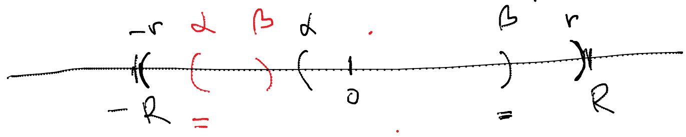
Uniformna konv. stepenih redova

(T) Stepeni red $\sum_{k=0}^{\infty} a_k x^k$ je uniformno konvergentan na $[-r,r]$ gde je $0<r<R$, a R radijus konvergencije.

$[(-R,R)]$ (K) $\qquad [-r,r]$ (UK) $, r<R$

(T) Ako je R poluprečnik konv. i $-R<\alpha<\beta<R$ onda je $S(x)=\sum_{k=0}^{\infty} a_k x^k$ neprekidna f-ja na $[\alpha,\beta]$.

(D)



$r=\max\{|\alpha|,|\beta|\}$

$[\alpha,\beta]\subseteq[-r,r]\subset \underbrace{(-R,R)}_{\text{interval konv.}}$

$\Downarrow$ (T)

$\underbrace{}$ uniformno (K)

$\Downarrow$

$\underbrace{}$ (UK)

1) $f_k(x)=a_kx^k$ neprekidne, $\sum_{k=0}^{\infty} f_k(x)$

2) $\sum f_k(x)$ UK na $[\alpha,\beta]$

preth. (T) $\Rightarrow$ $S(x)=\sum_{k=0}^{\infty} f_k(x)$ neprekidna $\boxtimes$

(T) Ako je R poluprečnik konv. i $-R<\alpha<\beta<R$, stepeni red $\sum_{k=0}^{\infty} a_k x^k$ na $[\alpha,\beta]$ može da se integrali i diferencira proizvoljan broj puta. Pri tome, svi dobijeni redovi imaju isti poluprečnik konvergencije R.

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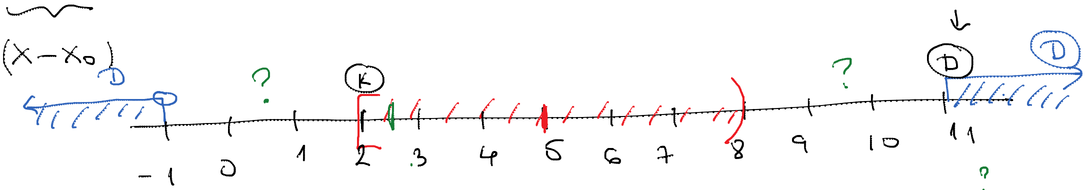
$$S(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$\int_\alpha^\beta S(x)\,dx = \int_\alpha^\beta \sum_{k=0}^{\infty} a_k x^k dx = \sum_{k=0}^{\infty} a_k \underbrace{\int_\alpha^\beta x^k\,dx}$$

$$S'(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right)' = \sum_{k=0}^{\infty} (a_k x^k)' = \underbrace{\sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}}_{\text{pp. k. } R}$$

$$S^{(n)}(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right)^{(n)} = \sum_{k=0}^{\infty} (a_k x^k)^{(n)} = \sum_{k=n}^{\infty} k\cdot(k-1)\cdot\ldots\cdot(k-n+1)\cdot a_k x^{k-n}$$

$$\sum_{n=1}^{\infty} \underbrace{(x-5)^n}_{(x-x_0)} \cdot a_n \quad , \quad x_0 = 5$$



$x = 2$ (K) : $R = 3$ $[2, 8)]$

$R = 4$ $[(1, 9)]$

$\boxed{x = 8: \; ?}$ $R = 3$, $[2, 8)$

$[2, 8]$

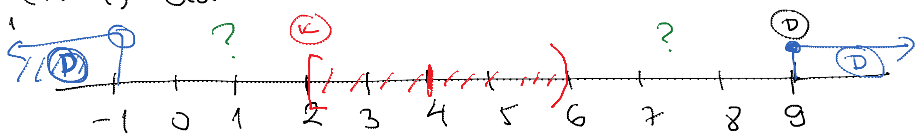
$R = 5$ $(0, 10)$

$6 > R > 3$ (K)

$\underline{x = -1}$? : $R = 6$ $[-1, 11)$ (K)

$(-1, 11)$ (D)

$$\sum_{n=1}^{\infty} (x-4)^n a_n$$



$$\sum_{n=0}^{\infty} a_n (x-2)^n$$

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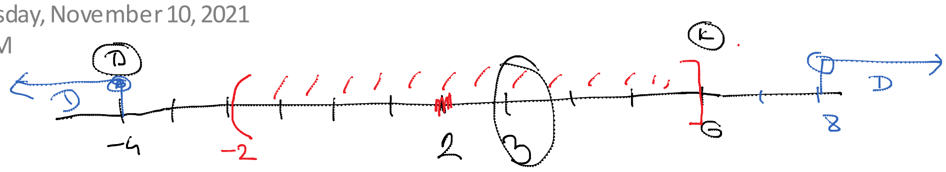

$\sum_{n=0}^{\infty} a_n$? K

$\sum_{n=0}^{\infty} a_n \underbrace{(x-2)^n}_{1}$, $x=3$: $\sum_{n=0}^{\infty} a_n (3-2)^n = \sum_{n=0}^{\infty} a_n \cdot 1^n = \sum_{n=0}^{\infty} a_n$, K

