ON MAXIMALITY OF THE CUP-LENGTH OF FLAG MANIFOLDS

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Abstract. In this paper we investigate which flag manifolds of the form $F(1,\dots,1,2,\dots,2,m)$ have $\mathbb{Z}_2$-cup-length equal to its dimension. We obtain a complete classification of such manifolds of the form $F(1,\dots,1,2,m)$ and $F(1,\dots,1,2,2,m)$. Additionally, we provide an infinite family of manifolds $F(1,\dots,1,2,\dots,2,m)$ which give the negative answer to a question from [J. Korbaš, J. Lörinc, *The $\mathbb{Z}_2$-cohomology cup-length of real flag manifolds*, Fund. Math. **178** (2003) 143–158].

1. Introduction

Let q and $n_1,n_2,\dots,n_q$ be positive integers. A *real flag of type* $(n_1,n_2,\dots,n_q)$ is a q-tuple $(V_1,V_2,\dots,V_q)$ of mutually orthogonal subspaces of $\mathbb{R}^n$, where $n=n_1+n_2+\cdots+n_q$, and $\dim V_i=n_i$, for $i\in\{1,2,\dots,q\}$. The space of all such flags is the *real flag manifold* $F(n_1,n_2,\dots,n_q)$, where the manifold structure comes from the natural identification $F(n_1,n_2,\dots,n_q)=O(n)/O(n_1)\times O(n_2)\times\cdots\times O(n_q)$. With this identification, $F(n_1,n_2,\dots,n_q)$ becomes a closed manifold of dimension $\dim F(n_1,n_2,\dots,n_q)=\sum_{1\le i<j\le q}n_in_j$ (in this paper dimension of a manifold M will be denoted by $\dim M$). From this definition, it is easy to see that for any permutation $\sigma\in\mathbb{S}_q$, one has $F(n_1,n_2,\dots,n_q)\approx F(n_{\sigma(1)},n_{\sigma(2)},\dots,n_{\sigma(q)})$. Consequently, it suffices to consider flag manifolds $F(n_1,n_2,\dots,n_q)$ with $n_1\le n_2\le\cdots\le n_q$. Note that flag manifolds with $q=2$ are in fact *Grassmann manifolds*. One other important and well-studied class of flag manifolds are the *complete flag manifolds* – these are the flag manifolds with $n_1=n_2=\ldots=n_q=1$.

2010 *Mathematics Subject Classification.* 57N65, 14M15, 55M30.

Key words and phrases. flag manifolds, Stiefel-Whitney classes, cup-length, Lyusternik-Schnirelmann category.

The first author was partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia Project #174032. The second author was partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia Project #174034. The third author was partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia Project #174008.

Over the real flag manifold $F(n_1, n_2, \ldots, n_q)$ there are q canonical vector bundles – in this paper we will denote them by $\gamma_1, \gamma_2, \ldots, \gamma_q$, where $\dim \gamma_i = n_i$, for $i \in \{1, 2, \ldots, q\}$.

For a commutative ring R, the R-cup-length of a path connected space X, denoted by $\operatorname{cup}_R X$, is the supremum of all integers d such that there exist classes $a_1, a_2, \ldots, a_d \in \widetilde{H}^*(X; R)$ with nonzero cup product. In this paper we will only work with mod 2 cohomologies, so we will write "cup-length" for "$\mathbb{Z}_2$-cup-length", and $\operatorname{cup} X$ for $\operatorname{cup}_{\mathbb{Z}_2} X$.

An important property of $\operatorname{cup} X$ is that it provides a lower bound for the *Lyusternik-Shnirelmann category* of X (denoted by $\operatorname{cat} X$). Additionally, for a manifold M one has

$$1 + \dim M \geq \operatorname{cat} M \geq 1 + \operatorname{cup} M.$$

It is clear that $\operatorname{cup} M \leq \dim M$, and if $\operatorname{cup} M = \dim M$, then $\operatorname{cat} M = \dim M + 1$.

We adopt the notation from [5]:

$$a^{\cdots k} = \underbrace{a, a, \ldots, a}_{k}.$$

For a general real flag manifold the cup-length is not known; it is not known even for a general Grassmann manifold. In [2] and [7] the authors obtained cup-length of Grassmann manifolds $F(2, n)$, $F(3, n)$ and $F(4, n)$; in [3] and [4] some bounds for the cup-length of oriented Grassmann manifolds are obtained; in [5] and [6] the cup-length of some real flag manifolds of the form $F(1^{\cdots j}, 2^{\cdots d}, n)$ is obtained.

In [5] Korbaš and Lörinc investigated for which flag manifolds the cup-length is maximal (i.e., equal to the dimension of the manifold). They established that this holds for $F(1^{\cdots k}, m)$ $(k, m \geq 1)$ and that a sufficient condition for $\operatorname{cup} F(1^{\cdots j}, 2^{\cdots d}, m) = \dim F(1^{\cdots j}, 2^{\cdots d}, m)$ $(j, d \geq 1,\ m \geq 2)$ is the inequality $j \geq 2^{t+d} - m - 2d + 1$, where t is the integer such that $2^t \leq m < 2^{t+1}$. They also asked whether this condition is necessary as well ([5, p. 148]) and gave some examples which indicate that this could be true.

In this paper we prove that the answer to the proposed question, although positive for $d = 1$ (Proposition 2.3), is negative in general. Actually, for $d \geq 2$, in Theorem 2.4 we present a weaker condition which guaranties the maximality of the cup-length for the manifolds $F(1^{\cdots j}, 2^{\cdots d}, m)$, and establish that in the case $d = 2$ this condition is also necessary (Corollary 2.4.1). Finally, in Proposition 2.5, we prove that the condition of Theorem 2.4, although necessary for $d = 2$, is not necessary in general.

At the end of the paper, we show that $\operatorname{cup} F(1, 1, 1, 3, 4) = \dim F(1, 1, 1, 3, 4)$, which is an indication that the general problem of detecting all flag manifolds $F(n_1, n_2, \ldots, n_q)$ with maximal cup-length might be very difficult.

The method that we are using for calculating the cup-length is the one presented in [5, p. 154].

2. Evaluation of cup-length

Let $n \geq 2$ and observe the complete flag manifold $F(1^{\cdots n})$. Denote by $e_i := w_1(\gamma_i)$ the first Stiefel-Whitney class of the canonical line bundle γ_i over $F(1^{\cdots n})$, $1 \leq i \leq n$. For an n-tuple $(a_1, a_2, \ldots, a_n)$ of nonnegative integers, the monomial $e_1^{a_1} e_2^{a_2} \cdots e_n^{a_n} \in H^*(F(1^{\cdots n}); \mathbb{Z}_2)$ will be abbreviated to $E^{(a_1, a_2, \ldots, a_n)}$. The statement of the following lemma is well known (see [5, 7]).

