Nehomogene LDJ II reda s KK

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$$y'' + p\cdot y' + q\cdot y = f(x) \qquad (\star)$$

$$y'' + p\cdot y' + q\cdot y = 0 \qquad \text{(homogeni deo)} \qquad (\heartsuit)$$

(T) Ako je y_P jedno partikularno rešenje jednačine $(\star)$ i ako je y_H opšte rešenje jednačine $(\heartsuit)$ onda je $y = y_P + y_H$ opšte rešenje jednačine $(\star)$.

(D) 1) y rešenje od $(\star)$?

$$(\star):\quad L(y) = L(y_P + y_H) = (y_P + y_H)'' + p\cdot(y_P+y_H)' + q\cdot(y_P + y_H) \overset{???}{=} f(x)$$

$$= y_P'' + y_H'' + p\cdot y_P' + p\cdot y_H' + q\cdot y_P + q\cdot y_H$$

$$= \underbrace{(y_P'' + p\cdot y_P' + q\cdot y_P)} + \underbrace{(y_H'' + p\cdot y_H' + q\cdot y_H)}$$

$$= f(x) + 0$$

(y_P je rešenje od $\star$) (y_H je rešenje od $\heartsuit$)

$= f(x)$ ✓ y jeste rešenje $\star$

2) y opšte rešenje od $\star$??

pp. $\bar{y}$ je OR od $(\star)$

posmatramo $\bar{y} - y_P$

$$(\star):\quad L(\bar{y} - y_P) = (\bar{y} - y_P)'' + p\cdot(\bar{y} - y_P)' + q\,(\bar{y} - y_P)$$

$$= \bar{y}'' - y_P'' + p\cdot\bar{y}' - p\cdot y_P' + q\,\bar{y} - q\cdot y_P$$

$$= (\bar{y}'' + p\cdot\bar{y}' + q\cdot\bar{y}) - (y_P'' + p\cdot y_P' + q\cdot y_P)$$

$$= f(x) - f(x) = 0$$

(jer $\bar{y}$ rešenje $\star$) (y_P je rešenje od $\star$)

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$L(\underbrace{\bar{y} - y_p}) = 0$

je rešenje jednačine ♡

$\Rightarrow \bar{y} - y_p = y_H$

$\Rightarrow \underset{\sim}{\bar{y}} = y_H + y_p = y$

opšte rešenje od ☆ (polazna pp.) ☒

OR. (☆): $y = y_H + y_p$

y_H ↙ znamo ☺ ; y_p ↘ ne znamo (još uvek) ☹

$y_p = ???$

$\boxed{1}$ Metoda neodređenih koeficijenata

Ako je $f(x) = e^{\alpha x}[P_n(x) \cdot \cos\beta x + Q_l(x) \cdot \sin\beta x]$

α, β - const, P_n i Q_l - polinomi stepena n i l

Tada y_p tražimo u obliku:

$y_p = x^m \cdot e^{\alpha x}[(a_p x^p + a_{p-1} \cdot x^{p-1} + \dots + a_0) \cdot \cos\beta x + (b_p x^p + \dots + b_0) \cdot \sin\beta x]$

$p = \max\{n, l\}$, a broj m se određuje na sl. način:

- ako $\alpha + i\beta$ nije koren karakteristične jednačine (♡) onda je $m = 0$
- ako $\alpha + i\beta$ jeste jednostruki koren k. jed. (♡) (prvog reda) $\Rightarrow m = 1$
- ako $\alpha + i\beta$ je dvostruki koren kar. jne (♡) $\Rightarrow m = 2$

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$y'' - y = \underbrace{x^2 - x + 1}_{\neq 0} \Rightarrow$ nehomogena $\qquad y_{OR} = \underset{\smile}{y_H} + \underset{\smile}{y_P}$

y_H : homogeni deo (☺) : $y'' - y = 0$

kar. jedn. : $\lambda^2 - 1 = 0$

$\lambda^2 = 1 \Rightarrow \lambda_{1,2} = \pm 1$

$y_H = C_1 \cdot e^{\lambda_1 x} + C_2 \cdot e^{\lambda_2 x} = C_1 e^{x} + C_2 \cdot e^{-x}$ OR(☺)

y_P : $f(x) = x^2 - x + 1 \overset{?}{=} \underbrace{e^{0\cdot x}}_{1} \Big[\underbrace{x^2 - x + 1}_{P_n} \cdot \underbrace{\cos(0 \cdot x)}_{1} + \underbrace{0}_{Q_\ell} \cdot \underbrace{\sin(0\cdot x)}_{0} \Big]$