Predstavljanje f-je u obliku stepenog reda

$$\sum_{k=0}^{\infty} f_k(x) \xrightarrow{?} \underbrace{S(x)}_{?} = \sum_{k=0}^{\infty} f_k(x)$$

$$f(x) \xrightarrow{?} f(x) = \sum_{k=0}^{\infty} f_k(x)$$

$$f(x) \overset{?}{=} \sum_{k=0}^{\infty} a_k x^k \qquad (x-x_0)^k$$

$e^{0.05} = ?$ (4 decimale)

pp. $R \neq 0$, $(-R, R)$, $\sum_{k=0}^{\infty} a_k x^k = S(x)$

$$\exists S^{(n)}(x) = \sum_{k=n}^{\infty} \underbrace{k(k-1)\cdots(k-n+1)\cdot a_k}_{\frac{k!}{(k-n)!}\cdot a_k} \cdot x^{k-n}, \quad x \in (-R, R)$$

$\sum a_k x^k$ konv. za $x=0$

$$\exists S(0) = a_0 \cdot \underbrace{0^0}_{1} + \underbrace{a_1 \cdot 0^1}_{0} + \underbrace{\cdots}_{0} = a_0$$

n=k

$$\exists S^{(k)}(0) = \frac{k!}{(k-k)!} \cdot a_k \cdot 0^{k-k} = \frac{k!}{0!} \cdot a_k \cdot \underbrace{0^0}_{1} = k! \cdot a_k$$

$\underbrace{f(x)}_{\text{dato}}$, $\exists f^{(k)}(0)$, $f(x) \overset{?}{=} \sum a_k x^k$, $x \in (-R, R)$

$f^{(k)}(0) = k! \cdot a_k$

$$\Rightarrow a_k = \frac{f^{(k)}(0)}{k!} \Rightarrow \boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \cdot x^k}$$

(beskonačni) razvoj f-je f u okolini 0=x

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(T) Ako $\exists$ brojevi M $(0<M<\infty)$ i r $(0<r<\infty)$ takvi da $\forall k \in \mathbb{N}_0$ važi $|f^{(k)}(x)| \le M$, $x \in [-r,r]$ tada funkcija $f(x)$ može da se predstavi sa

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \cdot x^k$$

Tejlorov razvoj u $x=0$ na $[-r,r]$.

Skica: $$f(x) = \underbrace{S_n(x)}_{1,\dots,n} + \underbrace{R_n(x)}_{n+1,\dots,\infty}$$

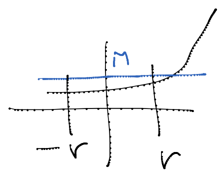
$$R_n(x) = \underbrace{f(x) - S_n(x)}$$

$$\lim_{n\to\infty} R_n(x) = 0 \quad \Rightarrow \quad f(x) - S_n(x) = 0$$
$$f(x) = S_n(x)$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} \cdot (x-x_0)^k, \quad x \in [x_0 - r, x_0 + r]$$

Tejlorov razvoj u okolini $x = x_0$

$e^{0.05}$. e^x, $x_0 = 0$



$f(x) = e^x$
$f'(x) = e^x$
$f^{(k)}(x) = e^x$
$f^{(k)}(0) = e^0 = 1.$

$$\underset{x\in[-r,r]}{|f^{(k)}(x)| = |e^x| \le e^r = M} \quad \Rightarrow \text{smem Tejlora}$$

$$f(x) = e^x \overset{\text{smem}}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \cdot x^k = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{0.05} = 1.05\underbrace{13}_{k=0}\dots\dots$$

$$e^{0.05} \underset{\substack{x=0.05 \\ (\text{Tejlor})}}{=} \underbrace{\frac{0.05^0}{0!}}_{1} + \overset{k=1}{\underbrace{\frac{0.05^1}{1!}}_{0.05}} + \underbrace{\frac{0.05^2}{2!}}_{0.0013} + \underbrace{\frac{0.05^3}{3!}}_{0.00002}$$

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$f(x) = \sin(x)$, $x_0 = 0$

$$\left.\begin{aligned} f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f'''(x) &= -\cos(x) \\ f^{IV}(x) &= \sin x \end{aligned}\right\} \quad \begin{aligned} f^{(2n)}(x) &= \pm\sin(x) \\ f^{(2n+1)}(x) &= \pm\cos(x) \end{aligned}$$

$$\left.\begin{aligned} -1 \le \cos(x) \le 1 \\ 1 \le \sin(x) \le 1 \end{aligned}\right\} \Rightarrow |f^{(k)}(x)| \le \underline{1} = M$$

✓ sue Teilor

$f^{(2n)}(0) = \pm\sin(0) = 0$

$f^{(2n+1)}(0) = \pm\cos(0) = \left\{ \begin{array}{ll} -1 & ; 2 \mid n \\ 1 & ; 2 \nmid n \end{array} \right\} = (-1)^n$

$$\Rightarrow f(x) = \sin(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \cdot x^k = \underbrace{\sum_{\substack{\text{parni} \\ k}}}_{=0} + \underbrace{\sum_{\substack{\text{neparnih} \\ k}}}$$

$$\boxed{\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1}}$$

$$\sin(0.05) = \underset{k=0}{\ldots} + \underset{k=1}{\ldots} + \underset{k=2}{\ldots}$$