Lemma 2.1. *A top dimensional monomial $E^{(a_1, a_2, \ldots, a_n)}$ (i.e., such that $a_1 + a_2 + \cdots + a_n = \dim F(1^{\cdots n}) = \binom{n}{2}$) is nonzero in $H^{\binom{n}{2}}(F(1^{\cdots n}); \mathbb{Z}_2) \cong \mathbb{Z}_2$ if and only if $(a_1, a_2, \ldots, a_n)$ is a permutation of the n-tuple $(n-1, n-2, \ldots, 1, 0)$.*

Now, let $n_1, n_2, \ldots, n_q$ $(q \geq 2)$ be positive integers, $\nu_i = n_1 + n_2 + \cdots + n_i$, $0 \leq i \leq q$ (it is understood that $\nu_0 = 0$), and $n = \nu_q = n_1 + n_2 + \cdots + n_q$. For the flag manifold $F(n_1, n_2, \ldots, n_q)$ we have the map $p : F(1^{\cdots n}) \to F(n_1, n_2, \ldots, n_q)$, given by

$$p(S_1, \ldots, S_{n_1}, \ldots, S_{\nu_{q-1}+1}, \ldots, S_n) = (S_1 \oplus \cdots \oplus S_{n_1}, \ldots, S_{\nu_{q-1}+1} \oplus \cdots \oplus S_n).$$

Our calculation of the cup-length will be based on the following observation from [5, p. 154].

Lemma 2.2. *If $F = F(n_1, n_2, \ldots, n_q)$, $u \in H^{\dim F}(F; \mathbb{Z}_2)$ and*

$$v = E^{(n_1-1, n_1-2, \ldots, 1, 0, n_2-1, n_2-2, \ldots, 1, 0, \ldots, n_q-1, n_q-2, \ldots, 1, 0)} \in H^*(F(1^{\cdots n}); \mathbb{Z}_2),$$

then $p^*(u) \cdot v \in H^{\binom{n}{2}}(F(1^{\cdots n}); \mathbb{Z}_2)$ and

$$u \neq 0 \quad \Longleftrightarrow \quad p^*(u) \cdot v \neq 0.$$

In [5, p. 155] the authors also note the following fact. If $w_{i,k}$ is the k-th Stiefel-Whitney class of the canonical bundle γ_i over $F(n_1, n_2, \ldots, n_q)$, $1 \leq k \leq n_i$, $1 \leq i \leq q$, then $p^*(w_{i,k})$ is the k-th elementary symmetric polynomial in variables $e_{\nu_{i-1}+1}, e_{\nu_{i-1}+2}, \ldots, e_{\nu_i}$. For example,

$$p^*(w_{i,1}) = e_{\nu_{i-1}+1} + e_{\nu_{i-1}+2} + \cdots + e_{\nu_i}. \tag{2.1}$$

Now we restrict our attention to the flag manifolds $F(1^{\cdots j}, 2^{\cdots d}, m)$. The following proposition gives a purely arithmetic condition on integers j, d, m which guaranties that $\operatorname{cup} F(1^{\cdots j}, 2^{\cdots d}, m) = \dim F(1^{\cdots j}, 2^{\cdots d}, m)$, and consequently, $\operatorname{cat} F(1^{\cdots j}, 2^{\cdots d}, m) = 1 + \dim F(1^{\cdots j}, 2^{\cdots d}, m)$.

Proposition 2.1. *Let $F = F(1^{\cdots j}, 2^{\cdots d}, m)$, $j \geq 0$, $d \geq 1$, $m \geq 2$. Suppose that there exist pairwise different integers*

$$\lambda_1, \lambda_2, \ldots, \lambda_d, \mu_1, \mu_2, \ldots, \mu_d \in \{m, m+1, \ldots, m+j+2d-1\}$$

which satisfy the following conditions:

(i) *if* $\{\alpha_1,\alpha_2,\dots,\alpha_d,\beta_1,\beta_2,\dots,\beta_d\} = \{\lambda_1,\lambda_2,\dots,\lambda_d,\mu_1,\mu_2,\dots,\mu_d\}$ *and* $\alpha_i+\beta_i=\lambda_i+\mu_i$ *for all* $i\in\{1,2,\dots,d\}$*, then* $\{\alpha_i,\beta_i\}=\{\lambda_i,\mu_i\}$ *for all* $i\in\{1,2,\dots,d\}$ *(in other words, there is a unique way for the set* $\{\lambda_1,\lambda_2,\dots,\lambda_d,\mu_1,\mu_2,\dots,\mu_d\}$ *to be partitioned in* d *pairs such that the sums of the pairs are exactly* $\lambda_i+\mu_i$*,* $1\le i\le d$*);*

(ii) $\binom{\lambda_1+\mu_1}{\lambda_1}\binom{\lambda_2+\mu_2}{\lambda_2}\cdots\binom{\lambda_d+\mu_d}{\lambda_d}\equiv 1 \pmod 2$.

Proof. Let

$$\{a_1,a_2,\dots,a_j\}=\{m,m+1,\dots,m+j+2d-1\}\setminus\{\lambda_1,\lambda_2,\dots,\lambda_d,\mu_1,\mu_2,\dots,\mu_d\}.$$

Observe the monomial

$$u=w_{1,1}^{a_1}w_{2,1}^{a_2}\cdots w_{j,1}^{a_j}w_{j+1,1}^{\lambda_1+\mu_1-1}w_{j+2,1}^{\lambda_2+\mu_2-1}\cdots w_{j+d,1}^{\lambda_d+\mu_d-1}\in H^*(F;\mathbb{Z}_2).$$

The sum of the exponents in u is equal to

$$\begin{aligned}a_1+a_2+\cdots+a_j+\lambda_1+\lambda_2+\cdots+\lambda_d+\mu_1+\mu_2+\cdots+\mu_d-d&=\\ &=m+(m+1)+\cdots+(m+j+2d-1)-d\\ &=m(j+2d)+\tbinom{j}{2}+2jd+4\tbinom{d}{2}\\ &=\dim F,\end{aligned}$$

and so, it suffices to prove that $u\neq 0$. Since $w_{i,1}\in H^1(F;\mathbb{Z}_2)$, $1\le i\le j+d$, we also have that $u\in H^{\dim F}(F;\mathbb{Z}_2)$, and, according to Lemma 2.2, we need to show that $p^*(u)\cdot E^{(0\cdots j,1,0,1,0,\dots,1,0,m-1,m-2,\dots,2,1,0)}\neq 0$ in $H^{\binom{n}{2}}(F(1\cdots n);\mathbb{Z}_2)$, where $n=m+j+2d$ and $p:F(1\cdots n)\to F$ is the previously defined map. If $z\in H^{\binom{n}{2}}(F(1\cdots n);\mathbb{Z}_2)\cong\mathbb{Z}_2$ is the generator, we set

$$p^*(u)\cdot E^{(0\cdots j,1,0,1,0,\dots,1,0,m-1,m-2,\dots,2,1,0)}=\theta\cdot z,\quad \theta\in\{0,1\}.$$

