An introduction to Model theory with application in Computer science

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Course description

In this course some theoretical aspects of computer science and real use of computers will be presented. The following themes will be covered:

Model theory with emphasis on finite model theory. Basics of this theory will be given (first order languages and theories, models, satisfaction relation, Completeness and Compactness theorems). Some proofs will be outlined and decision problem of first order theories discussed.

Random graphs and an introduction to the theory of these structures (Erdös, Rénei). On the basis of this theory the so called 0-1 Low (Glebski, Kogan, Liagonki, Talanov[1969], Fagin[1976]) will be inferred and presented its application.

Course description

Free algebras and equational logic. Examples and constructions of free Boolean algebras. Birkhoff HSP theorem and Completeness theorem for Equational logic. Word problem (a decision problem for algebraic identities in mathematics and computer science). Free Boolean algebras and their applications in parallel computing.

Programming systems Programming platforms for symbolic computations such as Wolfram *Mathematica*, Mace4 and TBA for finding finite structures will be presented. The framing language of TBA is Python, while the core of the program has parallel implementation and it is designed for execution on graphic cards.

During this course students would be asked to study and use the programming language **Python**.

Bibliography

Books:

1. Ž. Mijajlović, An introduction to model theory, Univ. of Novi Sad, 1987, http://elibrary.matf.bg.ac.rs
2. Gerald Sacks, Saturated Model Theory, Reading, Mass, 1972.
3. C.C. Chang, J.H. Keisler, Model theory, North Holland, 1990.
4. D. Marker, Model Theory: An Introduction, Springer, 2002.
5. G. Kreisel, J.L. Krivine, Elements of Mathematical Logic (Model Theory), North Holland, 1967.
6. E. Grädel et all, Finite Model Theory and Its Applications, Springer, 2007.
7. A lot more on the address http://gen.lib.rus.ec

Programming systems:

1. Python version 3.3 or later.
2. Prover9-Mace4, version 0.5, 2007, or later. https://www.cs.unm.edu/∼mccune/mace4
3. Wolfram Mathematica version 8 or later.

Model theory: Preliminaries

These lectures are designed as an excursion through the main topics of classical model theory. The most important constructions and theorems of model theory and their proofs are outlined.

Boolean algebras play an important role in this book. The use of Boolean algebras in model theory is prolific. We have applied them in many model-theoretic constructions, but we also have applied model theory in the proofs of certain properties of Boolean algebras.

Basic constructions of models are presented such as the method of constants, elementary chains of models and types. A few words are devoted to abstract model theory.

However, we could not cover all the important topics in model theory, but there are books of an encyclopedic nature on this subject and the reader is directed to consult them whenever he needs more details.

We suppose that the student is acquainted with some parts of the naive set theory. This includes the basic properties of ordinal and cardinal numbers and partially, their arithmetic.

We have adopted Von Neuman representation of ordinals, so we have taken that every ordinal is the set of all the smaller ordinals. Therefore

$$0 = \emptyset,\ 1 = \{0\},\ 2 = \{0,1\},\ \ldots,\ \omega_0 = \{0,1,2,\ldots\},\ \ldots \tag{0.1}$$

Here $\emptyset$ denotes the empty set. The set of all natural numbers is denoted by ω_0, i.e. $\omega = \omega_0 = \{0,1,2,\ldots\}$. We do not distinguish ordinal numbers ω_α and cardinal numbers $\aleph_\alpha$.

If $f\colon A \to B$ is a mapping from a set A into a set B and $X \subseteq A$, then

- $f|X$ denotes the restriction of f to the set X,
- $f[X] = \{f(x)\colon x \in X\}$, but sometimes we write $f(X)$ for $f[X]$ as well,
- $f\mathbf{x}$ or $f(\mathbf{x})$ stands for the sequence $fx_1, fx_2, \ldots, fx_n$, where $\mathbf{x}$ denotes the sequence $x_1, x_2, \ldots, x_n$.

The cardinal number of a set X is denoted by $|X|$, and the set of all subsets of X by $P(X)$.

Our metatheory is based on the ZFC set theory, and we shall not point out explicitly when we use, for example, the Axiom of Choice or its equivalents. However, all exceptions will be indicated, as the use of the Continuum Hypothesis or some weaker variants of the Axiom of Choice.

Final remarks are on usage and signs. The word "iff" is often used instead of the phrase "if and only if". The end of a proof is indicated by $\square$.

Model theory: Introduction

Model theory is often defined as a union of formal logic and universal algebra. More detailed analysis shows that model theory is the study of the relationship between syntactical objects on the one hand and the structures of a set-theoretical nature on the other hand, or in other words, between formal languages and their interpretations.

Therefore, two areas of logic, syntax and semantics, both have a role to play in this subject.

While syntax is concerned mainly with the formation rules of formulas, sentences and other syntactical objects, semantics bears on the meaning of these notions.

Model theory: Introduction

One of the most important concepts of model theory is the **satisfaction relation**, denoted by $\models$, a relation between mathematical structures and sentences. Model theory was recognized as a separate subject during the thirties of the XX century in the works of Thoralf Skolem (1887-1963, Norwegian), Alfred Tarski (1901-1983, Polish), Kurt Gödel (1906–1978, Austrian), Anatoly Malcev (1909–1967, Russian) and their followers.

Since then, this field has developed vigorously, and was applied in many other branches of mathematics: algebra, set-theory, nonstandard analysis, computer science and even mathematical economy.

We can speak of model theory of any kind of logic, but we shall study model theory of first-order predicate calculus.

Model theory: First-order languages

A first order language is any set L of constant symbols, function symbols and relation symbols. Each of the relation and function symbols has some definite, finite number of argument places. Sometimes it is convenient to consider constant symbols as function symbols with zero argument places. According to our classification, we have

$$L = \mathrm{Fnc}_L \cup \mathrm{Rel}_L \cup \mathrm{Const}_L, \text{ where} \tag{0.2}$$

- $\mathrm{Fnc}_L = \{s \in L \colon s \text{ is a function symbol of } L\}$,
- $\mathrm{Rel}_L = \{s \in L \colon s \text{ is a relation symbol of } L\}$,
- $\mathrm{Const}_L = \{s \in L \colon s \text{ is a constant symbol of } L\}$,

If L and L' are first order languages, and $L \subseteq L'$, then L' is called an expansion of the language L, while L is called a reduct of L'. If $L' \backslash L$ is a set of constant symbols, then we say that L' is a simple expansion of L.

All these three sets are pairwise disjoint, and each of them may be an empty set. Namely, we shall deal only with logic with equality.

The arity–function $\mathrm{ar}\colon L \to \omega$ assigns to each $s \in L$ its length, i.e. the number of argument places. By the remark above, if $s \in Const_L$, we define $\mathrm{ar}(s) = 0$, while for $s \in \mathrm{Fnc}_L \cup \mathrm{Rel}_L$, we have $\mathrm{ar}(s) \geqslant 1$.

In most cases it will be clear from the context what the lengths of the symbols of L are, so in such cases the arity function will not be mentioned explicitly. However, we take $\mathrm{Fnc}_L^k = \{F \in \mathrm{Fnc}_L \colon \mathrm{ar}(F) = k\}$. A similar meaning has Rel_L^k for relation symbols of L.

Example The language of ordered fields is $L = \{+, -, \cdot, \leqslant, 0, 1\}$. Here $\mathrm{Fnc}_L = \{+, -, \cdot,\}$, $\mathrm{Rel}_L = \{\leqslant\}$, $\mathrm{Const}_L = \{0, 1\}$.

Model theory: Terms and formulas

Terms and formulas of a first-order language L are special finite sequences of the symbols of L and the logical symbols of the first-order predicate calculus (which shall be abbreviated PR1).

Logical symbols of PR1 are logical connectives and quantifiers:

- $\wedge$ – and, $\vee$ – or signs, $\Rightarrow$ - implication, $\Leftrightarrow$ – equivalence,
- $\neg$ – negation and the equality sign $\equiv$,
- $\forall$ – universal quantifier and $\exists$ – existential quantifier.

Finally, we have an infinite sequence of variables $v_1, v_2, \ldots$.

The unique readability of terms and formulas must be provided, so some auxiliary symbols are used, the left and right parenthesis and the comma sign: ( ) ,.

Metavariables are $x, y, z, x_0, y_0, z_0, \ldots$, and they may denote any variable v_i, $i \in \omega$, i.e. the domain of metavariables is the set $\mathrm{Var} = \{v_1, v_2, \ldots\}$. Metaequality is another important such sign and it will be denoted by $=$.

Model theory: Terms and formulas

Terms, or algebraic expressions of a language L can be described inductively:

- Variables and constant symbols of L are terms.
- If $F \in \mathrm{Fnc}_L$ is of the length n, and $t_1 \ldots, t_n$ are terms of L, then $F(t_1, \ldots, t_n)$ is a term of L.
- Every term of L is obtained by a finite number of applications of the previous two rules.

A somewhat more formal (recursive) definition of this notion is:

$$T_0 = \mathrm{Var} \cup \mathrm{Const}_L,$$

$$T_{n+1} = T_n \cup \{F(t_1, \ldots, t_k)\colon t_1, \ldots, t_k \in T_n, F \in \mathrm{Fnc}_L^k, k \in \omega\},$$

$$\mathrm{Term}_L = \bigcup_{n \in \omega} T_n.$$

Then term of L is any element t of Term_L.

Model theory: Terms and formulas

Standard rules are applied on terms: rules about deleting parenthesis, special notation for binary function symbols, possible priority of function symbols, etc.

