

A.A. Logunov

HENRI POINCARÉ  
AND  
RELATIVITY THEORY

*Translated by G. Pontecorvo and V.O. Soloviev  
edited by V.A. Petrov*

**Logunov A. A.**

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The book presents ideas by H. Poincaré and H. Minkowski according to those the essence and the main content of the relativity theory are the following: the space and time form unique four-dimensional continuum supplied by the pseudo-Euclidean geometry. All physical processes take place just in this four-dimensional space. Comments to works and quotations related to this subject by L. de Broglie, P. A. M. Dirac, A. Einstein, V. L. Ginzburg, S. Goldberg, P. Langevin, H. A. Lorentz, L. I. Mandel'stam, H. Minkowski, A. Pais, W. Pauli, M. Planck, A. Sommerfeld and H. Weyl are given in the book. It is also shown that the special theory of relativity has been created not by A. Einstein only but even to greater extent by H. Poincaré.

The book is designed for scientific workers, post-graduates and upper-year students majoring in theoretical physics.

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*Devoted to 150th birthday of Henri Poincaré —  
the greatest mathematician, mechanist,  
theoretical physicist*

## **Preface**

**The special theory of relativity “resulted from the joint efforts of a group of great researchers: Lorentz, Poincaré, Einstein, Minkowski” (Max Born).**

**“Both Einstein and Poincaré, took their stand on the preparatory work of H. A. Lorentz, who had already come quite close to the result, without however quite reaching it. In the agreement between the results of the methods followed independently of each other by Einstein and Poincaré I discern a deeper significance of a harmony between the mathematical method and analysis by means of conceptual experiments (*Gedankenexperimente*), which rests on general features of physical experience” (W. Pauli, 1955).**

H. Poincaré, being based upon the relativity principle formulated by him for all physical phenomena and upon the Lorentz work, has discovered and formulated everything that composes the essence of the special theory of relativity. A. Einstein was coming to the theory of relativity from the side of relativity principle formulated earlier by H. Poincaré. At that he relied upon ideas by H. Poincaré on definition of the simultaneity of events occurring in different spatial points by means of the light signal. Just for this reason he introduced an additional postulate — the constancy of the velocity of light. This book presents a comparison of the article by A. Einstein of 1905 with the articles by H. Poincaré and clarifies what is the **new** content contributed by each of them. Somewhat later H. Minkowski further developed Poincaré’s approach. Since

Poincaré's approach was more general and profound, our presentation will precisely follow Poincaré.

According to Poincaré and Minkowski, the essence of relativity theory consists in the following: **the special theory of relativity is the pseudo-Euclidean geometry of space-time. All physical processes take place just in such a space-time.** The consequences of this postulate are energy-momentum and angular momentum conservation laws, the existence of inertial reference systems, the relativity principle for all physical phenomena, Lorentz transformations, the constancy of the velocity of light in Galilean coordinates of the inertial frame, the retardation of time, the Lorentz contraction, the possibility to exploit non-inertial reference systems, the clock paradox, the Thomas precession, the Sagnac effect, and so on. Series of fundamental consequences have been obtained on the base of this postulate and the quantum notions, and the quantum field theory has been constructed. The preservation (form-invariance) of physical equations in all inertial reference systems means that all **physical processes** taking place in these systems under the same conditions are **identical**. Just for this reason all **natural standards** are **the same** in all inertial reference systems.

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*A.A. Logunov*  
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## 1. Euclidean geometry

In the third century BC **Euclid published a treatise on mathematics, the “Elements”**, in which he summed up the preceding development of **mathematics in antique Greece**. It was precisely in this work that the geometry of our three-dimensional space — Euclidean geometry — was formulated.

This happened to be a most important step in the development of both mathematics and physics. The point is that geometry originated from observational data and practical experience, i. e. it arose via the study of Nature. But, since all natural phenomena take place in space and time, the importance of geometry for physics cannot be overestimated, and, moreover, geometry is actually a part of physics.