We want to prove that $\theta=1$. Now, by (2.1) we know that

$$\begin{aligned}p^*(u)&=e_1^{a_1}e_2^{a_2}\cdots e_j^{a_j}(e_{j+1}+e_{j+2})^{\lambda_1+\mu_1-1}\cdots(e_{j+2d-1}+e_{j+2d})^{\lambda_d+\mu_d-1}\\ &=\sum_{\substack{\alpha_i+\beta_i=\lambda_i+\mu_i-1\\ 1\le i\le d}}\binom{\lambda_1+\mu_1-1}{\alpha_1}\cdots\binom{\lambda_d+\mu_d-1}{\alpha_d}E^{(a_1,\dots,a_j,\alpha_1,\beta_1,\dots,\alpha_d,\beta_d,0\cdots m)},\end{aligned}$$

where the sum is taken over all $2d$-tuples of integers $(\alpha_1,\beta_1,\alpha_2,\beta_2,\dots,\alpha_d,\beta_d)$ such that $\alpha_i+\beta_i=\lambda_i+\mu_i-1$ for all $i\in\{1,2,\dots,d\}$. Therefore,

$$\begin{aligned}&p^*(u)\cdot E^{(0\cdots j,1,0,1,0,\dots,1,0,m-1,m-2,\dots,2,1,0)}=\\ &=\sum_{\substack{\alpha_i+\beta_i=\lambda_i+\mu_i-1\\ 1\le i\le d}}\binom{\lambda_1+\mu_1-1}{\alpha_1}\cdots\binom{\lambda_d+\mu_d-1}{\alpha_d}E^{(a_1,\dots,a_j,\alpha_1+1,\beta_1,\dots,\alpha_d+1,\beta_d,m-1,\dots,1,0)}\\ &=\sum_{\substack{\alpha_i+\beta_i=\lambda_i+\mu_i\\ 1\le i\le d}}\binom{\lambda_1+\mu_1-1}{\alpha_1-1}\cdots\binom{\lambda_d+\mu_d-1}{\alpha_d-1}E^{(a_1,\dots,a_j,\alpha_1,\beta_1,\dots,\alpha_d,\beta_d,m-1,\dots,1,0)}.\end{aligned}$$

By Lemma 2.1, the monomial $E^{(a_1,\dots,a_j,\alpha_1,\beta_1,\dots,\alpha_d,\beta_d,m-1,\dots,1,0)}$ is nonzero (i.e., equal to z) for those (and only for those) $2d$-tuples $(\alpha_1,\beta_1,\alpha_2,\beta_2,\dots,\alpha_d,\beta_d)$ which satisfy $\{\alpha_1,\alpha_2,\dots,\alpha_d,\beta_1,\beta_2,\dots,\beta_d\} = \{\lambda_1,\lambda_2,\dots,\lambda_d,\mu_1,\mu_2,\dots,\mu_d\}$ (and, of course, $\alpha_i+\beta_i=\lambda_i+\mu_i$, $1\le i\le d$). Therefore, the condition (i) implies that the nontrivial summands are exactly those with the property that for each $i\in\{1,2,\dots,d\}$ either $\alpha_i=\lambda_i$ or $\alpha_i=\mu_i$. Since $\binom{\lambda_i+\mu_i-1}{\mu_i-1}=\binom{\lambda_i+\mu_i-1}{\lambda_i}$, we have that

$$\begin{aligned}\theta &\equiv \sum_{(i_1,\dots,i_d)\in\{0,1\}^d}\binom{\lambda_1+\mu_1-1}{\lambda_1-i_1}\cdots\binom{\lambda_d+\mu_d-1}{\lambda_d-i_d}\\ &= \left(\binom{\lambda_1+\mu_1-1}{\lambda_1}+\binom{\lambda_1+\mu_1-1}{\lambda_1-1}\right)\cdots\left(\binom{\lambda_d+\mu_d-1}{\lambda_d}+\binom{\lambda_d+\mu_d-1}{\lambda_d-1}\right)\\ &= \binom{\lambda_1+\mu_1}{\lambda_1}\cdots\binom{\lambda_d+\mu_d}{\lambda_d} \pmod 2.\end{aligned}$$

The condition (ii) now finishes the proof. □

Hence, the existence of the integers λ_i,μ_i, $1\le i\le d$, with the specified properties is a sufficient condition for maximality of the cup-length. Let us now show that in the cases $d=1$ and $d=2$ this condition is necessary as well.

Proposition 2.2. *Let $j\ge 0$ and $m\ge 2$.*

(a) $\operatorname{cup} F(1^{\cdots j},2,m)=\dim F(1^{\cdots j},2,m)$ *if and only if there exist integers* $\lambda,\mu\in\{m,m+1,\dots,m+j+1\}$ *such that* $\binom{\lambda+\mu}{\lambda}\equiv 1 \pmod 2$.

(b) $\operatorname{cup} F(1^{\cdots j},2,2,m)=\dim F(1^{\cdots j},2,2,m)$ *if and only if there exist pairwise different integers* $\lambda_1,\lambda_2,\mu_1,\mu_2\in\{m,m+1,\dots,m+j+3\}$ *such that* $\lambda_1+\mu_1\ne\lambda_2+\mu_2$ *and* $\binom{\lambda_1+\mu_1}{\lambda_1}\binom{\lambda_2+\mu_2}{\lambda_2}\equiv 1 \pmod 2$.

Proof. The "if parts" of (a) and (b) follow directly from Proposition 2.1. For the opposite implications we use the following observation (see [5, p. 155]): every cohomology class $u\in H^*(F(n_1,n_2,\dots,n_q);\mathbb{Z}_2)$ can be expressed in terms of the Stiefel-Whitney classes of the first $q-1$ canonical vector bundles $\gamma_1,\gamma_2,\dots,\gamma_{q-1}$.