The complexity function $\mathrm{co}\colon \mathrm{Term}_L \to \omega$ of terms is a measure of the complexity of the terms. It is defined in the following way:

If $t \in T_0$, then $\mathrm{co}(t) = 0$.

If $t \in T_n \setminus T_{n-1}$, then $\mathrm{co}(t) = n$, $n \in \omega \setminus \{0\}$.

The complexity of terms can be visualized from the following diagram. Letters F and G here are the binary function symbols.

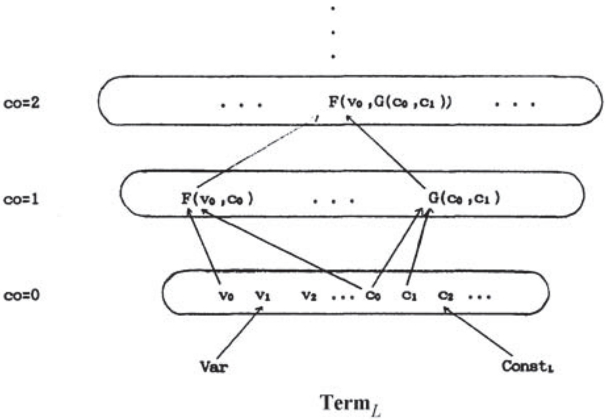

Figure: Terms of a first order language L

Model theory: Terms and formulas

Formulas of the first-order language L are defined in a similar manner. First, the atomic formulas are defined:

A string φ is an **atomic formula** of a language L, if and only if φ has one of the following forms:

$u \equiv v$, u, v are terms of L,

$R(t_1, \dots, t_n)$, $R \in \mathrm{Rel}_L^n$ and $t_1, \dots, t_n$ are terms of L.

Let At_L denote the set of the atomic formulas of L. Then by the previous definition we have

$$\begin{aligned}\mathrm{At}_L = \{&u \equiv v \colon u, v \in \mathrm{Term}_L\} \cup \\ &\{R(t_1, \dots, t_n\} \colon n \in \omega, R \in \mathrm{Rel}_L^n, t_1, \dots, t_n \in \mathrm{Term}_L\}.\end{aligned}$$

Formulas of a language L are also defined inductively by the use of an auxiliary sequence $\mathcal{F}_n$, $n \in \omega$, of sets of strings of L:

$$
\begin{aligned}
\mathcal{F}_0 &= \mathrm{At}_L,\\
\mathcal{F}_{n+1} &= \mathcal{F}_n \cup \{(\varphi \wedge \psi)\colon \varphi,\psi \in \mathcal{F}_n\} \cup\\
&\quad \{(\varphi \vee \psi)\colon \varphi,\psi \in \mathcal{F}_n\} \cup\\
&\quad \{(\neg\varphi)\colon \varphi \in \mathcal{F}_n\} \cup\\
&\quad \{(\varphi \Rightarrow \psi)\colon \varphi,\psi \in \mathcal{F}_n\} \cup\\
&\quad \{(\varphi \Leftrightarrow \psi)\colon \varphi,\psi \in \mathcal{F}_n\} \cup\\
&\quad \{(\forall x\varphi)\colon x \in \mathrm{Var}, \varphi \in \mathcal{F}_n\} \cup\\
&\quad \{(\exists x\varphi)\colon x \in \mathrm{Var}, \varphi \in \mathcal{F}_n\}, \quad n \in \omega,
\end{aligned}
$$

$$\mathrm{For}_L = \bigcup_{n\in\omega} \mathcal{F}_n$$

Then the elements of the set For_L are defined as formulas of the language L. It is not difficult to see that the formulas satisfy the following conditions:

- Atomic formulas are formulas.
- If φ and ψ are formulas of L , and x is a variable, then $(\varphi \wedge \psi), (\varphi \vee \psi), (\neg\varphi), (\varphi \Rightarrow \psi), (\varphi \Leftrightarrow \psi), (\exists x\varphi), (\forall x\varphi)$ are also formulas of L.
- Every formula of L is obtained by a finite number of use of the previous two rules,

In order to measure the **complexity of formulas**, we shall extend the complexity function co to formulas as well. Therefore, $\mathrm{co}\colon \mathrm{For}_L \to \omega$ is defined inductively in the following way:

- If $\varphi \in \mathrm{At}_L$, then $\mathrm{co}(\varphi) = 0$,
- If $\varphi \in \mathrm{For}_n \backslash \mathrm{For}_{n-1}$, $n \in \omega\backslash\{0\}$, then $\mathrm{co}(\varphi) = n$.

Model theory: Terms and formulas

As in the case of terms, we suppose that the reader is familiar with the basic conventions about formulas: the use of rules on deleting parenthesis, priority of logical connectives, universal quantifiers at the outermost level can be omitted, etc.

In addition, we shall shrink blocks of quantifiers, for example instead of $\forall x_0 \forall x_1 \dots \forall x_n \varphi$ we shall write $\forall x_0 x_1 \dots x_n \varphi$ whenever appropriate.

The notion of a **free occurrence of variables** allows us to describe precisely the variables of a formula φ which are not in the scope of the quantifiers.

Definition The set $\mathrm{Fv}(\varphi)$ of variables which have free occurrences in a formula φ of L is introduced inductively by the complexity of φ in the following way:

- If $\varphi \in \mathrm{At}_L$, then $\mathrm{Fv}(\varphi)$ is the set of variables which occur in φ.
- $\mathrm{Fv}(\neg\varphi) = \mathrm{Fv}(\varphi)$.
- $\mathrm{Fv}(\varphi \wedge \psi) = \mathrm{Fv}(\varphi \vee \psi) = \mathrm{Fv}(\varphi \Rightarrow \psi) = \mathrm{Fv}(\varphi \Leftrightarrow \psi) = \mathrm{Fv}(\varphi) \cup \mathrm{Fv}(\psi)$,
- $\mathrm{Fv}(\exists x\varphi) = \mathrm{Fv}(\forall x\varphi) = \mathrm{Fv}(\varphi) \backslash \{x\}$.

The elements of the set $\mathrm{Fv}(\varphi)$ are called free variables of the formula φ, while the other variables which occur in φ are called bounded.

Example If $\varphi = (\neg x \equiv 0 \Rightarrow \exists y(x \cdot y \equiv 1))$ then $\mathrm{Fv}(\varphi) = \{x\}$, so x is a free variable of φ and y is a bounded variable of φ.

Model theory: Terms and formulas

If $\varphi \in \mathrm{For}_L$, then the notation $\varphi(x_0, \dots, x_n)$, or $\varphi x_0 \dots x_n$ is used to denote that free variables of φ are some of the variables $x_0, \dots, x_n$.

Formulas φ which do not contain free variables, i.e. $\mathrm{Fv}(\varphi) = 0$, are called **sentences**. The formulas

$$0 \equiv 1, \quad \forall x(\neg x \equiv 0 \Rightarrow \exists y(x \cdot y \equiv 1))$$

are examples of sentences of the language $L = \{\cdot, 0, 1\}$, where $\cdot$ is a binary function symbol. The set of all sentences of L is denoted by Sent_L. The cardinal number of For_L is denoted by $||L||$, therefore $||L|| = |\mathrm{For}_L|$. It is not difficult to see that for every first-order language L we have

$$||L|| = \max\{|L|, \aleph_0\}. \tag{0.3}$$

The definition of the notion of a **first-order theory** is simple:

A theory of a first order language L is any set of sentences of L.

Therefore, a set T is a theory of L iff $T \subseteq \mathrm{Sent}_L$. In this case elements of T are called **axioms** of T.

The main notion connected to the concept of a theory is the **notion of a proof** in the first-order logic. There are several approaches to the formalization of the notion of a proof. For example, Gentzen's systems are very useful for the analysis of the proof-theoretical strength of mathematical theories. The emphasis in Gentzen's approach is on deduction rules, as distinct from Hilbert-oriented systems, where the stress is on the axioms.

Hilbert style formal systems are more convenient in model theory, so we shall confine our attention to them. We present the logical axioms and rules of inference for a first order language L:

- **Sentential axioms**. These axioms are derived from propositional tautologies by the simultaneous substitution of propositional letters by formulas of L.
- **Identity axioms**. If $\varphi \in \mathrm{For}_L$, $t \in \mathrm{Term}_L$, $x \in \mathrm{Var}$, then $\varphi(t/x)$ denotes the formula obtained from φ by substituting the term t for each free occurrence of x in φ. Sometimes, we shall use the abridged form $\varphi(t)$ or φt, instead of $\varphi(t/x)$. Now we shall list the identity axioms:
  - $x \equiv x$. If $t \in \mathrm{Term}_L$ an $n \in \omega$, then
  - $x_1 \equiv y_1 \wedge \ldots \wedge x_n \equiv y_n \Rightarrow t x_1 x_2 \ldots x_n \equiv t y_1 y_2 \ldots y_n$,
  - $x_1 \equiv y_1 \wedge \ldots \wedge x_n \equiv y_n \Rightarrow (\varphi x_1 x_2 \ldots x_n \Leftrightarrow \varphi y_1 y_2 \ldots y_n)$.
- **Quantifier axioms**. If $\varphi \in \mathrm{For}_L$, $t \in \mathrm{Term}_L$, $x \in \mathrm{Var}$ then

  $$\forall x \varphi x \Rightarrow \varphi t, \quad \varphi t \Rightarrow \exists x \varphi x,$$

  where φt is obtained from φx by freely substituting each free occurrence of x in φx by the term t.

Rules of inferences. Let φ and ψ be formulas of L.

