The Intrinsic Geometry of Perceptual Space: Its Metrical, Affine and Projective Properties

James T. Todd[1], Stijn Oomes[1], Jan J. Koenderink[2] & Astrid M. L. Kappers[2]
[1]Ohio State University, Columbus, OH [2]Universiteit Utrecht, the Netherlands

Abstract

Two experiments were performed to measure the internal consistency of observers' colinearity and bisection judgments. In the colinearity task, observers were presented with two pairs of stakes on a visible ground surface to define the endpoints of two line segments in perceptual space. They were asked to adjust a 5th stake so that it appeared at the point of intersection between these two line segments. Although the observers' settings were systematically distorted relative to the true intersection points in physical space, they satisfied a basic theorem of projective geometry first proved by Pappus around 340 AD. In the bisection task, observers were presented with a single pair of stakes on a visible ground surface to define a single line segment in perceptual space, and were asked to adjust a 3rd stake so that it appeared to bisect that line segment. These judgments were also systematically distorted relative to the true bisection points in physical space, but they satisfied a basic theorem of affine geometry that was first introduced by Pierre Varignon around 1700. These findings suggest that perceptual space has an internally consistent affine and projective geometry.

A fundamental problem for the study of human perception is to identify the data structures by which visible objects in the natural environment are perceptually represented. It is important to keep in mind when evaluating this issue that there are many possible attributes of an object's structure, which could, in principle, form the primitive components of an observer's perceptual knowledge. There is at present no general consensus about the relative perceptual salience of these attributes, nor is there even a clear formulation of the space of possibilities that needs to be considered.

One important factor in evaluating potential primitives for the perceptual representation of 3D form is their relative stability. When an object is transformed in the natural environment, it is generally the case that only some of its properties will be altered, while others remain invariant. Consider, for example, some possible transformations of a right triangle. If a triangle is moved rigidly from one location to another, its position changes, but its size and shape remain invariant. If the triangle is expanded in a uniform manner, its size will change but its shape will not. If a shadow of the triangle is projected onto a planar surface, the resulting image will still be a triangle though not necessarily a right triangle. If the shadow is projected onto a curved surface, then each edge will lie along a geodesic of that surface that covers the shortest distance between two vertices, but the sum of the angles will no longer equal 180 degrees.

While considering the phenomenon of invariance under change, it is interesting to note its historical importance to the development of modern geometry. In 1872, the German mathematician Felix Klein gave a lecture at Erlangen University, in which he outlined a general principle for constructing different geometries that is now known as the Erlanger Programm. His basic idea was to consider arbitrary groups of transformations, and to investigate the properties of objects they leave invariant. Using this principle, it is possible to build a hierarchy of geometries (i.e., Euclidean, affine, projective, etc.) in which structural properties can be stratified with respect to their stability in a formally precise way.

The research described in the present article was designed to investigate whether a similar type of stratification might also be useful for understanding the geometry of perceived space. It is important to recognize when evaluating this issue that the "geometry" of perceived space can be construed in two different ways. One possibility is to consider the *extrinsic* structure of observers' perceptions relative to the physical environment. From an extrinsic point of view, the structure of perceptual space (Ψ) is determined by its formal relation to physical space (Φ), such that $\Psi = f(\Phi)$. Within this context, the geometry of perceived space is defined by the particular set of properties that are invariant over the transformation $f(\Phi)$. Thus, if observers could make veridical judgments about projective properties, such as colinearity, then their perceptions would be extrinsically projective, and if they could make accurate judgments about affine properties, such as parallelism or bisection, then their perceptions would be extrinsically affine (see Koenderink & van Doorn, 1991; Todd & Bressan, 1990).

