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11:25 AM

$$L(y_1)=0 \Leftrightarrow (y_1'' + p\cdot y_1' + q\cdot y = 0)$$

y_1 rešenje $\Rightarrow L(y_1)=0$

y_2 rešenje $\Rightarrow L(y_2)=0$

$\overset{?}{\Rightarrow} c_1 y_1 + c_2 y_2$ rešenje

$$L(c_1y_1+c_2y_2) \overset{?}{=} 0$$

$$L(c_1y_1+c_2y_2) = (c_1y_1+c_2y_2)'' + p\cdot(c_1y_1+c_2y_2)' + q(c_1y_1+c_2y_2)$$

$$= c_1\cdot y_1'' + c_2y_2'' + p\cdot c_1\cdot y_1' + p\cdot c_2 y_2' + q\, c_1 y_1 + q\cdot c_2 y_2$$

$$= c_1(\underbrace{y_1''+p\cdot y_1'+q\cdot y_1}_{L(y_1)=0}) + c_2(\underbrace{y_2''+p\cdot y_2'+q\cdot y_2}_{L(y_2)=0})$$

$$= c_1\cdot 0 + c_2\cdot 0 = 0$$

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}, \quad \lambda_1, \lambda_2 \in \mathbb{R}, \ \lambda_1 \neq \lambda_2$$

lin. nez. ??

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1\cdot e^{\lambda_1 x} & \lambda_2\cdot e^{\lambda_2 x} \end{vmatrix} = e^{\lambda_1 x}\cdot\lambda_2\cdot e^{\lambda_2 x} - e^{\lambda_2 x}\cdot\lambda_1\cdot e^{\lambda_1 x}$$

$$= \lambda_2\cdot e^{(\lambda_1+\lambda_2)x} - \lambda_1 e^{(\lambda_1+\lambda_2)x} = \underbrace{e^{(\lambda_1+\lambda_2)x}}_{\neq 0}\underbrace{(\lambda_2-\lambda_1)}_{\substack{\neq 0 \\ \lambda_1\neq\lambda_2}} \overset{?}{=} 0 \quad \neq 0 \ ?$$

$W(y_1,y_2) \neq 0, \ \forall x$

$\Rightarrow y_1$ i y_2 lin. nezavisni

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$y'' - y' - 6y = 0 \qquad y_1 = e^{\lambda x}$

$\lambda^2 - \lambda - 6 = 0$ karakt. jednačina

$\lambda_{1,2} = \dfrac{1 \pm \sqrt{1+24}}{2} = \dfrac{1 \pm 5}{2} \nearrow 3 = \lambda_1, \searrow -2 = \lambda_2$ I slučaj $(\lambda_1 \neq \lambda_2 \in \mathbb{R})$

$\Downarrow$

$y_1 = e^{3x}, \ y_2 = e^{-2x}$

lin. nez. (već pokazano $\forall \lambda_1 \neq \lambda_2$)

$\Rightarrow$ OR $y = C_1 \cdot y_1 + C_2 y_2 = C_1 e^{3x} + C_2 e^{-2x}$

$y_1 = e^{\lambda x}$ rešenje ✓

$y_2 = x \cdot e^{\lambda x}$ rešenje ???

$y_2' = e^{\lambda x} + x \cdot \lambda \cdot e^{\lambda x} =$

$y_2'' = \lambda \cdot e^{\lambda x} + \lambda \cdot e^{\lambda x} + \lambda \cdot x \cdot \lambda e^{\lambda x}$

$= 2\lambda e^{\lambda x} + \lambda^2 \cdot x \cdot e^{\lambda x}$

$L(y) = 0, \quad y'' + p \cdot y' + q \cdot y = 0$

$\underbrace{2\lambda e^{\lambda x} + \lambda^2 \cdot x e^{\lambda x}}_{y_2''} + p \cdot (\underbrace{e^{\lambda x} + x \cdot \lambda \cdot e^{\lambda x}}_{y_2'}) + q \cdot x \cdot e^{\lambda x} \overset{???}{=} 0$

$e^{\lambda x}(2\lambda + \lambda^2 x + p + p \cdot \lambda \cdot x + q \cdot x) \overset{???}{=} 0$

$\underbrace{e^{\lambda x}}_{\neq 0}(\overbrace{2\lambda + p}^{\lambda = 5} + x \cdot (\lambda^2 + p\lambda + q)) \overset{???}{=} 0$ ✓ $\Rightarrow y_2 = x \cdot e^{5x}$ jeste rešenje (☺)

$\lambda = 5$ (koren karakt. jednačine)

$\underbrace{\underbrace{25 - 25}_{=0} \qquad \underbrace{}_{=0}}_{0 + 0 = 0}$

Vijetove formule : $p = -25$

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$y_1 = e^{sx}$, $y_2 = x \cdot e^{sx}$ lin. nez?