- *Modus Ponens*:

$$\frac{\varphi, \quad \varphi \Rightarrow \psi}{\psi}$$

- *Generalization rules*. Provided x is not free in φ and θ:

$$\frac{\varphi \Rightarrow \psi}{\varphi \Rightarrow \forall x \psi}$$

$$\frac{\varphi \Rightarrow \theta}{\exists x \varphi \Rightarrow \theta}$$

A proof of a sentence φ in a theory T of a language L is every sequence $\psi_1, \psi_2, \ldots, \psi_n$ of formulas of the language L such that $\varphi = \psi_n$ and each formula ψ_i, $i = 1, \ldots, n$, is a logical axiom, or an axiom of T, or it is derived by inference rules from the preceding members of the sequence.

Model theory: First-order theories

If there exists a proof of φ in T, then φ is called a **theorem** of T, and in this case we use the notation $T \vdash \varphi$. The relation $\vdash$ between theories and formulas of a language L is the **provability relation**. If $T = \emptyset$, then we simply write $\vdash \varphi$ instead of $\emptyset \vdash \varphi$, and φ is called a theorem of the first-order predicate calculus.

If φ is not a theorem of T, then we write $\sim T \vdash \varphi$ or $T \nvdash \varphi$ for short.

Formulas of the form $\varphi \wedge \neg\varphi$ are called **contradictions**.

A theory T is **consistent** if a contradiction is not provable in T, i.e. there is no contradiction ψ such that $T \vdash \psi$.

Another important property which theories may have is **completeness**. A theory T of a language L is complete if for each sentence φ of L either $T \vdash \varphi$ or $T \vdash \neg\varphi$.
Finally, T is **deductively closed** if T contains all its theorems.

There is a group of first-order notions which are related to the effective computability. We shall suppose that the reader has some basic ideas of the effective computability and arithmetical coding.

So, for this purpose, if $\varphi \in For_L$, then $\lceil \varphi \rceil$ denotes the code of formula φ. We remind that the code $\lceil \varphi \rceil$ is a unique positive integer assigned to φ A similar notation is applied to other syntactical objects (terms, elements of L, etc.).

A first order language L is **recursive**, if the set $\lceil L \rceil = \{\lceil s \rceil : s \in L\}$ is recursive. Similarly, L is recursively enumerable if $\lceil L \rceil$ is a recursively enumerable set.

A theory T of the language L is **finitely axiomatizable**, if T has a finite set of axioms. A generalization of this notion is the concept of an **axiomatic theory**.

A theory T is axiomatic or recursive if T i.e. $\{\lceil\varphi\rceil : \varphi \in T\}$ is a recursive set of sentences.

The definitions of notions introduced in this way can be broadened. Namely, two theories T and S of the same language L are equivalent, if they have the same theorems. Then a theory T is considered to be also finitely axiomatizable (axiomatic), if there is a theory S equivalent to T which has a finite set of axioms.

It is interesting that the assumption of recursive enumerability does not bring a generalization, as the following theorem shows.

Theorem (Craig's trick). Suppose T is a theory of a language L with a recursively enumerable set of axioms. Then there is a recursive theory S of the language L equivalent to T.
Proof Since T is recursively enumerable, there exists a map $\tau\colon \omega \to Sent_L$ such that $T = \{\tau_n\colon n \in \omega\}$ and $f\colon n \mapsto \lceil \tau_n \rceil$ is a recursive function.

Let $\psi\colon \omega \to \mathrm{Sent}_L$ be defined by

$$\psi_n = \tau_0 \wedge \tau_1 \wedge \ldots \wedge \tau_n, \quad n \in \omega$$

and $S = \{\psi_n\colon n \in \omega\}$. Then T and S have the same theorems, i.e. T and S are equivalent theories.

Furthermore, the mapping $g\colon n \mapsto \ulcorner \psi_n \urcorner$ is also a recursive function, because we may take, for example,

$$\ulcorner \psi_n \urcorner = 2^{\ulcorner \tau_0 \urcorner} 3^{\ulcorner \tau_1 \urcorner} \cdots p_n^{\ulcorner \tau_n \urcorner}$$

where p_n is the n-th prime.

Also, g is a monotonously increasing function, since $n \leqslant m$ obviously implies $\ulcorner \psi_n \urcorner \leqslant \ulcorner \psi_m \urcorner$. Yet, from elementary recursion theory, it is well known that the set of all values of a monotonous increasing recursive function is a recursive set, therefore,

$$S = \{g_n \colon n \in \omega\} = \{\ulcorner \psi_n \urcorner \colon n \in \omega\}$$

is a recursive set. $\square$

A first-order theory T is **decidable**, if the set of all theorems of T is decidable (i.e. recursive) set, otherwise T is **undecidable**. The most interesting mathematical theories are undecidable. However, the following proposition gives a test of decidability for certain theories.

Theorem Suppose T is an axiomatic and complete theory of a recursive language L. Then T is decidable.

Proof Let T' be the set of all the theorems of T. Since T is complete, for each $\varphi \in Sent_L$ we have $\varphi \in T'$ or $\neg\varphi \in T'$. If for some sentence φ it holds that $\varphi, \neg\varphi \in T'$, then $T' = Sent_L$, and since $Sent_L$ is a recursive set, it follows T' is recursive as well.

Suppose the second, more interesting case holds, i.e. that T is a consistent theory.

Since T is recursive, the set (of all the codes) of proofs may be effectively listed. By the completeness of T, for each sentence φ of L either φ or $\neg\varphi$ should occur as the last member of a proof in the list. In the first case, φ is a theorem of T, while in the second case, φ is not a theorem of T by the consistency of T.

The just described property of T defines an algorithm for decidability of $T \vdash \varphi$, where $\varphi \in \mathrm{Sent}_L$:

Generate all the proofs of T, and look at the end of each proof until one of the formulas φ, $\neg\varphi$ appears. If φ occurs then $T \vdash \varphi$, otherwise $T \vdash \neg\varphi$.

This search will stop since either $T \vdash \varphi$, or $T \vdash \neg\varphi$. $\square$

Here are several elementary, but important, theorems from logic without proofs.

Deduction Theorem Suppose T is a theory of a language L and $T \vdash \varphi$ where $\varphi \in \mathrm{For}_L$. Then, there are sentences $\theta_0, \theta_1, \ldots, \theta_n \in T$ such that

$$\vdash \theta_0 \wedge \theta_1 \wedge \ldots \wedge \theta_n \Rightarrow \varphi.$$

As a consequence of this theorem we have that a first order theory T is consistent iff every finite subset of T is consistent.

Lemma on the new constant Let T be a theory of a language L, and assume c is a constant symbol not occurring in L. Then for every formula $\varphi(x)$ of L we have:

if $T \vdash \varphi(c)$, then $T \vdash \forall x \varphi(x)$.

The proof of this lemma is very easy:

If in the proof of $\varphi(c)$ from T, the constant symbol c is replaced by a variable y, which does not occur in that proof, then we shall obtain a proof of $\varphi(y)$ from T. By the inference rule of generalization, the lemma then follows at once. $\square$

A formula φ of a first order language L is in a **prenex normal form**, if φ is of the form $Q_1 y_1 Q_2 y_2 \dots Q_n y_n \psi$, where ψ is a formula without quantifiers, and $Q_1, Q_2, \dots, Q_n$ are some of the quantifiers $\forall$, $\exists$. In this case the formula ψ is called a matrix.

Prenex Normal Form Theorem (PNF Theorem) For every formula φ of a first order language L, there exists a formula ψ of L in a prenex normal form, such that $T \vdash \varphi \Leftrightarrow \psi$.

Another important notion is related to the last theorem. This is the so-called proof-theoretical hierarchy of formulas of a language L.

Definition Let L be a first order language. Then:

- $\sum_0^0 = \prod_0^o = \{\varphi \in \mathrm{For}_L \colon \varphi \text{ does not contain quantifiers}\}$,
- $\sum_{n+1}^0 = \{\exists x_1 \ldots x_k \varphi \colon k \in \omega, \varphi \in \prod_n \}$,
- $\prod_{n+1}^0 = \left\{\forall x_1 \ldots x_k \varphi \colon k \in \omega, \varphi \in \sum_n^0 \right\}$.

If $\varphi \in \sum_n^0$ then φ is called a $\sum_n^0$-formula, and if $\varphi \in \prod_n^0$, then φ is a $\prod_n^0$ -formula.

If φ is a $\sum_1^0$ -formula, then φ is also called an **existential formula**, while if φ is a $\prod_1^0$ -formula, then φ is called a **universal formula**.

The sequences $\sum_n^0$ and $\prod_n^0$ formulas of L define a **proof-theoretical hierarchy of formulas** of L.

By PNF Theorem every formula φ of L is equivalent to a formula ψ such that either $\psi \in \sum_n^0$ or $\psi \in \prod_n^0$. Then φ is called a $\sum_n^0$-formula and respectively $\prod_n^0$ - formula. If φ is a formula of L and for some $n \in \omega$ there is a $\psi \in \sum_n^0$ and a $\theta \in \prod_n^0$ both equivalent to φ then φ is called a Δ_n^0 - formula.

The main properties of the proof-theoretical hierarchy are described in the following diagram.