It is also possible, however, to investigate the *intrinsic* geometry of perceptual space, without making any comparisons at all to the corresponding structure of the external environment. To better appreciate the distinction between intrinsic and extrinsic geometry, it is useful to consider how an intelligent ant might determine whether the surface on which it lives is a plane or a sphere. One possibility, originally devised by the Greek mathematician Eratosthenes around 200 BC, would be to measure the angle of the sun at different locations at the same moment in time. If the surface were planar, then the angle of the sun would be the same at all locations, but if it were curved, then the angle would vary as a function of position. Suppose, on the other hand, that there were no external landmarks available. Could the ant still determine the curvature of its environment using only intrinsic measures? As it turns out, there are several different procedures by which this could be accomplished. Let us assume that our intelligent ant has access to a long piece of string, which can be securely anchored to points on the surface. One possible procedure would be to pull the string tightly around three anchor points to form a triangle, and then measure the three angles. If the surface were planar, then the sum of the three angles would be 180°; if it were spherical then the sum would be greater than 180°; and if it were saddle-shaped then the sum would be less than 180°. An alternative procedure would be to anchor a string of length (r) at a single pivot point, and then pace off the circumference of a circle around that point. If the surface were planar, then the circumference should equal $2\pi r$. It would be less than $2\pi r$ for a spherical surface and greater than $2\pi r$ for a saddle shaped surface. The key thing to note in both of these examples is that it is possible to measure the geometry of a space without having to make use of any entity that is outside the space.

In the study of human vision, there have been numerous attempts to measure the intrinsic geometry of perceived space using procedures similar to the ones described above. One of the earliest experiments to address this issue was performed by Blumenfeld (1913). He asked observers to perform two tasks: one in which they aligned rows of lights in depth so that they appeared parallel, and another in which they aligned the lights so that the corresponding positions in each row would all appear to be separated by the same distance. If the intrinsic geometry of perceptual space were Euclidean, then these tasks should have been identical to one another.

The empirical results revealed, however, that parallel alleys were consistently constructed so that they diverged away from the equidistant alleys at greater and greater distances. Because this is the pattern of results that would be expected on a saddle-shaped surface, this finding was interpreted by Luneberg (1947) and his followers (e.g., Blank, 1958, 1961) as evidence that perceptual space has a negatively curved non-Euclidean distance metric. Several other procedures have also been performed to measure the intrinsic curvature of perceived space (e.g., see Battro, Netto & Rozenstraten, 1976; Norman, Todd, Perotti & Tittle, 1996). However, the results of these experiments have often been inconsistent with one another, and there can be large variations among different observers and different viewing contexts.

An implicit assumption of all of these studies is that distances between visible points are the primary component of perceived 3D structure, and that the central issue that needs to be addressed in the study of visual space perception is to determine the distance metric. Given the inconsistency of the results, however, it might be reasonable to question the validity of these assumptions (cf. Foley, 1964, 1972). An important insight from the Klein Erlanger Programm is that coherent geometries can be constructed without necessarily introducing the concept of distance. Perhaps it might be possible therefore to investigate the geometry of perceived space using other more basic geometric primitives. In the present article we will describe two different procedures by which this can be accomplished: One based entirely on colinearity judgments to measure the intrinsic projective structure of perceived space, and another based entirely on bisection judgments to measure the intrinsic affine structure.

In order to investigate the internal consistency of observers' colinearity judgments, it is useful to employ a theorem first introduced by Pappus of Alexandria around 340 AD. Let points P_1, P_2 and P_3 be colinear on one line and points Q_1, Q_2 and Q_3 be colinear on a different line. Consider a point R_1 at the intersection of P_1Q_2 with P_2Q_1, a point R_2 at the intersection of P_1Q_3 with P_3Q_1, and a point R_3 at the intersection of P_2Q_3 with P_3Q_2. According to the Pappus theorem, the points R_1, R_2 and R_3 must be colinear (see Figure 1).

