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## On asymptotic solutions of Friedmann equations

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### ABSTRACT

Our main aim is to apply the theory of regularly varying functions to the asymptotical analysis at infinity of solutions of Friedmann cosmological equations. A new constant  $\Gamma$  is introduced related to the Friedmann cosmological equations. Determining the values of  $\Gamma$  we obtain the asymptotical behavior of the solutions, i.e. of the expansion scale factor  $a(t)$  of a universe. The instance  $\Gamma < \frac{1}{4}$  is appropriate for both cases, the spatially flat and open universe, and gives a sufficient and necessary condition for the solutions to be regularly varying. This property of Friedmann equations is formulated as the *generalized power law principle*. From the theory of regular variation it follows that the solutions under usual assumptions include a multiplicative term which is a slowly varying function.

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### 1. Introduction

In this paper<sup>1</sup> we describe conditions under which the Friedmann equations [5] have regularly varying solutions. Strictly speaking, we found a necessary and sufficient condition for Friedmann equations, expressed by the values of a constant  $\Gamma$ , to have regularly varying solutions. We formulate this description as the *generalized power law principle for Friedmanns equation*. The physical formulation of this condition is that a certain form of the equation of state  $p \sim w\rho c^2$  must hold. Hence, our discussion is mainly about a universe filled with the perfect fluid with constant barotropic equation of state  $p = w\rho c^2$ . The sufficiency of this condition is well known, e.g. Liddle and Lyth [9], Coles and Lucchin [4], Narlikar [16] and Islam [8]. However, we have not found in the literature the necessity part of the power law principle.

It appears that the mentioned constant  $\Gamma$  related to the Friedmann acceleration equation plays the crucial role in this analysis. Its values determine the asymptotical behavior of the solutions of the Friedmann equations, i.e. of the scale factor  $a(t)$  as time  $t$  tends to  $\infty$ . Our solution is also valid for non-zero cosmological constant  $\Lambda$  if the pressureless spatially flat universe is assumed. This was possible due to a formula of Carroll et al. [3] for the predicted age of the universe. In the course of this analysis, mathematical singularities appearing in the solutions are classified and are clearly distinguished from those arising from the physical constrains. All solutions we found are in agreement with the results widely found in the literature on standard cosmological model.

The background for our analysis is the theory of regularly varying functions which could be considered as the mathematical counterpart of the general form of the power law, the term often used in physics. A good presentation of this subject can be found in Bingham et al. [2] and Seneta [17]. Another tool we used is the theory of regularly varying solutions of differential equations. A good source for this theory is Marić [13]. The theory of regular variation provides additional means in the asymptotical analysis of the solutions of the second order linear differential equations as (3), but it seems it has not been

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much applied in cosmology and in astrophysics. There are few such applications, e.g. Molchanov et al. [15], Stern [18] and Mijajlović et al. [14].

By  $\mathbb{R}$  we denote the set of real numbers. As usually, for two real functions  $f$  and  $g$ ,  $f(x) \sim g(x)$  (or  $f \sim g$ ) means that  $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$ .

The paper is organized as follows. In the first section the history of the problem is explained and physical (Friedmann equations) and mathematical (regular variation) background is given. The main results of the paper are presented in Sections 2 and 3.

### 1.1. Friedmann equations

The scale factor  $a(t)$  is defined by Friedmann–Lemaître–Robertson–Walker (FLRW) metric. The FLRW 4-dimensional line element in spherical comoving coordinates is given by

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right]. \quad (1)$$

This metric is an exact solution of Einstein's field equations of general relativity and it describes a homogeneous, isotropic expanding or contracting universe. In this paper we shall discuss only the expanding universe. The scale factor  $a(t)$  is a solution of the Friedmann equations. These equations are derived from the Einstein field equations; they are the following three differential equations. The term Friedmann equation is usually reserved for the first one.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}. \quad (2)$$

The Friedmann acceleration equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \quad (3)$$

while the fluid equation is

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0. \quad (4)$$

The solutions of these equations are three fundamental parameters, the scale factor  $a = a(t)$ , the energy density  $\rho = \rho(t)$  and  $p = p(t)$ , the pressure of the material in the universe. Here  $k$  is the curvature index with possible values  $-1$  (elliptic geometry),  $0$  (spatially flat geometry) and  $1$  (hyperbolic geometry). The symbol  $G$  denotes the gravitational constant and  $c$  is the speed of light. Eqs. (2)–(4) are not independent. Eq. (3) follows from (2) and (4), while Eqs. (2) and (3) yield (4).

We shall use Karamata theory of regularly varying functions, as applied to differential equations in Maric and Tomic [12] and Marić [13]. This theory generalizes the power law in physics and we shall use it to obtain the asymptotic analysis of solutions of Friedmann equations.

In our study of the asymptotic solutions of Friedmann equations, the acceleration equation will have the central point for several reasons. First, it does not contain explicitly the curvature index  $k$ . Secondly, the theory of regularly varying solutions of such type of equations can be applied successfully, regardless if the cosmological constant  $\Lambda$  is added in (2) and (3):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda}{3}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}. \quad (5)$$

Namely, under the transformations  $\rho' = \rho + \Lambda/(8\pi G)$ ,  $p' = p - \Lambda/(8\pi G)$  Eqs. (5) yield (2) and (3), but now with respect to the parameters  $\rho'$  and  $p'$ . The fluid equation is not affected by the parameter  $\Lambda$ . Therefore, our discussion will be concentrated further on the solutions of the Friedmann equations in their basic form (2)–(4) if it is not otherwise stated.

From now on, we shall assume that the functions  $a(t)$ ,  $p(t)$  and  $\rho(t)$  satisfy all three Friedmann equations. We shall also assume that all appearing functions are continuous in their domains and have the sufficient number of derivatives, at least that they have the continuous second derivative.

### 1.2. Regular variation

In this section we shall review the basic notions related to the regular variation necessary for our analysis. In particular we shall need properties of regularly varying solutions of the second order differential equation

$$\ddot{y} + f(t)y = 0, \quad f(t) \text{ is continuous on } [\alpha, \infty]. \quad (6)$$

Observe that the acceleration equation (3) has the form (6). In short, the notion of a regular variation is related to the power law distributions, described by the following relationship between quantities  $F$  and  $t$ :

$$F(t) = t^r(\alpha + o(1)), \quad \alpha, r \in \mathbb{R}. \quad (7)$$

It is said that two quantities  $y$  and  $t^r$  satisfy the power law if they are related by a proportion,<sup>2</sup> i.e. there is a constant  $\alpha$  so that  $y = \alpha t^r$ . This definition of power law can be naturally extended by use of the notion of slowly varying function.

A real positive continuous function<sup>3</sup>  $L(t)$  defined for  $x > x_0$  which satisfies

$$\frac{L(\lambda t)}{L(t)} \rightarrow 1 \quad \text{as } t \rightarrow \infty, \quad \text{for each real } \lambda > 0 \tag{8}$$

is called a slowly varying function.

**Definition.** A physical quantity  $F(t)$  is said to satisfy the generalized power law if

$$F(t) = t^r L(t), \tag{9}$$

where  $L(t)$  is a slowly varying function and  $r$  is a real constant.

Examples of slowly varying functions are  $\ln(x)$  and iterated logarithmic functions  $\ln(\dots \ln(x) \dots)$ . More complicated examples (cf. Marić [13]) are provided by:

$$L_1(x) = \frac{1}{x} \int_a^x \frac{dt}{\ln t}, \quad L_2(x) = \exp((\ln x)^{1/3} \cos(\ln x)^{1/3}). \tag{10}$$

We note that  $L_2(x)$  varies infinitely between 0 and  $\infty$ .

A positive continuous function  $F$  defined for  $t > t_0$ , is the regularly varying function of the index  $r$ , if and only if it satisfies

$$\frac{F(\lambda t)}{F(t)} \rightarrow \lambda^r \quad \text{as } t \rightarrow \infty, \quad \text{for each } \lambda > 0. \tag{11}$$

It immediately follows that a regularly varying function  $F(t)$  has the form (9). So to say that  $F(t)$  is regularly varying is the same as  $F(t)$  to satisfy the generalized power law. By Proposition 7 in [13], if a function  $F(x)$  is asymptotically equivalent to a regularly varying function, it is a regularly varying function. Hence, we may define the generalized power law also by

$$F(x) \sim t^\alpha L(t), \quad \text{as } t \rightarrow \infty. \tag{12}$$

The class of regularly varying functions of index  $\alpha$  we shall denote by  $\mathcal{R}_\alpha$ . Hence  $\mathcal{R}_0$  is the class of all slowly varying functions. By  $\mathcal{Z}_0$  we shall denote the class of zero functions at  $\infty$ , i.e.  $\varepsilon \in \mathcal{Z}_0$  if and only if  $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$ .