Theorem

$$\begin{tikzcd}[column sep=small]
 & \Sigma_1^0 \arrow[dr, phantom, sloped, "\subseteq"] & & \Sigma_2^0 \arrow[dr, phantom, sloped, "\subseteq"] & & \\
\Sigma_0^0 = \Pi_0^0 \subseteq \Delta_1^0 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta_2^0 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta_3^0 & \cdots \\
 & \Pi_1^0 \arrow[ur, phantom, sloped, "\subseteq"] & & \Pi_2^0 \arrow[ur, phantom, sloped, "\subseteq"] & &
\end{tikzcd}$$

Figure: Proof theoretical hierarchy of first-order formulas

We shall give several examples of first-order theories. Most examples are from working mathematics, and we shall consider some cases in greater detail.

For every example, we shall exhibit explicitly the corresponding language L in which the axioms of the theory are written down.

Example Pure predicate calculus with identity, J_0. For this theory we have: $L = \emptyset$, $T = \emptyset$.

Theorems of J_0 are exactly the theorems of PR1 which contain only logical symbols. Here are several interesting examples of sentences which can be written in L:

$$\sigma_1 = \exists x_1 \forall x (x \equiv x_1),$$
$$\sigma_2 = \exists x_1 x_2 (\neg x_1 \equiv x_2 \wedge \forall x (x \equiv x_1 \vee x \equiv x_2)),$$
$$\vdots$$
$$\sigma_n = \exists x_1 \ldots x_n ((\bigwedge_{1 \leqslant i < j \leqslant n} (\neg x_i \equiv x_j) \wedge \forall x \bigvee_{1 \leqslant i \leqslant n} (x \equiv x_i)).$$
$$\tau_1 = \exists x_1 (x_1 = x_1),$$
$$\tau_2 = \exists x_1 x_2 (\neg x_1 \equiv x_2),$$
$$\vdots$$
$$\tau_n = \exists x_1 \ldots x_n (\bigwedge_{i<j} (\neg x_i \equiv x_j)).$$

σ_n – "There are exactly n elements".
τ_n – "There are at least n elements".

Model theory: Examples of theories

Example The theory of linear ordering, LO. In this case we have: $L_{\mathrm{LO}} = \{\leqslant\}$, $\leqslant$ is a binary relation symbol. Axioms of T are:

$LO.1$	$x \leqslant x$,	reflexivity,
$LO.2$	$x \leqslant y \wedge y \leqslant z \Rightarrow x \leqslant z$,	transitivity,
$LO.3$	$x \leqslant y \wedge y \leqslant y \Rightarrow x \equiv y$,	antisimetricity,
$LO.4$	$x \leqslant y \vee y \leqslant x$,	linearity.

A theory PO whose axioms are LO.I-3 is called a theory of partial ordering.
The binary relation symbol $<$ is introduced by the definition axiom: $x < y \Leftrightarrow x \leqslant y \wedge x \not\equiv y$.

Example The theory of dense linear ordering without endpoints, DLO. The language of this theory is the same as in the case of LO, while the axioms DLO are the axioms of LO plus the following sentences:

$\exists x \exists y\, x \not\equiv y, \quad \forall x \exists y\, x < y, \quad \forall x \exists y\, y < x,$

$\forall x \forall y \exists z\, (x < y \Rightarrow x < z \wedge z < y).$

It is not difficult to see that for each $n \in \omega \setminus \{0\}$, $\mathrm{DLO} \vdash \tau_n$, where τ_n is the sentence from Example J_0.

Example The theory of Abelian groups, Ab. In this case we have: $\mathrm{Rel}_{\mathrm{Ab}} = \emptyset$, $\mathrm{Fnc}_L = \{+, -\}$, where $+$ is a binary function symbol, and $-$ is a unary function symbol. Further, $\mathrm{Const}_L = \{0\}$. The axioms of Ab are the following formulas:

$$
\begin{array}{lll}
Ab.1 & (x+y)+z \equiv x+(y+z), & \text{associative identity,} \\
Ab.2 & x+y \equiv y+x, & \text{commutative identity,} \\
Ab.3 & x+0 \equiv x, & \text{identity of a neutral element,} \\
Ab.4 & x+(-x) \equiv 0, & \text{identity of an inverse element.}
\end{array}
$$

It is easy to prove by induction on the complexity of terms:
If $t \in \mathrm{Term}_L$, then there is $k \in \omega$ and integers $m_1, m_2, \ldots, m_k$ such that

$\mathrm{Ab} \vdash t\mathbf{x} \equiv m_1 x_1 + \ldots + m_k x_k$, where $x_1, x_2, \ldots, x_k$ are variables.

Example Field theory, **F**. The language of this theory is the language of Abelian groups plus some additional symbols, i.e. $L_{\mathbf{F}} = L_{\mathrm{Ab}} \cup \{\cdot, I\}$ where $\cdot$ is a binary function symbol and 1 is a constant symbol. Axioms of **F** are those of Ab plus the following sentences:

$$
\begin{array}{ll}
(x \cdot y) \cdot z \equiv x \cdot (y \cdot z), & x \cdot y \equiv y \cdot x, \\
x \not\equiv 0 \Rightarrow \exists y\, x \cdot y = 1, & x \cdot (y + z) \equiv (x \cdot y) + (x \cdot z), \\
\quad x \cdot 1 \equiv x & 0 \not\equiv 1,
\end{array}
$$

It is possible to introduce a new function symbol $^{-1}$ in the theory **F** by the following defining axiom:

$$\forall xy (x \not\equiv 0 \Rightarrow (x \cdot y \equiv 1 \Leftrightarrow y \equiv x^{-1})).$$

Then **F** proves: $\forall x\, (x \not\equiv 0 \Rightarrow x \cdot x^{-1} \equiv 1)$.

An important extension of **F** is the theory of **closed fields**, **CF**. It is obtained from **F** by adding an infinite set of axioms which say that every polynomial of positive degree has a root.

Example The theory of ordered fields, FO. The language of this theory is $L_{\mathrm{FO}} = L_{\mathrm{LO}} \cup L_{\mathbf{F}}$, while the axioms are the axioms of theory **F** plus the following formulas:

$$x \leqslant y \Rightarrow x + z \leqslant y + z, \quad x \leqslant y \wedge 0 \leqslant z \Rightarrow x \cdot z \leqslant y \cdot z.$$

We note that the formula

$$x_1^2 + \ldots + x_n^2 = 0 \Rightarrow x_1 \equiv 0 \wedge \ldots \wedge x_n \equiv 0$$

is a theorem of the theory FO.

An important extension of FO is the theory of **ordered real closed fields**, RCF. It is obtained from FO by adding an axiom which says that every positive element has a root and an infinite set of axioms which say that every polynomial of odd degree has a root.

Example The theory of Boolean algebras, BA. The language is $L_{\mathrm{BA}} = \{+, \cdot, ', \leqslant, 0, 1\}$, where $+$ and $\cdot$ are binary function symbols, $'$ is a unary function symbol, $\leqslant$ is a binary relation symbol, while 0, 1 are constant symbols. The axioms of BA:

1.	$(x+y)+z \equiv x+(y+z)$,	$1'$.	$(x \cdot y) \cdot z \equiv x \cdot (y \cdot z)$,
2.	$x+y \equiv y+x$,	$2'$.	$x \cdot y \equiv y \cdot x$,
3.	$x+0 \equiv x$,	$3'$.	$x \cdot 1 \equiv x, ,$
4.	$x+x' \equiv 1$,	$4'$.	$x \cdot x' \equiv 0$,
5.	$x+(x \cdot y) \equiv x$,	$5'$.	$x \cdot (x+y) \equiv x$,
6.	$x+(y \cdot z) \equiv (x+y) \cdot (x+z)$,	$6'$.	$x \cdot (y+z) \equiv (x \cdot y)+(x \cdot z)$,
7.	$0 \not\equiv 1$,	$7'$.	$x \leqslant y \Leftrightarrow x \equiv x \cdot y$.

The following notation is also used for Boolean operations. Namely, the symbols $\vee$ and $\wedge$ are often used for $+$ and $\cdot$ respectively. The sign $'$ is used unchanged, but $\bar{x}$ is also used for $'$. For example, the term $(x' \cdot y) + z$ in the new notation may be written as $(\bar{x} \wedge y) \vee z$.

It is easy to infer in BA the axioms of partial order in respect to the relation symbol $\leqslant$. We have the following important theorems of the theory BA:

- Let sup and inf denote the order supremum and infimum in respect to $\leqslant$. Then the next identities are theorems of BA:

$$\sup\{x_1, x_2, \ldots, x_n\} \equiv \sum_{i \leqslant n} x_i, \quad \inf\{x_1, x_2, \ldots, x_n\} \equiv \prod_{i \leqslant n} x_i.$$

- For each $t \in Term_{L_{BA}}$,

$$\mathrm{BA} \vdash t(x_0, \ldots, x_{n-1}) \equiv \sum_{\alpha \in 2^n} t(\alpha_0, \ldots, \alpha_{n-1}) x_0^{\alpha_0} \ldots x_{n-1}^{\alpha_{n-1}}$$

where $2^n = \{\alpha |\, \alpha \colon n \to 2\}$ and $x^0 = x'$, $x^1 = x$. This property of Boolean terms is proved by induction on the complexity of terms.

Model theory: Examples of theories

Example Peano arithmetic, PA. This theory has the same language as theory BA, i.e. $L_{\mathrm{PA}} = L_{\mathrm{BA}}$. Axioms of PA are the following formulas:

1.	$x' \not\equiv 0$,	6.	$x'y \equiv xy + y$,
2.	$x' \equiv y' \Rightarrow x \equiv y$,	7.	$\neg(x < 0)$,
3.	$x + 0 \equiv x$,	8.	$x < y' \Rightarrow x < y \vee x \equiv y$,
4.	$x + y' \equiv (x + y)'$,	9.	$x < y \vee x \equiv y \vee y < x$,
5.	$x \cdot 0 \equiv 0$,	10.	$1 \equiv 0'$.