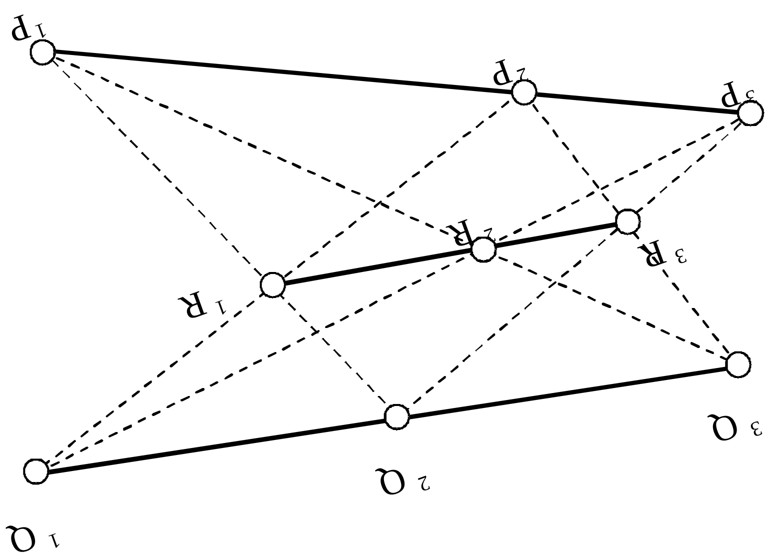

Figure 1 -- A pappus configuration similar to those used in the present experiments to investigate the projective structure of perceptual space.

We have recently performed a series of experiments to investigate whether observers' colinearity judgments satisfy this theorem. The procedure was as follows: On each trial, observers were presented with two pairs of stakes on a stereoscopically defined bumpy ground surface to define the endpoints of two line segments in perceptual space. They were asked to adjust a 5th stake so that it appeared at the point of intersection between these two line segments. The positions of the points corresponding to P_1, P_3, Q_1 and Q_3 were fixed prior to the experiment, but all other points in the configuration were obtained from observers' settings. They first marked a point P_2 that appeared to be colinear with P_1 and P_3, and an additional point pair that was not part of the Pappus configuration. The location of Q_2 was determined in a similar manner. Next the observers made settings for the points R_1, R_2 and R_3 so that they appeared to lie at the appropriate intersections. Finally we tested whether the points R_1, R_2 and R_3 appeared to be colinear with one another. This was accomplished by having observers mark a point T_1 at the intersection of R_1R_3 with Q_1P_3, and a point T_2 at the intersection of R_1R_3 with P_1Q_3. For observers' judgments to satisfy the Pappus Theorem, the points R_2, T_1 and T_2 must all be coincident within measurement error.

Six naïve observers participated in the experiment. Each observer made judgments for four different Pappus configurations, and each point within these configurations was judged on 10 separate trials. Two representative patterns of responses are shown in Figure 2. The lines in this figure show the actual Pappus configuration in physical space, with depth represented in the vertical direction. The small ellipses mark the mean positions of the observers' settings, and the axis lengths of these ellipses in different directions are six times the standard error. (They were made that large so that the ellipses would be visible.) The three ellipses in the center of the configuration show the judged positions of R_2, T_1 and T_2, respectively. Note in each case that the observers' judgments are systematically distorted relative to the true intersection point in physical space. Nevertheless, the fact that R_2, T_1 and T_2 overlap one another indicates that the Pappus theorem is satisfied despite these distortions.

