Friday, October 29, 2021
9:37 AM

(D) $a_n \searrow \quad \lim_{n\to\infty} a_n = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ (K)

$$S_n = \sum_{k=1}^{n} (-1)^{k+1} \underbrace{a_k}_{\geq 0}$$

$$S_{2n} = \sum_{k=1}^{2n} (-1)^{k+1} a_k = \overbrace{(a_1 - a_2)}^{\geq 0} + \overbrace{(a_3 - a_4)}^{\geq 0} + \ldots + \ldots + \underbrace{(a_{2n-1} - a_{2n})}_{\geq 0}$$

$$S_{2n+1} = S_{2n} + a_{2n+1}$$

a_n nerastući $\Rightarrow a_k \geq a_{k+1}$

$\Rightarrow a_1 \geq a_2, \; a_2 \geq a_3, \; a_3 \geq a_4, \ldots, a_{2n-1} \geq a_{2n}$

$S_2 \leq S_4 \leq S_6 \leq \ldots$

$\Rightarrow S_{2n} \nearrow \quad (\heartsuit)$

$$S_{2n} = a_1 - a_2 + a_3 - a_4 \ldots - a_{2n-2} + a_{2n-1} - a_{2n}$$

$$= \underbrace{a_1}_{\geq 0} - \underbrace{(a_2 - a_3)}_{\geq 0} - \underbrace{(a_4 - a_5)}_{\geq 0} - \ldots - \underbrace{(a_{2n-2} - a_{2n-1})}_{\geq 0} - \underbrace{a_{2n}}_{\geq 0}$$

$S_{2n} \leq a_1 \Rightarrow$ ograničen odozgo $(\star)$

$(\heartsuit)$ i $(\star)$ $\Rightarrow S_{2n}$ (K) $\Rightarrow \exists \lim_{n\to\infty} S_{2n} = S$

$S_{2n+1} = S_{2n} + a_{2n+1} \quad / \lim$

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + a_{2n+1})$$

$$= \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1}$$

$$= S + 0$$

$$= S$$

$$\lim_{n\to\infty} S_{2n} = S = \lim_{n\to\infty} S_{2n+1} \Rightarrow \lim_{n\to\infty} S_n = S$$

$$S = \sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ (K)}$$

$\boxtimes$

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9:37 AM

D) $\sum_{k=1}^{\infty} a_k$ (AK) $\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ (K)

Formiramo pomoćne redove $\sum_{k=1}^{\infty} b_k$ i $\sum_{k=1}^{\infty} c_k$

$$b_k = \frac{1}{2}\left(|a_k| + \underbrace{a_k}_{\le |a_k|}\right) \quad , \quad c_k = \frac{1}{2}\left(|a_k| \underbrace{- a_k}_{\le |a_k|}\right)$$

$$b_k - c_k = \frac{1}{2}\left(\cancel{|a_k|} + a_k - \cancel{|a_k|} + a_k\right) = a_k \qquad (\square)$$

$a_k \le |a_k| \Rightarrow b_k \geq 0$ (*)

$$b_k = \frac{1}{2}\left(|a_k| + \underbrace{a_k}_{\le |a_k|}\right) \le \frac{1}{2}\left(|a_k| + |a_k|\right) = |a_k| \qquad (✿)$$

$-a_k \le |a_k| \Rightarrow c_k \geq 0$ (**)

$$c_k = \frac{1}{2}\left(|a_k| \underbrace{- a_k}_{\le |a_k|}\right) \le \frac{1}{2}\left(|a_k| + |a_k|\right) = |a_k| \qquad (✿✿)$$

(*) i (**) $\sum b_k$ i $\sum c_k$ su redovi sa pozitivnim članovima

i $\underbrace{b_k \le |a_k|}_{(♡)}$ i $\underbrace{c_k \le |a_k|}_{(\Delta)}$ (✿) i (✿✿)

$$\sum_{k=1}^{\infty} a_k \text{ (AK)} \overset{\text{def.}}{\Rightarrow} \sum_{k=1}^{\infty} |a_k| \text{ (K)} \xrightarrow[(♡)]{\text{poredbeni}} \sum_{k=1}^{\infty} b_k \text{ (K)}$$

$$\xrightarrow[(\Delta)]{} \sum_{k=1}^{\infty} c_k \text{ (K)}$$

$$\Rightarrow \sum_{k=1}^{\infty} \underbrace{b_k - c_k}_{a_k} \text{ (K)} \overset{(\square)}{\Rightarrow} \sum_{k=1}^{\infty} a_k \text{ (K)}$$

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$$\sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{2n-1} n} = 1 + \frac{1}{1\cdot 3} + \frac{1}{1\cdot 3\cdot 5} + \frac{1}{1\cdot 3\cdot 5\cdot 7} + \dots = \sqrt{\frac{\pi e}{2}}$$

$$\frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^3 - 2k} = \ln 2$$

$$\frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103 + 26390k)}{(k!)^4 \cdot 396^{4k}} = \frac{1}{\pi}$$

$$1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 - 13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \dots = \frac{2}{\pi}$$

Ramanujan ☺