$$

We now prove the main result of this section—that $\mathrm{charrank}(\widetilde{\gamma}_{k,n})$ increases with k.

THEOREM 3.1. *If* $k \ge 2$ *and* $n \ge 1$*, then*

$$\mathrm{charrank}(\widetilde{\gamma}_{k-1,n}) \le \mathrm{charrank}(\widetilde{\gamma}_{k,n}).$$

Proof. Let $l := \mathrm{charrank}(\widetilde{\gamma}_{k,n})$.

If $l = \dim \widetilde{G}_{k,n} = kn$, then $\mathrm{charrank}(\widetilde{\gamma}_{k-1,n}) \le \dim \widetilde{G}_{k-1,n} = (k-1)n < l$, and the conclusion holds.

If $l < kn$, then by (3.2) the kernel of $H^{l+1}(G_{k,n}) \xrightarrow{w_1} H^{l+2}(G_{k,n})$ is nontrivial. According to Theorem 2.1, there is a nonzero polynomial

$$p = W_k^{\alpha_1} + \cdots + W_k^{\alpha_r} \in H^{l+1}(G_{k,n})$$

such that $|\alpha_i| \le n$ for all $i \in \{1, \ldots, r\}$, and $w_1 \cdot p = 0$ in $H^{l+2}(G_{k,n})$. By definition of the function F (from the previous section), the property (2.5), Theorem 2.11 and Proposition 2.8, we also have

$$w_1 \cdot F(p) = F(w_1) \cdot F(p) = F(w_1 \cdot p) = 0 \quad \text{in } H^{l+2-s(p)}(G_{k-1,n}).$$

Now, according to (2.4), $F(p)$ is a nonzero polynomial (considered as an element of $\mathbb{Z}_2[w_1,\dots,w_{k-1}]$). However, Theorem 2.1 implies that it is also nonzero in $H^{l+1-s(p)}(G_{k-1,n})$, since, by (2.3) and the definition of F, all its monomials have degrees at most n.

Therefore, $H^{l+1-s(p)}(G_{k-1,n}) \xrightarrow{w_1} H^{l+2-s(p)}(G_{k-1,n})$ is not a monomorphism. From (3.2) we conclude that

$$\operatorname{charrank}(\widetilde{\gamma}_{k-1,n}) \le l - s(p) \le l,$$

and the proof is complete. ∎

Remark 3.2. Actually, $\operatorname{charrank}(\widetilde{\gamma}_{k,n})$ increases with n as well. This is immediate from Theorem 3.1 and from $\operatorname{charrank}(\widetilde{\gamma}_{k,n}) = \operatorname{charrank}(\widetilde{\gamma}_{n,k})$, which is easy to verify. Indeed, if $h : \widetilde{G}_{k,n} \to \widetilde{G}_{n,k}$ is the homeomorphism which maps an oriented k-dimensional subspace $V \subset \mathbb{R}^{n+k}$ to its orthogonal complement $V^\perp$, oriented in such a way that the induced orientation on the direct sum $V \oplus V^\perp = \mathbb{R}^{n+k}$ coincides with the standard orientation of $\mathbb{R}^{n+k}$, then $h^*(\widetilde{\gamma}_{n,k}) \cong \widetilde{\gamma}_{k,n}^\perp$, where $\widetilde{\gamma}_{k,n}^\perp$ is the (n-dimensional) orthogonal complement of the vector bundle $\widetilde{\gamma}_{k,n}$. Therefore, $\operatorname{charrank}(\widetilde{\gamma}_{n,k}) = \operatorname{charrank}(\widetilde{\gamma}_{k,n}^\perp)$. On the other hand, $\widetilde{\gamma}_{k,n} \oplus \widetilde{\gamma}_{k,n}^\perp$ is a trivial $(n+k)$-dimensional bundle, and so for the total Stiefel–Whitney classes we have $w(\widetilde{\gamma}_{k,n}) \cdot w(\widetilde{\gamma}_{k,n}^\perp) = 1$, implying that all Stiefel–Whitney classes of one of these two bundles are expressible as polynomials in the Stiefel–Whitney classes of the other. This means that $\operatorname{charrank}(\widetilde{\gamma}_{k,n}^\perp) = \operatorname{charrank}(\widetilde{\gamma}_{k,n})$.

4. A lower bound for $\operatorname{charrank}(\widetilde{\gamma}_{k,n})$**.** Let us now fix positive integers k and n such that $k \le n$. The main result of this section is based on the following result from [6], a proposition which proved useful for obtaining lower bounds for the characteristic rank of $\widetilde{\gamma}_{k,n}$ in many cases (see [6, 9, 10]).

Let $\rho : \mathbb{Z}_2[w_1,\dots,w_k] \to \mathbb{Z}_2[w_1,\dots,w_k]/(w_1) \cong \mathbb{Z}_2[w_2,\dots,w_k]$ be reduction modulo w_1. For $r \ge 1$, we thus have a polynomial $\rho(\overline{w}_r) \in \mathbb{Z}_2[w_2,\dots,w_k]$ (see (2.1)), and the corresponding polynomial in the Stiefel–Whitney classes $w_2,\dots,w_k$ in $H^r(G_{k,n})$ is again denoted by the same symbol.

Proposition 4.1 ([6]). *For $x \in \mathbb{N}_0$, let $N_x(G_{k,n}) \subset H^{n+1+x}(G_{k,n})$ be the set defined by*

$$N_x(G_{k,n}) := \bigcup_{i=0}^{k-1} \{w_2^{b_2} w_3^{b_3} \cdots w_k^{b_k} \rho(\overline{w}_{n+1+i}) : 2b_2 + 3b_3 + \cdots + kb_k = x - i\}.$$

If $x \le n-1$ and the set $N_x(G_{k,n})$ is linearly independent in $H^{n+1+x}(G_{k,n})$, then

$$H^{n+x}(G_{k,n}) \xrightarrow{w_1} H^{n+1+x}(G_{k,n})$$

is a monomorphism.

We use this proposition to prove the following crucial lemma.

LEMMA 4.2. *If $s \in \{2, \ldots, k\}$ is an integer such that for every $m \in \{0, 1, \ldots, k-1\}$ the polynomial $\overline{w}_{n+1+m} \in \mathbb{Z}_2[w_1, \ldots, w_k]$ contains a monomial of the form $w_s^{a_s} w_{s+1}^{a_{s+1}} \cdots w_k^{a_k}$, then*

$$\operatorname{charrank}(\widetilde{\gamma}_{k,n}) \geq n + s - 1.$$

Proof. By Theorem 2.1, $H^j(G_{k,n}) \xrightarrow{w_1} H^{j+1}(G_{k,n})$ is a monomorphism for $j \leq n-1$, and therefore, by (3.2), it is enough to prove that

$$H^{n+x}(G_{k,n}) \xrightarrow{w_1} H^{n+1+x}(G_{k,n})$$

is a monomorphism for all $x \in \{0, 1, \ldots, s-1\}$.

So, let $0 \leq x \leq s-1$. Then $x \leq k-1 \leq n-1$. By Proposition 4.1 it is now sufficient to show that $N_x(G_{k,n})$ is linearly independent in $H^{n+1+x}(G_{k,n})$. Suppose that some linear combination of elements of $N_x(G_{k,n})$ vanishes. Then in $H^{n+1+x}(G_{k,n})$ one has the equality

$$\sum_{i=0}^{k-1} p_{x-i} \rho(\overline{w}_{n+1+i}) = 0, \tag{4.1}$$

where p_{x-i} $(0 \leq i \leq k-1)$ are polynomials in the Stiefel–Whitney classes $w_2, \ldots, w_k$, and the dimension of p_{x-i} is $x - i$ (hence, $p_{x-i} = 0$ for $i > x$). Interpreting the polynomials p_{x-i} and $\rho(\overline{w}_{n+1+i})$ $(0 \leq i \leq k-1)$ as elements of $\mathbb{Z}_2[w_2, \ldots, w_k]$, it is easy to see that this equality holds in $\mathbb{Z}_2[w_2, \ldots, w_k]$ as well. Indeed, this follows from Theorem 2.1, since for a monomial $w_2^{b_2} \cdots w_k^{b_k}$ on the left-hand side of the equality one has $2(b_2 + b_3 + \cdots + b_k) \leq 2b_2 + 3b_3 + \cdots + kb_k = n + x + 1 \leq 2n$ (since $x \leq n-1$).