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{sx} & x \cdot e^{sx} \\ se^{sx} & e^{sx} + x \cdot s \cdot e^{sx} \end{vmatrix}$$

$$= e^{sx} \cdot (e^{sx} + x \cdot s \cdot e^{sx}) - x \cdot e^{sx} \cdot s \cdot e^{sx}$$

$$= e^{sx} (e^{sx} + \cancel{x \cdot s \cdot e^{sx}} - \cancel{s \cdot x \cdot e^{sx}})$$

$$= e^{2sx} \neq 0 \Rightarrow y_1, y_2 \text{ lin. nez.}$$

$$y'' + 4y' + 4y = 0$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{16-16}}{2} = -2$$

(II slučaj $\lambda_1 = \lambda_2 = -2 = s$

$\Downarrow$

$y_1 = e^{sx} = e^{-2x}$ rešenje ✓

$y_2 = x e^{-2x}$ rešenje ✓

(pokazali !)

da li su lin. nez? ✓ (pokazali)

$$\Rightarrow \text{OR} \quad y = C_1 y_1 + C_2 y_2 = C_1 \cdot e^{-2x} + C_2 \cdot x \cdot e^{-2x}$$

$$= e^{-2x} (C_1 + C_2 x)$$

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$$w(x) = u(x) \pm i \cdot v(x)$$

$$L(w) = 0 \overset{???}{\Rightarrow} \begin{matrix} L(u) = 0 \\ L(v) = 0 \end{matrix}$$

$$L(w) = (u(x) + i \cdot v(x))'' + p \cdot (u(x) + i \cdot v(x))' + g\,(u(x) + i \cdot v(x)) = 0$$

$$= u''(x) + i \cdot v''(x) + p \cdot u'(x) + i \cdot p \cdot v'(x) + g \cdot u(x) + g \cdot i \cdot v(x) = 0$$

$$[u''(x) + p \cdot u'(x) + g \cdot u(x)] + i \cdot \underbrace{[v''(x) + p \cdot v'(x) + g \cdot v(x)]} = 0$$

$$\underbrace{a(x)}_{Re} + i \cdot \underbrace{b(x)}_{Im} = 0 \quad \text{akko Re i Im deo} = 0$$

$0 + 0i$

$$\Rightarrow u''(x) + p u'(x) + g u(x) = 0 \quad i \quad v''(x) + p \cdot v'(x) + g \cdot v(x) = 0$$

$$\Rightarrow L(u) = 0 \ ✓ \quad i \quad L(v) = 0 \ ✓ \qquad \boxtimes$$

$$\left.\begin{matrix} u(x) = e^{\alpha x} \cos \beta x & \text{rešenje} ✓ \\ v(x) = e^{\alpha x} \sin \beta x & \text{rešenje} ✓ \end{matrix}\right\} \text{??? lin. nez.}$$

$$W(u,v) = \begin{vmatrix} e^{\alpha x} \cos \beta x & e^{\alpha x} \sin \beta x \\ \alpha \cdot e^{\alpha x} \cdot \cos \beta x + e^{\alpha x} \cdot \beta \cdot (-\sin \beta x) & \alpha \cdot e^{\alpha x} \sin \beta x + e^{\alpha x} \cdot \beta \cdot \cos \beta x \end{vmatrix}$$

$$= e^{\alpha x} \cos \beta x \cdot (\alpha \cdot e^{\alpha x} \sin \beta x + e^{\alpha x} \beta \cos \beta x) - e^{\alpha x} \sin \beta x \cdot (\alpha \cdot e^{\alpha x} \cos \beta x + e^{\alpha x} \beta (-\sin \beta x))$$

$$= e^{2\alpha x} \cos \beta x (\alpha \sin \beta x + \beta \cos \beta x) - e^{2\alpha x} \sin \beta x (\alpha \cos \beta x - \beta \cdot \sin \beta x)$$

$$= e^{2\alpha x} [\cancel{\alpha \sin \beta x \cos \beta x} + \beta \cdot \cos^2 \beta x - \cancel{\alpha \sin \beta x \cos \beta x} + \beta \sin^2 \beta x]$$

$$= e^{2\alpha x} \beta \underbrace{(\cos^2 \beta x + \sin^2 \beta x)}_{=1} = \underbrace{e^{2\alpha x}}_{\neq 0} \cdot \underbrace{\beta}_{\neq 0 \text{ jer } \lambda_{1,2} = \alpha \pm i\beta} \neq 0$$

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1:04 PM

$$y'' + 2y' + 2y = 0$$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$= \alpha \pm \beta i \Rightarrow \alpha = -1, \quad \beta = 1$$

$$\lambda_1 = -1 + i \qquad \cancel{(\alpha = -1, \beta = 1)}$$

$$\lambda_2 = -1 - i \qquad \cancel{(\alpha = -1, \beta = -1)}$$

$$y_1(x) = u(x) = e^{\alpha x} \cos \beta x = e^{-x} \cdot \cos x$$

$$y_2(x) = v(x) = e^{\alpha x} \cdot \sin \beta x = e^{-x} \cdot \sin x$$

lin. nez ? (pokažemo)

$\Rightarrow$ OR $\quad y = C_1 \cdot e^{-x} \cdot \cos x + C_2 e^{-x} \sin x$

$$= e^{-x}(C_1 \cos x + C_2 \sin x)$$