$\alpha = 0$; $\beta = 0$, $P_n = x^2 - x + 1$, $Q_\ell(x) = 0$
$n = 2$ $\qquad \ell = 0$

$p = \max\{n, \ell\} = \max\{2, 0\} = 2$

m : $\alpha + i\beta$ da li je nula / koren od $\lambda^2 - 1$?

$0 + i \cdot 0 = 0$ nije koren k. jedn.

$\Rightarrow m = 0$

$\Rightarrow y_P = x^m \cdot e^{\alpha x} \left[(a_p x^p + \dots + a_0) \cos\beta x + (b_p x^p + \dots + b_0) \cdot \sin\beta x \right]$

$= \underbrace{x^0}_{1} \cdot \underbrace{e^{0\cdot x}}_{1} \Big[(a_2 x^2 + a_1 x + a_0) \cdot \underbrace{\cos 0 \cdot x}_{1} + \underbrace{(b_2 x^2 + b_1 x + b_0) \cdot \underbrace{\sin(0x)}_{0}}_{0} \Big]$

$= a_2 x^2 + a_1 x + a_0 \qquad a_0, a_1, a_2 = ?$

$y_P' = 2a_2 x + a_1$, $y_P'' = 2a_2$ (uvatimo u ✡)

$y_P'' - y_P = 2a_2 - a_2 x^2 - a_1 x - a_0 = x^2 - x + 1$

$a_2 = -1 \qquad a_1 = 1 \qquad 2a_2 - a_0 = 1$

$2 \cdot (-1) - a_0 = 1$

$a_0 = -3$

$y_P = -1 \cdot x^2 + 1 \cdot x - 3$

$y_{OR} = C_1 e^{x} + C_2 e^{-x} + (-x^2 + x - 3)$ od ✡

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$$y'' - 2y' + y = \underbrace{4e^x}_{f(x)} \qquad (\star)$$

$$y'' - 2y' + y = 0 \qquad (\heartsuit)$$

$$\lambda^2 - 2\lambda + 1 = 0 \quad , \; \dots \; , \quad \lambda_{1,2} = 1 \qquad (\text{slučaj } 2)$$

$$\Rightarrow y_H = C_1 \cdot e^x + C_2 \cdot x \cdot e^x$$

$$4e^x = e^{1 \cdot x} \cdot \Big[\frac{4}{P_n} \cdot \underbrace{\cos(0 \cdot x)}_{1} + \frac{0}{Q_l} \cdot \underbrace{\sin(0 \cdot x)}_{0} \Big]$$

$\alpha = 1 \;,\; \beta = 0 .$

$P_n = 4 \;, \quad Q_l = 0$
$n = 0 \qquad\quad l = 0$

$p = \max\{n, l\} = 0$

$m : \; \alpha + i\beta = 1 + i \cdot 0 = 1$ da li je 1 koren $\lambda^2 - 2\lambda + 1$?

jeste! dvostruka! $\Rightarrow m = 2$

$$y_p = x^m \cdot e^{\alpha x} [(a_p x^p + \dots + a_0) \cos \beta x + (b_p x^p + \dots + b_0) \sin \beta x]$$

$$= x^2 \cdot e^{1 \cdot x} [a_0 \cdot \underbrace{\cos(0 \cdot x)}_{1} + \underbrace{b_0 \cdot \underbrace{\sin(0 \cdot x)}_{=0}}_{=0}]$$

$$= a_0 \cdot x^2 \cdot e^x$$

$a_0 = ?$

$y_p' = \dots \quad , \quad y_p'' = \dots \qquad$ (vratimo u $\star$)

$$\dots \qquad a_0 = 2$$

OR od $\star$: $\; y = c_1 e^x + c_2 x e^x + 2x^2 e^x$

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$\boxed{2}$ $f(x)$ je zbir nekoliko f-ja oblika $\boxed{1}$

$$f(x) = \underbrace{f_1(x)}_{\text{može } \boxed{1}} + \underbrace{f_2(x)}_{\text{može } \boxed{1}} + \dots + \underbrace{f_n(x)}_{\text{može } \boxed{1}}$$

(T) Neka je $y'' + py' + qy = f_1(x) + f_2(x)$ $(\square)$.