Induction scheme (I). Let $\varphi x y_1 y_2 \ldots y_n$ be a formula of L_{PA}. Then the universal closure of

$$\varphi 0 y_1 y_2 \ldots y_n \wedge \forall x(\varphi x y_1 y_2 \ldots y_n \Rightarrow \varphi x' y_1 y_2 \ldots y_n) \Rightarrow \forall x \varphi x y_1 y_2 \ldots y_n$$

is an axiom of PA.

Model theory: Examples of theories

This theory is also called the formal arithmetic. It contains several interesting subtheories. At this moment we shall mention two of them.

The first theory is P^-. This theory consists of the axioms 1–10. Therefore, $\mathrm{PA} = P^- + (I)$.

Another example is the Presburger arithmetic. It consists of those axioms of PA which are expressed in the language $\{+,', 0\}$, i.e. in the language L_{PA} without the symbols $\cdot, \leqslant, 1$.

Model theory: Examples of theories

All the examples we have listed are axiomatic theories, i.e. with recursive sets of axioms. Also, all except the last example, are finitely axiomatizable theories.

Theories J_0, LO, DLO, Ab, BA are decidable, while F, FO and PA are not. PA is certainly the most famous example of an undecidable theory.

The Pressburger arithmetic is also decidable and a complete theory. The first example of a program not dealing with numbers but with the symbols only was the implementation of the decision procedure for Presburger arithmetic (Martin Davis, around 1955).

Model theory: Models

We have dealt in the previous sections mainly with syntactical notions. On the other hand, the most important concept in model theory is the idea of an operational-relational structure, or simply a **model of a first-order language** L.

Customary mathematical structures such as groups, fields, ordered fields, and the structure of natural numbers, are examples of models. When studying the properties of models, a distinctively important role is played by the concept of formal language used to make precise the set of symbols and rules used to build formulas and sentences.

The main reason for introducing formulas is to describe properties of models. Therefore, it is not astonishing that some properties of models are often consequences of the structure of sentences or classes of sentences. The proofs of such features of models are usually called **model-theoretical proofs**.

Model theory: Models

Using the methods of model theory many open mathematical problems have been solved. One such famous problem is the consistent foundation of Leibnitz Analysis, a problem which stood open for 300 years. Abraham Robinson gave a simple but ingenious solution, and thanks to him there is now a whole new methodology which is equally well applied to topology, algebra, probability theory, and practically to all mathematical fields where infinite objects appear.

Definition A model is every structure $\mathbf{A} = (A, \mathcal{F}, \mathcal{R}, \mathcal{C})$ where A is a nonempty set (the domain of $\mathbf{A}$), $\mathcal{F}$ is a family of operations over A, $\mathcal{R}$ is a set of relations over A and $\mathcal{C}$ is a set of constants of A.

By this definition of a model we have.

If $F \in \mathcal{F}$, then there is an $n \in \omega$ such that $F\colon A^n \to A$, i.e. F is an n-ary operation over A. The length of F is denoted by $\mathrm{ar}(F)$.

If $R \in \mathcal{R}$, then there is $n \in \omega$, such that $R \subseteq A^n$, i.e. R is a relation over A of a length n. The length of R is denoted by $\mathrm{ar}(R)$.

Finally, $C \subseteq A$.

If $\mathcal{R}$, $\mathcal{F}$, $\mathcal{C}$ are finite sets, for example

$$\mathcal{F} = \{F_1, F_2, \ldots, F_m\}, \quad \mathcal{R} = \{R_1, R_2, \ldots, R_n\}, \quad \mathcal{C} = \{a_1, a_2, \ldots, a_k\},$$

then **A** may be denoted as

$$\mathbf{A} = (A, F_1, F_2, \ldots, F_m, R_1, R_2, \ldots, R_n, a_1, a_2, \ldots, a_k).$$

If these sets are indexed, i.e. $\mathcal{F} = \langle F_i \colon i \in I\rangle$, $\mathcal{R} = \langle R_j \colon j \in J\rangle$, $\mathcal{C} = \langle a_k \colon k \in K\rangle$, we can also use the notation:

$$\mathbf{A} = (A, F_i, R_j, a_k)_{i\in I, j\in J, k\in K}.$$

Example.

1. The ordered field of real numbers: $\mathbf{R} = (R, +, \cdot, -, \leqslant, 0, I)$.
   Here, $\mathcal{F} = \{+, \cdot, -\}$, $\mathcal{R} = \{\leqslant\}$, $\mathrm{ar}(\leqslant) = 2$,
   $\mathrm{ar}(+) = ar(\cdot) = 2$, $\mathrm{ar}(-) = 1$ and $\mathcal{C} = \{0, I\}$.
2. The structure of natural numbers: $\mathbf{N} = (N, +, \cdot, ', \leqslant, 0)$.
3. The field of all subsets of a set X:
   $\mathbf{P}(X) = (P(X), \cup, \cap, {}^c, \subseteq, X)$, where $P(X) = \{Y : Y \subseteq X\}$,
   and for $Y \in P(X)$, $Y^c = X \setminus Y$.

Models are interpretations of first-order languages. Let L be a first-order language and A a non-empty set. An **interpretation** of L into the domain A is every mapping I defined on L, and with values determined as follows:

If $F \in \mathrm{Fnc}_L$, then $I(F)$ is an operation of A with the length $\mathrm{ar}(F)$.
If $R \in \mathrm{Rel}_L$, then $I(R)$ is a relation of A with the length $\mathrm{ar}(R)$.
If $c \in \mathrm{Const}_L$ then $I(c) \in A$.

Therefore, every interpretation I of a language L into a domain A determines a unique model $\mathbf{A} = (A, I(\mathrm{Rel}_L), I(Fnc_L), I(\mathrm{Const}_L))$.

For so introduced notion of a model we write simply

$$\mathbf{A} = (A, I), \quad \text{or} \quad \mathbf{A} = (A, s^{\mathbf{A}})_{s \in L},$$

where for $s \in L$, $s^{\mathbf{A}} = I(s)$.

We see that in the last example $\mathbf{R}$ is a model of the language of ordered fields, while $\mathbf{N}$ is a model of the language of Peano arithmetic and finally $\mathbf{P}(X)$ is a model of the language of the theory of Boolean algebras.

From now on the letters $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}, \ldots$ will be reserved for models and A, B, $C, \ldots$ for their domains.

If L is a language and $\mathbf{A}$ is a model of L, then $s \in L$ and $s^{\mathbf{A}}$ denote objects of a different nature. However, if the context allows, we shall use the same sign to denote a symbol of L and its interpretation in $\mathbf{A}$. Therefor the superscript $^{\mathbf{A}}$ will be often omitted from $s^{\mathbf{A}}$. The circumstance under which s appears will determine if $s \in L$ or s is in fact an interpretation of a symbol of L.

Very often a structure $\mathbf{A}$ is introduced without explicit mention of the related language. But, from the definition of the structure $\mathbf{A}$ it will be clear what is the corresponding language and in that case we shall denote the language in question by $L_{\mathbf{A}}$ and it will be called the **language of the model A**.

A similar situation may appear for a theory T; the corresponding language will be denoted by L_T and it will be called the **language of the theory** T.

Assume $L \subseteq L'$ are first-order languages, and let $\mathbf{A}$ be a model of L'. Omitting $s^{\mathbf{A}}$ for $s \in L' \backslash L$ from the model $\mathbf{A}$, we obtain a new model $\mathbf{B}$ of L with the domain $B = A$. In this case, $\mathbf{A}$ is called an **expansion** of the model $\mathbf{B}$, while $\mathbf{B}$ is called a **reduct** of the model $\mathbf{A}$. If I and I' are interpretations which determine $\mathbf{B}$ and $\mathbf{A}$, respectively, we see that $I = I'|L$.

Definition Let **A** and **B** be models of a language L. Then **B** is a **submodel** of **A**, if and only if:

- if $B \subseteq A$ and $R \in \mathrm{Rel}_L^k$ then $R^{\mathbf{B}} = R^{\mathbf{A}} \cap B^k$,
- if $F \in \mathrm{Fnc}_L^k$ then $F^{\mathbf{B}} = F^{\mathbf{A}} | B^k$,
- if $c \in \mathrm{Const}_L$ then $c^{\mathbf{B}} = c^{\mathbf{A}}$.

The fact that **B** is a submodel of **A**, we shall denote by $\mathbf{B} \subseteq \mathbf{A}$. For example $(N, +, \cdot, \leqslant, O, I) \subseteq (R, +, \cdot, \leqslant, O, I)$, but for $Y \subset X$, $Y \neq X$, it is not true that

$$(P(Y), \cup, \cap, ^c, \emptyset, Y) \subseteq (P(X), \cup, \cap, ^c, \emptyset, X).$$

Algebras are special types of models; they are models of a languages L with $\mathrm{Rel}_L = \emptyset$. As in the case of algebras, it is possible to introduce notions of a **homomorphism** and an **isomorphism** for models, too.