Karamata introduced in [10] the concept of regularly varying functions continuing the works of G.H. Hardy, J.L. Littlewood and E. Landau in the asymptotic analysis of real functions. The following two theorems describe fundamental properties of this class of functions.

**Theorem 1.1** ([10] Representation theorem).  $L \in \mathcal{R}_0$  if and only if there are measurable functions  $h(x)$  and  $\varepsilon \in \mathcal{Z}_0$  and  $b \in \mathbb{R}$  so that

$$L(x) = h(x) e^{\int_b^x \frac{\varepsilon(t)}{t} dt}, \quad x \geq b, \tag{13}$$

and  $h(x) \rightarrow h_0$  as  $x \rightarrow \infty$ ,  $h_0$  is a positive constant.

The function  $\varepsilon(t)$  in the above theorem is not uniquely determined. If  $h(x)$  is a constant function, then  $L(x)$  is called normalized. We denote by  $\mathcal{N}$  the class of normalized slowly varying functions. We note the following important fact for  $\mathcal{N}$ -functions. If  $L \in \mathcal{N}$  and there is  $\dot{L}$ , then  $\varepsilon$  in (13) has the first order derivative  $\dot{\varepsilon}$ . This follows from the identity  $\varepsilon(t) = t\dot{L}(t)/L(t)$ .

There is also an appropriate definition of regular variation at 0 and  $\infty$  and various generalizations such as the rapidly varying functions. Even if such solutions of Friedmann equation are possible, we will not discuss these types of solutions in this article, so we omit these definitions.

For our study of Friedmann equations we need several results on solutions of (6). There are various conditions for  $f(t)$  that ensure that regularly varying solutions of  $\ddot{y} + f(t)y = 0$  exist. We shall particularly use the following result, see Howard and Maric [7] and Marić [13] the Theorems 1.10 and 1.11:

**Theorem 1.2.** Let  $-\infty < \Gamma < 1/4$ , and let  $\alpha_1 < \alpha_2$  be two roots of the equation

$$x^2 - x + \Gamma = 0. \tag{14}$$

Further let  $L_i, i = 1, 2$  denote two normalized slowly varying functions. Then there are two linearly independent regularly varying solutions of  $\ddot{y} + f(t)y = 0$  of the form

$$y_i(t) = t^{\alpha_i} L_i(t), \quad i = 1, 2, \tag{15}$$

if and only if  $\lim_{x \rightarrow \infty} x \int_x^\infty f(t) dt = \Gamma$ . Moreover,  $L_2(t) \sim \frac{1}{(1-2\alpha_1)L_1(t)}$ .  $\square$

<sup>2</sup> This relation is usually denoted by  $y \propto t^r$ .

<sup>3</sup> More generally it may be assumed that  $L(t)$  is a measurable function, but in this article we are dealing only with continuous functions anyway.

The limit of the integral in the theorem is not always easy to compute. As  $\lim_{t \rightarrow \infty} t^2 f(t) = \Gamma$  implies  $\lim_{x \rightarrow \infty} x \int_x^\infty f(t) dt = \Gamma$ , we see that

$$\lim_{t \rightarrow \infty} t^2 f(t) = \Gamma \quad (16)$$

gives a useful sufficient condition for the existence of regular solutions of the equation  $\ddot{y} + f(t)y = 0$  as described in the previous theorem.

## 2. Regularly varying solutions of acceleration equations

As noted, the acceleration equation obviously has the form (6) so under appropriate assumptions, i.e. that the functions we encounter are continuously differentiable as many times as necessary, the analysis of the previous section, in particular Theorem 1.2, can be applied to it. For this reason, we shall write from now on the acceleration equation (5) in the form

$$\ddot{a} + \frac{\mu(t)}{t^2} a = 0, \quad (17)$$

where

$$\mu(t) = \frac{4\pi G}{3} t^2 \left( \rho + \frac{3p}{c^2} \right). \quad (18)$$

Our approach in the next analysis is as follows. Obviously  $\mu$  is a function of  $\rho$  and  $p$ . We assumed that  $\rho$  and  $p$  are solutions of Friedmann equations, hence  $\mu(t)$  is a well-defined function. Under this assumption, the theory of regular variation applied to Eq. (17) yields the asymptotic expansions of  $\mu(t)$  and  $a(t)$  and exact conditions on  $\mu(t)$  under which these expansions exist. Using the identity (18) we will be able then to find the asymptotical expansions for  $\rho$ ,  $p$  and other cosmological parameters.

In the next the crucial role will play the following integral limit:

$$\lim_{x \rightarrow \infty} \int_x^\infty \frac{\mu(t)}{t^2} dt = \Gamma. \quad (19)$$

Let us denote by  $\mathcal{M}_r$  the class of functions  $\mu$  that satisfy the integral condition (19). Further, let

$$\mathcal{M} = \bigcup_{r \in \mathbb{R}} \mathcal{M}_r.$$

Marić introduced the integral condition (19) (cf. [13]), accordingly we shall call the class  $\mathcal{M}$  also as Marić class of functions. Obviously,  $\mathcal{M}$  is a vector space over  $\mathbb{R}$  and the map  $\mathbf{M} : \mathcal{M} \rightarrow \mathbb{R}$  defined by

$$\mathbf{M}(u) = \lim_{x \rightarrow \infty} \int_x^\infty \frac{u(t)}{t^2} dt$$

is a linear functional, i.e.  $\mathbf{M}(\alpha u + \beta v) = \alpha \mathbf{M}(u) + \beta \mathbf{M}(v)$ ,  $\alpha, \beta \in \mathbb{R}$ ,  $u, v \in \mathcal{M}$ . It is easy to see that  $\mathbf{M}(\varepsilon) = 0$  for  $\varepsilon \in \mathcal{Z}_0$ . By the note regarding (16), we immediately have

**Proposition 2.1.** *If  $\lim_{t \rightarrow \infty} u(t) = r$  then  $\mathbf{M}(u) = r$ .*

Now we prove a useful representation theorem for Marić class of functions.

**Theorem 2.2** (Representation theorem for  $\mathcal{M}$ -functions).  *$u \in \mathcal{M}_r$  if and only if there are  $\varepsilon, \eta \in \mathcal{Z}_0$  such that  $u(t) = r - t\dot{\varepsilon}(t) + \eta(t)$ . If  $r < 1/4$  then  $\varepsilon$  is that one appearing in the representation (13) of  $a(t)$ , with  $h(t)$  constant.*

**Proof.** ( $\Rightarrow$ ) Suppose  $u \in \mathcal{M}_r$ , and  $r < \frac{1}{4}$ . By Theorem 1.2 the equation  $\ddot{y} + \frac{u(t)}{t^2} y = 0$  has a solution  $a(t) = t^\alpha L(t)$  where  $\alpha$  is a root of the equation  $x^2 - x + r = 0$  and  $L \in \mathcal{N}$ . By Theorem 1.1 there are  $a_0, b \in \mathbb{R}$  and  $\varepsilon \in \mathcal{Z}_0$  so that  $a(t) = a_0 t^\alpha e^{\int_b^\alpha \frac{\varepsilon(t)}{t} dt}$ . As  $\dot{L}(t) = \frac{\varepsilon(t)}{t} L(t)$  and  $r = -\alpha(\alpha - 1)$ , we have

$$\ddot{a}(t) = (-r + t\dot{\varepsilon} - (1 - 2\alpha)\varepsilon + \varepsilon^2)L(t)t^{\alpha-2}.$$

Since  $-\frac{u(t)}{t^2} = \frac{\ddot{a}(t)}{a(t)}$  it follows  $u(t) = r - t\dot{\varepsilon}(t) + \eta(t)$  where  $\eta = \varepsilon^2 - (1 - 2\alpha)\varepsilon$ .

Suppose  $r \geq \frac{1}{4}$ . As  $\mathbf{M}(\frac{1}{8r}u) = \frac{1}{8r}$ , taking  $\frac{1}{8r}u$  instead of  $u$  in the previous proof, we have  $\frac{1}{8r}u(t) = \frac{1}{8} - t\dot{\varepsilon}(t) + \eta(t)$  for some  $\varepsilon, \eta \in \mathcal{Z}_0$ , hence  $u(t) = r - t\dot{\varepsilon}_1(t) + \eta_1(t)$  where  $\varepsilon_1 = 8r\varepsilon$  and  $\eta_1 = 8r\eta$ .