(a) If $\operatorname{cup} F(1^{\cdots j},2,m)=\dim F(1^{\cdots j},2,m)$, then there is a monomial $u=w_{1,1}^{a_1}w_{2,1}^{a_2}\cdots w_{j,1}^{a_j}w_{j+1,1}^{b}\ne 0$ in the top dimension of $H^*(F(1^{\cdots j},2,m);\mathbb{Z}_2)$. According to Lemma 2.2, this means that $p^*(u)\cdot E^{(0^{\cdots j},1,0,m-1,m-2,\dots,1,0)}\ne 0$, where $p: F(1^{\cdots m+j+2})\to F(1^{\cdots j},2,m)$ is the well-known map. On the other hand, by (2.1)

$$\begin{aligned}p^*(u)\cdot E^{(0^{\cdots j},1,0,m-1,\dots,1,0)} &= e_1^{a_1}e_2^{a_2}\cdots e_j^{a_j}(e_{j+1}+e_{j+2})^b\cdot E^{(0^{\cdots j},1,0,m-1,\dots,1,0)}\\ &= \sum_{k=0}^{b}\binom{b}{k}E^{(a_1,a_2,\dots,a_j,k+1,b-k,m-1,\dots,1,0)}. \qquad (2.2)\end{aligned}$$

Since this class is nonzero, by Lemma 2.1 we conclude that $a_1,a_2,\dots,a_j$ are pairwise different integers form the set $\{m,m+1,\dots,m+j+1\}$. Let $\lambda,\mu\in$

$\{m, m+1, \dots, m+j+1\}$ be such that

$$\{\lambda, \mu\} = \{m, m+1, \dots, m+j+1\} \setminus \{a_1, a_2, \dots, a_j\}.$$

By the nontriviality of the sum (2.2) and Lemma 2.1, we also have that there exists an integer k with the property $\{\lambda, \mu\} = \{k+1, b-k\}$, which means that $\lambda + \mu = k + 1 + b - k = b + 1$.

Now, according to Lemma 2.1 again, the nonzero summands in (2.2) are just those for $k = \lambda - 1$ and $k = \mu - 1$. Therefore, the coefficient

$$\binom{b}{\lambda-1} + \binom{b}{\mu-1} = \binom{\lambda+\mu-1}{\lambda-1} + \binom{\lambda+\mu-1}{\mu-1} = \binom{\lambda+\mu-1}{\lambda-1} + \binom{\lambda+\mu-1}{\lambda} = \binom{\lambda+\mu}{\lambda}$$

(considered modulo 2) is nonzero.

(b) The proof of this part is similar. If $\operatorname{cup} F(1^{\cdots j}, 2, 2, m)$ is maximal, then there is a monomial $u = w_{1,1}^{a_1} w_{2,1}^{a_2} \cdots w_{j,1}^{a_j} w_{j+1,1}^{b} w_{j+2,1}^{c} \neq 0$ in the top dimension of $H^*(F(1^{\cdots j}, 2, 2, m); \mathbb{Z}_2)$. Now we have that

$$\begin{aligned} 0 &\neq p^*(u) \cdot E^{(0^{\cdots j},1,0,1,0,m-1,m-2,\dots,1,0)} \\ &= e_1^{a_1} e_2^{a_2} \cdots e_j^{a_j} (e_{j+1} + e_{j+2})^b (e_{j+3} + e_{j+4})^c \cdot E^{(0^{\cdots j},1,0,1,0,m-1,\dots,1,0)} \\ &= \sum_{k=0}^{b} \sum_{l=0}^{c} \binom{b}{k}\binom{c}{l} E^{(a_1,a_2,\dots,a_j,k+1,b-k,l+1,c-l,m-1,\dots,1,0)}. \end{aligned} \tag{2.3}$$

All summands for $k = l$ are zero (by Lemma 2.1). If b was equal to c, then the rest of the summands would cancel out in pairs (a pair of summands is obtained by interchanging k and l), and the sum would be trivial. So we conclude that $b \neq c$.

Reasoning as in the proof of part (a), we take $\lambda_1, \lambda_2, \mu_1, \mu_2$ to be the four integers from the set $\{m, m+1, \dots, m+j+3\} \setminus \{a_1, a_2, \dots, a_j\}$. According to (2.3) and Lemma 2.1, there exist nonnegative integers k and l such that $\{\lambda_1, \mu_1, \lambda_2, \mu_2\} = \{k+1, b-k, l+1, c-l\}$, and therefore, we can take that $\lambda_1 + \mu_1 = b+1$ and $\lambda_2 + \mu_2 = c+1$. Since $b \neq c$, we have that $\lambda_1 + \mu_1 \neq \lambda_2 + \mu_2$, and we are left to prove that $\binom{\lambda_1+\mu_1}{\lambda_1}\binom{\lambda_2+\mu_2}{\lambda_2} \equiv 1 \pmod 2$.

By Lemma 2.1, if $E^{(a_1,a_2,\dots,a_j,k+1,b-k,l+1,c-l,m-1,\dots,1,0)} \neq 0$, then $k+1$ must be one of the integers $\lambda_1, \mu_1, \lambda_2, \mu_2$, and likewise for $b-k$. However, $k+1 \in \{\lambda_2, \mu_2\}$ implies that $\lambda_1 + \mu_1 = b + 1 = (k+1) + (b-k)$ is equal to the sum of some other two elements of the set $\{\lambda_1, \mu_1, \lambda_2, \mu_2\}$, and this is impossible since these are four distinct integers and $\lambda_1 + \mu_1 \neq \lambda_2 + \mu_2$. We conclude that k can take the values $\lambda_1 - 1$ and $\mu_1 - 1$ only, and similarly, l must be either $\lambda_2 - 1$ or $\mu_2 - 1$. This means that the coefficients of the nonzero summands in the (nontrivial) sum (2.3) are precisely $\binom{\lambda_1+\mu_1-1}{\lambda_1-1}\binom{\lambda_2+\mu_2-1}{\lambda_2-1}$, $\binom{\lambda_1+\mu_1-1}{\lambda_1-1}\binom{\lambda_2+\mu_2-1}{\mu_2-1}$, $\binom{\lambda_1+\mu_1-1}{\mu_1-1}\binom{\lambda_2+\mu_2-1}{\lambda_2-1}$ and $\binom{\lambda_1+\mu_1-1}{\mu_1-1}\binom{\lambda_2+\mu_2-1}{\mu_2-1}$. It is now routine to check that the sum of this coefficients is $\binom{\lambda_1+\mu_1}{\lambda_1}\binom{\lambda_2+\mu_2}{\lambda_2}$, which completes the proof of the proposition. □

We now prove a technical lemma.

Lemma 2.3. *Let λ and μ be positive integers with the binary expansions $\lambda = \sum_{i\geq 0} a_i 2^i$ and $\mu = \sum_{i\geq 0} b_i 2^i$ ($a_i, b_i \in \{0,1\}$). Then the following equivalence holds:*

$$\binom{\lambda+\mu}{\lambda} \equiv 1 \pmod 2 \qquad \Longleftrightarrow \qquad a_i b_i = 0 \text{ for all } i \geq 0.$$

Proof. Let $\lambda+\mu = \sum_{i\geq 0} c_i 2^i$ be the binary expansion of $\lambda+\mu$. If $a_i b_i = 0$ for all $i \geq 0$, then $a_i + b_i \in \{0,1\}$, and so $c_i = a_i + b_i$, for all $i \geq 0$. Since $a_i + b_i \geq a_i$, by Lucas formula we obtain that $\binom{\lambda+\mu}{\lambda} \equiv 1 \pmod 2$.