Definition. Let **A** and **B** be models of a language L, and $f : A \to B$. The map f is a homomorphism from **A** into **B**, what is denoted by $f : \mathbf{A} \to \mathbf{B}$, if and only if:

- For $F \in Fnc_L^k$, for all $a_1, a_2, \dots, a_k \in A$,
  $$f(F^{\mathbf{A}}(a_1, a_2, \dots, a_k)) = F^{\mathbf{B}}(fa_1, fa_2, \dots, fa_k).$$
  In this case we say that f is concurrent with operations $F^{\mathbf{A}}$ and $F^{\mathbf{B}}$.
- For $R \in \mathrm{Rel}_L^k$ and for all $a_1, a_2, \dots, a_k \in A$,
  $$R^{\mathbf{A}}(a_1, a_2, \dots, a_k) \text{ implies } R^{\mathbf{B}}(fa_1, fa_2, \dots, fa_k).$$
  In this case we say that f is concurrent with relations $R^{\mathbf{A}}$ and $R^{\mathbf{B}}$.
- For $c \in \mathrm{Const}_L$, $f(c^{\mathbf{A}}) = c^{\mathbf{B}}$.

Model theory: Models

Similarly to the case of algebraic structures, we have the following classification of homomorphisms.

- f is an **embedding**, if f is 1–1.
- f is an onto-homomorphism (or epimorphism), if f is onto.
- f is a **strong homomorphism**, if for every $R \in \mathrm{Rel}_L^k$, and $a_1, a_2, \ldots, a_k \in A$,

  $R^{\mathbf{A}}(a_1, a_2, \ldots, a_k)$ holds iff $R^{\mathbf{B}}(fa_1, fa_2, \ldots, fa_k)$ holds.
- f is an **isomorphism**, if f is 1-1 and a strong epimorphism.
- f is an **automorphism**, if f is an isomorphism and $\mathbf{A} = \mathbf{B}$.

The set of all the automorphisms of a model $\mathbf{A}$ is denoted by $\mathrm{Aut}\mathbf{A}$. It is not difficult to see that $\mathrm{Aut}\mathbf{A}$ is a group under function multiplication; this group will be denoted by **AutA**.

Suppose $f: \mathbf{A} \to \mathbf{B}$ is a homomorphism. Then we shall use the following conventions:

- If f is an embedding, we shall say that $\mathbf{A}$ is embedded into $\mathbf{B}$.
- If f is an onto map, we shall say that $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$, and we shall occasionally denote this fact by $\mathbf{B} = f(\mathbf{A})$.
- If f is an isomorphism between models $\mathbf{A}$ and $\mathbf{B}$, then we shall write $f: \mathbf{A} \approx \mathbf{B}$. The notation $\mathbf{A} \approx \mathbf{B}$ is used to indicate that there is an isomorphism $f: \mathbf{A} \approx \mathbf{B}$, and in this case we shall say that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic.

The set of all automorphisms of a countable model has the following interesting property.

Theorem (Kueker) Let $\mathbf{A}$ be a countable model. Then $|\mathrm{Aut}\mathbf{A}| > \aleph_0$ implies $|\mathrm{Aut}\mathbf{A}| = 2^{\aleph_0}$.

Model theory: Satisfaction relation

When introducing syntactical objects of PR1, as the terms, formulas and sentences are, we had in mind certain meanings related to these notions. Alfred Tarski's definition of the **satisfaction relation** $\models$ determines these ideas precisely.

The introduction of this relation also solves the **problem of mathematical truth**. Namely, a sentence φ will be true in a structure $\mathbf{A}$, if $\mathbf{A} \models \varphi$. Finally, this formalization of the mathematical truth enables a mathematical analysis of metamathematical notions.

We shall first define the values of the terms in models. Let $\mathbf{A}$ be a model of a first-order language L. A **valuation** or an assignment of the domain A is every map $\mu\colon \mathrm{Var} \to A$. Hence, valuations assign values to variables. The value of a term $u(x_0, \ldots, x_n) \in \mathrm{Term}_L$ in a model $\mathbf{A}$, denoted by $u^{\mathbf{A}}[\mu]$, is defined by induction on the complexity of terms, assuming that $\mu(v_i) = a_i$, $i \in \omega$.

Value of a term $u \in \mathrm{Term}_L$. If $\mathrm{co}(u) = 0$, then we distinguish two cases:

- If u is a variable v_i , then $u^{\mathbf{A}}[\mu] = a_i$.
- If u is a constant symbol c, then $u^{\mathbf{A}}[\mu] = c^{\mathbf{A}}$.

Suppose now $\mathrm{co}(u) = n + 1$, and assume that the values of the terms of the complexity $\leqslant n$ are determined. Then there is $F \in Fnc_L^k$ such that $u = F(u_1, u_2, \ldots, u_k)$ where $u_1, u_2, \ldots, u_k$ are terms of complexity $\leqslant n$. Then, by definition,

$$u^{\mathbf{A}}[\mu] = F^{\mathbf{A}}(u_1^{\mathbf{A}}[\mu], u_2^{\mathbf{A}}[\mu], \ldots, u_k^{\mathbf{A}}[\mu]).$$

Instead of $u^{\mathbf{A}}[\mu]$, it is common to write $u^{\mathbf{A}}[a_1, a_2, \ldots, a_r]$ or $u[a_1, a_2, \ldots, a_r]$, or $u(a_1, a_2, \ldots, a_r)$, if it is clear which model is in question. Here, r is the number of distinct variables appearing in u.

If **A** is a model of a language L, an operation F of domain A is **derived** if there is $t(x_1, x_2, \ldots, x_n) \in \mathrm{Term}_L$ such that

$$F(a_1, a_2, \ldots, a_n) = t^{\mathbf{A}}(a_1, a_2, \ldots, a_n), \quad a_1, a_2, \ldots, a_n \in A.$$

The following proposition says that homomorphisms of a model remain concurrent with respect to the derived operations.

Theorem. Let **A** and **B** be models of a language L, and $h\colon \mathbf{A} \to \mathbf{B}$ a homomorphism. Then for every term $u(x_1, x_2, \ldots, x_n)$ of L and all $a_1, a_2, \ldots, a_n \in A$ the following holds:

$$h(u^{\mathbf{A}}[a_1, a_2, \ldots, a_n]) = u^{\mathbf{B}}[ha_1, ha_2, \ldots, ha_n].$$

Proof The proof is done by induction on the complexity of terms. So, let $u \in \mathrm{Term}_L$, and suppose that the variables $v_0, v_1, \ldots$ have the values $a_0, a_1, \ldots$ under valuation μ. First assume $\mathrm{co}(u) = 0$, We have two cases:

- $u \in \mathrm{Const}_L$. Then: $h(u^{\mathbf{A}}[\mu]) = h(u^{\mathbf{A}}) = u^{\mathbf{B}} = u^{\mathbf{B}}[\mu]$.
- u is a variable x_i, Then:
  $h(u^{\mathbf{A}}[\mu]) = h(a_i) = u^{\mathbf{B}}[ha_1, ha_2, \ldots, ha_n]$.

Now suppose the statement is true for some fixed $n \in \omega$, and let $\mathrm{co}(u) = n + l$. Then there is an $F \in Fnc_L^k$ of and there are some terms $u_1, u_2, \ldots, u_k$ such that $u = F(u_1, u_2, \ldots, u_k)$. Then the terms u_i are of complexity $\leqslant n$ and hence, by the inductive hypothesis, we have

Model theory: Satisfaction relation

$$\begin{aligned} h(u^{\mathbf{A}}[\mu]) &= hF^{\mathbf{A}}(u_1^{\mathbf{A}}[\mu], \ldots, u_k^{\mathbf{A}}[\mu]) \\ &= F^{\mathbf{B}}(hu_1^{\mathbf{A}}[\mu], \ldots, hu_k^{\mathbf{A}}[\mu]) \\ &= F^{\mathbf{B}}(u_1^{\mathbf{B}}[h\mu], \ldots, u_k^{\mathbf{B}}[h\mu]). \end{aligned} \qquad \square$$

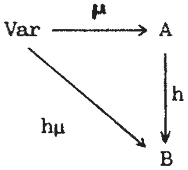

Figure: Homomorphism theorem for terms

Note. This theorem can be obviously restated as follows: For every valuation $\mu\colon \mathrm{Var} \to A$ the displayed diagram commutes, i.e.

$$hu^{\mathbf{A}}[\mu] = u^{\mathbf{B}}[h\mu].$$

An algebraic identity of a language L is every formula $u \equiv v$, where $u, v, \in \mathrm{Term}_L$. We say that an algebra of L satisfies the identity $u \equiv v$, if and only if for all $a_1, a_2, \dots, a_n \in A$, $u^{\mathbf{A}}[a_1, a_2, \dots, a_n] = v^{\mathbf{A}}[a_1, a_2, \dots, a_n]$.

Corollary Let **A** and **B** be algebras of a language L, and assume that **B** is a homomorphic image of **A**. Then every identity true in **A** also holds in **B**.

Proof Let $h \colon A \to B$ be onto, and suppose an identity $u = v$ holds in **A**. Then, for arbitrary $b_1, b_2, \dots, b_n \in B$, there are $a_1, a_2, \dots, a_n \in A$ such that $ha_1 = b_1, \dots, ha_n = b_n$, So

$$\begin{aligned} u^{\mathbf{B}}[b_1, b_2, \dots, b_n] &= u^{\mathbf{B}}[ha_1, ha_2, \dots, ha_n] && = hu^{\mathbf{A}}[a_1, a_2, \dots, a_n] \\ &= hv^{\mathbf{A}}[a_1, a_2, \dots, a_n] && = v^{\mathbf{B}}[ha_1, ha_2, \dots, ha_n] \\ &= v^{\mathbf{B}}[b_1, b_2, \dots, b_n]. && \square \end{aligned}$$

Model theory: Satisfaction relation

This corollary is an example of a **preservation theorem**. Namely, it says that some properties are preserved under homomorphisms and in this case these properties are those which can be described by identities.