**In the modern language of mathematics the essence of Euclidean geometry is determined by the Pythagorean theorem.** In accordance with the Pythagorean theorem, the distance of a point with Cartesian coordinates  $x, y, z$  from the origin of the reference system is determined by the formula

$$\ell^2 = x^2 + y^2 + z^2, \quad (1.1)$$

or in differential form, the distance between two infinitesimally close points is

$$(d\ell)^2 = (dx)^2 + (dy)^2 + (dz)^2. \quad (1.2)$$

Here  $dx, dy, dz$  are differentials of the Cartesian coordinates. Usually, the proof of the Pythagorean theorem is based on Euclid’s axioms, but it turns out to be that it can actually be considered a definition of Euclidean geometry. Three-dimensional space, determined by Euclidean geometry, possesses the properties of homogeneity and isotropy. This means that there exist no singular

points or singular directions in Euclidean geometry. By performing transformations of coordinates from one Cartesian reference system,  $x, y, z$ , to another,  $x', y', z'$ , we obtain

$$\ell^2 = x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2. \quad (1.3)$$

This means that the square distance  $\ell^2$  is an invariant, while its projections onto the coordinate axes are not. We especially note this obvious circumstance, since it will further be seen that such a situation also takes place in four-dimensional space-time, so, consequently, depending on the choice of reference system in space-time the projections onto spatial and time axes will be relative. Hence arises the relativity of time and length. But this issue will be dealt with later.

Euclidean geometry became a composite part of Newtonian mechanics. For about two thousand years Euclidean geometry was thought to be the unique and unchangeable geometry, in spite of the rapid development of mathematics, mechanics, and physics.

**It was only at the beginning of the 19-th century that the Russian mathematician Nikolai Ivanovich Lobachevsky made the revolutionary step — a new geometry was constructed — the Lobachevsky geometry. Somewhat later it was discovered by the Hungarian mathematician Bolyai.**

About 25 years later Riemannian geometries were developed by the German mathematician Riemann. Numerous geometrical constructions arose. As new geometries came into being the issue of the geometry of our space was raised. What kind was it? Euclidean or non-Euclidean?

## 2. Classical Newtonian mechanics

All natural phenomena proceed in space and time. Precisely for this reason, in formulating the laws of mechanics in the 17-th century, Isaac Newton first of all defined these concepts:

*“Absolute Space, in its own nature, without regard to any thing external, remains always similar and immovable”.*

*“Absolute, True, and Mathematical Time, of itself, and from its own nature flows equably without regard to any thing external, and by another name is called Duration”.*

As the geometry of three-dimensional space Newton actually applied Euclidean geometry, and he chose a Cartesian reference system with its origin at the center of the Sun, while its three axes were directed toward distant stars. Newton considered precisely such a reference system to be “motionless”. The introduction of absolute motionless space and of absolute time turned out to be extremely fruitful at the time.

The first law of mechanics, or the law of inertia, was formulated by Newton as follows:

*“Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon”.*

The law of inertia was first discovered by Galileo. If, in motionless space, one defines a Cartesian reference system, then, in accordance with the law of inertia, a solitary body will move along a trajectory determined by the following equations:

$$x = v_x t, \quad y = v_y t, \quad z = v_z t. \quad (2.1)$$

Here,  $v_x, v_y, v_z$  are the constant velocity projections, their values may, also, be equal to zero.

In the book “Science and Hypothesis” H. Poincaré formulated the following general principle:

*“The acceleration of a body depends only on its position and that of neighbouring bodies, and on their velocities. Mathematicians would say that the movements of all the material molecules of the universe depend on differential equations of the second order. To make it clear that this is really a generalisation of the law of inertia we may again have recourse to our imagination. The law of inertia, as I have said above, is not imposed on us a priori; other laws would be just as compatible with the principle of sufficient reason. If a body is not acted upon by a force, instead of supposing that its velocity is unchanged we may suppose that its position or its acceleration is unchanged. Let us for moment suppose that one of these two laws is a law of nature, and substitute it for the law of inertia: what will be the natural generalisation? moment’s reflection will show us. In the first case, we may suppose that the velocity of a body depends only on its position and that of neighbouring bodies; in the second case, that the variation of the acceleration of a body depends only on the position of the body and of neighbouring bodies, on their velocities and accelerations; or, in mathematical terms, the differential equations of the motion would be of the first order in the first case and of the third order in the second”.*