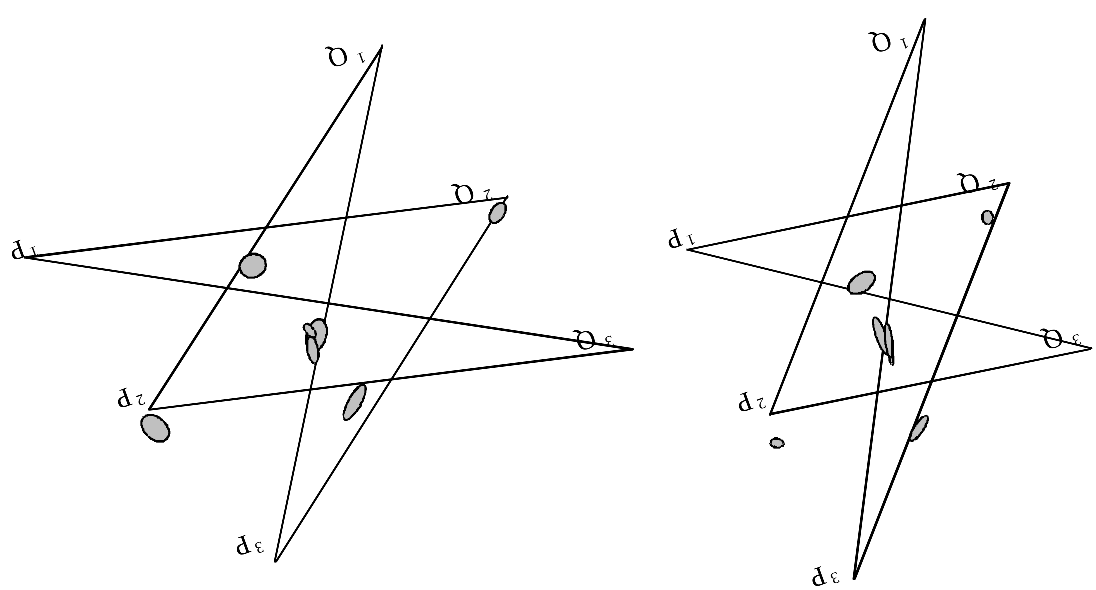

Figure 2 -- A representative pattern of results from a series of line intersection judgments. The left and right configurations show data for two different observers.

We also performed a second experiment to evaluate the internal consistency of observers' bisection judgments, based on a theorem that was first proven by Pierre Varignon around 1700. Let P_1, P_2, P_3 and P_4 be arbitrarily selected points. Consider a point Q_1 that bisects P_1P_2, a point Q_2 that bisects P_2P_3, a point Q_3 that bisects P_3P_4, and a point Q_4 that bisects P_4P_1. Now consider the point T_1 that bisects Q_1Q_3, and the point T_2 that bisects Q_2Q_4. In an affine space, the points T_1 and T_2 must be coincident with one another (see Figure 3).

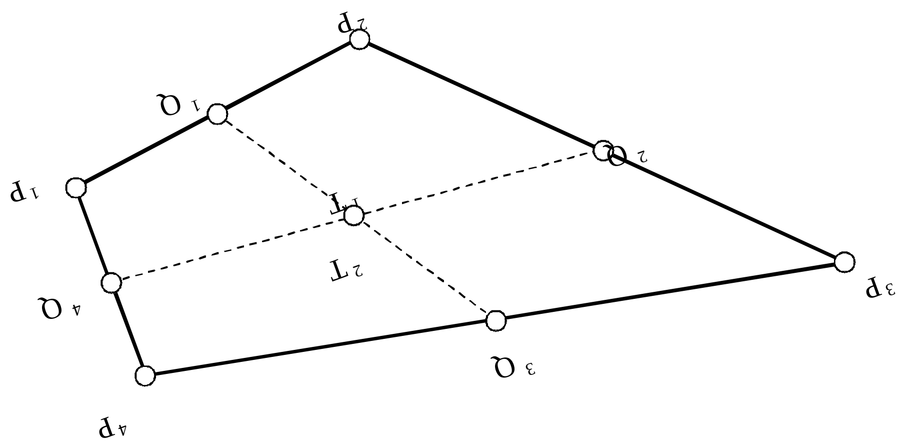

Figure 3 -- A Varignon configuration similar to those used in the present experiments to investigate the affine structure of perceptual space.

Our experimental procedure was as follows: On each trial, an observer was presented with a single pair of stakes on a stereoscopically defined bumpy ground surface, and was asked to adjust a 3rd stake so that it appeared to bisect the other two. The positions of the points corresponding to P_1, P_2, P_3 and P_4 were fixed prior to the experiment, but all other points in the configuration were obtained from observers' settings. In the first phase of the experiment they were required to mark the apparent locations of Q_1, Q_2, Q_3 and Q_4, and these settings were then used in a second phase to determine the apparent locations of T_1 and T_2.

Six naïve observers participated in the experiment. Each observer made judgments for four different configurations, and each point within these configurations was judged on 10 separate trials. Two representative patterns of responses are shown in Figure 4. The lines in this figure show the actual Varignon configuration in physical space, with depth represented in the vertical direction. The small ellipses mark the mean positions of the observers' settings, and the axis lengths of these ellipses in different directions are six times the standard error. The two ellipses in the center of the configuration show the judged positions of T_1 and T_2, respectively. Note in each case that the observers' judgments are systematically distorted relative to the actual bisections in physical space. Nevertheless, the fact that T_1 and T_2 overlap one another indicates that the pattern of perceived bisections are internally consistent with one another.