Some examples are the associativity and the commutativity of algebraic operations. This is probably one of the places where one can see the **algebraic nature of model theory**.

Now we shall turn to the most important concept of model theory. This is the notion of the **satisfaction relation**, or the definition of the mathematical truth.

Definition Let $\mathbf{A}$ be a model of a language L. We define the relation $\mathbf{A} \models \varphi[\mu]$ for all formulas φ of L and all valuations $\mu = \langle a_i \colon i \in \omega \rangle$ of the domain A by induction on the complexity of formulas φ:

Model theory: Satisfaction relation

- If φ is $u \equiv v$, $u, v \in \mathrm{Term}_L$, then
  $\mathbf{A} \models \varphi[\mu]$ iff $u^{\mathbf{A}}[\mu] = v^{\mathbf{A}}[\mu]$,
- If φ is $R(u_1, u_2, \ldots, u_n)$, $R \in \mathrm{Rel}_L^n$, $u_1, u_2, \ldots, u_n \in \mathrm{Term}_L$, then $\mathbf{A} \models \varphi[\mu]$ iff $(u_1^{\mathbf{A}}[\mu], u_2^{\mathbf{A}}[\mu], \ldots, u_n^{\mathbf{A}}[\mu]) \in R^{\mathbf{A}}$, i.e.
  $R^{\mathbf{A}}(u_1^{\mathbf{A}}[\mu], u_2^{\mathbf{A}}[\mu], \ldots, u_n^{\mathbf{A}}[\mu])$.
- If φ is $\neg\psi$, then $\mathbf{A} \models \varphi[\mu]$ iff **not** $\mathbf{A} \models \psi[\mu]$.
- If φ is $\psi \wedge \theta$, then $\mathbf{A} \models \varphi[\mu]$ iff $\mathbf{A} \models \psi[\mu]$ **and** $\mathbf{A} \models \theta[\mu]$.
- If φ is $\psi \vee \theta$, then $\mathbf{A} \models \varphi[\mu]$ iff $\mathbf{A} \models \psi[\mu]$ **or** $\mathbf{A} \models \theta[\mu]$.
- If φ is $\psi \Rightarrow \theta$, then
  $\mathbf{A} \models \varphi[\mu]$ iff **not** $\mathbf{A} \models \psi[\mu]$ **or** $\mathbf{A} \models \theta[\mu]$.
- If φ is $\psi \Leftrightarrow \theta$, then
  $\mathbf{A} \models \varphi[\mu]$ iff $\mathbf{A} \models \psi[\mu]$ **if and only if** $\mathbf{A} \models \theta[\mu]$.
- If φ is $\exists v_k \psi(v_1, v_2, \ldots, v_n)$, $k \leqslant n$, then
  $\mathbf{A} \models \varphi[\mu]$ iff **exists** $a \in A$ **so that** $\mathbf{A} \models \psi[\mu(k/a)]$.
- If φ is $\forall v_k \psi(v_1, v_2, \ldots, v_n)$, $k \leqslant n$, then
  $\mathbf{A} \models \varphi[\mu]$ iff **for all** $a \in A$ **it is valid** $\mathbf{A} \models \psi[\mu(k/a)]$.

Model theory: Satisfaction relation

By the definition of the satisfaction relation, we see that the value of $\mathbf{A} \models \varphi[\mu]$ depends only on the free variables which occur in φ. A rigorous proof of this fact can be derived by induction on the complexity of formulas.

This property enables us to introduce the following conventions.

If $\varphi = \varphi(v_0, \dots, v_n)$ and $\mu = \langle a_i \colon i \in \omega \rangle$, then we shall simply write $\mathbf{A} \models \varphi[a_0, \dots, a_n]$ instead of $\mathbf{A} \models \varphi[\mu]$.

Sentences do not have free variables, so their values do not depend on the choice of a valuation, i.e. if $\varphi \in \mathrm{Sent}_L$ and $\mathbf{A} \models \varphi[\mu]$, then for all valuations σ we have $\mathbf{A} \models \varphi[\sigma]$. Thus, we shall use the abbreviated form $\mathbf{A} \models \varphi$ instead of $\mathbf{A} \models \varphi[\mu]$.

The **theory of a model A** of L:

$$\mathrm{Th}\mathbf{A} = \{\varphi \in \mathrm{Sent}_L \colon \mathbf{A} \models \varphi\}.$$

is another useful model-theoretic concept.

Model theory: Satisfaction relation

It is easy to see that for each formula φ of L and every valuation μ either $\mathbf{A} \models \varphi[\mu]$ or $\mathbf{A} \models \neg\varphi[\mu]$, thus, $\mathrm{Th}\mathbf{A}$ is a **complete theory**.

For example, the theory of the structure of natural numbers, $\mathrm{Th}\mathbf{N}$, is complete, and so it is called a **complete arithmetic**. As **N** is a model of the theory PA, it follows that PA$\subseteq \mathrm{Th}\mathbf{N}$.

On the other hand, the **Gödel's Second Incompleteness** Theorem states the set of theorems of PA is a proper subset of $\mathrm{Th}\mathbf{N}$. Moreover, $\mathrm{Th}\mathbf{N}$ is not an axiomatic theory, i.e. it does not have a recursive set of axioms.

One of the tasks of model theory is to solve the problem whether a given theory is **axiomatic**.

Model theory: Satisfaction relation

Let T be a theory of a language L. A model **A** of L is a **model of the theory** T, if every axiom of T holds in **A**, i.e. $T \subseteq \mathrm{Th}\mathbf{A}$. In such a case, we write $\mathbf{A} \models T$.

For example, every ordered field, like an ordered field of rational numbers or real numbers, is a model of theory FO.

Similarly, every Boolean algebra is a model of theory BA.

Every model **A** of a language L satisfies all the axioms of the first-order logic (predicate calculus) for L. Rules of inferences (Modus Ponens and Generalization rules) are preserved by the satisfaction relation, i.e. if μ is a valuation of domain A and $\mathbf{A} \models \varphi_1[\mu], \ldots, \varphi_n[\mu]$, where $\varphi_1, \varphi_2, \ldots, \varphi_n \in \mathrm{For}_L$ and ψ is derived by applications of these rules, then $\mathbf{A} \models [mu]$.

Therefore, the following theorem is easily proved by induction on the length of proofs in T.

Model theory: Satisfaction relation

Soundness theorem Assume $\mathbf{A}$ is a model of a language L and T is a theory of L. If $\mathbf{A} \models T$ and $T \vdash \varphi$, where $\varphi \in \mathrm{Sent}_L$, then $\mathbf{A} \models \varphi$.

Two models $\mathbf{A}$ and $\mathbf{B}$ of a language L are **elementary equivalent** if $\mathbf{A}$ and $\mathbf{B}$ satisfy the same sentences of L, i.e. $\mathrm{Th}\mathbf{A} = \mathrm{Th}\mathbf{B}$. This relation between models is denoted by $\mathbf{A} \equiv \mathbf{B}$. It is also said that $\mathbf{A}$ and $\mathbf{B}$ have the same first-order properties.

By induction on the complexity of formulas it is easy to show:

Theorem Let $g \colon \mathbf{A} \approx \mathbf{B}$ be an isomorphism of models $\mathbf{A}$ and $\mathbf{B}$ of a language L. Then, for every formula $\varphi v_0 \ldots v_n$ of L and every valuation $\mu = \langle a_i \colon i \in \omega \rangle$ of the domain A, the following holds:

$$\mathbf{A} \models \varphi[a_0, \ldots, a_n] \text{ if and only if } \mathbf{B} \models \varphi[ga_0, \ldots, ga_n].$$

Since the value of a sentence in a model does not depend on the choice of a valuation, we have the following consequence.

Model theory: Satisfaction relation

Corollary If $\mathbf{A}$ and $\mathbf{B}$ are isomorphic models of a language L, then $\mathbf{A} \equiv \mathbf{B}$.

Therefore isomorphisms preserve first-order properties.

Elementary embeddings of models are embeddings which preserve first-order properties. Hence, an elementary embedding between models $\mathbf{A}$ and $\mathbf{B}$ of a language L is every map $g\colon A \to B$, such that for all $\varphi \in \mathrm{For}_L$, all valuations   of domain A, it satisfies

$$\mathbf{A} \models \varphi[a_0, \ldots, a_n] \quad \text{if and only if} \quad \mathbf{B} \models \varphi[ga_0, \ldots, ga_n].$$

In this case we use the notation $g\colon \mathbf{A} \xrightarrow{\prec} \mathbf{B}$.

If $\mathbf{A} \subseteq \mathbf{B}$ and the inclusion map $i_A\colon \mathbf{A} \to \mathbf{B}$, $i_A\colon x \mapsto x$, $x \in A$, is elementary, then we write $\mathbf{A} \prec \mathbf{B}$.
Observe that $\mathbf{A} \prec \mathbf{B}$ implies $\mathbf{A} \equiv \mathbf{B}$.

Model theory: Satisfaction relation

A class of $\mathfrak{M}$ of models of a language L is **axiomatic** if there is a theory T of L such that $\mathfrak{M} = \{\mathbf{A} \colon \mathbf{A} \models T\}$. For example, the class of all ordered fields is axiomatic and so is the class of all Boolean algebras.

The class of all cyclic groups is not an axiomatic class.

Also, if a theory T has infinitely many non-isomorphic finite models, then the class of all finite models of T is not an axiomatic class.

The class of all models of a theory T is denoted by $\mathfrak{M}(T)$. The central theorem of model theory says:

For every consistent theory T, $\mathfrak{M}(T) \neq \emptyset$.