( $\Leftarrow$ ) Suppose  $u(t) = r - t\dot{\varepsilon}(t) + \eta(t)$  where  $\varepsilon, \eta \in \mathcal{Z}_0$ . Then

$$\mathbf{M}(u) = \mathbf{M}(r) - \mathbf{M}(t\dot{\varepsilon}) + \mathbf{M}(\eta) = r - \mathbf{M}(t\dot{\varepsilon}).$$

Further, taking  $v(t) = t\dot{\varepsilon}$ ,

$$\int_x^\infty \frac{v(t)}{t^2} dt = \int_x^\infty \frac{d\varepsilon}{t} = -\frac{\varepsilon}{x} + \int_x^\infty \frac{\varepsilon}{t^2} dt = -\frac{\varepsilon}{x} + o\left(\frac{1}{x}\right).$$

Hence,  $\mathbf{M}(t\dot{\varepsilon}) = \lim_{x \rightarrow \infty} x \int_x^\infty \frac{v(t)}{t^2} dt = \lim_{x \rightarrow \infty} (-\varepsilon + o(1)) = 0$ , so  $\mathbf{M}(u) = r$ .  $\square$

**Corollary 2.3.** Assume  $u \in \mathcal{M}_r$ . Then  $\lim_{t \rightarrow \infty} u(t) = r$  if and only if  $\lim_{t \rightarrow \infty} t\dot{\varepsilon}(t) = 0$  in above representation of  $u$ .

**Example.** Let  $\varepsilon(t) = \frac{\sin(t^3)}{t}$ . Then for

$$\mu(t) = \frac{1}{8} - t\dot{\varepsilon}(t) - \varepsilon(t) = \frac{1}{8} - 3t^2 \cos(t^3).$$

$\mathbf{M}(\mu) = \frac{1}{8}$  and all ultimately positive solutions of (17) are regularly varying, but  $\lim_{t \rightarrow \infty} \mu(t)$  does not exist. Note that it follows  $\lim_{x \rightarrow \infty} x \int_x^\infty \cos(t^3) dt = 0$ .  $\square$

The next proposition will be useful in our further analysis. It also gives the  $\varepsilon$ -representation of the logarithmic derivative  $H(t) \equiv \dot{a}(t)/a(t)$  of  $a(t)$ .

**Proposition 2.4.** Suppose  $\mu \in \mathcal{M}$  and  $\Gamma \equiv \mathbf{M}(\mu) < \frac{1}{4}$  holds for Eq. (17). Then any ultimately positive solution  $a(t)$  of (17) is a normalized regularly varying function, i.e. there are  $L \in \mathcal{N}$  and  $\alpha \in \mathbb{R}$ , so that  $a(t) = t^\alpha L(t)$ .

If  $L(t)$  has the  $\varepsilon$ -representation as in Theorem 1.1, where  $h(t)$  is a positive constant, then  $H(t) = \alpha/t + \varepsilon/t$ .

**Proof.** Suppose  $a(t)$  is positive at  $\infty$ . By Theorem 1.2 there are  $L_1, L_2 \in \mathcal{N}$  and  $\alpha_1, \alpha_2 \in \mathbb{R}$  so that

$$a(t) = c_1 L_1 t^{\alpha_1} + c_2 L_2 t^{\alpha_2}, \quad c_1, c_2 \in \mathbb{R}, \tag{20}$$

where  $\alpha_1, \alpha_2$  are the roots of Eq. (14). Since  $\Gamma < 1/4$  we have  $\alpha_1 \neq \alpha_2$ , so we may assume  $\alpha_1 > \alpha_2$ . Suppose  $c_1 \neq 0$ . Hence, the term  $c_1 L_1 t^{\alpha_1}$  dominates  $c_2 L_2 t^{\alpha_2}$ , so there is  $t_0 > 0$  so that  $a(t) > 0$  for  $t > t_0$ . Let  $\delta = \alpha_2 - \alpha_1$ ,  $c_0 = c_1/c_2$  and  $L_0 = L_1/L_2$ . Note that  $L_0 \in \mathcal{R}_0$ . By Representation Theorem 1.1 and as  $L_1, L_2$  are normalized, there are constants  $h_1, h_2, b \in \mathbb{R}$  and  $\varepsilon_1, \varepsilon_2 \in \mathcal{Z}_0$  so that

$$L_i(x) = h_i e^{\int_b^x \frac{\varepsilon_i(t)}{t} dt}, \quad x \geq b, \quad i = 1, 2.$$

As  $\dot{L}_i(t) = \frac{\varepsilon_i(t)}{t} L_i(t)$ , taking the logarithmic derivative  $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$  of  $a(t)$  and  $\alpha \equiv \alpha_1$  we obtain

$$H(t) = \frac{\alpha}{t} (1 + \varepsilon_1/\alpha) \frac{1 + c_0 \frac{\alpha_2 + \varepsilon_2}{\alpha_1 + \varepsilon_1} L_0(t) t^\delta}{1 + c_0 L_0(t) t^\delta}.$$

Since  $L_0$  is slowly varying and  $\delta < 0$ , it follows  $L_0 t^\delta \rightarrow 0$  as  $t \rightarrow \infty$ . Hence, there is  $\varepsilon \in \mathcal{Z}_0$  so that

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\alpha}{t} + \frac{\varepsilon(t)}{t}. \tag{21}$$

By integration of this relation we have immediately

$$a(x) = a_0 e^{\int_b^x \frac{\varepsilon(t)}{t} dt}, \quad x \geq b, \quad \text{where } b = t_0, \quad a_0 = a(t_0).$$

Hence, by Theorem 1.1,  $a(t)$  is a normalized slowly varying function.  $\square$

### 3. Asymptotic solutions of Friedmann equations

We proceed to the analysis of solutions of Friedman equations taking into account the physical constraints. We remind that besides the acceleration equation (17) with  $\mu(t)$  defined by (18), the scale factor  $a(t)$  also satisfies the other two Friedmann Eqs. (2) and (4). Note that  $a(t)$  is a function of time which represents the relative expansion of the universe. This function relates the proper distance between a pair of objects, e.g. two galaxies, moving with the Hubble flow in a FLWR universe at any arbitrary time  $t$  to their distance at some reference time  $t_0$ . Thus,  $d(t) = a(t)d(t_0)$  where  $d(t)$  is the proper distance at epoch  $t$ . Hence  $a(t) > 0$ . Therefore, we shall consider only positive solutions  $a(t)$  of Friedmann equations.

#### 3.1. Cosmological parameters

The Hubble parameter  $H(t)$  and the deceleration parameter  $q(t)$  are defined in Cosmology by

$$H(t) = \frac{\dot{a}(t)}{a(t)}, \quad q(t) = -\frac{\ddot{a}(t)}{a(t)} \cdot \frac{1}{H(t)^2} \quad (22)$$

where  $a(t)$  is the scale factor. Then obviously we have the following identity.

$$\mu(t) = q(t)(H(t)t)^2. \quad (23)$$

Observe that  $\mu(t)$  is a dimensionless parameter. We shall assume that  $\mu(t)$  is continuous. From the physical point of view, it means that scenarios such as Big Crunch, or Big Rip are not included in our analysis. That is, in finite time  $t$ ,  $a(t) \neq 0$ , neither  $\dot{a}(t)$  becomes infinite. If  $\mu(t)$  is an  $\mathcal{M}$ -function then the real constant  $\Gamma$  is defined by  $\Gamma = \mathbf{M}(\mu)$ .

Let us remind that the density parameter  $\Omega(t)$  and the density parameter for the cosmological constant  $\Lambda$  are defined by

$$\Omega = \Omega(t) = \frac{\rho(t)}{\rho_c}, \quad \Omega_\Lambda = \Omega_\Lambda(t) = \frac{\Lambda}{3H(t)^2},$$

where  $\rho_c$  is the critical density.

**Proposition 3.1.** *If the limit  $H_\infty = \lim_{t \rightarrow \infty} H(t)$  exists, then*

$$\Gamma = \lim_{t \rightarrow \infty} \left( (H(t) - H_\infty) - \int_t^\infty H(t)^2 dt \right).$$

**Proof.** As  $\mu(t) = -\frac{\ddot{a}}{a} t^2$  by use of partial integration we have:

$$\int \frac{\mu(t)}{t^2} dt = -\frac{\dot{a}}{a} - \int \frac{\dot{a}^2}{a^2} dt = -H(t) - \int H(t)^2 dt$$

and the statement follows as  $\Gamma = \lim_{x \rightarrow \infty} x \int_x^\infty \frac{\mu(t)}{t^2} dt$ .  $\square$

Therefore, if the limit (19) exists then  $\Gamma$  depends solely on the behavior of the Hubble parameter  $H(t)$  at  $\infty$ .