If we show that $p_{x-i} = 0$ in $\mathbb{Z}_2[w_2, \ldots, w_k]$ for all i, then all coefficients of these polynomials are zero, i.e., all coefficients in the starting linear combination vanish, and the proof is complete.

Assume to the contrary that some of the polynomials p_{x-i}, $0 \leq i \leq k-1$, are nonzero in $\mathbb{Z}_2[w_2, \ldots, w_k]$. Let $m \in \{0, 1, \ldots, k-1\}$ be the largest integer such that $p_{x-m} \neq 0$. But a monomial of the form $w_s^{a_s} w_{s+1}^{a_{s+1}} \cdots w_k^{a_k}$ occurs in $\overline{w}_{n+1+m}$ with nonzero coefficient, and since $s \geq 2$, it occurs in the reduction $\rho(\overline{w}_{n+1+m})$ as well. Therefore, $w_s^{a_s} \cdots w_k^{a_k}$ divides a monomial in $p_{x-m} \rho(\overline{w}_{n+1+m})$ (since $p_{x-m} \neq 0$), and hence by (4.1) it must divide a monomial in $p_{x-i} \rho(\overline{w}_{n+1+i})$ for some $i < m$. However, the dimension of p_{x-i} is $x - i \leq x \leq s - 1 < s$, and so $w_s^{a_s} \cdots w_k^{a_k}$ divides a monomial in $\rho(\overline{w}_{n+1+i})$; but this is impossible since the dimension of $w_s^{a_s} \cdots w_k^{a_k}$ is $n + 1 + m > n + 1 + i$. ∎

REMARK 4.3. Note that for $s = 2$ the condition of the lemma is simply $\rho(\overline{w}_{n+1+m}) \neq 0$ for all $m \in \{0, 1, \ldots, k-1\}$. Since this condition is satisfied if $k \geq 5$ [4, Lemma 2.3(iii)], we have another proof of the fact that $\operatorname{charrank}(\widetilde{\gamma}_{k,n}) \geq n + 1$ when $k \geq 5$ [4, Theorem 2.1(3)].

For simplicity, we write $a \equiv_2 b$ instead of $a \equiv b \pmod 2$. The integer part of a number α is denoted by $\lfloor \alpha \rfloor$.

THEOREM 4.4. *Let k and n be positive integers.*

(i) *If $n \geq k \geq 6$, then*

$$\operatorname{charrank}(\widetilde{\gamma}_{k,n}) \geq n + \lfloor k/3 \rfloor - 1.$$

(ii) *If $n \geq 6 \cdot \lfloor k/5 \rfloor$, then*

$$\operatorname{charrank}(\widetilde{\gamma}_{k,n}) \geq n + 2 \cdot \lfloor k/5 \rfloor - 1.$$

Proof. (i) Let $s := \lfloor k/3 \rfloor$. Then $2 \leq s \leq k/3 < k$. By Lemma 4.2, it suffices to find a monomial of the form $w_s^{a_s} w_{s+1}^{a_{s+1}} \cdots w_k^{a_k}$ in $\overline{w}_{n+1+m}$ for every $m \in \{0, 1, \ldots, k-1\}$. For that purpose, we shall divide the integer $n+m-s$ by $2s$ (note that $n+m-s > 0$ since $s < k \leq n$). Thus we obtain the quotient $q \geq 0$ and the remainder $r \in \{0, 1, \ldots, 2s-1\}$ and we have $n+m-s = 2sq+r$. If we add $s+1$ to both sides of this equality, then we get the following representation of the integer $n+1+m$:

$$n+1+m = 2sq+r+s+1 = s \cdot 2q + (s+r+1) \cdot 1, \qquad q \geq 0,\ 0 \leq r \leq 2s-1.$$

Since $2 \leq s < s+r+1 \leq 3s \leq k$, we have the monomial $w_s^{2q} w_{s+r+1}$ in dimension $n+1+m$. Moreover, by (2.1) this monomial appears in $\overline{w}_{n+1+m}$ because its coefficient in $\overline{w}_{n+1+m}$ is a multinomial coefficient

$$[0, \ldots, 0, 2q, 0, \ldots, 0, 1, 0, \ldots, 0] = \binom{2q+1}{2q}\binom{1}{1} \equiv_2 1.$$

This proves (i).

(ii) The inequality $\operatorname{charrank}(\widetilde{\gamma}_{k,n}) \geq n-1$ is well known (and easily verified from (3.2) and Theorem 2.1). So, (ii) is true for $k \leq 4$.

Now, suppose that $k \geq 5$ and let $t := \lfloor k/5 \rfloor \geq 1$. As $2 \leq 2t \leq 2 \cdot k/5 < k$, we can apply Lemma 4.2 to $s := 2t$, and so, for $0 \leq m \leq k-1$, it remains to prove that the polynomial $\overline{w}_{n+1+m}$ contains a monomial of the form $w_{2t}^{a_{2t}} w_{2t+1}^{a_{2t+1}} \cdots w_k^{a_k}$. In order to find such a monomial, we divide $n+m$ by $2t$, obtaining the remainder $r \in \{0, 1, \ldots, 2t-1\}$ and the quotient q, which we write in the form $q = 4l + 2d + j$ with $l \in \mathbb{N}_0$ and $d, j \in \{0, 1\}$. So, we have a representation

$$n+1+m = 2t(4l+2d+j) + r + 1,$$

where $l \geq 0$, $0 \leq d \leq 1$, $0 \leq j \leq 1$ and $0 \leq r \leq 2t-1$. We distinguish three cases.

Case 1: $j = 1$. Then

$$n+1+m = 2t(4l+2d+1) + r + 1 = 2t \cdot (4l+2d) + (2t+r+1) \cdot 1,$$

$2 \leq 2t < 2t+r+1 \leq 4t < 5t \leq k$, so the monomial $w_{2t}^{4l+2d} w_{2t+r+1}$ has dimension $n+1+m$. Moreover, it appears in $\overline{w}_{n+1+m}$ according to (2.1), since the corresponding multinomial coefficient is

$$[0,\dots,0,4l+2d,0,\dots,0,1,0,\dots,0]=\tbinom{4l+2d+1}{4l+2d}\tbinom{1}{1}\equiv_2 1.$$

Case 2: $j=0$, $d=1$. First observe that in this case we must have $l\geq 1$. Indeed, $l=0$ implies $n+1+m=4t+r+1\leq 6t$, which is impossible since $n\geq 6t$ (this is the assumption of part (ii) of the theorem). Now,

$$n+1+m=2t(4l+2)+r+1=2t\cdot(4l-4)+(2t+r+1)\cdot 1+5t\cdot 2,$$

and also $2\leq 2t<2t+r+1\leq 4t<5t\leq k$, which means that we have the monomial $w_{2t}^{4l-4}w_{2t+r+1}w_{5t}^2$ in dimension $n+1+m$. This monomial is in $\overline{w}_{n+1+m}$ since the associated multinomial coefficient is

$$[0,\dots,0,4l-4,0,\dots,0,1,0,\dots,0,2,0,\dots,0]=\tbinom{4l-1}{4l-4}\tbinom{3}{1}\tbinom{2}{2}\equiv_2 1.$$

Case 3: $j=d=0$. Now we have

$$n+1+m=2t\cdot 4l+r+1=2t\cdot(4l-4)+(2t+r+1)\cdot 1+3t\cdot 2,$$

and the desired monomial is $w_{2t}^{4l-4}w_{2t+r+1}w_{3t}^2$ (as in the previous case, we have $l\geq 1$). Namely, its dimension is obviously $n+1+m$ and its coefficient in $\overline{w}_{n+1+m}$ is:

- $[0,\dots,0,4l-4,0,\dots,0,1,0,\dots,0,2,0,\dots,0]=\tbinom{4l-1}{4l-4}\tbinom{3}{1}\tbinom{2}{2}\equiv_2 1$ if $r<t-1$;
- $[0,\dots,0,4l-4,0,\dots,0,3,0,\dots,0]=\tbinom{4l-1}{4l-4}\tbinom{3}{3}\equiv_2 1$ if $r=t-1$;
- $[0,\dots,0,4l-4,0,\dots,0,2,0,\dots,0,1,0,\dots,0]=\tbinom{4l-1}{4l-4}\tbinom{3}{2}\tbinom{1}{1}\equiv_2 1$ if $r>t-1$. ■

REMARK 4.5. Part (i) of the theorem improves [4, Theorem 2.1(3)] for all $k\geq 9$ and $n\geq k$. For $5\leq k\leq 8$ we have $\lfloor k/5\rfloor=1$, and hence for all $n\geq k$, $(k,n)\neq(5,5)$, part (ii) gives $\operatorname{charrank}(\widetilde{\gamma}_{k,n})\geq n+1$, which coincides with the bound from [4, Theorem 2.1(3)].

However, according to Theorem 3.1, if $k\geq 5$, then a lower bound for $\operatorname{charrank}(\widetilde{\gamma}_{k,n})$ is $\operatorname{charrank}(\widetilde{\gamma}_{4,n})$, which we determine in Section 6. It turns out that this lower bound is better in many cases (especially for small k). For example, by Theorems 3.1 and 6.6,

$$\operatorname{charrank}(\widetilde{\gamma}_{5,5})\geq\operatorname{charrank}(\widetilde{\gamma}_{4,5})=7.$$

5. Application of $\mathcal{G}_{k,n}$ to obtain $\operatorname{charrank}(\widetilde{\gamma}_{k,n})$**.** In this section we show how the Gröbner basis $\mathcal{G}_{k,n}$ (of the ideal $J_{k,n}$ in $\mathbb{Z}_2[w_1,\dots,w_k]$) can be used to obtain $\operatorname{charrank}(\widetilde{\gamma}_{k,n})$.

Let $\mu=(m_2,\dots,m_k)\in\mathbb{N}_0^{k-1}$ with $|\mu|\leq n$. Then $w_1\mid W_k^{\overline{\mu}}$ (see Proposition 2.6). Furthermore, $W_k^{\overline{\mu}}=w_1W_k^{\widetilde{\mu}}$, where $\widetilde{\mu}=(n-|\mu|,m_2,\dots,m_k)$. Since $g_\mu\in J_{k,n}$, in $H^*(G_{k,n})\cong\mathbb{Z}_2[w_1,\dots,w_k]/J_{k,n}$ we have

$$0=g_\mu=w_1(W_k^{\widetilde{\mu}}+q_\mu)+\rho(g_\mu),\quad\text{i.e.,}\quad w_1W_k^{\widetilde{\mu}}=w_1q_\mu+\rho(g_\mu),\tag{5.1}$$

where q_μ is a polynomial in $w_1,\dots,w_k$. Moreover, by Proposition 2.6, each monomial of q_μ is regular, and each monomial of $\rho(g_\mu)$ is in $D_{k,n}$ (see Theorem 2.1 and the following sentence). Actually, $w_1q_\mu+\rho(g_\mu)$ is the representation of $w_1W_k^{\widetilde{\mu}}$ in the additive basis $D_{k,n}$.

Example 5.1. In Example 2.4 we calculated $g_{(1,0,2)}$ for $k=4$ and $n=6$. By that calculation, in $H^{14}(G_{4,6})$ one has

$$w_1\cdot w_1^3w_2w_4^2=w_1^4w_2w_4^2=w_1(\underbrace{w_1^2w_3w_4^2+w_1w_2^2w_4^2+w_1w_4^3}_{q_{(1,0,2)}})+\underbrace{w_2^3w_4^2+w_3^2w_4^2}_{\rho(g_{(1,0,2)})}.$$

Theorem 5.2. *For $x\in\mathbb{N}_0$, let $T_{k,n}^x=\{\mu\in\mathbb{N}_0^{k-1}:|\mu|\le n,\ \|\mu\|=x\}\subset S_{k,n}$ and $\mathcal{G}_{k,n}^x=\{\rho(g_\mu):\mu\in T_{k,n}^x\}\subset H^{n+x+1}(G_{k,n})$. Then*

$$H^{n+x}(G_{k,n})\xrightarrow{w_1}H^{n+x+1}(G_{k,n})$$

is a monomorphism if and only if the set $\mathcal{G}_{k,n}^x$ is linearly independent in $H^{n+x+1}(G_{k,n})$ and $\rho(g_{\mu_1})\neq\rho(g_{\mu_2})$ for all distinct $\mu_1,\mu_2\in T_{k,n}^x$.

Proof. Assume that $\sum_{i=1}^r\rho(g_{\mu_i})=0$ for some $r\ge 1$ and some pairwise different $\mu_1,\dots,\mu_r\in T_{k,n}^x$. By (5.1),

$$0=\sum_{i=1}^r\big(w_1(W_k^{\widetilde{\mu}_i}+q_{\mu_i})+\rho(g_{\mu_i})\big)=w_1\sum_{i=1}^r(W_k^{\widetilde{\mu}_i}+q_{\mu_i})\quad\text{in } H^{n+x+1}(G_{k,n}).$$

However, all monomials in q_{μ_i}, $1\le i\le r$, are regular, and $W_k^{\widetilde{\mu}_i}$, $1\le i\le r$, are pairwise different singular monomials. According to Theorem 2.1, this means that $\sum_{i=1}^r(W_k^{\widetilde{\mu}_i}+q_{\mu_i})$ is a nontrivial element in the kernel of $H^{n+x}(G_{k,n})\xrightarrow{w_1}H^{n+x+1}(G_{k,n})$, so this map is not a monomorphism.

Conversely, suppose that $H^{n+x}(G_{k,n})\xrightarrow{w_1}H^{n+x+1}(G_{k,n})$ is not a monomorphism. Then there exists $p\in H^{n+x}(G_{k,n})$, $p\neq 0$, such that $w_1p=0$. Consider the representation of p in the additive basis $D_{k,n}$. Since $p\neq 0$ and $w_1p=0$, by Theorem 2.1 this representation must contain singular monomials. Observe also that every singular monomial $W_k^\alpha\in H^{n+x}(G_{k,n})$ is equal to $W_k^{\widetilde{\mu}}$ for a (unique) $\mu\in T_{k,n}^x$ (if $\alpha=(a_1,\dots,a_k)$, then $\mu=(a_2,\dots,a_k)$). This means that we have a representation

$$p=\sum_{i=1}^r W_k^{\widetilde{\mu}_i}+\sum_{j=1}^s W_k^{\beta_j},$$

where $r\ge 1$, $\mu_1,\dots,\mu_r\in T_{k,n}^x$ are pairwise different, and $W_k^{\beta_j}$, $1\le j\le s$, are regular. Now, using (5.1), we obtain

$$0=w_1p=\sum_{i=1}^r w_1W_k^{\widetilde{\mu}_i}+\sum_{j=1}^s w_1W_k^{\beta_j}=\sum_{i=1}^r(w_1q_{\mu_i}+\rho(g_{\mu_i}))+\sum_{j=1}^s w_1W_k^{\beta_j}.$$

All monomials on the right-hand side are in $D_{k,n}$, so they must cancel out. However, the polynomials $\rho(g_{\mu_i})$, $1 \leq i \leq r$, do not contain the variable w_1, and thus $\sum_{i=1}^{r} \rho(g_{\mu_i}) = 0$. Since $r \geq 1$ and $\mu_1, \ldots, \mu_r \in T_{k,n}^{x}$ are pairwise different, the proof is complete. ∎

The following corollary is a straightforward consequence of Theorem 5.2 and (3.2).

