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$\mathbb{N}: \quad 1, 2, 3, \ldots$

$\mathbb{Q}:$

$$\begin{array}{ccccccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \ldots \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \ldots \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \ldots \\ \frac{4}{1} & \frac{4}{2} & & & & \\ \vdots & & & & & \end{array}$$

$$S = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \ldots = ? \quad = -\frac{1}{12} \quad ?!?$$

$$S_1 = \sum_{n=1}^{\infty} (-1)^{n+1} = \underbrace{1 - 1} + \underbrace{1 - 1} + 1 \ldots$$

$$S_2 = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n = 1 - 2 + 3 - 4 + 5 - \ldots$$

$\cdot 2$

$$S_1 = \underbrace{(1-1)}_{0} + \underbrace{(1-1)}_{0} + \underbrace{(1-1)}_{0} + \ldots = 0$$

$$\cancel{S_1 =} 1 + \underbrace{(-1+1)}_{0} + \underbrace{(-1+1)}_{0} + \ldots = 1 + 0 = 1$$

$(+)$

$$2S_1 = 1 \Rightarrow S_1 = 1/2$$

$$2 \cdot S_2 = 1 - 2 + 3 - 4 + 5 - 6 + 7 - \ldots$$
$$+ \quad 1 - 2 + 3 - 4 + 5 - 6 + \ldots$$

$$= 1 - 1 + 1 - 1 + 1 - \ldots$$
$$= S_1$$
$$= 1/2 \Rightarrow S_2 = 1/4$$

$S_1 = 1$
$S_2 = -1$
$S_3 = 2$
$S_4 = -2$
$S_5 = 3$
$S_6 = -3$

$S - S_2 = 4S$

$S - \frac{1}{4} = 4S$

$3S = -\frac{1}{4}$

$$\boxed{S = -\frac{1}{12}}$$

$$S - S_2 = 1 + 2 + 3 + 4 + 5 + \ldots$$
$$- [\cancel{1 - 2 +} 3 - 4 + 5 - \ldots]$$

$$= 0 + 4 + 0 + 8 + 0 + 12 + \ldots = 4(1 + 2 + 3 + 4 + 5 + \ldots) = 4 \cdot S$$

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(D) $\sum_{k=1}^{\infty} a_k$ (C) $\Rightarrow \sum_{k=1}^{\infty} c \cdot a_k$ (C)

$$\begin{cases} S_n = a_1 + \dots + a_n \\ \exists \lim_{n\to\infty} S_n = S \end{cases}$$

$$\overline{S}_n = c \cdot a_1 + \dots + c \cdot a_n = c \cdot (\underbrace{a_1 + \dots + a_n}) = c \cdot S_n$$

$$\lim_{n\to\infty} \overline{S}_n = \lim_{n\to\infty} c \cdot S_n = c \cdot \underbrace{\lim_{n\to\infty} S_n} = \underbrace{c \cdot S} \quad \text{(C)}$$

QED

(D) $\sum_{k=1}^{\infty} a_k, \sum_{k=1}^{\infty} b_k$ (C) $\Rightarrow \sum_{k=1}^{\infty} (a_k + b_k)$ (C)

$$\left. \begin{aligned} S_n' &= a_1 + \dots + a_n \\ S_n'' &= b_1 + \dots + b_n \end{aligned} \right\} \Rightarrow \begin{aligned} &\exists \lim_{n\to\infty} S_n' = S' \\ &\exists \lim_{n\to\infty} S_n'' = S'' \end{aligned}$$

$$S_n = \sum_{k=1}^{n} a_k + b_k = (a_1 + b_1) + (a_2 + b_2) + \dots + (a_n + b_n)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} [a_1 + b_1 + a_2 + b_2 + \dots + a_n + b_n]$$

$$= \lim_{n\to\infty} [\underbrace{(a_1 + \dots + a_n)}_{S_n'} + \underbrace{(b_1 + \dots + b_n)}_{S_n''}]$$

$$= \lim_{n\to\infty} [S_n' + S_n'']$$

$$= \lim_{n\to\infty} S_n' + \lim_{n\to\infty} S_n''$$

$$= S' + S''$$

$$= S$$

QED

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(D) $\sum_{k=1}^{\infty} a_k$ (K) $\Rightarrow \lim_{n\to\infty} a_n = 0$

$S_n = a_1 + \dots + \underline{a_n}$, $\exists \lim_{n\to\infty} S_n = S$

$S_{n-1} = a_1 + \dots + a_{n-1}$

$\overline{} \ominus$

$S_n - S_{n-1} = a_n$

$m = n-1$
$n = \infty \Rightarrow m = \infty$
$\lim_{m\to\infty} S_m$

$\underline{\lim_{n\to\infty} a_n} = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$

$= S - S = \underline{0}$ $\boxtimes$

PRIMER

$\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{k=1}^{\infty} a_k$ (K)

$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, $a_k = \frac{1}{\sqrt{k}} \xrightarrow{k\to\infty} 0$, $\exists \lim_{k\to\infty} a_k = 0$

$\hookrightarrow S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}}$

$> \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \dots + \frac{1}{\sqrt{n}}$

$= n \cdot \frac{1}{\sqrt{n}} = \sqrt{n} \not\rightarrow 0$

$S_1 = \sqrt{1}$, $S_2 = \sqrt{1} + \sqrt{2}$, $S_3 = \sqrt{1} + \sqrt{2} + \sqrt{3}$, ...