The next theorem describes the main property of the scale factor  $a(t)$  for the non-oscillatory universe. Namely, it gives the necessary and sufficient condition for  $a(t)$  to satisfy the generalized power law.

**Theorem 3.2** (Generalized power law for the scale factor  $a(t)$ ). *Let  $a(t)$  be the scale factor, a solution of Friedmann equations, and  $\alpha \in \mathbb{R}$ . Then*

1. If  $\mu \in \mathcal{M}_\Gamma$  and  $\Gamma < 1/4$  then there is  $L \in \mathcal{N}$  so that  $a(t) = t^\alpha L(t)$ , where  $\alpha$  is a root of the polynomial  $x^2 - x + \Gamma$ .
2. If there is  $L \in \mathcal{N}$  so that  $a(t) = t^\alpha L(t)$  then  $\mu \in \mathcal{M}_\Gamma$ ,  $\alpha^2 - \alpha + \Gamma = 0$  and  $\Gamma \leq 1/4$ .

**Proof**

1. This assertion follows immediately from Proposition 2.4.
2. The next proof follows the ideas presented in [13, Section 1.4]. So, suppose  $a(t) = t^\alpha L(t)$ ,  $L \in \mathcal{N}$ . By Representation theorem 1.1 there is  $\varepsilon \in \mathcal{Z}_0$  so that  $\dot{L} = \varepsilon L$ , hence

$$t \frac{\dot{a}(t)}{a(t)} = \varepsilon(t) + \alpha, \quad \left( t \frac{\dot{a}(t)}{a(t)} \right)^2 = \eta(t) + \alpha^2, \quad \eta \in \mathcal{Z}_0. \quad (24)$$

Using  $\frac{\ddot{a}}{a} = -\frac{\mu}{t^2}$  and by integration of the identity  $\frac{\ddot{a}}{a} = \left(\frac{\dot{a}}{a}\right)' + \left(\frac{\dot{a}}{a}\right)^2$  we obtain after multiplying by  $x$

$$-x \frac{\dot{a}(x)}{a(x)} + x \int_x^\infty \left( t \frac{\dot{a}(t)}{a(t)} \right)^2 t^{-2} dt + x \int_x^\infty \frac{\mu(t)}{t^2} dt = 0.$$

By (24), the last identity and applying  $x \rightarrow \infty$ , we infer  $\alpha^2 - \alpha + \Gamma = 0$ . Since  $\alpha$  is a real number, for the discriminant  $\Delta = 1 - 4\Gamma$  of the polynomial  $x^2 - x + \Gamma$  must be  $\Delta \geq 0$ , i.e.  $\Gamma \leq 1/4$ .  $\square$

**Remark.** Under certain conditions Theorem 3.2.1 also holds for  $\Gamma = 1/4$ , i.e.  $\alpha = 1/2$ . This case will be discussed in Section 3.4.

**Theorem 3.3.** *Assume  $\mu \in \mathcal{M}_\Gamma$  where  $\Gamma < 1/4$ . Let  $a(t) = t^\alpha L(t)$  be the corresponding scale factor, where  $\alpha \neq 0$  and  $L \in \mathcal{N}$ , with  $L$  having the  $\varepsilon$ -representation as in Theorem 1.1, where  $h(t)$  is a positive constant. Then*

1. The possible values of the curvature index  $k$  are 0 and  $-1$ , i.e. the Friedmann model of the universe is non-oscillatory.
2. The Hubble parameter  $H(t)$  has the following representation

$$H(t) = \frac{\alpha}{t} + \frac{\varepsilon}{t}. \tag{25}$$

3. The deceleration parameter  $q(t)$  has the following representations

$$q(t) = \frac{\mu(t)}{\alpha^2} (1 + \eta), \tag{26}$$

$$q(t) = \frac{1 - \alpha}{\alpha} - \frac{t\dot{\varepsilon}}{\alpha^2} (1 + \eta) + \tau, \quad \eta, \tau \in \mathbb{Z}_0. \tag{27}$$

$$q(t) = \frac{1 - \alpha}{\alpha} - t\dot{\zeta} + \zeta, \quad \zeta, \zeta \in \mathbb{Z}_0 \tag{28}$$

**Proof**

1.  $\mathcal{N}$ -functions belong to so called Zygmund class (Bojanić and Karamata, see [2]), hence, since  $\alpha \neq 0$ , the scale factor  $a(t)$  is ultimately monotonous function. Thus, the universe is non-oscillatory, hence  $k = 0$  or  $k = -1$ .
2. The representation (25) follows from Proposition 2.4
3.  $q(t) = -\frac{\ddot{a}}{a} \cdot \frac{1}{H^2} = \frac{\mu}{t^2} \cdot \frac{1}{(\alpha/t + \varepsilon/t)^2} = \frac{\mu}{\alpha^2} (1 + \varepsilon/\alpha)^{-2} = \frac{\mu(t)}{\alpha^2} (1 + \eta(t))$  for some  $\eta \in \mathbb{Z}_0$ , i.e. (26) holds. Further, by  $\varepsilon$ -representation for  $\mu(t)$ , Theorem 2.2, there is  $\delta \in \mathbb{Z}_0$  so that  $q(t) = (\Gamma - t\dot{\varepsilon} + \delta)(1 + \eta)/\alpha^2$ . As  $\Gamma = \alpha(1 - \alpha)$  we obtain (27) taking  $\tau = \Gamma/\alpha^2 + \delta(1 + \eta)/\alpha^2$ .

Finally we show that  $q \in \mathcal{M}_{(1-\alpha)/\alpha}$ . According to Theorem 2.2 this will prove the representation (28). So, we have

$$q(t) = \frac{\mu}{\alpha^2} (1 + \varepsilon/\alpha)^{-2} = \frac{\Gamma - t\dot{\varepsilon} + \delta}{\alpha^2} (1 + \varepsilon/\alpha)^{-2},$$

hence for  $v(t) = (1 + \varepsilon(t)/\alpha)^2$  we have

$$\mathbf{M}(q) = \frac{\Gamma}{\alpha^2} \mathbf{M}(1/v) - \frac{1}{\alpha^2} \mathbf{M}(t\dot{\varepsilon}/v) + \frac{1}{\alpha^2} \mathbf{M}(\delta/v).$$

Further,  $\mathbf{M}(1/v) = 1$  since  $x \int_x^\infty \frac{1}{(1+\varepsilon/x)^2} \cdot \frac{dt}{t^2} \rightarrow 1$  as  $x \rightarrow \infty$ .

$\mathbf{M}(t\dot{\varepsilon}/v) = 0$  since  $x \int_x^\infty \frac{t\dot{\varepsilon}}{v} \cdot \frac{dt}{t^2} = -x\alpha \int_x^\infty \frac{1}{t} d\frac{1}{1+\varepsilon/x} = \frac{\alpha}{1+\varepsilon(x)/\alpha} - x\alpha \int_x^\infty \frac{1}{1+\varepsilon/x} \cdot \frac{1}{t^2} dt = \frac{\alpha}{1+\varepsilon(x)/\alpha} - \alpha + o(1) \rightarrow 0$  as  $x \rightarrow \infty$ .

$\mathbf{M}(\delta/v) = 0$  since  $\delta(t)(1 + \varepsilon(t)/\alpha)^{-2}$  is a  $\mathcal{Z}_0$ -function.

Therefore  $\mathbf{M}(q) = \Gamma/\alpha^2 = (1 - \alpha)/\alpha$ .  $\square$

Now we introduce a new constant  $w$  related to the scale factor  $a(t)$  which satisfy the generalized power law. It will appear that  $w$  is in fact the equation of state parameter. So assume  $a(t) = t^L L(t)$ ,  $L \in \mathcal{N}$  and  $\alpha \neq 0$ . We define  $w$  by

$$w \equiv w_\alpha = \frac{2}{3\alpha} - 1 \quad (\text{equation of state parameter}). \tag{29}$$

Note that  $w \neq -1$ . As  $\Gamma = \alpha(1 - \alpha)$ , we have the following statement:

**Proposition 3.4**

1.  $\Gamma = \frac{2}{3} \cdot \frac{1+3w}{(1+w)^2}$ .
2.  $w = \frac{1-3\Gamma+\sigma_\alpha\sqrt{1-4\Gamma}}{3\Gamma}$ , where  $\sigma_\alpha \in \{1, -1\}$ .