COROLLARY 5.3. *If $\mu \in \mathbb{N}_0^{k-1}$ is such that $|\mu| \leq n$ and $\rho(g_\mu) = 0$, then*

$$\operatorname{charrank}(\widetilde{\gamma}_{k,n}) \leq n + \|\mu\| - 1.$$

To end this section, let us briefly compare Theorem 5.2 and Proposition 4.1. It is evident that Theorem 5.2 has two big advantages: it can be applied to all x (whereas Proposition 4.1 can only be used when $x \leq n-1$) and it gives a condition that is both necessary and sufficient for $H^{n+x}(G_{k,n}) \xrightarrow{w_1} H^{n+x+1}(G_{k,n})$ to be a monomorphism (whereas Proposition 4.1 gives a sufficient condition only). Nevertheless, when applied, Proposition 4.1 proved to be much more useful for obtaining lower bounds for $\operatorname{charrank}(\widetilde{\gamma}_{k,n})$ (see [6, 9, 10], and also Sections 4 and 6.2 below). On the other hand, it seems that Theorem 5.2 is better suited for obtaining upper bounds for $\operatorname{charrank}(\widetilde{\gamma}_{k,n})$. Indeed, to prove that $\operatorname{charrank}(\widetilde{\gamma}_{k,n}) \leq n + x - 1$, it is enough to find a nonempty subset T of $T_{k,n}^{x}$ such that $\sum_{\mu \in T} \rho(g_\mu) = 0$. As a matter of fact, the simplest application of this theorem (more precisely, Corollary 5.3) essentially occurs in [9], where $\operatorname{charrank}(\widetilde{\gamma}_{3,n})$ was determined, and it will occur in Section 6.1 below in calculating $\operatorname{charrank}(\widetilde{\gamma}_{4,n})$.

6. Characteristic rank of vector bundles over $\widetilde{G}_{4,n}$.

In this section we obtain the exact values of $\operatorname{charrank}(\widetilde{\gamma}_{4,n})$ for all $n \geq 4$.

6.1. Upper bound.

We shall begin with an arithmetic lemma. As before, we abbreviate $a \equiv b \pmod{m}$ to $a \equiv_m b$. Recall that for nonnegative integers α and β one has

$$\binom{\alpha}{\beta} \equiv_2 \binom{a_r}{b_r}\binom{a_{r-1}}{b_{r-1}} \cdots \binom{a_1}{b_1}\binom{a_0}{b_0},$$

where $\alpha = \sum_{i=0}^{r} a_i 2^i$ and $\beta = \sum_{i=0}^{r} b_i 2^i$, $a_i, b_i \in \{0,1\}$, are the binary expansions of these integers.

LEMMA 6.1. *Let $t \geq 2$ be an integer. There are no nonnegative integers α, β and γ such that $2\alpha + \beta + \gamma = 2^t - 3$ and $\binom{\alpha}{\beta}\binom{\beta}{\gamma} \equiv_2 1$.*

Proof. Assume to the contrary that we have integers α, β and γ with $\alpha, \beta, \gamma \geq 0$, $2\alpha + \beta + \gamma = 2^t - 3$ and $\binom{\alpha}{\beta}\binom{\beta}{\gamma} \equiv_2 1$.

Note first that β is odd. Indeed, if β were even, then γ would have to be even too (since $\binom{\beta}{\gamma} \equiv_2 1$), and that would imply that $2\alpha + \beta + \gamma$ is also even, which is not the case ($2\alpha + \beta + \gamma = 2^t - 3$).

Now, from the fact $\binom{\alpha}{\beta} \equiv_2 1$ we conclude that α is odd as well.

Moreover, let us prove that $\alpha \equiv_4 \beta \equiv_4 3$. Since α is odd, it follows that $2\alpha \equiv_4 2$. If $\beta \equiv_4 1$, then the relation $2\alpha + \beta + \gamma = 2^t - 3 \equiv_4 1$ implies $\gamma \equiv_4 2$, which is a contradiction because $\binom{\beta}{\gamma} \equiv_2 1$. Therefore, $\beta \equiv_4 3$, and since $\binom{\alpha}{\beta} \equiv_2 1$, we have $\alpha \equiv_4 3$.

Now, consider the binary expansions of the integers α and β:

$$\alpha = \sum_{i=0}^{r} a_i 2^i, \quad \beta = \sum_{i=0}^{r} b_i 2^i, \qquad a_i, b_i \in \{0,1\}.$$

We know that $\alpha \equiv_4 \beta \equiv_4 3$, which means that $a_0 = a_1 = b_0 = b_1 = 1$. Let $m := \min\{i : a_i = 0\}$ and $l := \min\{i : b_i = 0\}$. Then $\binom{\alpha}{\beta} \equiv_2 1$ implies $m \geq l \geq 2$, and we have

$$\alpha = 1 + 2 + \cdots + 2^{m-1} + 2^{m+1}\widetilde{\alpha} \quad \text{and} \quad \beta = 1 + 2 + \cdots + 2^{l-1} + 2^{l+1}\widetilde{\beta}$$

for some nonnegative integers $\widetilde{\alpha}$ and $\widetilde{\beta}$. Now

$$\begin{aligned} 2\alpha + \beta &= 2 + 4 + 8 + \cdots + 2^m + 2^{m+2}\widetilde{\alpha} + 1 + 2 + 4 + \cdots + 2^{l-1} + 2^{l+1}\widetilde{\beta} \\ &= 1 + 4 + 8 + \cdots + 2^{l-1} + 2^{l+1}\delta \end{aligned}$$

for some positive integer δ (it is understood that $1 + 4 + 8 + \cdots + 2^{l-1} = 1$ for $l = 2$). Since

$$2\alpha + \beta + \gamma = 2^t - 3 = 1 + 4 + 8 + \cdots + 2^{t-1},$$

we have $\gamma = 2^l + 2^{l+1}\varepsilon$, for some integer $\varepsilon \geq 0$. But this contradicts $\binom{\beta}{\gamma} \equiv_2 1$ (since $b_l = 0$). ■

Proposition 6.2. *Let $n \geq 4$ be an integer. If $t \geq 3$ is the unique integer such that $2^{t-1} < n + 4 \leq 2^t$, then*

$$\operatorname{charrank}(\widetilde{\gamma}_{4,n}) \leq \min\{4n - 3 \cdot 2^{t-1} + 11, 2^t - 5\}.$$

Proof. First, suppose that $2^t - 5 \leq 4n - 3 \cdot 2^{t-1} + 11$, that is, $n + 4 \geq 2^{t-1} + 2^{t-3}$. In this case, we need to prove that $\operatorname{charrank}(\widetilde{\gamma}_{4,n}) \leq 2^t - 5$.