For the opposite implication, assume that $a_i b_i = 1$ for some $i \geq 0$, and let $t = \min\{i \mid a_i b_i = 1\}$. This means that $a_t = b_t = 1$, $c_i = a_i + b_i$ for $0 \leq i < t$, and $c_t = 0$. Since $c_t < a_t$, we conclude that $\binom{\lambda+\mu}{\lambda} \equiv 0 \pmod 2$, again by Lucas formula. □

In [5, Theorem 3.1.3] it was proved that $j \geq 2^{t+1} - m - 1$ implies that the cup-length of $F(1^{\cdots j}, 2, m)$ is maximal (where t is the integer such that $2^t \leq m < 2^{t+1}$). In the following proposition we give an alternative proof of this fact and we show that the opposite implication holds as well.

Proposition 2.3. *Let $j \geq 0$, $m \geq 2$ and let $t \geq 1$ be such that $2^t \leq m < 2^{t+1}$. Then $\operatorname{cup} F(1^{\cdots j}, 2, m) = \dim F(1^{\cdots j}, 2, m)$ if and only if $j \geq 2^{t+1} - m - 1$.*

Proof. If $j \geq 2^{t+1} - m - 1$, then we have that $m < 2^{t+1} \leq m + j + 1$. Therefore, $2^{t+1} - 1, 2^{t+1} \in \{m, m+1, \ldots, m+j+1\}$. Since $2^{t+1} - 1 = 1 + 2 + 2^2 + \cdots + 2^t$, by Proposition 2.2 (a) and Lemma 2.3 we conclude that $\operatorname{cup} F(1^{\cdots j}, 2, m) = \dim F(1^{\cdots j}, 2, m)$.

Conversely, if $j < 2^{t+1} - m - 1$, then we have that $2^t \leq m < m + j + 1 < 2^{t+1}$. This means that binary expansions of all integers from the set $\{m, m+1, \ldots, m+j+1\}$ have the form $\sum_{i=0}^{t-1} a_i 2^i + 2^t$. Lemma 2.3 then tells us that we cannot find integers $\lambda, \mu \in \{m, m+1, \ldots, m+j+1\}$ such that $\binom{\lambda+\mu}{\lambda} \equiv 1 \pmod 2$. Proposition 2.2 (a) now finishes the proof. □

Note that for $j = 0$ Proposition 2.3 reduces to the well-known fact about Grassmannians: $\operatorname{cup} F(2, m) = \dim F(2, m)$ if and only if $m = 2^{t+1} - 1$ for some $t \geq 1$ ([1, Theorem 1·1]).

Now, for $d \geq 2$ we extend the class (obtained in [5]) of flag manifolds $F(1^{\cdots j}, 2^{\cdots d}, m)$ with maximal cup-length, and thus give the negative answer to the question of Korbaš and Lörinc.

Theorem 2.4. *Let $F = F(1^{\cdots j}, 2^{\cdots d}, m)$, $j \geq 0$, $d \geq 1$, $m \geq 2$, and let $t \geq 1$ be such that $2^t \leq m < 2^{t+1}$.*

(a) *For d odd, say $d = 2l - 1$ ($l \geq 1$), we have the following implications:*
   (a1) *if $2^t \leq m \leq 2^{t+1} - 3$ and $j \geq 2^{t+l} - m - 2d + 1$, then $\operatorname{cup} F = \dim F$ and $\operatorname{cat} F = 1 + \dim F$;*

(a2) *if* $2^{t+1}-2 \leq m < 2^{t+1}$ *and* $j \geq 2^{t+l}-m-2d+2$, *then* $\operatorname{cup} F = \dim F$ *and* $\operatorname{cat} F = 1 + \dim F$.

(b) *For* d *even, say* $d = 2l$ $(l \geq 1)$, *we have the following implications:*

(b1) *if* $2^t \leq m \leq 2^{t+1}-3$ *and* $j \geq 2^{t+l}-m-2d+2$, *then* $\operatorname{cup} F = \dim F$ *and* $\operatorname{cat} F = 1 + \dim F$*;*

(b2) *if* $2^{t+1}-2 \leq m < 2^{t+1}$ *and* $j \geq 2^{t+l+1}-m-2d+1$, *then* $\operatorname{cup} F = \dim F$ *and* $\operatorname{cat} F = 1 + \dim F$.

Proof. In all cases, it suffices to find the integers $\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_d, \mu_d$ with the properties specified in Proposition 2.1.

Since $m \leq 2^{t+1}-1$ and $2^{t+l} \leq m+j+2d-1$ (in all four cases), for any integer i such that $2 \leq i \leq l-1$ the numbers $2^{t+i}-3$, $2^{t+i}-2$, 2^{t+i} and $2^{t+i}+1$ are in the set $\{m, m+1, \ldots, m+j+2d-1\}$. So, for $2 \leq i \leq l-1$ we can define:

$$\lambda_{2i-1} := 2^{t+i}-3, \;\; \mu_{2i-1} := 2^{t+i}, \;\; \lambda_{2i} := 2^{t+i}-2, \;\; \mu_{2i} := 2^{t+i}+1.$$

Hence, we now have the integers $\lambda_3, \mu_3, \lambda_4, \mu_4, \ldots, \lambda_{2l-2}, \mu_{2l-2}$. Note that $\lambda_k < \mu_k$, $3 \leq k \leq 2l-2$, and that the least of these integers is $\lambda_3 = 2^{t+2}-3$ and the greatest is $\mu_{2l-2} = 2^{t+l-1}+1$. Using Lemma 2.3, it is routine to check that $\binom{\lambda_k+\mu_k}{\lambda_k} \equiv 1 \pmod 2$, $3 \leq k \leq 2l-2$.

Let us now fix an integer $k \in \{3, 4, \ldots, 2l-2\}$ and show that if $\alpha, \beta \in A = \{\lambda_3, \mu_3, \lambda_4, \mu_4, \ldots, \lambda_{2l-2}, \mu_{2l-2}\}$ are two distinct integers such that $\alpha+\beta = \lambda_k+\mu_k$, then $\{\alpha, \beta\} = \{\lambda_k, \mu_k\}$. Assume to the contrary that α and β are some other integers from the set A, and that e.g., $\alpha < \beta$. Observe first that the set A contains only one integer between λ_k and μ_k. This means that we must have $\alpha < \lambda_k < \mu_k < \beta$ (since $\alpha+\beta = \lambda_k+\mu_k$). This is a contradiction if either $k = 3$ or $k = 2l-2$, so let us assume that $3 < k < 2l-2$.

