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Uvod u interaktivno dokazivanje teorema

Vežbe 05

Zadatak 1 *Svojstva funkcija*

Pokazati da je slika unije, unija pojedinačnih slika.
Savet: Razmotriti teoreme *image-def* i *vimage-def*.

```
lemma image-union:
  shows f ` (A ∪ B) = f ` A ∪ f ` B
proof
  show f ` (A ∪ B) ⊆ f ` A ∪ f ` B
  proof
    fix y
    assume y ∈ f ` (A ∪ B)
    then have ∃ x. x ∈ A ∪ B ∧ f x = y by auto
    then obtain x where x ∈ A ∪ B f x = y by auto
    then have x ∈ A ∨ x ∈ B by auto
    then have f x ∈ f ` A ∨ f x ∈ f ` B by auto
    with ‹f x = y› show y ∈ f ` A ∪ f ` B by auto
  qed
next
  show f ` A ∪ f ` B ⊆ f ` (A ∪ B)
  proof
    fix y
    assume y ∈ f ` A ∪ f ` B
    then have y ∈ f ` A ∨ y ∈ f ` B by simp
    then show y ∈ f ` (A ∪ B)
    proof
      assume y ∈ f ` A
      then have ∃ x. x ∈ A ∧ f x = y by auto
      then obtain x where x ∈ A f x = y by auto
      then have x ∈ A ∪ B by simp
      then have f x ∈ f ` (A ∪ B) by simp
      with ‹f x = y› show y ∈ f ` (A ∪ B) by auto
    next
      assume y ∈ f ` B
      then have ∃ x. x ∈ B ∧ f x = y by auto
      then obtain x where x ∈ B f x = y by auto
      then have x ∈ A ∪ B by simp
      then have f x ∈ f ` (A ∪ B) by simp
      with ‹f x = y› show y ∈ f ` (A ∪ B) by auto
    qed
  qed
qed
```

Neka je funkcija f injektivna. Pokazati da je slika preseka, presek pojedinačnih slika.
Savet: Razmotriti teoremu *inj-def*.

```
lemma image-inter:
  assumes inj f
    shows f ‘ (A ∩ B) = f ‘ A ∩ f ‘ B
proof
  show f ‘ (A ∩ B) ⊆ f ‘ A ∩ f ‘ B
  proof
    fix y
    assume y ∈ f ‘ (A ∩ B)
    then have ∃ x ∈ A ∩ B. f x = y by auto
    then obtain x where x ∈ A ∩ B f x = y by auto
    then have x ∈ A ∧ x ∈ B by auto
    then have f x ∈ f ‘ A ∧ f x ∈ f ‘ B by auto
    with ‹f x = y› show y ∈ f ‘ A ∩ f ‘ B by auto
  qed
next
  show f ‘ A ∩ f ‘ B ⊆ f ‘ (A ∩ B)
  proof
    fix y
    assume y ∈ f ‘ A ∩ f ‘ B
    then have y ∈ f ‘ A y ∈ f ‘ B by auto
    from ‹y ∈ f ‘ A› obtain xa where xa ∈ A f xa = y by auto
    moreover
    from ‹y ∈ f ‘ B› obtain xb where xb ∈ B f xb = y by auto
    ultimately
    have xa = xb using assms by (simp add: inj-def)
    with ‹xa ∈ A› have xb ∈ A by auto
    with ‹xb ∈ B› have xb ∈ A ∧ xb ∈ B by auto
    then have xb ∈ A ∩ B by auto
    then have f xb ∈ f ‘ (A ∩ B) by auto
    with ‹f xb = y› show y ∈ f ‘ (A ∩ B) by auto
  qed
qed
```

Savet: Razmotriti teoremu *surj-def* i *surjD*.

```
lemma surj-image-vimage:
  assumes surj f
    shows f ‘ (f −‘ B) = B
proof
  show f ‘ f −‘ B ⊆ B
  proof
    fix y
    assume y ∈ f ‘ f −‘ B
    then obtain x where x ∈ f −‘ B f x = y by auto
    then have f x ∈ B by auto
    with ‹f x = y› show y ∈ B by auto
  qed
next
  show B ⊆ f ‘ f −‘ B
  proof
    fix y
    assume y ∈ B
```

```
    with assms obtain x where f x = y using surjD by metis
    with ‹y ∈ B› have x ∈ f −‘ B by auto
    then have f x ∈ f ‘ (f −‘ B) by auto
    with ‹f x = y› show y ∈ f ‘ f −‘ B by auto
  qed
qed
```

Pokazati da je kompozicija injektivna ako su pojedinačne funkcije injektivne.
Savet: Razmotrite teoremu *inj-eq*.

```
lemma comp-inj:
  assumes inj f
      and inj g
    shows inj (f ∘ g)
proof
  fix x y
  assume (f ∘ g) x = (f ∘ g) y
  then have f (g x) = f (g y) by auto
  with ‹inj f› have g x = g y by (simp add: inj-eq)
  with ‹inj g› show x = y by (simp add: inj-eq)
qed

lemma
  assumes inj f
    shows x ∈ A ⟷ f x ∈ f ‘ A
proof
  assume x ∈ A
  then show f x ∈ f ‘ A by auto
next
  assume f x ∈ f ‘ A
  then obtain x' where x' ∈ A f x = f x' by auto
  with ‹inj f› have x = x' by (simp add: inj-eq)
  with ‹x' ∈ A› show x ∈ A by auto
qed

lemma inj-vimage-image:
  assumes inj f
    shows f −‘ (f ‘ A) = A
proof
  show f −‘ f ‘ A ⊆ A
  proof
    fix x
    assume x ∈ f −‘ (f ‘ A)
    then obtain y where y ∈ f ‘ A f x = y by auto
    then obtain x' where x' ∈ A f x' = y by auto
    with ‹f x = y› have f x = f x' by auto
    with assms have x = x' by (simp add: inj-eq)
    with ‹x' ∈ A› show x ∈ A by auto
  qed
next
  show A ⊆ f −‘ f ‘ A
  proof
```

```
    fix x
    assume x ∈ A
    then have f x ∈ f ` A by auto
    then show x ∈ f -` f ` A by auto
  qed
qed
```

Kompozicija je surjekcija ako su pojedinačne funkcije surjekcije.

```
lemma comp-surj:
  assumes surj f
      and surj g
    shows surj (f ∘ g)
  unfolding surj-def
proof
  fix z
  from ‹surj f› obtain y where ‹z = f y› by auto
  moreover
  from ‹surj g› obtain x where ‹y = g x› by auto
  ultimately
  have z = f (g x) by auto
  then show ∃x. z = (f ∘ g) x by auto
qed
```

```
lemma vimage-compl:
  shows f -` (- B) = - (f -` B)
proof
  show f -` (- B) ⊆ - f -` B
  proof
    fix x
    assume x ∈ f -` (- B)
    then obtain y where y ∈ - B f x = y by auto
    then have y ∉ B by auto
    with ‹f x = y› have f x ∉ B by auto
    then have x ∉ f -` B by auto
    then show x ∈ - f -` B by auto
  qed
next
  show - f -` B ⊆ f -` (- B)
  proof
    fix x
    assume x ∈ - f -` B
    then have x ∉ f -` B by auto
    then have f x ∉ B by auto
    then have f x ∈ -B by auto
    then show x ∈ f -` (- B) by auto
  qed
qed
```