SEEMOUS 2022
MAY 27 - JUNE 1, PALIĆ, SERBIA

Language: **English**

Sunday, May 29, 2022

Problem 1.

Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be such that $AB^2A = AB$. Prove that:

a) $(AB)^2 = AB$.

b) $(AB - BA)^3 = O_n$.

Problem 2.

Let $a, b, c \in \mathbb{R}$ be such that

$$a + b + c = a^2 + b^2 + c^2 = 1, \quad a^3 + b^3 + c^3 \neq 1.$$

We say that a function f is a *Palić function* if $f : \mathbb{R} \to \mathbb{R}$, f is continuous and satisfies

$$f(x) + f(y) + f(z) = f(ax + by + cz) + f(bx + cy + az) + f(cx + ay + bz)$$

for all $x, y, z \in \mathbb{R}$.
Prove that any Palić function is infinitely many times differentiable and find all Palić functions.

Problem 3.

Let $\alpha \in \mathbb{C} \setminus \{0\}$ and $A \in \mathcal{M}_n(\mathbb{C})$, $A \neq O_n$, be such that

$$A^2 + (A^*)^2 = \alpha A \cdot A^*,$$

where $A^* = \left(\overline{A}\right)^T$. Prove that $\alpha \in \mathbb{R}$, $|\alpha| \leqslant 2$, and $A \cdot A^* = A^* \cdot A$.

Problem 4.

Let $\mathcal{F}$ be the family of all nonempty finite subsets of $\mathbb{N} \cup \{0\}$. Find all positive real numbers a for which the series

$$\sum_{A \in \mathcal{F}} \frac{1}{\sum\limits_{k \in A} a^k}$$

is convergent.

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Time: 5 hours
Each problem is worth 10 points