Let $m := 2^t - n - 4$. We have $m \geq 0$ (since $n + 4 \leq 2^t$). Also, for $t \geq 4$, $m \leq 2^t - 2^{t-1} - 2^{t-3} = 2^{t-1} - 2^{t-3} \leq 2^{t-1} + 2^{t-3} - 4 \leq n$, while for $t = 3$ we find that n must be equal to 4, and the inequality $m \leq n$ is obvious. This means that the triple $\mu := (m, 0, 0)$ satisfies the condition $|\mu| = m \leq n$. Moreover, by (2.2) we have

$$g_\mu = \sum_{a+2b+3c+4d=n+1+m} \binom{a+b+c+d-m}{a}\binom{b+c+d}{b}\binom{c+d}{c} w_1^a w_2^b w_3^c w_4^d,$$

where the sum is taken over all $(a,b,c,d) \in \mathbb{N}_0^4$ such that $a + 2b + 3c + 4d = n + 1 + \|\mu\| = n + 1 + m = 2^t - 3$. This means that the reduction of g_μ

modulo w_1 is given by the formula

$$\rho(g_\mu) = \sum_{2b+3c+4d=2^t-3} \binom{b+c+d}{b}\binom{c+d}{c} w_2^b w_3^c w_4^d.$$

But $\rho(g_\mu) = 0$, since there is no $(b,c,d) \in \mathbb{N}_0^3$ such that $2b+3c+4d = 2^t - 3$ and $\binom{b+c+d}{b}\binom{c+d}{c} = \binom{b+c+d}{c+d}\binom{c+d}{d} \equiv_2 1$. This follows from Lemma 6.1 if we take $\alpha := b+c+d$, $\beta := c+d$ and $\gamma := d$. Finally, by Corollary 5.3,

$$\operatorname{charrank}(\widetilde{\gamma}_{4,n}) \le n + \|\mu\| - 1 = n + m - 1 = 2^t - 5.$$

Assume now that $4n - 3\cdot 2^{t-1} + 11 \le 2^t - 5$ (i.e., $n + 4 \le 2^{t-1} + 2^{t-3}$), and let us prove that $\operatorname{charrank}(\widetilde{\gamma}_{4,n}) \le 4n - 3\cdot 2^{t-1} + 11$.

Define $l := n+4-2^{t-1}$. Then $2^{t-1} < n+4$ implies $l > 0$, while $2^{t-1}-3 \le n \le 2^{t-1} + 2^{t-3} - 4$ and $t \ge 3$ imply $l \le 2^{t-3} \le 2^{t-1} - 3 \le n$. So, the triple $\lambda := (0,0,l)$ satisfies $|\lambda| = l \le n$. As in the first part of the proof, we have

$$\rho(g_\lambda) = \sum_{2b+3c+4d=n+1+3l} \binom{b+c+d-l}{b}\binom{c+d-l}{c} w_2^b w_3^c w_4^d,$$

where the sum is taken over all triples $(b,c,d) \in \mathbb{N}_0^3$ such that $2b+3c+4d = n+1+\|\lambda\| = n+1+3l$.

Suppose that some coefficient $\binom{b+c+d-l}{b}\binom{c+d-l}{c}$ is odd. Let us first prove that $d-l \ge 0$. The inequality $d-l < 0$ would imply that $c+d-l < 0$ (since $c+d-l < c$ and $\binom{c+d-l}{c} \ne 0$), and for a similar reason ($\binom{b+c+d-l}{b} \ne 0$) we would have $b+c+d-l < 0$. But this would mean that $n+1+3l = 2b+3c+4d \le 4(b+c+d) < 4l$, i.e., $l > n+1$, which contradicts $l \le n$ established before.

Therefore, $d - l \ge 0$. However, since

$$\binom{b+c+d-l}{c+d-l}\binom{c+d-l}{d-l} = \binom{b+c+d-l}{b}\binom{c+d-l}{c} \equiv_2 1$$

and

$$2(b+c+d-l)+(c+d-l)+(d-l) = 2b+3c+4d-4l = n+1-l = 2^{t-1}-3,$$

we have nonnegative integers $\alpha := b+c+d-l$, $\beta := c+d-l$ and $\gamma := d-l$ such that $2\alpha+\beta+\gamma = 2^{t-1}-3$ and $\binom{\alpha}{\beta}\binom{\beta}{\gamma} \equiv_2 1$, which do not exist according to Lemma 6.1.

We conclude that all coefficients in the above sum are even, that is, $\rho(g_\lambda) = 0$. Corollary 5.3 now gives

$$\operatorname{charrank}(\widetilde{\gamma}_{4,n}) \le n + \|\lambda\| - 1 = n + 3l - 1 = 4n - 3\cdot 2^{t-1} + 11,$$

and the proof is complete. ∎

6.2. Lower bound. The polynomials $\overline{w}_i$, $i \ge 0$ (see (2.1)), depend on k (they are polynomials in $\mathbb{Z}_2[w_1,\dots,w_k]$), and likewise for their reductions $\rho(\overline{w}_i) \in \mathbb{Z}_2[w_2,\dots,w_k]$, $i \ge 0$, modulo w_1. In this subsection, for $k = 4$ the polynomials $\rho(\overline{w}_i) \in \mathbb{Z}_2[w_2,w_3,w_4]$ will be abbreviated to ρ_i, while for

$k=3$ the reductions $\rho(\overline{w}_i)\in\mathbb{Z}_2[w_2,w_3]$ will be denoted by τ_i. Also, as usual, $\mathbb{Z}_2[w_2,w_3]$ will be considered as a subalgebra of $\mathbb{Z}_2[w_2,w_3,w_4]$.

In $\mathbb{Z}_2[w_2,w_3,w_4]$, for all $i\geq 4$ the following recurrence relation holds (see [4]):

$$\rho_i=w_2\rho_{i-2}+w_3\rho_{i-3}+w_4\rho_{i-4}. \tag{6.1}$$

In matrix form,

$$\begin{pmatrix}\rho_i\\ \rho_{i-1}\\ \rho_{i-2}\\ \rho_{i-3}\end{pmatrix}=\begin{pmatrix}0 & w_2 & w_3 & w_4\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\rho_{i-1}\\ \rho_{i-2}\\ \rho_{i-3}\\ \rho_{i-4}\end{pmatrix},$$

and so, for all integers $s>0$ and $i\geq s+3$,

$$\begin{pmatrix}\rho_i\\ \rho_{i-1}\\ \rho_{i-2}\\ \rho_{i-3}\end{pmatrix}=\begin{pmatrix}0 & w_2 & w_3 & w_4\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}^s\begin{pmatrix}\rho_{i-s}\\ \rho_{i-s-1}\\ \rho_{i-s-2}\\ \rho_{i-s-3}\end{pmatrix}. \tag{6.2}$$

In what follows we use the following notation:

$$A=\begin{pmatrix}0 & w_2 & w_3 & w_4\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix}0 & w_4 & 0 & 0\\ 0 & 0 & w_4 & 0\\ 0 & 0 & 0 & w_4\\ 1 & 0 & w_2 & w_3\end{pmatrix}.$$

It is straightforward to verify that $AB=w_4I=BA$, where I is the identity matrix, and so $A^sB^s=w_4^sI$ for all $s>0$.

Note that if we define $\rho_{-3}=\rho_{-2}=\rho_{-1}=0$, then the recurrence formula (6.1) holds for all $i\geq 1$ (it is easy to see that $\rho_0=1$, $\rho_1=0$, $\rho_2=w_2$ and $\rho_3=w_3$). Furthermore, (6.1) leads to the following relation (see [4, (2.6)]):

$$\rho_i=w_2^{2^s}\rho_{i-2\cdot 2^s}+w_3^{2^s}\rho_{i-3\cdot 2^s}+w_4^{2^s}\rho_{i-4\cdot 2^s} \tag{6.3}$$

where $s\geq 0$ and $i\geq 4\cdot 2^s-3$.

We will need the following lemma from [4, Lemma 2.3(ii)].

Lemma 6.3. *If $i\geq 0$, then $\rho_i=0$ if and only if $i=2^r-3$ for some $r\geq 2$.*

Let us briefly compare the polynomials $\rho_i\in\mathbb{Z}_2[w_2,w_3,w_4]$ with the polynomials $\tau_i\in\mathbb{Z}_2[w_2,w_3]\subset\mathbb{Z}_2[w_2,w_3,w_4]$ for $i\geq 0$. Similarly to (6.1), we have the following recurrence formula:

$$\tau_i=w_2\tau_{i-2}+w_3\tau_{i-3},\quad i\geq 3.$$

Since $\rho_0 = \tau_0 = 1$, $\rho_1 = \tau_1 = 0$ and $\rho_2 = \tau_2 = w_2$, an easy induction shows that τ_i is the reduction of ρ_i modulo w_4.

We now prove some additional properties of the polynomials ρ_i.