Model theory: Method of new constants

Introduction of new linguistic constants is a dual procedure to the process of interpretations. Namely, to every nonempty set A there corresponds a certain language L_A.

- If g is an n-ary operation over domain A, we can introduce a function symbol $\underline{g} \in L_A$ of arity k.
- If R is a k-ary relation over A, then let $\underline{R}$ be a relation symbol of length k which belongs to L_A.
- If $a \in A$ then $\underline{a} \in \mathrm{Const}_{L_A}$.

The symbols $\underline{g}$, $\underline{R}$, $\underline{a}$ are called **names** of g, R, a, respectively. We have a natural interpretation of the language L_A so defined:

If $s \in L_A$, then $\underline{s}^A = s$.

In this way we have built a model $\mathbf{A} = (\mathbf{A}, \mathcal{F}, \mathcal{R}, \mathbf{C})$, where $\mathcal{F}$ is the set of all operations with domain A, $\mathcal{R}$ is the set of all relations over A, and $\mathbf{C} = A$.

It is not always necessary to consider the full expansion of set A. For example, if A is any model of a language L, and $a_1, a_2, \ldots, a_n \in A$, then $\mathbf{A}' = (A, a_1, \ldots, a_n)$ is a **simple expansion** of $\mathbf{A}$, and $\mathbf{A}'$ is a model of the language $L' = L \cup \{\underline{a}_1, \ldots, \underline{a}_n\}$.

Note that $\varphi \underline{a}_1 \underline{a}_2 \ldots \underline{a}_n$ is a sentence of $L \cup \{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$.

The following proposition is interesting for two reasons. The first one relates to the inductive nature of the satisfaction relation. Secondly, this proposition shows that the satisfaction relation can be defined only for sentences if the starting model is modified.

Theorem (Satisfaction relation theorem on sentences) Let $\mathbf{A}$ be a model of a language L and $\varphi v_0 v_1 v_2 \ldots v_n \in \mathrm{For}_L$ Then, for all $a_0, a_1, a_2, \ldots, a_n \in A$, we have

$$\mathbf{A} \models \varphi[a_0, a_1, a_2, \ldots, a_n] \quad \text{iff} \quad (\mathbf{A}, \underline{a_0}, \underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n) \models \varphi[\underline{a_0}, \underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n]. \quad (0.4)$$

Proof of this theorem is simple, but long and tedious.

Step 1 If $t(v_0, v_1, v_2, \dots, v_n) \in \mathrm{Term}_L$, $\mathbf{A}' = (\mathbf{A}, a_0, a_1, a_2, \dots, a_n)$, by induction on the complexity of the terms one proves:

$$t^{\mathbf{A}'}(\underline{a}_0, \underline{a}_1, \underline{a}_2, \dots, \underline{a}_n) = t^{\mathbf{A}}[a_0, a_1, a_2, \dots, a_n].$$

Step 2 By induction on the complexity of formulas one proves (0.4).

For example we prove the induction step $\varphi = \exists v_i \varphi$. We take $i = 0$, $\varphi = \varphi v_1 v_2 \dots v_n$ and $\psi = \psi(v_0, v_1, v_2, \dots, v_n)$. Then

$$\begin{array}{lll}
\mathbf{A} \models \varphi[a_1, a_2, \dots, a_n] & \text{iff} & \text{for some } b \in A, \quad \mathbf{A} \models \psi[b, a_1, a_2, \dots, a_n] \\
 & & \text{using inductive hypothesis} \\
 & \text{iff} & \text{for some } b \in A, \quad (\mathbf{A}', b) \models \psi \underline{b} \underline{a}_1 \underline{a}_2 \dots \underline{a}_n \\
 & \text{iff} & \text{for some } b \in A, \quad \mathbf{A}' \models \theta[b] \\
 & & \text{where } \theta x = \psi x \underline{a}_1 \underline{a}_2 \dots \underline{a}_n, \text{ so} \\
 & \text{iff} & \mathbf{A}' \models \exists x \theta x \\
 & \text{iff} & \mathbf{A}' \models \varphi \underline{a}_1 \underline{a}_2 \dots \underline{a}_n \quad \Box
\end{array}$$

We shall apply the previous proposition in the following theorem which says that there is no quite satisfactory model theory for finite structures. The reason is that the relation of elementary equivalence and the isomorphisms of models coincide for finite structures.

Theorem Let $\mathbf{A}$ and $\mathbf{B}$ be models of a language L. If A is finite and $\mathbf{A} \equiv \mathbf{B}$, then $\mathbf{A} \approx \mathbf{B}$.

Proof Assume $|A| = n$. Then $\mathbf{A} \models \sigma_n$, where

$\sigma_n =$ "There are exactly n elements".

But $\mathbf{A}$ and $\mathbf{B}$ are elementary equivalent, so $B \models \sigma_n$.

Therefore, $\mathbf{A}$ and $\mathbf{B}$ have the same number of elements.

Now prove the following fact:

Claim If $\mathbf{A}$ and $\mathbf{B}$ are finite models and $\mathbf{A} \equiv \mathbf{B}$, then for each $a \in A$ there is a $b \in B$ such that $(\mathbf{A}, a) \equiv (\mathbf{B}, b)$.

Proof of Claim Fix $a \in A$ and let $B = \{b_1, b_2, \ldots, b_n\}$. Assume there is no $b \in B$ such that $(\mathbf{A}, a) \equiv (\mathbf{B}, b)$, and choose a constant symbol $c \notin L$, the so-called **new constant symbol**.

Then, for all $i \leqslant n$ there is a formula $\varphi_i x$ of the language L and there is $b_i \in B$ such that $(\mathbf{A}, a) \models \varphi_i c$ and $(\mathbf{B}, b_i) \models \neg\varphi_i c$, where c is interpreted by a in (A, a) and by b_i in $(\mathbf{B}, b_i)$.

Hence, $(\mathbf{A}, a) \models \bigwedge_{j \leqslant n} \varphi_j c$, so $\mathbf{A} \models \exists x \bigwedge_{j \leqslant n} \varphi_j x$.
Since $\mathbf{A} \equiv \mathbf{B}$, we have $\mathbf{B} \models \exists x \bigwedge_{j \leqslant n} \varphi_j x$. Thus for some $k \leqslant n$, $\mathbf{B} \models \bigwedge_{j \leqslant n} \varphi_j [b_k]$. By previous theorem it follows that $(\mathbf{B}, b_k) \models \bigwedge_{j \leqslant n} \varphi_j \underline{b}_k$, hence, $(\mathbf{B}, b_k) \models \bigwedge_{j \leqslant n} \varphi_j c$ if c is interpreted by b_k. This is a contradiction to the choice of the formula φ_k. $\square$

By repeated use of Claim, we can find an enumeration $a_1, a_2, \dots, a_n$ of domain A, so that

- $(\mathbf{A}, a_1, a_2, \dots, a_n) \equiv (\mathbf{B}, b_1, b_2, \dots, b_n)$, where
- $(\mathbf{A}, a_1, a_2, \dots, a_n)$ and $(\mathbf{B}, b_1, b_2, \dots, b_n)$ are models of a language $L \cup \{c_1, c_2, \dots, c_n\}$.

Then the map $f\colon A \to B$ defined by $f\colon a_i \mapsto b_i$, $i \leqslant n$, is an isomorphism of models $\mathbf{A}$ and $\mathbf{B}$. For example, if $*$ is a binary operation symbol of L then:

If some $a_i, a_j, a_k \in A$ satisfy $a_k = a_i *^{\mathbf{A}} a_j$ then
$(\mathbf{A}, a_1, a_2, \dots, a_n) \models \underline{a}_k = \underline{a}_i * \underline{a}_j$, so
$(\mathbf{B}, b_1, b_2, \dots, b_n) \models \underline{b}_k = \underline{b}_i * \underline{b}_j$. Hence $b_k = b_i *^{\mathbf{B}} b_j$.

Therefore we proved $f(a_i *^{\mathbf{A}} a_j) = f(a_i) *^{\mathbf{B}} f(a_j)$, i.e. f is concurrent in respect to the operations $*^{\mathbf{A}}$ and $*^{\mathbf{B}}$.

In a similar way one can show that f is concurrent with relations of models **A** and **B**. Obviously, f is onto. This map is also 1–1, since

$$(\mathbf{A}, a_1, a_2, \dots, a_n) \models \underline{a}_i \equiv \underline{a}_j \quad \text{iff} \quad (\mathbf{B}, b_1, b_2, \dots, b_n) \models \underline{b}_i \equiv \underline{b}_j.$$

Thus $f\colon \mathbf{A} \approx \mathbf{B}$. □

The idea of constructing an isomorphism as in the previous theorem is often used in model theory. It is summarized as follows.

Theorem Let **A** and **B** be models of a language L, $A = \{a_i \colon i \in I\}$, $B = \{b_i \colon i \in I\}$, and $\mathbf{A}' = (\mathbf{A}, a_i)_{i \in I}$, $\mathbf{B}' = (\mathbf{B}, b_i)_{i \in I}$ be models of a language $L \cup \{c_i \colon i \in I\}$ with c_i interpreted in $\mathbf{A}'$ by a_i and in $\mathbf{B}'$ by b_i. Then,

$$(\mathbf{B}, b_i)_{i \in I} \equiv (\mathbf{B}, b_i)_{i \in I} \quad \text{implies} \quad \mathbf{A} \approx \mathbf{B}.$$

As expected, $f\colon \mathbf{A} \approx \mathbf{B}$ where $f\colon a_i \mapsto b_i$, $i \in I$.