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$f(x_1, \dots, x_n)$

$e_1 = (1, 0, \dots, 0)$

$g: \mathbb{R} \to \mathbb{R}$ (jedna prava)

$$g(t) = f(\overset{x_1}{t}, \underbrace{x_2, \dots, x_n}_{\text{fiksirano}})$$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$g'(x_1) = \lim_{h \to 0} \frac{g(x_1+h) - g(x_1)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_1+h, \overbrace{x_2, \dots, x_n}^{\text{fix.}}) - f(x_1, \overbrace{x_2, \dots, x_n})}{h}$$

$$= \lim_{h \to 0} \frac{f((x_1, x_2, \dots, x_n) + (h, 0, \dots, 0)) - f(x_1, \dots, x_n)}{h}$$

$(x_1, \dots, x_n) = x$

$(h, 0, \dots, 0) = h \cdot (1, 0, \dots, 0)$
$= h \cdot e_1$

$$= \lim_{h \to 0} \frac{f(x + h \cdot e_1) - f(x)}{h}$$

$$= D_{e_1} f(x)$$

(izvod f-je g u x_1
"
izvod f-je f u x u pravcu vektora e_1)

$e_k = (0, \dots, 0, \underset{\substack{\uparrow \\ k}}{1}, 0, \dots, 0)$

$$g(t) = (\underbrace{x_1, \dots, x_{k-1}}_{\text{fix}}, t, \underbrace{x_{k+1}, \dots, x_n}_{\text{fix}})$$

$$g'(x_k) = D_{e_k} f(x) = \underbrace{\frac{\partial f(x)}{\partial x_k}} = f'_{x_k}(x)$$

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$$f(x,y) = x \cdot y$$

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x$$

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h \to 0} \frac{(x+h)\cdot y - x \cdot y}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x \cdot y} + \cancel{h} \cdot y - \cancel{x \cdot y}}{\cancel{h}} = y$$

Parcijalni izvodi višeg reda

$$\frac{\partial f}{\partial x_k}(x)$$

$$f : D \to \mathbb{R}, \quad D \subseteq \mathbb{R}^2$$

$$f(x,y)$$

$$\frac{\partial f}{\partial x}(x,y) \qquad \frac{\partial f}{\partial y}(x,y) \quad (\text{I reda})$$

II reda

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

po x, po y

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

· Mešoviti

(T) Neka je $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ otvoren skup. Ako f-ja f ima na D parcijalne izvode I reda i mešovite parcijalne izvode II reda i ako su mešoviti izvodi neprekidne f-je onda su oni jednaki:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$, $f(x_1 \dots, x_n)$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \overset{\text{def}}{=} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) \qquad \left(\begin{array}{l}\text{parc.}\\ \text{reda } 2\end{array}\right)$$

$$\frac{\partial^k f}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_m^{k_m}} \qquad k_1 + \dots + k_m = k, \quad m \leq n \qquad \left(\begin{array}{l}\text{parc.}\\ \text{reda } k\end{array}\right)$$

(T) Neka je $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$. Ako f-ja f ima sve parcijalne izvode zaključno sa izvodima reda k na D i ako ima mešovite par. izvode reda $k+1$ i ako su oni neprekidni, onda su oni i jednaki (ne zavise od redosleda promenljivih)

Gradijent

$f(x_1 \dots, x_n) \to \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$

Def: Neka je $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$ definisana u nekoj okolini tačke A i neka postoje $\frac{\partial f}{\partial x_k}(A)$, $\forall k = 1, \dots, n$.
Vektor $\left(\frac{\partial f}{\partial x_1}(A), \dots, \frac{\partial f}{\partial x_n}(A)\right) = \nabla f(A)$ zove se gradijent f-je f u tački A.
(Gradijent je pravac u kom f-ja najbrže raste)
iz A

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$f(x,y) = 4 - 2x^2 - y^2$

$\frac{\partial f}{\partial x} = -4x \quad , \quad \frac{\partial f}{\partial y} = -2y$

$\nabla f(x,y) = (-4x, -2y)$

$A = (1,2) \quad : \quad \nabla f(1,2) = (-4\cdot 1, -2\cdot 2) = (-4,-4) = -4\cdot(1,0) - 4\cdot(0,1)$

$\nabla f(x,y) = \ldots \cdot \underset{(1,0,0)}{i} + \ldots \cdot \underset{(0,1,0)}{j} + \ldots \cdot \underset{(0,0,1)}{k}$

T $f,g : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$, $x \in D$. Ako f i g imaju gradijent u tački x onda:

1) $\nabla (k\cdot f(x)) = k \cdot \nabla f(x)$

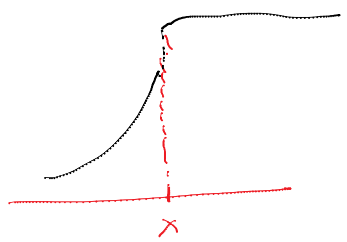
2) $\nabla (f(x) \pm g(x)) = \nabla f(x) + \nabla g(x)$

3) $\nabla (f(x)\cdot g(x)) = g(x)\cdot \nabla f(x) + f(x)\cdot \nabla g(x)$

4) $\nabla \left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot \nabla f(x) - f(x)\cdot \nabla g(x)}{g^2(x)}$

Diferencijabilnost

$f(x)$ postojanje izvoda značilo je neprekidnost f-je



$f(x_1, \ldots, x_n)$ mogu postojati svi $\frac{\partial f}{\partial x_k}(A)$ ali da ona nije neprekidna u A