The sign  $\sigma_\alpha$  is determined as follows. Suppose  $\Gamma \neq 1/4$ . Then the polynomial  $x^2 - x + \Gamma$  has two different roots  $\alpha, \beta$ . As  $\alpha + \beta = 1$ , we see that  $\alpha > \beta$  if and only if  $\alpha > 1/2$ . Since  $w$  is decreasing in  $\alpha$  we have:

Case  $\alpha > 1/2$ . Then: if  $1/4 > \Gamma > 0$  then  $\sigma_\alpha = -1$ ; if  $\Gamma < 0$  then  $\sigma_\alpha = +1$ .

Case  $\alpha < 1/2$ . Then: if  $1/4 > \Gamma > 0$  then  $\sigma_\alpha = +1$ ; if  $\Gamma < 0$  then  $\sigma_\alpha = -1$ .

According to Theorem 3.3 we have also the following statement.

**Theorem 3.5.** Under the assumptions of Theorem 3.3 there are the following relations

$$\alpha = \frac{2}{3(1+w)}, \quad a(t) = a_0 L(t) t^{\frac{2}{3(1+w)}}, \quad H(t) \sim \frac{2}{3(1+w)t}, \quad \mathbf{M}(q) = \frac{1+3w}{2}. \quad (30)$$

For determination of energy density  $\rho(t)$  and pressure  $p(t)$  more information on the geometry of the universe are needed. We proceed to study cosmological parameters of the universe with the specific curvature index  $k$ .

### 3.2. Asymptotic solution for universe with curvature index $k = 0$

In this subsection we shall discuss cosmological parameters for spatially flat universe. Hence  $k = 0$  where  $k$  is the curvature index. We also assume that the scale factor  $a(t)$  satisfies the generalized power law. This allows us to estimate at  $\infty$  parameters  $\rho = \rho(t)$  and  $p = p(t)$ . The symbol  $w$  denotes the equation of state parameter as defined in the previous subsection.

**Theorem 3.6.** Assume  $\mu \in \mathcal{M}_\Gamma$  where  $\Gamma < 1/4$ . Let  $a(t) = t^\alpha L(t)$  be the corresponding scale factor, where  $\alpha \neq 0$  and  $L \in \mathcal{N}$ , with  $L$  having the  $\varepsilon$ -representation as in Theorem 1.1. Then

$$1. \rho = \frac{1}{6\pi G(1+w)^2 t^2} + \frac{\eta}{t^2}, \quad \eta \in \mathcal{Z}_0. \quad 2. \mathbf{M}\left(\frac{p}{\rho c^2}\right) = w.$$

#### Proof

- As  $k = 0$ , the Friedmann equation (2) becomes  $H^2 = 8\pi G\rho/3$ . As  $H(t) = \alpha/t + \varepsilon/t$  and  $w = \frac{2}{3\alpha} - 1$ , the statement follows if  $\eta = \frac{3(2\varepsilon + \varepsilon^2)}{8\pi G}$ .
- By (18), Theorem 2.2 and the above representation of  $\rho$ , we have

$$\frac{p}{\rho c^2} = \frac{2\mu}{3\alpha^2(1 + \varepsilon/\alpha)^2} - \frac{1}{3}, \quad \mu = \Gamma - t\dot{\varepsilon} + \eta, \quad \eta \in \mathcal{Z}_0.$$

Let us take  $v(t) = (1 + \varepsilon(t)/\alpha)^2$ . Then

$$\mathbf{M}(p/\rho c^2) = \frac{2\Gamma}{3\alpha^2} \mathbf{M}(1/v) - \frac{2}{3\alpha^2} \mathbf{M}(t\dot{\varepsilon}/v) + \mathbf{M}(\eta/v) - \frac{1}{3}.$$

As in the proof of Theorem 3.3.3, we have  $\mathbf{M}(1/v) = 1$ ,  $\mathbf{M}(t\dot{\varepsilon}/v) = 0$  and  $\mathbf{M}(\eta/v) = 0$ . Hence,  $\mathbf{M}(p/\rho c^2) = 2\Gamma/3\alpha^2 - 1/3 = 2/3\alpha - 1/3 = w$ .  $\square$

**Corollary 3.7.** Under assumptions of Theorem 3.6 there are  $\xi, \zeta \in \mathcal{Z}_0$  so that  $p = \hat{w}\rho c^2$ , where  $\hat{w}(t) = w - t\dot{\xi} + \zeta$ .

Hence, the assumption that the scale factor  $a(t)$  satisfies the generalized power law implies a certain form of equation of state,  $p = \hat{w}\rho c^2$ . If  $\Gamma = \lim_{t \rightarrow \infty} \mu(t)$  exists, then  $\mathbf{M}(\mu) = \Gamma$  and by the proof of Theorem 3.6 it follows  $\hat{w} = w$ , i.e. the classical form of the equation of state is valid. In the next subsection we shall see that the assumption of the existence of  $\Gamma = \lim_{t \rightarrow \infty} \mu(t)$  leads to the classical formulas for cosmological parameters.

### 3.3. Solution for $\mu(t)$ constant at $\infty$

In this section we shall discuss conditions under which the parameter  $\mu(t)$  introduced by (23) is constant at  $\infty$  and how this relates to the solutions of the Friedmann equations. Therefore we assume  $\lim_{t \rightarrow \infty} \mu(t) = \Gamma$ . Hence, by Proposition 2.1,  $\mathbf{M}(\mu) = \Gamma$ . So, all up to now derived properties of cosmological parameters related to the scale factor  $a(t)$  which satisfies the generalized power law are valid. By Theorem 3.2 this will be the case if  $\Gamma < 1/4$  and under additional assumptions if  $\Gamma = 1/4$ . In this subsection we shall assume  $\Gamma < 1/4$ .

#### 3.3.1. Case $k = 0$ , spatially flat universe

By Corollary 2.3  $\lim_{t \rightarrow \infty} \mu(t) = \Gamma$  if and only if  $\lim_{t \rightarrow \infty} t\dot{\varepsilon} = 0$  in  $\varepsilon$ -representation of  $a(t)$  described by Theorem 1.1. Hence, according to Theorems 3.3, 3.5, and 3.6 we immediately obtain:

$$\alpha = \frac{2}{3(1+w)}, \quad a(t) = a_0 L_\alpha(t) t^{\frac{2}{3(1+w)}}, \quad \rho(t) \sim \frac{1}{6\pi G(1+w)^2 t^2}, \quad p(t) \sim w c^2 \rho, \quad H(t) \sim \frac{2}{3(1+w)t}, \quad q(t) \sim \frac{1+3w}{2} \quad (31)$$

and Eqs. (3), (4) and (2) are satisfied.

First we suppose that  $\alpha$  is greater of the roots of the polynomial  $x^2 - x + \Gamma$ , hence  $\alpha > 1/2$ . Then by (31) immediately follows  $\frac{1}{3} > w > -1$ , hence the set of admissible values of  $w$  is the interval



$$I_\alpha = (-1, 1/3). \tag{32}$$

The value  $w = -1$  yields singularity; for such  $w$  there is no corresponding  $\alpha$  neither  $\Gamma$ . If  $p = -\rho c^2$  is anyway assumed, then by fluid equation we have  $\dot{\rho} = 0$ , i.e.  $\rho$  is constant. This case corresponds to the cosmological constant, so  $\rho = \rho_\Lambda = \frac{\Lambda}{8\pi G}$ . The constant  $\Lambda$  has a negative effective pressure, and as the universe expands, work is done on the cosmological constant fluid. Hence energy density remains constant in spite of the fact that universe expands.

If  $w = 1/3$  then  $\alpha = \beta = 1/2$ ,  $\Gamma = 1/4$  and in this case (20) is not the general solution for Friedmann equations. This case will be discussed later.

If  $w = -1/3$ , then  $\alpha = 1$ ,  $\Gamma = 0$  and the acceleration equation reduces to  $\ddot{a} \sim 0$ . If  $\rho$  is computed using the acceleration equation, assuming the asymptotic value for  $a(t)$  in (31), then the following asymptotic formula for  $\rho$  holds, except for  $w = -1/3$ ,

$$\rho \sim \frac{3\Gamma}{4\pi G(1+3w)} \cdot \frac{1}{t^2}.$$

Hence,  $w = -1/3$  is a kind of singularity, but not the proper one, as it can be replaced by the second formula for  $\rho$  in (31).