To summarize, the research described in the present paper has examined the structure of perceptual space at a more primitive geometric level than has previously been investigated. As in many previous studies, the results reveal that the extrinsic mapping between physical and perceived space is non-veridical, such that curved lines in the environment can appear perceptually to be straight. The results also indicate, however, that the intrinsic structure of perceptual space has an internally consistent affine and projective geometry.

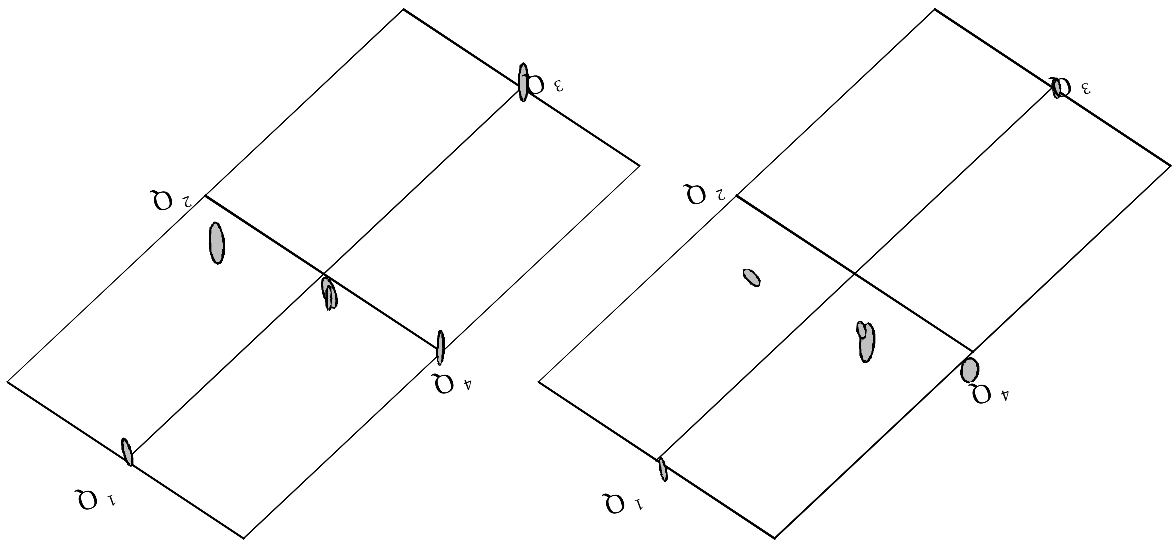

Figure 4 -- A representative pattern of results from a series of bisection judgments. The left and right configurations show data from two different observers.

References

Battro, A. M., Netto, S. P., & Rozestraten, R. J. A. (1976). Riemannian geometries of variable curvature in visual space: visual alleys, horopters, and triangles in big open fields. *Perception, 5,* 9-23.

Blank, A. A. (1958). Analysis of experiments in binocular space perception. *Journal of the Optical Society of America, 48,* 911-925.

Blank, A. A. (1961). Curvature of binocular visual space: An experiment. *Journal of the Optical Society of America, 51,* 335-339.

Blumenfeld, W. (1913). Untersuchungen über die scheinbare Grösse in Sehraume. *Zeitschrift für Psychologie und Physiologie der Sinnesorgane, 65,* 241-404.

Foley, J. M. (1964). Desarguesian property in visual space. *Journal of the Optical Society of America, 54,* 684-692.

Foley, J. M. (1972). The size-distance relation and intrinsic geometry of visual space: implications for processing. *Vision Research, 12,* 323-332.

Koenderink, J. J., & van Doorn, A. J. (1991). Affine structure from motion. *Journal of the Optical Society of America A, 8,* 377-385.

Luneberg, R. K. (1947). *Mathematical analysis of binocular vision*. Princeton, NJ: Princeton University Press.

Norman, J. F., Todd, J. T., Perotti, V. J., & Tittle, J. S. (1996). The visual perception of 3D length. *Journal of Experimental Psychology: Human Perception and Performance, 22,* 173-186.

Todd, J. T., & Bressan, P. (1990). The perception of 3-dimensional affine structure from minimal apparent motion sequences. *Perception & Psychophysics, 48,* 419-430.