If k is odd, say $k = 2i-1$ $(3 \leq i \leq l-1)$, then $\mu_k = 2^{t+i}$ and $\beta \in \{2^{t+i}+1, 2^{t+i+1}-3, 2^{t+i+1}-2, \ldots\}$, and since $2^{t+i+1}-3 = \lambda_k+\mu_k = \alpha+\beta > \beta$ we conclude that $\beta = 2^{t+i}+1$. But $\lambda_k = 2^{t+i}-3$, and so $\alpha \leq 2^{t+i-1}+1$, which implies that

$$\alpha+\beta \leq 2^{t+i-1}+1+2^{t+i}+1 = 3\cdot 2^{t+i-1}+2 < 4\cdot 2^{t+i-1}-3 = 2^{t+i+1}-3 = \lambda_k+\mu_k$$

(since $2^{t+i-1} \geq 2^{t+2} \geq 2^3 > 5$), contradicting the assumption $\alpha+\beta = \lambda_k+\mu_k$.

If k is even, say $k = 2i$ $(2 \leq i \leq l-2)$, then $\mu_k = 2^{t+i}+1$ and $\beta \geq 2^{t+i+1}-3$. However, since $\alpha \geq \lambda_3 = 2^{t+2}-3 \geq 2^3-3 = 5$, we obtain that

$$\alpha+\beta \geq 5+2^{t+i+1}-3 = 2^{t+i+1}+2 > 2^{t+i+1}-1 = \lambda_k+\mu_k,$$

which is a contradiction.

Let us now define the remaining integers $\lambda_1, \mu_1, \lambda_2, \mu_2, \lambda_{2l-1}, \mu_{2l-1}$, and, if d is even, λ_{2l} and μ_{2l}, from the set $\{m, m+1, \ldots, m+j+2d-1\}$. Actually, when $l = 1$ (i.e., $d \in \{1, 2\}$) we only need to define λ_1 and μ_1 (if $d = 1$), that is, $\lambda_1, \mu_1, \lambda_2$ and μ_2 (if $d = 2$), and so, for $l = 1$ the "excess" in the following definitions should be ignored. We now distinguish the four cases:

(a1) $\lambda_1 := 2^{t+1}-3$, $\mu_1 := 2^{t+1}$, $\lambda_2 := 2^{t+1}-2$, $\mu_2 := 2^{t+1}+1$, $\lambda_{2l-1} := 2^{t+l}-3$, $\mu_{2l-1} = 2^{t+l}$ (in this case $m \le 2^{t+1}-3$ and $2^{t+l} \le m+j+2d-1$);

(a2) $\lambda_1 := 2^{t+1}-1$, $\mu_1 := 2^{t+1}$, $\lambda_{2l-1} := 2^{t+l}-3$, $\mu_{2l-1} = 2^{t+l}$, $\lambda_2 := 2^{t+l}-2$, $\mu_2 := 2^{t+l}+1$ (in this case $m \le 2^{t+1}-1$ and $2^{t+l}+1 \le m+j+2d-1$);

(b1) $\lambda_1 := 2^{t+1}-3$, $\mu_1 := 2^{t+1}$, $\lambda_2 := 2^{t+1}-2$, $\mu_2 := 2^{t+1}+1$, $\lambda_{2l-1} := 2^{t+l}-3$, $\mu_{2l-1} = 2^{t+l}$, $\lambda_{2l} := 2^{t+l}-2$, $\mu_{2l} = 2^{t+l}+1$ (in this case $m \le 2^{t+1}-3$ and $2^{t+l}+1 \le m+j+2d-1$);

(b2) $\lambda_1 := 2^{t+1}-1$, $\mu_1 := 2^{t+1}$, $\lambda_{2l-1} := 2^{t+l}-3$, $\mu_{2l-1} = 2^{t+l}$, $\lambda_{2l} := 2^{t+l}-2$, $\mu_{2l} = 2^{t+l}+1$, $\lambda_2 := 2^{t+l+1}-3$, $\mu_2 := 2^{t+l+1}$ (in this case $m \le 2^{t+1}-1$ and $2^{t+l+1} \le m+j+2d-1$).

Arguing as before, it is easy to verify that $\binom{\lambda_k+\mu_k}{\lambda_k} \equiv 1 \pmod 2$ for all $k \in \{1,2,\dots,d\}$, and that the separation of the set $\{\lambda_1,\lambda_2,\dots,\lambda_d,\mu_1,\mu_2,\dots,\mu_d\}$ into the pairs $\{\lambda_k,\mu_k\}$, $1 \le k \le d$, is the only one such that the sums of the pairs are exactly $\lambda_k+\mu_k$, $1 \le k \le d$. A minor difficulty occurs in the cases (a2) and (b2) when $t = 1$ (i.e., $m \in \{2,3\}$) and $l \ge 2$. Then we have the following situation: $\lambda_1 = 3$, $\mu_1 = 4$, $\lambda_3 = 5$, $\mu_3 = 8$, $\lambda_4 = 6$, $\mu_4 = 9$, $\lambda_5 = 13$ etc. The difficulty is the fact that $\lambda_3+\mu_3 = 13 = 4+9 = \mu_1+\mu_4$, but if we pair $\mu_1 = 4$ and $\mu_4 = 9$, then we cannot form a pair $\{\alpha,\beta\}$ from the remaining integers with the property $\alpha+\beta = \lambda_1+\mu_1 = 7$.

Therefore, in all cases the integers $\lambda_1,\mu_1,\lambda_2,\mu_2,\dots,\lambda_d,\mu_d$ satisfy the conditions (i) and (ii) of Proposition 2.1, and we conclude that $\operatorname{cup} F = \dim F$ (and consequently, $\operatorname{cat} F = 1 + \dim F$). □

Let us now show that the inequalities from the part (b) of this proposition, that provide maximality of the cup-length, are the sharpest possible in the case $d = 2$.

Corollary 2.4.1. *Let $j \ge 0$, $m \ge 2$ and let $t \ge 1$ be such that $2^t \le m < 2^{t+1}$.*

1. *For $2^t \le m \le 2^{t+1}-3$ we have the equivalence: $\operatorname{cup} F(1^{\cdots j},2,2,m) = \dim F(1^{\cdots j},2,2,m)$ if and only if $j \ge 2^{t+1}-m-2$.*
2. *For $2^{t+1}-2 \le m < 2^{t+1}$ we have the equivalence: $\operatorname{cup} F(1^{\cdots j},2,2,m) = \dim F(1^{\cdots j},2,2,m)$ if and only if $j \ge 2^{t+2}-m-3$.*

Proof. The "if parts" have already been proved in Theorem 2.4 (b). We now prove the "only if" parts.