LEMMA 6.4. *For $r \geq 1$ we have:*

(i) $\rho_{2^r-2} = \tau_{2^r-2}$*;*
(ii) $\rho_{2^r-1} = \tau_{2^r-1}$*;*
(iii) $\gcd(\rho_{2^r-2}, \rho_{2^r-1}) = 1$ *for $r \geq 2$;*
(iv) $w_4^{2^{r-2}}$ *is a monomial of the polynomial ρ_{2^r} for $r \geq 2$.*

Proof. (i) It is enough to show that ρ_{2^r-2} does not contain monomials divisible by w_4. We prove this by induction on $r \geq 1$. For $r = 1$ and $r = 2$, we have $\rho_0 = 1$ and $\rho_2 = w_2$.

So, assume that $r \geq 3$ and that $\rho_{2^{r-2}-2}$ and $\rho_{2^{r-1}-2}$ do not contain monomials divisible by w_4. We want to prove that ρ_{2^r-2} does not contain such monomials either. By (6.3) applied to $i = 2^r - 2$ and $s = r-2$, we have

$$\begin{aligned}\rho_{2^r-2} &= w_2^{2^{r-2}} \rho_{2^r-2-2\cdot 2^{r-2}} + w_3^{2^{r-2}} \rho_{2^r-2-3\cdot 2^{r-2}} + w_4^{2^{r-2}} \rho_{2^r-2-4\cdot 2^{r-2}} \\ &= w_2^{2^{r-2}} \rho_{2^{r-1}-2} + w_3^{2^{r-2}} \rho_{2^{r-2}-2},\end{aligned}$$

which completes the proof.

(ii) The proof is by induction on $r \geq 1$. For $r = 1$ and $r = 2$, we have $\rho_1 = 0$ and $\rho_3 = w_3$. Furthermore, by (6.3) applied to $i = 2^r - 1$ and $s = r - 2$, we have

$$\begin{aligned}\rho_{2^r-1} &= w_2^{2^{r-2}} \rho_{2^r-1-2\cdot 2^{r-2}} + w_3^{2^{r-2}} \rho_{2^r-1-3\cdot 2^{r-2}} + w_4^{2^{r-2}} \rho_{2^r-1-4\cdot 2^{r-2}} \\ &= w_2^{2^{r-2}} \rho_{2^{r-1}-1} + w_3^{2^{r-2}} \rho_{2^{r-2}-1},\end{aligned}$$

and the conclusion follows as in part (i).

(iii) By [10, Lemma 2.5], for $r \geq 2$ we have

$$\gcd(\tau_{2^r-2}, \tau_{2^r-1}) = 1,$$

and hence the result follows from (i) and (ii).

(iv) By (6.3) applied to $i = 2^r$ and $s = r - 2$, we have

$$\begin{aligned}\rho_{2^r} &= w_2^{2^{r-2}} \rho_{2^r-2\cdot 2^{r-2}} + w_3^{2^{r-2}} \rho_{2^r-3\cdot 2^{r-2}} + w_4^{2^{r-2}} \rho_{2^r-4\cdot 2^{r-2}} \\ &= w_2^{2^{r-2}} \rho_{2^{r-1}} + w_3^{2^{r-2}} \rho_{2^{r-2}} + w_4^{2^{r-2}}.\end{aligned}$$

Since neither $w_2^{2^{r-2}} \rho_{2^{r-1}}$ nor $w_3^{2^{r-2}} \rho_{2^{r-2}}$ contains $w_4^{2^{r-2}}$, the proof is complete. ■

We are ready to prove the main result of this subsection.

PROPOSITION 6.5. *Let $n \geq 4$ be an integer. If $t \geq 3$ is the unique integer such that $2^{t-1} < n + 4 \leq 2^t$, then*

$$\operatorname{charrank}(\widetilde{\gamma}_{4,n}) \geq \min\{4n - 3 \cdot 2^{t-1} + 11, 2^t - 5\}.$$

Proof. Let $\delta = \min\{4n-3\cdot 2^{t-1}+11, 2^t-5\}$. By (3.2) we need to prove that $H^j(G_{4,n}) \xrightarrow{w_1} H^{j+1}(G_{4,n})$ is a monomorphism if $0 \le j \le \delta$. This is immediate from Theorem 2.1 for $0 \le j \le n-1$. So, it is enough to show that

$$H^{n+x}(G_{4,n}) \xrightarrow{w_1} H^{n+1+x}(G_{4,n})$$

is a monomorphism for all $x \in \{0,1,\dots,\delta-n\}$. This will be done by using Proposition 4.1, since $x \le \delta - n$ implies $x \le n-1$. Namely, if $n \ge 2^{t-1}-2$, then $x \le \delta - n \le 2^t - 5 - n \le n-1$; and if $n = 2^{t-1}-3$, then $x \le \delta - n \le 4n - 3\cdot 2^{t-1} + 11 - n = 2 < n-1$. So, it remains to prove that $N_x(G_{4,n})$ is linearly independent in $H^{n+1+x}(G_{4,n})$ for all $x \in \{0,1,\dots,\delta-n\}$.

Let $0 \le x \le \delta - n$ and suppose that some linear combination of elements of $N_x(G_{4,n})$ vanishes. Then, as in the proof of Lemma 4.2, in $H^{n+1+x}(G_{4,n})$ we have the relation

$$q_{x-3}\rho_{n+4} + q_{x-2}\rho_{n+3} + q_{x-1}\rho_{n+2} + q_x\rho_{n+1} = 0, \tag{6.4}$$

where $q_{x-j} \in H^{x-j}(G_{4,n})$, $0 \le j \le 3$, are some polynomials in w_2, w_3 and w_4. Arguing as in the proof of Lemma 4.2, it now suffices to show that $q_{x-3} = q_{x-2} = q_{x-1} = q_x = 0$, where q_{x-3}, q_{x-2}, q_{x-1} and q_x are considered as elements of the polynomial algebra $\mathbb{Z}_2[w_2,w_3,w_4]$.

So, from now on $q_{x-j} \in \mathbb{Z}_2[w_2,w_3,w_4]$, $0 \le j \le 3$. If $w_2^b w_3^c w_4^d$ is a monomial of (cohomological) dimension $n+1+x$, then $x \le n-1$ implies $2(b+c+d) \le 2b+3c+4d = n+1+x \le 2n$, i.e., $b+c+d \le n$. From Theorem 2.1 we now conclude that (6.4) holds in $\mathbb{Z}_2[w_2,w_3,w_4]$ as well.

Let $s = n+4-2^{t-1}$ (then $s > 0$) and

$$(p_{x+s-3}\ \ p_{x+s-2}\ \ p_{x+s-1}\ \ p_{x+s}) = (q_{x-3}\ \ q_{x-2}\ \ q_{x-1}\ \ q_x)A^s. \tag{6.5}$$

Note that the dimension of $p_{x+s-j} \in \mathbb{Z}_2[w_2,w_3,w_4]$ is $x+s-j$, $0 \le j \le 3$. Multiplying (6.5) by the column $(\rho_{2^{t-1}}\ \ \rho_{2^{t-1}-1}\ \ \rho_{2^{t-1}-2}\ \ \rho_{2^{t-1}-3})^T$, and using (6.2) (for $i = n+4$) and (6.4), we get

$$p_{x+s-3}\rho_{2^{t-1}} + p_{x+s-2}\rho_{2^{t-1}-1} + p_{x+s-1}\rho_{2^{t-1}-2} + p_{x+s}\rho_{2^{t-1}-3} = 0 \tag{6.6}$$

in $\mathbb{Z}_2[w_2,w_3,w_4]$.