Newton formulated the second law of mechanics as follows:

*“The alteration of motion is ever proportional to the motive force impressed; and is made in the di-*

*rection of the right line in which that force is impressed”.*

And, finally, the Newton’s third law of mechanics:

*“To every Action there is always opposed an equal Reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts”.*

On the basis of these laws of mechanics, in the case of central forces, the equations for a system of two particles in a reference system “at rest” are:

$$\begin{aligned} M_1 \frac{d^2 \vec{r}_1}{dt^2} &= F(|\vec{r}_2 - \vec{r}_1|) \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}, \\ M_2 \frac{d^2 \vec{r}_2}{dt^2} &= -F(|\vec{r}_2 - \vec{r}_1|) \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}. \end{aligned} \quad (2.2)$$

Here  $M_1$  and  $M_2$  are the respective masses of the first and second particles,  $\vec{r}_1$  is the vector radius of the first particle,  $\vec{r}_2$  is the vector radius of the second particle. The function  $F$  reflects the character of the forces acting between bodies.

In Newtonian mechanics, mostly forces of two types are considered: of gravity and of elasticity.

For the forces of Newtonian gravity

$$F(|\vec{r}_2 - \vec{r}_1|) = G \frac{M_1 M_2}{|\vec{r}_2 - \vec{r}_1|^2}, \quad (2.3)$$

$G$  is the gravitational constant.

For elasticity forces Hooke’s law is

$$F(|\vec{r}_2 - \vec{r}_1|) = k|\vec{r}_2 - \vec{r}_1|, \quad (2.4)$$

$k$  is the elasticity coefficient.

Newton's equations are written in vector form, and, consequently, they are independent of the choice of three-dimensional reference system. From equations (2.2) it is seen that the momentum of a closed system is conserved.

As it was earlier noted, Newton considered equations (2.2) to hold valid only in a reference system at rest. But, if one takes a reference system moving with respect to the one at rest with a constant velocity  $\vec{v}$

$$\vec{r}' = \vec{r} - \vec{v}t, \quad (2.5)$$

it turns out that equations (2.2) are not altered, i. e. **they remain form-invariant, and this means that no mechanical phenomena could permit to ascertain whether we are in a state of rest or of uniform and rectilinear motion. This is the essence of the relativity principle first discovered by Galileo. The transformations (2.5) have been termed Galilean.**

Since the velocity  $\vec{v}$  in (2.5) is arbitrary, there exists an infinite number of reference systems, in which the equations retain their form. This means, that in each reference system the law of inertia holds valid. If in any one of these reference systems a body is in a state of rest or in a state of uniform and rectilinear motion, then in any other reference system, related to the first by transformation (2.5), it will also be either in a state of uniform rectilinear motion or in a state of rest.

**All such reference systems have been termed inertial. The principle of relativity consists in conservation of the form of the equations of mechanics in any inertial reference system.** We are to emphasize that **in the base of definition of an inertial reference system lies the law of inertia by Galileo.** According to it in the absence of forces a body motion is described by linear functions of time.

But how has an inertial reference system to be defined? Newtonian mechanics gave no answer to this question. Nevertheless, the

reference system chosen as such an inertial system had its origin at the center of the Sun, while the three axes were directed toward distant stars.

In classical Newtonian mechanics time is independent of the choice of reference system, in other words, three-dimensional space and time are separated, they do not form a unique four-dimensional continuum.