Let us take into account the physical constraints on the parameters occurring in our calculations. For example, the universe is decelerating if and only if  $q > 0$  hence, by (27), this is equivalent to  $\alpha < 1$ . On the other hand,  $\alpha < 1$  is equivalent to  $w > -\frac{1}{3}$  by definition (29) of  $w$ . Therefore,

$$\frac{1}{2} < \alpha < 1 \quad \text{if and only if} \quad -\frac{1}{3} < w < \frac{1}{3} \tag{33}$$

and the universe decelerates in all cases. From  $p \sim wc^2\rho$  we see that the pressure  $p > 0$  if and only if  $w > 0$ . Hence, using (29), we see that  $p > 0$  if and only if  $\alpha < \frac{2}{3}$  and the interval for  $\alpha$  in (33) reduces to  $\frac{1}{2} < \alpha < \frac{2}{3}$ .

Let us consider the second fundamental solution in (20) of the acceleration equation with the index  $\beta \equiv \alpha_2 < 1/2$ . First we introduce the constant  $w_\beta$  by

$$w_\beta = \frac{2-3\beta}{3\beta}. \tag{34}$$

Then as  $\alpha + \beta = 1$  and the following symmetric identity holds:

$$w_\alpha + w_\beta + 3w_\alpha w_\beta = 1. \tag{35}$$

Let  $(31)_\beta$  be the set of parameters obtained from (31) by replacing  $w(=w_\alpha)$  by  $w_\beta$ . Using (35) one can show that  $b(t)$  satisfies all three Eqs. (3), (4) and (2). Now having  $\beta < \frac{1}{2}$  we can extend the interval for  $w$  in (33). As  $\beta < \frac{1}{2}$  we find from (34) that  $w \equiv w_\beta > \frac{1}{3}$  or  $w \equiv w_\beta < -1$ . Therefore, by  $(31)_\beta$  the set of admissible values for  $w = w_\beta$  is the set

$$I_\beta = (-\infty, -1) \cup \left(\frac{1}{3}, +\infty\right). \tag{36}$$

Putting together (32) and (36) we see that the set of all admissible values for  $w$  corresponding to all possible solutions of Eqs. (3), (4) and (2) is the set

$$I = \mathbb{R} \setminus \left\{-1, \frac{1}{3}\right\}, \quad \mathbb{R} \text{ is these to } f \text{ real numbers.} \tag{37}$$

The physical constraints on physical parameters for  $\beta < \frac{1}{2}$  give the narrower bound for  $w_\beta$ . By  $(31)_\beta$  the adiabatic sound speed  $v_s = \left(\frac{\partial p}{\partial \rho}\right)^{1/2} = w_\beta c$ , hence  $w_\beta < 1$  and so  $\beta > \frac{1}{3}$ . Therefore, including also our previous discussion on physical constraints for  $\alpha > \frac{1}{2}$ , we have the following bounds for  $\alpha$  and  $w$ :

$$\frac{1}{3} < \alpha < \frac{2}{3} \quad \text{and} \quad 0 < w < 1 \quad (\text{Zel'dovich interval}). \tag{38}$$

### 3.3.2. Case $k = -1$ , spatially open universe

We shall only briefly discuss this case. We remind that if we assume  $\Gamma = \lim_{t \rightarrow \infty} \mu(t)$  exists and  $\Gamma < 1/4$ , then  $t\dot{\epsilon}(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

**Lemma 3.8.** *Let  $a(t) = t^2 L(t)$  be a solution of Friedmann equations,  $L \in \mathcal{N}$ , with  $\epsilon$ -representation (13) and  $k = -1$  the spatial index. Then*

$$\frac{p}{\rho c^2} = \frac{2}{3} \cdot \frac{\alpha + \epsilon + kc^2 a^{-2} t^2 - t\dot{\epsilon}}{(\alpha + \epsilon)^2 + kc^2 a^{-2} t^2} - 1. \tag{39}$$

**Proof.** Taking the logarithmic derivative of  $\rho$  using (2) we obtain

$$\frac{\dot{\rho}}{\rho} = 2 \cdot \frac{H\dot{H} - kc^2 \dot{a} a^{-3}}{H^2 + kc^2 a^{-2}}.$$

Using  $\dot{a} = Ha$  and by (25),  $\dot{H} = \dot{\varepsilon}/t - (\alpha + \varepsilon)/t^2$ , we get

$$\frac{\dot{\rho}}{\rho} = -2H \cdot \frac{\alpha + \varepsilon + kc^2 a^{-2} t^2 - t\dot{\varepsilon}}{(\alpha + \varepsilon)^2 + kc^2 a^{-2} t^2}. \quad (40)$$

By fluid equation (4) we have  $p/\rho c^2 = -\frac{1}{3H} \cdot \frac{\dot{\rho}}{\rho} - 1$ , hence (39) follows.  $\square$

**Corollary 3.9.** Suppose  $k = -1$  and  $\alpha < 1$ . Then  $p/\rho c^2 \rightarrow -1/3$  as  $t \rightarrow \infty$ .

**Proof.** As  $\alpha < 1$  and  $L$  is slowly varying, we have  $a^{-2} t^2 = L(t) t^{2(1-\alpha)} \rightarrow \infty$  as  $t \rightarrow \infty$ . Since also  $\varepsilon, t\dot{\varepsilon} \rightarrow 0$  as  $t \rightarrow \infty$ , by (39) the assertion follows.  $\square$

**Theorem 3.10.** Suppose  $\Gamma = \lim_{t \rightarrow \infty} \mu(t)$  exists,  $\Gamma < 1/4$ , and  $a(t) = t^2 L(t)$ ,  $\alpha \neq 0$ , is a corresponding scale factor for an open universe (i.e.  $k = -1$ ). Then  $\alpha = 1$  and  $w = -1/3$ .

**Proof.** Suppose  $\alpha = 1$ . Then it easy to see that in this case, by (39),

$$\frac{p}{\rho c^2} = \frac{2}{3} \cdot \frac{1 + \varepsilon + kc^2 L^{-2} - t\dot{\varepsilon}}{(1 + \varepsilon)^2 + kc^2 L^{-2}} - 1 \rightarrow -\frac{1}{3}, \quad \text{as } t \rightarrow \infty.$$

Assume  $\alpha < 1$ . Then, by Corollary 3.9, we have  $p/\rho c^2 \rightarrow -1/3$ , as  $t \rightarrow \infty$ . Further, as  $a^{-2} t^2 \rightarrow \infty$  for  $t \rightarrow \infty$ , by (40) it follows  $\dot{\rho}/\rho \rightarrow -2H$ , as  $t \rightarrow \infty$ . Then by (25) it follows  $\rho(t) = L_1 t^{-2\alpha}$  for some  $L_1 \in \mathcal{N}$ . By (25), Friedman equation (2), some constants  $c_1, c_2$  and  $L_2 \in \mathcal{N}, L_2 = L^{-2}$ , it follows

$$(\alpha + \varepsilon)^2 = (c_1 L_1 + c_2 L_2) t^{2(1-\alpha)}. \quad (41)$$

Since  $2(1 - \alpha) > 0$  and  $L_1, L_2$  are slowly varying, it follows

$$L_1 t^{2(1-\alpha)}, L_2 t^{2(1-\alpha)} \rightarrow \infty \quad \text{as } t \rightarrow \infty,$$

contradicting the identity (41), as  $(\alpha + \varepsilon)^2 \rightarrow \alpha^2$  for  $t \rightarrow \infty$ . Thus,  $\alpha \geq 1$ .

Suppose  $\alpha > 1$  and  $t \rightarrow \infty$ . Then for the second fundamental solution  $u_\beta$  we would have  $\beta < 1$  as  $\alpha + \beta = 1$ . But this solution is impossible as it was just proved in the previous case  $\alpha < 1$ .

Therefore  $\alpha = 1$ .  $\square$

Hence, if the power law is assumed for the scale factor  $a(t)$  for an open universe, then  $a(t) \sim a_0 t$  as  $t \rightarrow \infty$ . Also,  $\Gamma = 0$  since  $\Gamma = \alpha(1 - \alpha)$ . Our analysis in this subsection leads to the following conclusions.