By Lemma 6.3 we have $\rho_{2^{t-1}-3} = 0$. Next, we show that $p_{x+s-3} = 0$. Suppose that this is not the case. Then, by Lemma 6.4 (iv), $p_{x+s-3}\rho_{2^{t-1}}$ contains a monomial divisible by $w_4^{2^{t-3}}$, and hence, by (6.6), one of $p_{x+s-2}\rho_{2^{t-1}-1}$ and $p_{x+s-1}\rho_{2^{t-1}-2}$ contains the same monomial. Since, by Lemma 6.4(i)&(ii), neither $\rho_{2^{t-1}-1}$ nor $\rho_{2^{t-1}-2}$ contains a monomial divisible by w_4, it follows that either p_{x+s-2} or p_{x+s-1} contains a monomial divisible by $w_4^{2^{t-3}}$. By comparing the dimensions of these polynomials, we conclude that $2^{t-1} \le x+s-1$. On the other hand, $x+s-1 \le \delta - n + n + 3 - 2^{t-1} \le 2^t - 5 + 3 - 2^{t-1} = 2^{t-1} - 2$, which contradicts the previous inequality.

So, (6.6) simplifies to $p_{x+s-2}\rho_{2^{t-1}-1} = p_{x+s-1}\rho_{2^{t-1}-2}$. By Lemma 6.4(iii), the polynomials $\rho_{2^{t-1}-2}$ and $\rho_{2^{t-1}-1}$ are coprime (they are also nonzero, by Lemma 6.3), so it follows that $\rho_{2^{t-1}-2}$ divides p_{x+s-2}. If $p_{x+s-2} \neq 0$, then by comparing the dimensions of $\rho_{2^{t-1}-2}$ and p_{x+s-2}, we have $2^{t-1} - 2 \leq x + s - 2 \leq \delta - n + s - 2 \leq 2^t - 5 - 2^{t-1} + 2 = 2^{t-1} - 3$. This contradiction proves that $p_{x+s-2} = 0$, and since $\rho_{2^{t-1}-2} \neq 0$, from (6.6) we conclude that $p_{x+s-1} = 0$.

Now, if we multiply (6.5) by the matrix B^s, we obtain

$$(0\ \ 0\ \ 0\ \ p_{x+s})B^s = (w_4^s q_{x-3}\ \ w_4^s q_{x-2}\ \ w_4^s q_{x-1}\ \ w_4^s q_x), \tag{6.7}$$

since $A^sB^s = w_4^s I$. For a matrix C over the ring $\mathbb{Z}_2[w_2, w_3, w_4]$, denote by $\overline{C}$ its (entrywise) reduction modulo w_4. So, $\overline{B}$ and $\overline{B^s}$ are matrices over the ring $\mathbb{Z}_2[w_2, w_3, w_4]/(w_4) \cong \mathbb{Z}_2[w_2, w_3]$. Since the reduction $\mathbb{Z}_2[w_2, w_3, w_4] \to \mathbb{Z}_2[w_2, w_3]$ is a ring homomorphism, we have $\overline{B^s} = \overline{B}^s$, and now an easy induction shows that

$$\overline{B^s} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ w_3^{s-1} & 0 & w_2 w_3^{s-1} & w_3^s \end{pmatrix}.$$

The bottom-right entry of the matrix B^s is thus equal to $w_3^s + w_4\widetilde{p}$ for some $\widetilde{p} \in \mathbb{Z}_2[w_2, w_3, w_4]$. From (6.7) it follows that $w_4^s q_x = p_{x+s}(w_3^s + w_4\widetilde{p})$, and so $w_4^s \mid p_{x+s}$. If $p_{x+s} \neq 0$, then, again by the dimension argument, we deduce that $4s \leq x + s \leq \delta - n + s \leq 3n - 3 \cdot 2^{t-1} + 11 + s = 3s - 1 + s = 4s - 1$. This contradiction leads to $p_{x+s} = 0$. Finally, (6.7) implies

$$q_{x-3} = q_{x-2} = q_{x-1} = q_x = 0,$$

which was to be proved. ∎

From Propositions 6.2 and 6.5 we obtain the exact value of $\operatorname{charrank}(\widetilde{\gamma}_{4,n})$ for all $n \geq 4$.

THEOREM 6.6. *Let $n \geq 4$ be an integer. If $t \geq 3$ is the unique integer such that $2^{t-1} < n + 4 \leq 2^t$, then*

$$\operatorname{charrank}(\widetilde{\gamma}_{4,n}) = \min\{4n - 3 \cdot 2^{t-1} + 11, 2^t - 5\}.$$

REMARK 6.7. This theorem improves [4, Theorem 2.1(2)].

REMARK 6.8. It seems to be difficult to adapt the methods used in this section to work for some $k \geq 5$ (and all $n \geq k$). There are several reasons for this. First, it is proven in [4, Lemma 2.3(iii)] that $\rho_i \neq 0$ for all $i \geq 2$ if $k \geq 5$. This affects the proofs of both Propositions 6.2 and 6.5. Also, it is not clear how to find consecutive indices i so that the polynomials ρ_i have properties similar to the ones in Lemma 6.4. Finally, by [1], the best known upper bound for $\operatorname{charrank}(\widetilde{\gamma}_{k,n})$ is $kn/2+1$ (at least when $n+k$ is odd). So, there are reasons to believe that for $k \geq 5$, there is some n such that $\operatorname{charrank}(\widetilde{\gamma}_{k,n}) > 2n-1$. On the other hand, the best lower bound for $\operatorname{charrank}(\widetilde{\gamma}_{k,n})$ that one can establish using Proposition 4.1 is $2n-1$.

6.3. Cup-length. The $\mathbb{Z}_2$-*cup-length* (or simply *cup-length*) of a path connected space X, denoted by $\operatorname{cup}(X)$, is the maximal r such that there exist classes $x_1, \ldots, x_r \in \widetilde{H}^*(X)$ with nontrivial cup product ($x_1x_2\cdots x_r \neq 0$). Also, the *height* of a class $x \in \widetilde{H}^*(X)$, denoted by $\operatorname{ht}(x)$, is the maximal r such that $x^r \neq 0$ in $\widetilde{H}^*(X)$.

Naolekar and Thakur [7] showed how characteristic rank can be used to obtain an upper bound for $\mathbb{Z}_2$-cup-length.

THEOREM 6.9 ([7]). *Let M be a smooth closed connected d-dimensional manifold. Let α be a real vector bundle over M and let $j \leq \operatorname{charrank}(\alpha)$ be an integer such that every monomial $w_{i_1}(\alpha)\cdots w_{i_s}(\alpha)$, $1 \leq i_1 \leq \cdots \leq i_s \leq j$, in dimension d vanishes. Then*

$$\operatorname{cup}(M) \leq 1 + \frac{d-j-1}{r},$$

where r is the smallest positive integer such that $H^r(M) \neq 0$.

In the case $M = \widetilde{G}_{4,n}$, we have $d = 4n$ and $r = 2$. It is also well known that the nonzero class in $H^{4n}(\widetilde{G}_{4,n}) \cong \mathbb{Z}_2$ is not a polynomial in the Stiefel–Whitney classes of the canonical bundle $\widetilde{\gamma}_{4,n}$ (see e.g. [4, p. 1171]). Therefore, for the bundle $\alpha := \widetilde{\gamma}_{4,n}$ we take $j := \operatorname{charrank}(\widetilde{\gamma}_{4,n})$, and Theorem 6.9 gives

$$\operatorname{cup}(\widetilde{G}_{4,n}) \leq 1 + \frac{4n - \operatorname{charrank}(\widetilde{\gamma}_{4,n}) - 1}{2}. \tag{6.8}$$

To obtain a lower bound we use the following result of Stong.