Da li su postojanje parc. izvoda
i neprekidnost "u vezi"?
NE

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$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \qquad A=(0,0)$$

$$\left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0) : \lim_{n\to\infty} \frac{\frac{1}{n}\cdot\frac{1}{n}}{(\frac{1}{n})^2 + (\frac{1}{n})^2} = \frac{1}{2}$$

$$\left(\frac{1}{n}, -\frac{1}{n}\right) \to (0,0) : \lim_{n\to\infty} \frac{\frac{1}{n}(-\frac{1}{n})}{(\frac{1}{n})^2 + (-\frac{1}{n})^2} = -\frac{1}{2}$$

$\neq$ fja nije neprekidna u $(0,0)$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+\underbrace{h}_{\neq 0},0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h\cdot 0}{h^2+0^2} - 0}{h} = \frac{0}{h} = 0$$

$\Rightarrow$ postoji parc. izvod po x u A

$f(x+\underbrace{h}_{\Delta x}) - f(x)$ $\to$ priraštaj fje $f: \mathbb{R} \to \mathbb{R}$

$f(x+h,y) - f(x,y)$ – prvi parc. priraštaj fje $f: \mathbb{R}^2 \to \mathbb{R}$ po x u tački (x,y)

$f(x,y+h) - f(x,y)$ – ——"—— po y

$f(x+h, y+k) - f(x,y)$ – totalni priraštaj fje $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x_1+h_1, x_2+h_2, \dots, x_n+h_n) - f(x_1, \dots, x_n) = \Delta f(x)$
↳ totalni priraštaj $f: \mathbb{R}^n \to \mathbb{R}$

Def: Neka $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$. Fja f je diferencijabilna u tački $x \in D$ ako postoji linearna funkcija

$$L(h) = L_1 \cdot h_1 + \dots + L_n \cdot h_n$$

takva da $\Delta f(x) = L(h) + o(\|h\|)$

$h = (h_1, \dots, h_n)$
$\|h\| \to 0$
$\|h\| = \sqrt{h_1^2 + \dots + h_n^2}$

Ako: $L_k = \frac{\partial f}{\partial x_k}(x) \Rightarrow L(h) = \frac{\partial f}{\partial x_1}(x)\cdot h_1 + \dots + \frac{\partial f}{\partial x_n}(x)\cdot h_n$

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$$\Delta f(x) - L(h) = o(\|h\|)$$

$$\lim_{\|h\|\to 0} \frac{f(x_1+h_1,\dots,x_n+h_n) - f(x_1,\dots,x_n) - \frac{\partial f}{\partial x_1}(x)\cdot h_1 - \dots - \frac{\partial f}{\partial x_n}(x)\cdot h_n}{\sqrt{h_1^2+\dots+h_n^2}} = 0$$

f je diferencijabilna u $x=(x_1,\dots,x_n)$ ako

$L(h)$ - diferencijal fje f u x

$$df(x) = \frac{\partial f}{\partial x_1}(x)\cdot dx_1 + \dots + \frac{\partial f}{\partial x_n}(x)\, dx_n$$

totalni diferencijal

Da li postojanje parc. izvoda $\Rightarrow$ diferencijabilnost?
NE

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0)\end{cases} \qquad A=(0,0)$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h\cdot 0}{\sqrt{h^2+0^2}} - 0}{h} = 0$$

$\exists$ oba parc. izvoda u A

$$\frac{\partial f}{\partial y}(0,0) = \dots = 0$$

$$\text{Diff: } \lim_{\substack{h\to 0\\ k\to 0}} \frac{f(x+h,y+k) - f(x,y) - \frac{\partial f}{\partial x}(x,y)\cdot h - \frac{\partial f}{\partial y}(x,y)\cdot k}{\sqrt{h^2+k^2}} \overset{??}{=} 0$$

$$\overset{(x,y)=(0,0)=A}{=} \lim_{\substack{h\to 0\\ k\to 0}} \frac{f(h,k) - \overbrace{f(0,0)}^{=0} - 0 - 0}{\sqrt{h^2+k^2}} = \lim_{\substack{h\to 0\\ k\to 0}} \frac{\frac{h\cdot k}{\sqrt{h^2+k^2}}}{\sqrt{h^2+k^2}}$$

$$= \lim_{\substack{k\to 0\\ h\to 0}} \frac{h\cdot k}{\sqrt{h^2+k^2}\cdot\sqrt{h^2+k^2}}$$

važi $\forall k,h$
$\Rightarrow$ važi i za $k=h$

$$\overset{k=h}{=} \lim_{h\to 0}\frac{h^2}{\sqrt{2h^2}\cdot\sqrt{2h^2}} = \frac{1}{2} \neq 0 \Rightarrow$$ nije diferenc.

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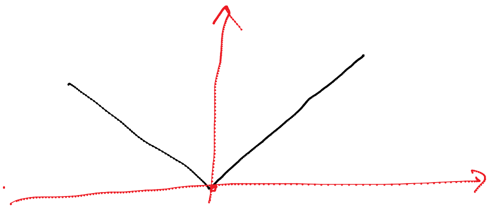
neprekidnost $\overset{?}{\Longrightarrow}$ $\exists$ izvoda NE

$f(x), f(x_1 \ldots, x_n)$

$f(x) = |x|$



(T) Ako je f diferencijabilna u nekoj tački onda je ona i neprekidna u toj tački.
Obrnuto ne mora da važi.

(D): $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$ je neprekidno diferencijabilna (glatka) na D ako u svim tačkama skupa D ima neprekidne parc. izvode.