Ako su y_{p1} i y_{p2} partikularna rešenja jednačina

$$y'' + p \cdot y' + q \cdot y = f_1(x) \quad \text{i} \quad y'' + py' + qy = f_2(x)$$

onda je $y_p = y_{p1} + y_{p2}$ partikularno rešenje od $(\square)$

(D) $L(y_{p1} + y_{p2}) \overset{???}{=} f_1(x) + f_2(x)$

$$\begin{aligned} L(y_{p1} + y_{p2}) &= (y_{p1} + y_{p2})'' + p \cdot (y_{p1} + y_{p2})' + q(y_{p1} + y_{p2}) \\ &= y_{p1}'' + y_{p2}'' + p \cdot y_{p1}' + p \cdot y_{p2}' + q \cdot y_{p1} + q \cdot y_{p2} \\ &= \underbrace{(y_{p1}'' + p \cdot y_{p1}' + q \cdot y_{p1})}_{} + \underbrace{(y_{p2}'' + p \cdot y_{p2}' + q \cdot y_{p2})}_{} \\ &= f_1(x) + f_2(x) \quad \checkmark \quad \boxtimes \end{aligned}$$

$\boxed{3}$ Ako $f(x)$ nije oblika ni $\boxed{1}$ ni $\boxed{2}$

Metoda varijacije konstanti

LDJ n-tog reda sKK

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(◇) $y^{(n)} + a_{n-1} \cdot y^{(n-1)} + \ldots + a_0 \cdot y = 0$ (homogena ....)

(o) $y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = f(x)$ (nehomogena ....)

(T) Ako su $y_1, \ldots, y_n$ linearno nezavisna partikularna rešenja od (◇) onda je $y = C_1 y_1 + \ldots + C_n y_n$ opšte rešenje od (◇)

Karakteristična jednačina: $\lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_0 = 0$

slučaj 1

λ različiti $\in \mathbb{R}$

slučaj 2

$s = \lambda_1 = \ldots = \lambda_m \in \mathbb{R}$

$y_1 = e^{sx}$

$y_2 = x \cdot e^{sx}$

$y_3 = x^2 \cdot e^{sx}$

$\vdots$

$y_m = x^{m-1} \cdot e^{sx}$

slučaj 3

parovi konj. komp.

$\lambda_{1,2} = \alpha \pm i\beta$ reda m

$\begin{cases} y_1 = e^{\alpha x} \cos \beta x \\ y_2 = e^{\alpha x} \sin \beta x \end{cases}$

$\begin{cases} y_3 = x \cdot e^{\alpha x} \cos \beta x \\ y_4 = x \cdot e^{\alpha x} \sin \beta x \end{cases}$

$\begin{cases} y_5 = x^2 e^{\alpha x} \cos \beta x \\ y_6 = x^2 e^{\alpha x} \sin \beta \end{cases}$

$\vdots$

$\begin{cases} y_{2m-1} = x^{m-1} e^{\alpha x} \cos \beta x \\ y_{2m} = x^{m-1} e^{\alpha x} \sin \beta x \end{cases}$

Rešavanje (o): Važi teorema kao kod (☆)

$y = y_p + \underbrace{y_H}_{C_1 y_1 + \ldots + C_n y_n}$

y_p: ?

$f(x) \rightsquigarrow$ metoda $\boxed{1}$ isto kao kod ☆

$\rightsquigarrow$ metoda $\boxed{2}$ —״—

$\rightsquigarrow$ metoda $\boxed{3}$ —״—