**1° The values of the equation of state parameter  $w$ .** Let us discuss the values of the parameter  $w$  excluded by (37). In the following we shall use the relation (35). We see that  $w = -1$  leads to the singularities in (31). Also,  $w_\alpha = -1$  if and only if  $w_\beta = -1$  and in this case there is no corresponding  $\Gamma$ . This case corresponds to cosmic inflation. The value  $w = -\frac{1}{3}$  yields a kind of singularity in (31), while the relation (35) is inconsistent. In this case  $w_\beta$  does not exist and the symmetry between  $w_\alpha$  and  $w_\beta$  is broken. Also  $\Gamma = 0$  and the corresponding  $w = -\frac{1}{3}$  appears in the solution for the open universe. If  $w = \frac{1}{3}$ , then  $\Gamma = \frac{1}{4}$ ,  $w_\alpha = w_\beta$  and  $\alpha = \beta$ . This case will be analyzed in the next subsection. Finally, let us consider the cases  $w = 0, 1$ , the values that appear as limits in (38). If  $w = 0$ , then  $\alpha = \frac{2}{3}$  and from the definition of  $w$  we see that  $p = 0$  (the matter dominated universe). If  $w = 1$ , then  $\alpha = \frac{1}{3}$  and this value of  $w$  corresponds to the universe with the mixture of dust and radiation. This is possible only if the integration constant  $c_2$  in (20) of the dominant fundamental solution is equal to 0.

**2°** By discussion in this subsection and the remarks in **1°** we arrive to the following conclusion: *For the spatially flat universe the assumption that  $\mu(t)$  is constant at  $\infty$  leads to the classic solution of the Friedmann equation.*

**3°** For the sake of completeness we give a rather short derivation of solution for the spatially flat universe assuming the equation of state  $p = wc^2 \rho$ ,  $w$  is nonsingular. Under this assumption the acceleration equation is reduced to  $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1 + 3w)\rho$ , while the Friedmann equation becomes  $(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho$ . Dividing the acceleration equation by the Friedmann equation, we obtain  $\frac{\ddot{a}}{a} = \lambda \frac{\dot{a}^2}{a^2}$  where  $\lambda = -\frac{1+3w}{2}$ . Hence  $\frac{d\dot{a}}{a} = \lambda \frac{da}{a}$ , i.e.  $\log \dot{a} = \log(c_0 a^\lambda)$  and so  $\frac{a^{1-\lambda}}{1-\lambda} = c_0 t + c_1$ , where  $c_0, c_1$  are the integration constants. Taking  $a(0) = 0$ , we find  $a(t) = a_0 t^{\frac{1}{1-\lambda}} = a_0 t^{\frac{2}{3(1+w)}}$  for some constant  $a_0$ .

**4° Generalized power law principle.** Putting together all results presented up to now, we see that the following are equivalent:

a. The integral limit  $\Gamma$  in (19) exists and  $\Gamma < \frac{1}{4}$ .

b. The solutions  $a(t)$  of the Friedmann equation satisfy the generalized power law with index  $\alpha \neq 1/2$ .

c. The equation of state holds at  $\infty$  as described by Corollary 3.7,  $w \neq -1, \frac{1}{3}$ . If  $\mu(t)$  is constant at  $\infty$  then  $p \sim wc^2\rho$ , as  $t \rightarrow \infty$ .

**5° Power law principle and cosmological constant  $\Lambda$ .** If  $\Lambda \neq 0$  is assumed, all the asymptotic formulas (31) for the cosmological parameters are valid, except for  $w = -1$ . This follows from the fact that by the appropriate substitutions the Friedmann equations with the parameter  $\Lambda$  transform to their basic form (2)–(4).

3.4. Case  $\Gamma = \frac{1}{4}$  adjacent case

In this subsection we shall assume  $\Gamma = \frac{1}{4}$  in the limit (19). Then the polynomial  $x^2 - x - \frac{1}{4}$  has the double root  $\alpha = \frac{1}{2}$ . In discussion of the acceleration equation (17) for this case we shall use the following criterion, see Marić [13, p. 37] and Kusano and Marić [11, Theorem 2.2]:

**Theorem 3.11.** Let  $\phi(x) = x \int_x^\infty \frac{\mu(t)}{t^2} dt - \frac{1}{4}$ , let the integral

$$\psi(x) = \int_x^\infty \frac{|\phi(t)|}{t} dt, \quad x > x_0 > 0, \text{ converge} \tag{42}$$

and assume

$$\int_x^\infty \frac{\psi(t)}{t} dt < \infty, \quad x > x_0. \tag{43}$$

Further, let  $L_1, L_2$  denote two normalized slowly varying functions. Then there exist two fundamental solutions of the acceleration equation (17):

$$u(t) = t^{\frac{1}{2}}L_1(t) \quad v(t) = t^{\frac{1}{2}}\log(t)L_2(t) \tag{44}$$

if and only if the condition (19) holds (for  $\Gamma = \frac{1}{4}$ ). Also  $L_1, L_2$  tend to a constant as  $t \rightarrow \infty$  and  $L_2(t) \sim 1/L_1(t)$ .  $\square$

As  $\log(t)L_2(t)$  is also a slowly varying, we see that both fundamental solutions  $u(t)$  and  $v(t)$  satisfy the general form of power law. Hence, each solution  $a(t) = c_1u(t) + c_2v(t)$  of the acceleration equation is regularly varying of index  $\frac{1}{2}$ . By the results in the previous subsection when  $\Gamma < \frac{1}{4}$  was assumed, we see, if the conditions (42) and (43) are satisfied, that  $a(t)$  is regularly varying of index  $\frac{1}{2}$  if and only if  $w \sim \frac{1}{3}$  as  $t \rightarrow \infty$ , i.e.  $p \sim \frac{1}{3}c^2\rho$  holds asymptotically. This is the second classic cosmological solution.

3.5. Asymptotic solution for spatially flat universe with matter-dominated evolution

We have seen that the constant  $\Gamma = \mathbf{M}(\mu)$  determine the asymptotical behavior at the  $\infty$  of the scale factor  $a(t)$ . If the matter-dominated evolution of the universe is assumed, i.e. dominated by some form of pressureless material after the certain time moment  $t_0$  then it appears that the expression  $H(t)t$  depends solely on the parameter  $\Omega$ . In this case we are able to estimate possible values of  $\Gamma$ . We shall discuss also the status of the constant  $\Gamma$  and the related asymptotic behavior of  $a(t)$  for the spatially flat universe including the cosmological constant  $\Lambda$ . Therefore, in this section we discuss asymptotic solutions and Friedmann equations, the related parameter  $\mu(t)$ , and the constant  $\Gamma$  assuming the pressureless spatially flat universe with the cosmological constant  $\Lambda$ .

Using the formula for the age of the spatially flat universe with the cosmological constant  $\Lambda$  Carroll et al. [3], see also Liddle and Lyth [9] and Narlikar [16], the expression  $H(t)t$  in this case is given by

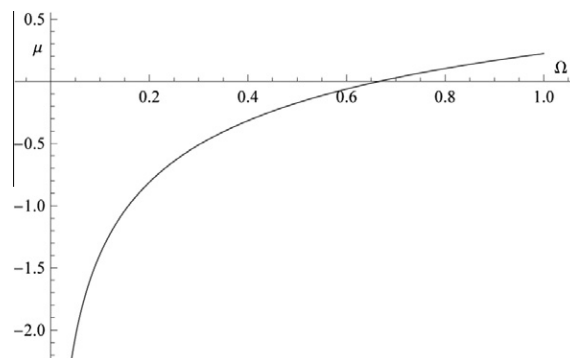


Fig. 1. Graph of  $\mu(\Omega)$ .

$$H(t)t = \frac{2}{3} \cdot \frac{1}{\sqrt{1-\Omega}} \ln \left( \frac{1 + \sqrt{1-\Omega}}{\sqrt{\Omega}} \right), \tag{45}$$

while the deceleration parameter  $q(t)$  is given by

$$q(t) = \frac{\Omega}{2} - \Omega_\Lambda. \tag{46}$$

In the model of the spatially flat universe we have  $\Omega + \Omega_\Lambda = 1$  hence,  $q(t) = \frac{3\Omega}{2} - 1$ . Therefore, by (23) and (45) it follows

$$\mu(t) = \frac{2}{9} \cdot \frac{3\Omega - 2}{1 - \Omega} \left( \ln \left( \frac{1 + \sqrt{1-\Omega}}{\sqrt{\Omega}} \right) \right)^2, \tag{47}$$

where  $\Omega = \Omega(t)$ . We see that the parameter  $\mu(t)$  in the model for the pressureless spatially flat universe depends solely on  $\Omega$ . The graph of the parameter  $\mu(t)$  is presented in Fig. 1, as a function of  $\Omega$ .

