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Uvod u interaktivno dokazivanje teorema

Vežbe 6

Zadatak 1 *Svojstva funkcija*

Pokazati da je slika unije, unija pojedinačnih slika.
Savet: Razmotriti teoreme *image-def* i *vimage-def*.

```
lemma image-union:
  shows f ` (A ∪ B) = f ` A ∪ f ` B
proof
  show f ` (A ∪ B) ⊆ f ` A ∪ f ` B
  proof
    fix y
    assume y ∈ f ` (A ∪ B)
    then have ∃ x. x ∈ A ∪ B ∧ f x = y by auto
    then obtain x where x ∈ A ∪ B f x = y by auto
    then have x ∈ A ∨ x ∈ B by auto
    then have f x ∈ f ` A ∨ f x ∈ f ` B by auto
    with ‹f x = y› show y ∈ f ` A ∪ f ` B by auto
  qed
next
  show f ` A ∪ f ` B ⊆ f ` (A ∪ B)
  proof
    fix y
    assume y ∈ f ` A ∪ f ` B
    then have y ∈ f ` A ∨ y ∈ f ` B by simp
    then show y ∈ f ` (A ∪ B)
    proof
      assume y ∈ f ` A
      then have ∃ x. x ∈ A ∧ f x = y by auto
      then obtain x where x ∈ A f x = y by auto
      then have x ∈ A ∪ B by simp
      then have f x ∈ f ` (A ∪ B) by simp
      with ‹f x = y› show y ∈ f ` (A ∪ B) by auto
    next
      assume y ∈ f ` B
      then have ∃ x. x ∈ B ∧ f x = y by auto
      then obtain x where x ∈ B f x = y by auto
      then have x ∈ A ∪ B by simp
      then have f x ∈ f ` (A ∪ B) by simp
      with ‹f x = y› show y ∈ f ` (A ∪ B) by auto
    qed
  qed
qed
```

Neka je funkcija f injektivna. Pokazati da je slika preseka, presek pojedinačnih slika.
Savet: Razmotriti teoremu *inj-def*.

```
lemma image-inter:
  assumes inj f
    shows f ‘ (A ∩ B) = f ‘ A ∩ f ‘ B
proof
  show f ‘ (A ∩ B) ⊆ f ‘ A ∩ f ‘ B
  proof
    fix y
    assume y ∈ f ‘ (A ∩ B)
    then have ∃ x ∈ A ∩ B. f x = y by auto
    then obtain x where x ∈ A ∩ B f x = y by auto
    then have x ∈ A ∧ x ∈ B by auto
    then have f x ∈ f ‘ A ∧ f x ∈ f ‘ B by auto
    with ‹f x = y› show y ∈ f ‘ A ∩ f ‘ B by auto
  qed
next
  show f ‘ A ∩ f ‘ B ⊆ f ‘ (A ∩ B)
  proof
    fix y
    assume y ∈ f ‘ A ∩ f ‘ B
    then have y ∈ f ‘ A y ∈ f ‘ B by auto
    from ‹y ∈ f ‘ A› obtain xa where xa ∈ A f xa = y by auto
    moreover
    from ‹y ∈ f ‘ B› obtain xb where xb ∈ B f xb = y by auto
    ultimately
    have xa = xb using assms by (simp add: inj-def)
    with ‹xa ∈ A› have xb ∈ A by auto
    with ‹xb ∈ B› have xb ∈ A ∧ xb ∈ B by auto
    then have xb ∈ A ∩ B by auto
    then have f xb ∈ f ‘ (A ∩ B) by auto
    with ‹f xb = y› show y ∈ f ‘ (A ∩ B) by auto
  qed
qed
```