(1) If $2^t \le m \le 2^{t+1}-3$, let us assume that $j < 2^{t+1}-m-2$. This means that $2^t \le m$ and $m+j+2 \le 2^{t+1}-1$, and therefore, arguing as in the proof of Proposition 2.3, we cannot find the integers $\lambda_1,\mu_1 \in \{m,m+1,\dots,m+j+2\}$ with the property $\binom{\lambda_1+\mu_1}{\lambda_1} \equiv 1 \pmod 2$. Consequently, we cannot find four distinct integers $\lambda_1,\mu_1,\lambda_2,\mu_2$ in the set $\{m,m+1,\dots,m+j+2,m+j+3\}$ such that $\binom{\lambda_1+\mu_1}{\lambda_1}\binom{\lambda_2+\mu_2}{\lambda_2} \equiv 1 \pmod 2$. By Proposition 2.2 (b), $\operatorname{cup} F(1^{\cdots j},2,2,m)$ is not maximal.

(2) If $2^{t+1}-2 \leq m < 2^{t+1}$, let us assume that $j < 2^{t+2}-m-3$. Suppose that $\lambda_1, \mu_1, \lambda_2, \mu_2 \in \{m, m+1, \ldots, m+j+3\}$ are four distinct integers such that $\binom{\lambda_1+\mu_1}{\lambda_1}\binom{\lambda_2+\mu_2}{\lambda_2} \equiv 1 \pmod 2$ (w.l.o.g. we may assume that $\lambda_1 < \mu_1$, $\lambda_2 < \mu_2$ and $\lambda_1 < \lambda_2$). Note that all of the integers $m+2, m+3, \ldots, m+j+3$ are in the interval $[2^{t+1}, 2^{t+2}-1]$, and so, for the same reason as before, m have to be equal to $2^{t+1}-2$, and it must be $\lambda_1 = 2^{t+1}-2$ and $\lambda_2 = 2^{t+1}-1$. Since $\lambda_2 = 2^{t+1}-1 = 1+2+2^2+\cdots+2^t$, by Lemma 2.3, the only integer μ in the interval $[2^{t+1}, m+j+3] \subseteq [2^{t+1}, 2^{t+2}-1]$ which satisfies $\binom{\lambda_2+\mu}{\lambda_2} \equiv 1 \pmod 2$ is 2^{t+1}, so $\mu_2 = 2^{t+1}$. Then we have that $\mu_1 \in [2^{t+1}+1, m+j+3] \subseteq [2^{t+1}+1, 2^{t+2}-1]$, and since $\lambda_1 = 2^{t+1}-2 = 2+2^2+\cdots+2^t$ and $\binom{\lambda_1+\mu_1}{\lambda_1} \equiv 1 \pmod 2$, it must be $\mu_1 = 2^{t+1}+1$ (by Lemma 2.3 again). Finally, we conclude that

$$\lambda_1+\mu_1 = 2^{t+1}-2+2^{t+1}+1 = 2^{t+2}-1 = 2^{t+1}-1+2^{t+1} = \lambda_2+\mu_2,$$

and Proposition 2.2 (b) finishes the proof. □

Having in mind the previous corollary, one might expect that the inequalities from Theorem 2.4 are the sharpest possible in general (i.e., for every $d \geq 2$). Unfortunately, this is not the case – in the following proposition we prove that these inequalities can be considerably improved for "almost all" $d \geq 3$.

Proposition 2.5. *Let* $F = F(1^{\cdots j}, 2^{\cdots d}, m)$, $j \geq 0$, $d \geq 3$, $m \geq 2$, *and let* $t \geq 1$ *be such that* $2^t \leq m < 2^{t+1}$.

(a) *If* $j \geq 2^{t+3}d^2 - m - 2d$, *then* $\operatorname{cup} F = \dim F$ *and* $\operatorname{cat} F = 1 + \dim F$.
(b) *If* $m \geq 2d^2$ *and* $j \geq 2^{t+3} - m - 2d$, *then* $\operatorname{cup} F = \dim F$ *and* $\operatorname{cat} F = 1 + \dim F$.

Proof. (a) Let s be the unique integer such that $2^{t+s} \leq m+j+2d < 2^{t+s+1}$, and $l = \lfloor \frac{t+s-2}{2} \rfloor$. For $i \in \{1, 2, \ldots, d\}$, we define

$$\lambda_i = 2^{t+s-2} + i, \qquad \mu_i = 2^{t+s-1} + i \cdot 2^l.$$

Let us prove that these numbers satisfy conditions of Proposition 2.1. Since $d \geq 3$, we have $m+j+2d \geq 2^{t+3}d^2 > 2^{t+6}$, and therefore $s \geq 6$. Additionally, $2^{t+s+1} > m+j+2d \geq 2^{t+3}d^2$, and so $d < 2^{\frac{s-2}{2}} \leq 2^{\frac{t+s-3}{2}} \leq 2^l \leq 2^{\frac{t+s-2}{2}}$. Hence,

$$m < 2^{t+1} < \lambda_1 < \lambda_2 < \cdots < \lambda_d < \mu_1 < \mu_2 < \cdots < \mu_d \leq 2^{t+s}-1 \leq m+j+2d-1.$$

Note that for $1 \leq i < j \leq d$ one has

$$\lambda_i + \lambda_j = 2^{t+s-1} + i + j < 2^{t+s-1} + 2d < 2^{t+s-1} + 2^{l+1} < \lambda_1 + \mu_1,$$

and (since $d \cdot 2^l < 2^{2l} \leq 2^{t+s-2}$)

$$\mu_i + \mu_j = 2^{t+s} + (i+j) \cdot 2^l \geq 2^{t+s} + 3 \cdot 2^l > 2^{t+s-2} + d + 2^{t+s-1} + d \cdot 2^l = \lambda_d + \mu_d.$$

Therefore, if the set $A = \{\lambda_1, \lambda_2, \ldots, \lambda_d, \mu_1, \mu_2, \ldots, \mu_d\}$ is partitioned into pairs, the only pair that can have sum equal to $\lambda_1 + \mu_1$ is $\{\lambda_1, \mu_1\}$ (for $1 \leq i < j \leq d$ one has $\lambda_i + \lambda_j < \lambda_1 + \mu_1 < \mu_i + \mu_j$; for $(i, j) \neq (1, 1)$ one has $\lambda_i + \mu_j > \lambda_1 + \mu_1$).

Similarly, the only pair from $A\setminus\{\lambda_1,\mu_1\}$ with sum equal to $\lambda_2+\mu_2$ is $\{\lambda_2,\mu_2\}$, the only pair from $A\setminus\{\lambda_1,\mu_1,\lambda_2,\mu_2\}$ with sum equal to $\lambda_3+\mu_3$ is $\{\lambda_3,\mu_3\}$, etc. We conclude that the set A satisfies part (i) of Proposition 2.1.