PROPOSITION 6.10 ([11]). *If $2^{t-1}+1 \leq n+4 \leq 2^{t-1}+2$, then in $H^*(G_{4,n})$ one has $w_1^{2^t-1}w_2^{2^{t-1}-7}w_3 \neq 0$. If $n+4 = 2^{t-1}+2^r+1+j$, where $1 \leq r \leq t-2$ and $0 \leq j \leq 2^r-1$, then in $H^*(G_{4,n})$ one has $w_1^{2^t-1}w_2^{2^{t-1}+2^{r+1}-7}w_3w_4^j \neq 0$.*

The first part of this proposition is formulated in [11] only for $n+4 = 2^{t-1}+1$ (not for $n+4 = 2^{t-1}+2$), but if $i : G_{4,2^{t-1}-3} \hookrightarrow G_{4,2^{t-1}-2}$ is the standard embedding, then $i^*(w_1^{2^t-1}w_2^{2^{t-1}-7}w_3) = w_1^{2^t-1}w_2^{2^{t-1}-7}w_3 \neq 0$ in $H^*(G_{4,2^{t-1}-3})$, and so $w_1^{2^t-1}w_2^{2^{t-1}-7}w_3 \neq 0$ in $H^*(G_{4,2^{t-1}-2})$ as well.

From the Gysin sequence (3.1) it is evident that a class of the form $w_2(\widetilde{\gamma}_{k,n})^{a_2}\cdots w_k(\widetilde{\gamma}_{k,n})^{a_k} = p^*(w_2^{a_2}\cdots w_k^{a_k})$ does not vanish (in $H^*(\widetilde{G}_{k,n})$) if and only if $w_2^{a_2}\cdots w_k^{a_k}$ is not a multiple of w_1 (in $H^*(G_{k,n})$). Together with Proposition 6.10 this will be the key observation leading to a lower bound for $\operatorname{cup}(\widetilde{G}_{4,n})$.

PROPOSITION 6.11. *For an integer $n \geq 4$, let $t \geq 3$ be the unique integer such that $2^{t-1} < n+4 \leq 2^t$.*

(i) *If $2^{t-1}+1 \leq n+4 \leq 2^{t-1}+2$, then*

$$2^{t-1}-5 \leq \mathrm{cup}(\widetilde{G}_{4,n}) \leq 2^{t-1}+2^{t-2}-5.$$

Now, for $2^{t-1}+3 \leq n+4 \leq 2^t$, let r and j be the (unique) integers such that $n+4 = 2^{t-1}+2^r+1+j$ with $1 \leq r \leq t-2$, $0 \leq j \leq 2^r-1$.

(ii) *If $2^{t-1}+3 \leq n+4 \leq 2^{t-1}+2^{t-3}$ (i.e., $1 \leq r \leq t-4$), then*

$$2^{t-1}+2^{r+1}+j-5 \leq \mathrm{cup}(\widetilde{G}_{4,n}) \leq 2^{t-1}+2^{t-2}-5.$$

(iii) *If $2^{t-1}+2^{t-3}+1 \leq n+4 \leq 2^t$ (i.e., $t-3 \leq r \leq t-2$), then*

$$2^{t-1}+2^{r+1}+j-5 \leq \mathrm{cup}(\widetilde{G}_{4,n}) \leq 2^{t-1}+2^{r+1}+2j-3.$$

Proof. If $n+4 \leq 2^{t-1}+2^{t-3}$, then $\mathrm{charrank}(\widetilde{\gamma}_{4,n}) = 4n-3\cdot 2^{t-1}+11$, and (6.8) gives $\mathrm{cup}(\widetilde{G}_{4,n}) \leq 3\cdot 2^{t-2}-5 = 2^{t-1}+2^{t-2}-5$.

If $n+4 \geq 2^{t-1}+2^{t-3}+1$, then $\mathrm{charrank}(\widetilde{\gamma}_{4,n}) = 2^t-5$. If r and j are as in the statement, then (6.8) yields $\mathrm{cup}(\widetilde{G}_{4,n}) \leq 2^{t-1}+2^{r+1}+2j-3$.

So, it remains to establish the stated lower bounds. For that purpose, note that $w_1^{2^t} = 0$ in $H^*(G_{4,n})$, since $\mathrm{ht}(w_1) = 2^t-1$ by [11].

Let $2^{t-1}+1 \leq n+4 \leq 2^{t-1}+2$. By Proposition 6.10, $w_1^{2^{t-1}-1}w_2^{2^{t-1}-7}w_3 \neq 0$, and since $w_1^{2^t} = 0$, we conclude that $w_2^{2^{t-1}-7}w_3$ is not divisible by w_1. So,

$$w_2(\widetilde{\gamma}_{4,n})^{2^{t-1}-7}w_3(\widetilde{\gamma}_{4,n}) \neq 0$$

in $H^*(\widetilde{G}_{4,n})$, and hence, by the Poincaré duality, there exists a class σ such that $\sigma \cdot w_2(\widetilde{\gamma}_{4,n})^{2^{t-1}-7}w_3(\widetilde{\gamma}_{4,n})$ is nonzero in $H^{4n}(\widetilde{G}_{4,n})$. This means that $\mathrm{cup}(\widetilde{G}_{4,n}) \geq 2^{t-1}-5$.

In the case $2^{t-1}+3 \leq n+4 \leq 2^t$ (with r and j as above), again by Proposition 6.10, we have $w_1^{2^{t-1}-1}w_2^{2^{t-1}+2^{r+1}-7}w_3w_4^j \neq 0$, and then the fact that $w_1^{2^t} = 0$ shows that $w_2^{2^{t-1}+2^{r+1}-7}w_3w_4^j$ is not a multiple of w_1. So,

$$w_2(\widetilde{\gamma}_{4,n})^{2^{t-1}+2^{r+1}-7}w_3(\widetilde{\gamma}_{4,n})w_4(\widetilde{\gamma}_{4,n})^j \neq 0$$

in $H^*(\widetilde{G}_{4,n})$, and the Poincaré duality gives a class σ with the property $\sigma \cdot w_2(\widetilde{\gamma}_{4,n})^{2^{t-1}+2^{r+1}-7}w_3(\widetilde{\gamma}_{4,n})w_4(\widetilde{\gamma}_{4,n})^j \neq 0$ in $H^{4n}(\widetilde{G}_{4,n})$. Finally, we conclude that $\mathrm{cup}(\widetilde{G}_{4,n}) \geq 2^{t-1}+2^{r+1}+j-5$. ∎

REMARK 6.12. Proposition 6.11 improves [4, Theorem 3.1(2)].

Acknowledgements. The first author is partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia Project #174032. The second author is partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia Project #174008.

References

[1] L'. Balko and J. Korbaš, *A note on the characteristic rank of null-cobordant manifolds*, Acta Math. Hungar. 140 (2013), 145–150.

[2] A. Borel, *La cohomologie mod 2 de certains espaces homogènes*, Comment. Math. Helv. 27 (1953), 165–197.

[3] J. Jaworowski, *An additive basis for the cohomology of real Grassmannians*, in: Lecture Notes in Math. 1474, Springer, Berlin, 1991, 231–234.

[4] J. Korbaš, *The characteristic rank and cup-length in oriented Grassmann manifolds*, Osaka J. Math. 52 (2015), 1163–1172.

[5] J. Korbaš, *The cup-length of the oriented Grassmannians vs a new bound for zero-cobordant manifolds*, Bull. Belg. Math. Soc. Simon Stevin 17 (2010), 69–81.

[6] J. Korbaš and T. Rusin, *On the cohomology of oriented Grassmann manifolds*, Homol. Homotopy Appl. 18 (2016), 71–84.

[7] A. C. Naolekar and A. S. Thakur, *Note on the characteristic rank of vector bundles*, Math. Slovaca 64 (2014), 1525–1540.

[8] Z. Z. Petrović, B. I. Prvulović and M. Radovanović, *Multiplication in the cohomology of Grassmannians via Gröbner bases*, J. Algebra 438 (2015), 60–84.

[9] Z. Z. Petrović, B. I. Prvulović and M. Radovanović, *Characteristic rank of canonical vector bundles over oriented Grassmann manifolds* $\widetilde{G}_{3,n}$, Topology Appl. 230 (2017), 114–121.

[10] T. Rusin, *A note on the characteristic rank of oriented Grassmann manifolds*, Topology Appl. 216 (2017), 48–58.

[11] R. E. Stong, *Cup products in Grassmannians*, Topology Appl. 13 (1982), 103–113.

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