Savet: Razmotriti teoremu *surj-def* i *surjD*.

```
lemma surj-image-vimage:
  assumes surj f
    shows f ‘ (f −‘ B) = B
proof
  show f ‘ f −‘ B ⊆ B
  proof
    fix y
    assume y ∈ f ‘ f −‘ B
    then obtain x where x ∈ f −‘ B f x = y by auto
    then have f x ∈ B by auto
    with ‹f x = y› show y ∈ B by auto
  qed
next
  show B ⊆ f ‘ f −‘ B
  proof
    fix y
    assume y ∈ B
```

    with *assms* **obtain** x **where** $f\ x = y$ **using** *surjD* **by** *metis*
    with ‹$y \in B$› **have** $x \in f -` B$ **by** *auto*
    then have $f\ x \in f\ `\ (f -` B)$ **by** *auto*
    with ‹$f\ x = y$› **show** $y \in f\ `\ f -` B$ **by** *auto*
  qed
qed

Pokazati da je kompozicija injektivna ako su pojedinačne funkcije injektivne.
Savet: Razmotrite teoremu *inj-eq*.

lemma *comp-inj*:
  assumes *inj f*
      and *inj g*
    shows *inj* $(f \circ g)$
proof
  fix $x\ y$
  assume $(f \circ g)\ x = (f \circ g)\ y$
  then have $f\ (g\ x) = f\ (g\ y)$ **by** *auto*
  with ‹*inj f*› **have** $g\ x = g\ y$ **by** (*simp add*: *inj-eq*)
  with ‹*inj g*› **show** $x = y$ **by** (*simp add*: *inj-eq*)
qed

lemma
  assumes *inj f*
    shows $x \in A \longleftrightarrow f\ x \in f\ `\ A$
proof
  assume $x \in A$
  then show $f\ x \in f\ `\ A$ **by** *auto*
next
  assume $f\ x \in f\ `\ A$
  then obtain x' **where** $x' \in A\ f\ x = f\ x'$ **by** *auto*
  with ‹*inj f*› **have** $x = x'$ **by** (*simp add*: *inj-eq*)
  with ‹$x' \in A$› **show** $x \in A$ **by** *auto*
qed

lemma *inj-vimage-image*:
  assumes *inj f*
    shows $f -` (f\ `\ A) = A$
proof
  show $f -` f\ `\ A \subseteq A$
  proof
    fix x
    assume $x \in f -` (f\ `\ A)$
    then obtain y **where** $y \in f\ `\ A\ f\ x = y$ **by** *auto*
    then obtain x' **where** $x' \in A\ f\ x' = y$ **by** *auto*
    with ‹$f\ x = y$› **have** $f\ x = f\ x'$ **by** *auto*
    with *assms* **have** $x = x'$ **by** (*simp add*: *inj-eq*)
    with ‹$x' \in A$› **show** $x \in A$ **by** *auto*
  qed
next
  show $A \subseteq f -` f\ `\ A$
  proof

```
    fix x
    assume x ∈ A
    then have f x ∈ f ‘ A by auto
    then show x ∈ f −‘ f ‘ A by auto
  qed
qed
```

Kompozicija je surjekcija ako su pojedinačne funkcije surjekcije.

```
lemma comp-surj:
  assumes surj f
      and surj g
    shows surj (f ∘ g)
  unfolding surj-def
proof
  fix z
  from ‹surj f› obtain y where ‹z = f y› by auto
  moreover
  from ‹surj g› obtain x where ‹y = g x› by auto
  ultimately
  have z = f (g x) by auto
  then show ∃ x. z = (f ∘ g) x by auto
qed
```

```
lemma vimage-compl:
  shows f −‘ (− B) = − (f −‘ B)
proof
  show f −‘ (− B) ⊆ − f −‘ B
  proof
    fix x
    assume x ∈ f −‘ (− B)
    then obtain y where y ∈ − B f x = y by auto
    then have y ∉ B by auto
    with ‹f x = y› have f x ∉ B by auto
    then have x ∉ f −‘ B by auto
    then show x ∈ − f −‘ B by auto
  qed
next
  show − f −‘ B ⊆ f −‘ (− B)
  proof
    fix x
    assume x ∈ − f −‘ B
    then have x ∉ f −‘ B by auto
    then have f x ∉ B by auto
    then have f x ∈ −B by auto
    then show x ∈ f −‘ (− B) by auto
  qed
qed
```