To prove that A satisfies part (ii) of Proposition 2.1, first note that for $i\in\{1,2,\dots,d\}$ the following inequalities hold

$$2^{t+s-2}<\lambda_i=2^{t+s-2}+i<2^{t+s-2}+2^l,$$

and

$$2^{t+s-1}+2^{t+s-2}>2^{t+s-1}+d\cdot 2^l\geq\mu_i=2^{t+s-1}+i\cdot 2^l=2^l(2^{t+s-l-1}+i).$$

Therefore, in the binary representation, the number λ_i can have nonzero digits only in positions 0 to $l-1$ and in position $t+s-2$ (if $\lambda_i=\sum_{j\geq 0}a_j2^j$, $a_j\in\{0,1\}$, then the digit a_k is in position k), while the number μ_i can have nonzero digits only in positions l to $t+s-3$ and in position $t+s-1$. Hence, by Lemma 2.3, we have $\binom{\lambda_i+\mu_i}{\lambda_i}\equiv 1 \pmod 2$, and the proof is completed.

(b) Let $l=\lfloor\frac{t+1}{2}\rfloor$. For $i\in\{1,2,\dots,d\}$, we define

$$\lambda_i=2^{t+1}+i,\qquad \mu_i=2^{t+2}+i\cdot 2^l.$$

Note that $2d^2\leq m<2^{t+1}\leq 2^{2l+1}$, i.e., $d<2^l$. Hence,

$$m<2^{t+1}<\lambda_1<\lambda_2<\cdots<\lambda_d<\mu_1<\mu_2<\cdots<\mu_d\leq 2^{t+3}-1\leq m+j+2d-1.$$

Additionally, for $1\leq i<j\leq d$ one has

$$\lambda_i+\lambda_j=2^{t+2}+i+j<2^{t+2}+2d<2^{t+2}+2^{l+1}<\lambda_1+\mu_1,$$

and (since $d\cdot 2^l<2^{2l}\leq 2^{t+1}$)

$$\mu_i+\mu_j=2^{t+3}+(i+j)\cdot 2^l\geq 2^{t+3}+3\cdot 2^l>2^{t+1}+d+2^{t+2}+d\cdot 2^l=\lambda_d+\mu_d.$$

As in part (a) of this proposition, these inequalities imply that the set $A=\{\lambda_1,\lambda_2,\dots,\lambda_d,\mu_1,\mu_2,\dots,\mu_d\}$ satisfies part (i) of Proposition 2.1.

To prove that A also satisfies part (ii) of Proposition 2.1, note that the following inequalities also hold

$$2^{t+1}<\lambda_i=2^{t+1}+i<2^{t+1}+2^l,$$

and

$$2^{t+2}+2^{t+1}>2^{t+2}+d\cdot 2^l\geq\mu_i=2^{t+2}+i\cdot 2^l=2^l(2^{t+2-l}+i).$$

The proof is now completed as in part (a) of this proposition. □

In [1] Berstein examined which Grassmann manifolds $F(k,n)$ have maximal cup-length and proved that all such manifolds are $F(1,n)$ for $n\geq 1$, and $F(2,2^{t+1}-1)$ for $t\geq 1$.

The result of Korbaš and Lörinc that $\operatorname{cup} F(1^{\cdots k},m)=\dim F(1^{\cdots k},m)$ for all $k\geq 1$ and Proposition 2.3 give a generalization of the Berstein's result to the manifolds of the form $F(1^{\cdots k},m)$ and $F(1^{\cdots j},2,m)$. Also, we know that there are manifolds of the form $F(1^{\cdots j}, 2^{\cdots d}, m)$ with maximal cup-length, but it seems that classifying all manifolds of this form with maximal cup-length is very difficult (see Theorem 2.4 and Proposition 2.5). On the other hand, we know that there are no Grassmannians $F(k,n)$ with maximal cup-length if $k, n \geq 3$ (by Berstein's result), so one might expect that $\operatorname{cup} F(n_1, n_2, \ldots, n_q) = \dim F(n_1, n_2, \ldots, n_q)$ only if $n_i \geq 3$ for at most one $i \in \{1, 2, \ldots, q\}$ (that is, only if $F(n_1, n_2, \ldots, n_q)$ is of the form $F(1^{\cdots j}, 2^{\cdots d}, m)$). However, the following example shows that this is not true.

Example 1. The dimension of the flag manifold $F(1,1,1,3,4)$ is 36. We shall prove that the class $u = w_{1,1}^6 w_{2,1}^7 w_{3,1}^8 w_{4,1}^{15} \in H^{36}(F(1,1,1,3,4); \mathbb{Z}_2)$ is nonzero, and so we will have that

$$\operatorname{cup} F(1,1,1,3,4) = \dim F(1,1,1,3,4).$$

We consider the class $p^*(u) \cdot E^{(0,0,0,2,1,0,3,2,1,0)} \in H^{45}(F(1^{\cdots 10}); \mathbb{Z}_2)$ (see Lemma 2.2). By (2.1) we know that

$$\begin{aligned} p^*(u) \cdot E^{(0,0,0,2,1,0,3,2,1,0)} &= e_1^6 e_2^7 e_3^8 (e_4 + e_5 + e_6)^{15} e_4^2 e_5 e_7^3 e_8^2 e_9 \\ &= \sum_{0 \leq l \leq k \leq 15} \binom{15}{k} \binom{k}{l} E^{(6,7,8,l+2,k-l+1,15-k,3,2,1,0)} \end{aligned}$$

It is clear that $\binom{15}{k} \equiv 1 \pmod 2$ for all $k \in \{0, 1, \ldots, 15\}$, and since the class $E^{(6,7,8,l+2,k-l+1,15-k,3,2,1,0)}$ is nontrivial only for $15 - k \in \{4, 5, 9\}$ (by Lemma 2.1), i.e., $k \in \{6, 10, 11\}$, we have that

$$\begin{aligned} p^*(u) \cdot E^{(0,0,0,2,1,0,3,2,1,0)} &= \sum_{l=0}^{6} \binom{6}{l} E^{(6,7,8,l+2,7-l,9,3,2,1,0)} + \\ &+ \sum_{l=0}^{10} \binom{10}{l} E^{(6,7,8,l+2,11-l,5,3,2,1,0)} + \sum_{l=0}^{11} \binom{11}{l} E^{(6,7,8,l+2,12-l,4,3,2,1,0)} \end{aligned}$$

It is now routine to check that there is exactly one nontrivial summand in the first sum (it is the one for $l = 2$), one in the second (also for $l = 2$), and one in the third (for $l = 3$). So $p^*(u) \cdot E^{(0,0,0,2,1,0,3,2,1,0)} \neq 0$, which implies that $u \neq 0$.

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