The limit value

$$\Omega_\infty = \lim_{t \rightarrow \infty} \Omega(t) \tag{48}$$

can be in principle any value in the interval  $[0, 1]$ . Let us introduce the parameter  $\bar{\mu}(\Omega)$  by the expression on the right hand side of (47). Hence  $\mu(t) = \bar{\mu}(\Omega(t))$  and

$$\Gamma = \lim_{t \rightarrow \infty} \mu(t) = \lim_{\Omega \rightarrow \Omega_\infty} \bar{\mu}(\Omega). \tag{49}$$

We see that  $\bar{\mu}(\Omega)$  is an increasing function in  $\Omega$  and that its values lay in the interval  $[-\infty, \frac{2}{9}]$ , as  $\lim_{\Omega \rightarrow 1-0} \bar{\mu}(\Omega) = \frac{2}{9}$ . Hence  $\mu(t) < \frac{2}{9}$ . Suppose the limit  $\Gamma = \lim_{t \rightarrow \infty} \mu(t)$  exists. Then  $\Gamma \leq \frac{2}{9} < \frac{1}{4}$ . Thus, assuming the pressureless spatially flat universe with the cosmological constant, all possible values of  $\Gamma$  are less than  $2/9$ , hence Theorem 3.2 can be applied. This analysis leads to the following conclusions for the solutions of Friedmann equations for the pressureless spatially flat universe with the cosmological constant  $\Lambda$ .

- 1° The scale factor  $a(t)$  satisfies the generalized power law. More specifically,  $a(t) = t^\alpha L(t)$  where  $L \in \mathcal{N}$  and  $\alpha$  is a root of the polynomial  $x^2 - x + \Gamma$ .
- 2° Suppose  $\Omega_\infty = 1$ . By the identity  $\Omega + \Omega_\Lambda = 1$  it follows  $\Omega_\Lambda \sim 0$  as  $t \rightarrow \infty$  and  $\Gamma = \frac{2}{9}$ . Then Eq. (14) becomes  $x^2 - x + \frac{2}{9} = 0$  and it has the solutions  $\alpha_1 = \frac{1}{3}$ ,  $\alpha_2 = \frac{2}{3}$ . According to the conclusion 1°,  $a(t)$  regularly varying of index  $\frac{2}{3}$  and by (45) and (46),  $H(t)t \sim \frac{2}{3}$  and  $q \sim \frac{1}{2}$  as  $t \rightarrow \infty$ . This result corresponds to the classic solution of the Friedmann equation for the pressureless spatially flat universe with the cosmological constant  $\Lambda = 0$ .
- 3° The formula (47) for  $\bar{\mu}$  shows that the evolution of the expansion scale factor  $a(t)$  depends only on the evolution of the density parameter  $\Omega$ . The nature of this evolution is determined by the constant  $\Gamma$  but in all instances it satisfies the power law represented by some regularly varying function. The introduction of the cosmological constant only changes the index of the regular variation with respect to the model with  $\Lambda = 0$ .
- 4° Let us consider the possible values of  $\Gamma$ . The value  $\Omega_0 = 0.3$  (e.g. [9]) for the present epoch is close to the value preferred by the observation. If we assume that the energy density  $\rho$  becomes lower as the age of the universe becomes

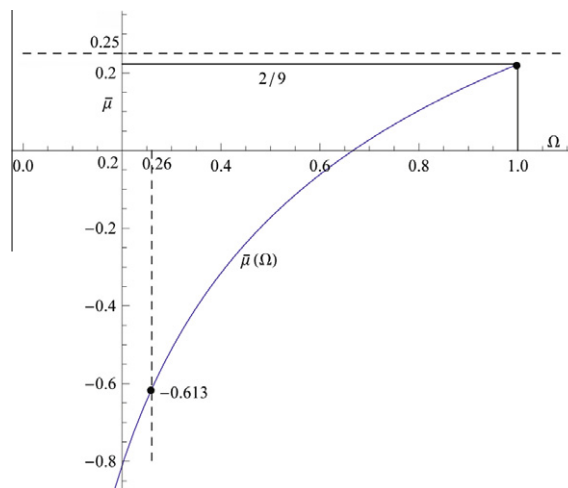


Fig. 2. Graph of  $\bar{\mu}(\Omega)$  in the preferred interval.

older, we may suppose that the possible range for the constant  $\Omega_\infty$  is the interval  $[0.3, 1]$ , i.e.  $0.3 \leq \Omega_\infty \leq 1$ . The graph of  $\bar{\mu}$  for this interval is presented in Fig. 2. We see that  $\Gamma \leq 2/9$ .

5° The solution  $a(t)$  is regularly varying of some index  $\alpha$ , i.e.  $a(xt)/a(t) \rightarrow x^\alpha$  as  $t \rightarrow \infty$ ,  $x > 0$ . So for relatively large<sup>4</sup>  $t_0$  with respect to  $x > 0$ , we have  $a(xt_0) \sim a(t_0)x^\alpha$  and we may take that the time instances  $t_0$  and  $xt_0$  belong to the same epoch in the evolution of the universe. So, taking  $t = xt_0$  and eliminating  $x$  from the last asymptotic relation, we find the asymptotic estimation for  $a(t)$  for an epoch with respect to the initial value  $a(t_0)$ :

$$a(t) \sim a(t_0) \left( \frac{t}{t_0} \right)^\alpha. \tag{50}$$

Also,  $a(t) = t^\alpha L(t)$  where  $L(t)$  is slowly varying. Hence  $L(t) \sim L(t_0)$  for an epoch, so it is hard to measure  $L(t_0)$  and the influence of  $L(t)$  on  $a(t)$ . However, the influence of  $L(t)$  on the large scale might be substantial, particularly if  $\alpha \approx 0$ , as the example (10) shows.

### 3.6. Case $\Gamma > \frac{1}{4}$

Assume  $\Gamma > \frac{1}{4}$  in the limit (19). Then the solution  $a(t)$  of the acceleration equation is oscillatory. This immediately follows from Hille’s classical theorem (see Hille [6] and Marić [13, Theorem 1.8]). Therefore, for these values of  $\Gamma$  the expansion scale factor  $a(t)$  of the universe does not satisfy the power law and the approach presented in the paper is not appropriate for this case. Since  $a(t)$  has in this case (infinitely many) zeros, then there is  $t_0$  such that  $\dot{a}(t_0) = 0$ . So, from the Friedmann equation (2) it follows that  $k > 0$ , i.e. the universe must be closed. This is obviously true even if the Friedmann equation is modified by adding the cosmological constant  $\Lambda > 0$ .

We see that the constant  $1/4$  plays an important role as a possible value of  $\Gamma$  in the limit (19). This constant provides a sharp “threshold,” or “cut-off point,” at which the oscillation of  $a(t)$  takes place.

## 4. Conclusion

It has been shown that a dimensionless constant  $\Gamma$  related to the Friedmann acceleration equation and the theory of regularly varying functions play the key role in the formulation of the power law principle for solutions  $a(t)$  of the Friedmann equations. The constant  $\Gamma$  is defined by

$$\Gamma = \lim_{x \rightarrow \infty} x \int_x^\infty \frac{\mu(t)}{t^2} dt, \tag{51}$$

where  $\mu(t) = q(t)(H(t)t)^2$  as  $t \rightarrow \infty$ ,  $q(t)$  is the deceleration parameter and  $H(t)$  is the Hubble parameter. We have shown that the generalized power law principle for the scale factor  $a(t)$  holds if and only if the integral limit (51) exists and  $\Gamma < \frac{1}{4}$ . The cosmological constants were also discussed under relaxed condition  $\lim_{t \rightarrow \infty} \mu(t) = \Gamma$  which implies (51). Under this condition we have shown that the power law principle is equivalent to the equation of state  $p \sim w c^2 p$ ,  $w \neq -1, 1/3$ . The values of  $\Gamma$  determine the asymptotical behavior of the scale factor  $a(t)$  as time  $t$  tends to  $\infty$ . The constant  $\Gamma$  also uniquely determines other cosmological parameters such as the Hubble parameter and the equation of state parameter  $w$ . Particularly is discussed the pressureless spatially flat universe with non-zero cosmological constant. Further, the value of  $\Gamma$  determines the type of the universe; for  $\Gamma < \frac{1}{4}$  the universe is spatially flat or open, while for  $\Gamma > \frac{1}{4}$  the universe is oscillatory. The boundary case  $\Gamma = \frac{1}{4}$  is also analyzed. All solutions we found are in agreement with the results found widely in the literature on standard cosmological model. As power law functions are the most frequently occurring type of the solutions of the Friedmann equation, the study of the constant  $\Gamma$  and the related function  $\mu(t)$  might be of a particular interest.

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<sup>4</sup> This notion can be made more precise by use of *nonstandard analysis*. There we can say that  $t_0$  is infinite, while  $x > 0$  is non-infinitesimal finite real number. For development and notions of nonstandard analysis one can see Stroyan and Luxemburg [19]. There are a lot of applications of nonstandard analysis in the theoretical physics, e.g. Albeverio et al. [1].

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