$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \dots + a_m}_{S_m} + \underbrace{a_{m+1} + \dots}_{\sum_{n=m+1}^{\infty} a_n = r_m}$

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(D) $\sum_{k=1}^{\infty} a_k$, $r_m = \sum_{k=m}^{\infty} a_k$, $n > m$

$\underbrace{\underbrace{a_1, \ldots, a_{m-1}}_{S_{m-1}}, \underbrace{a_m, a_{m+1}, \ldots, a_n}_{\overline{S}_{n-m+1}}}_{S_n}, a_{n+1}, \ldots\ldots$

$S_n = \sum_{k=1}^{n} a_k = a_1 + \ldots + a_n$ $\qquad S_{m-1} = a_1 + \ldots + a_{m-1}$

$\qquad = C$

$\overline{S}_{n-m+1} = a_m + a_{m+1} + \ldots + a_n$

$S_n = \underbrace{S_{m-1}} + \overline{S}_{n-m+1}$

$= C + \overline{S}_{n-m+1}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (C + \underline{\overline{S}_{n-m+1}})$

konačno = konačno

∞ $\qquad \infty$

$\boxtimes$

(T) $\sum_{k=1}^{\infty} a_k$ (K) $\iff$ $r_m \xrightarrow{m\to\infty} 0$

$\boxed{\Rightarrow}$ $\hookrightarrow \exists \lim_{n\to\infty} S_n = S = \sum_{k=1}^{\infty} a_k$

$= S_m + r_m$

$\Rightarrow r_m = S - S_m \qquad / \lim$

$\lim_{m\to\infty} r_m = \lim_{m\to\infty} (S - S_m)$

$= S - \lim_{m\to\infty} S_m$

$= S - S = 0$

$\boxed{\Leftarrow}$ $r_m \xrightarrow{m\to\infty} 0$

$\Rightarrow r_m = \sum_{k=m}^{\infty} a_k = a_m + a_{m+1} + \ldots$ (K)

pretp. teor. $\Rightarrow$ r_m i $\sum_{k=1}^{\infty} a_k$ istovremeno (K) $\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ (K)

$\boxtimes$

$0 < a_n \le b_n$

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(P) 1) $\sum b_n$ (K) $\Rightarrow$ $\sum a_n$ (K)

$S_n = \sum_{k=1}^{n} a_k$

$\bar{S}_n = \sum_{k=1}^{\infty} b_k \Rightarrow \exists \lim_{n\to\infty} \bar{S}_n = \bar{S} \in \mathbb{R}$

$S_n \nearrow$ $(a_n > 0)$ rastuć ogranićen odozg. $\Big\} \Rightarrow$

$a_n \le b_n \Rightarrow S_n \le \bar{S}_n \le \bar{S}$

$\Rightarrow S_n$ (K) $\Rightarrow \exists \lim_{n\to\infty} S_n = S \in \mathbb{R} \Rightarrow \sum_{n=1}^{\infty} a_n$ (K)

2) $\sum a_n$ (D) $\Rightarrow \sum b_n$ (D)

$\hookrightarrow a_k > 0 \Rightarrow \lim_{n\to\infty} S_n = +\infty$

$a_n \le b_n \Rightarrow S_n \le \bar{S}_n$

$\lim_{n\to\infty} S_n \le \lim_{n\to\infty} \bar{S}_n$

$\infty \le \lim_{n\to\infty} \bar{S}_n \Rightarrow \lim_{n\to\infty} \bar{S}_n = +\infty$

$\Rightarrow \sum_{k=1}^{\infty} b_k$ (D)

II pov: $\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}$

$\frac{a_2}{a_1} \le \frac{b_2}{b_1}, \quad \frac{a_3}{a_2} \le \frac{b_3}{b_2}, \quad \dots, \quad \frac{a_n}{a_{n-1}} \le \frac{b_n}{b_{n-1}}$

$a_n \le a_{n-1} \cdot \frac{b_n}{b_{n-1}} \le a_{n-2} \cdot \frac{\cancel{b_{n-1}}}{b_{n-2}} \cdot \frac{b_n}{\cancel{b_{n-1}}} \le a_{n-3} \cdot \dots$

$a_{n-1} \le a_{n-2} \cdot \frac{b_{n-1}}{b_{n-2}}$

$\dots \le \frac{a_1}{b_1} \cdot b_n$

$a_n \le \left(\frac{a_1}{b_1}\right)^{c} \cdot b_n \quad \Rightarrow \quad \frac{b_1}{a_1} \cdot a_n \le b_n$

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$$\sum_{k=1}^{\infty} b_k \; K \Rightarrow \sum_{k=1}^{\infty} c \cdot b_k \; K$$

$$\cancel{\exists \lim_{n\to\infty} S_n = \bar{S}}$$

$$a_n \le c \cdot b_n$$

$$\sum a_n \quad , \quad \underbrace{\sum c \cdot b_n}_{K}$$

$$\xrightarrow{\text{I por.}} \sum a_n \; K$$