Isaac Newton's ideas concerning absolute space and absolute motion were criticized in the 19-th century by Ernst Mach. Mach wrote:

*“No one is competent to predicate things about absolute space and absolute motion; they are pure things of thought, pure mental constructs, that cannot be produced in experience”.*

And further:

*“Instead, now, of referring a moving body  $K$  to space (that is to say to a system of coordinates) let us view directly its relation to the bodies of the universe, by which alone a system of coordinates can be determined.*

*... even in the simplest case, in which apparently we deal with the mutual action of only **two** masses, the neglecting of the rest of the world is **impossible**. ... If a body rotates with respect to the sky of motionless stars, then there arise centrifugal forces, while if it rotates around **another** body, instead of the sky of motionless stars, no centrifugal forces will arise. I have nothing against calling the first revolution **absolute**, if only one does not forget that this signifies nothing but revolution **relative** to the sky of motionless stars”.*

Therefore Mach wrote:

*“... there is no necessity for relating the Law of inertia to some special absolute space”.*

All this is correct, since Newton did not define the relation of an inertial reference system to the distribution of matter, and, actually, it was quite impossible, given the level of physics development at the time. By the way, Mach also did not meet with success. But his criticism was useful, it drew the attention of scientists to the analysis of the main concepts of physics.

Since we shall further deal with field concepts, it will be useful to consider the methods of analytical mechanics developed during the 18-th and 19-th centuries. Their main goal, set at the time, consisted in finding the most general formulation for classical mechanics. Such research turned out to be extremely important, since it gave rise to methods that were later quite readily generalized to systems with an infinite number of degrees of freedom. Precisely in this way was a serious theoretical start created, that was successfully used of in the 19-th and 20-th centuries.

In his “Analytic Mechanics”, published in 1788, Joseph Lagrange obtained his famous equations. Below we shall present their derivation. In an inertial reference system, Newton’s equations for a set of  $N$  material points moving in a potential field  $U$  have the form

$$m_\sigma \frac{d\vec{v}_\sigma}{dt} = -\frac{\partial U}{\partial \vec{r}_\sigma}, \quad \sigma = 1, 2, \dots, N. \quad (2.6)$$

In our case the force  $\vec{f}_\sigma$  is

$$\vec{f}_\sigma = -\frac{\partial U}{\partial \vec{r}_\sigma}. \quad (2.7)$$

To determine the state of a mechanical system at any moment of time it is necessary to give the coordinates and velocities of all

the material points at a certain moment of time. Thus, the state of a mechanical system is fully determined by the coordinates and velocities of the material points. In a Cartesian reference system Eqs. (2.6) assume the form

$$m_\sigma \frac{dv_\sigma^1}{dt} = f_\sigma^1, \quad m_\sigma \frac{dv_\sigma^2}{dt} = f_\sigma^2, \quad m_\sigma \frac{dv_\sigma^3}{dt} = f_\sigma^3. \quad (2.8)$$

If one passes to another inertial reference system and makes use of coordinates other than Cartesian, then it is readily seen that the equations written in the new coordinates differ essentially in form from equations (2.8). Lagrange found for Newton's mechanics such a covariant formulation for the equations of motion that they retain their form, when transition is made to new variables.

Let us introduce, instead of coordinates  $\vec{r}_\sigma$ , new **generalized coordinates**  $q^\lambda$ ,  $\lambda = 1, 2, \dots, n$ , here  $n = 3N$ . Let us assume relations

$$\vec{r}_\sigma = \vec{r}_\sigma(q_1, \dots, q_n, t). \quad (2.9)$$

After scalar multiplication of each equation (2.6) by vector

$$\frac{\partial \vec{r}_\sigma}{\partial q_\lambda} \quad (2.10)$$

and performing addition we obtain

$$m_\sigma \frac{d\vec{v}_\sigma}{dt} \cdot \frac{\partial \vec{r}_\sigma}{\partial q_\lambda} = - \frac{\partial U}{\partial \vec{r}_\sigma} \cdot \frac{\partial \vec{r}_\sigma}{\partial q_\lambda}, \quad \lambda = 1, 2, \dots, n. \quad (2.11)$$

Here summation is performed over identical indices  $\sigma$ .

We write the left-hand part of equation (2.11) as

$$\frac{d}{dt} \left[ m_\sigma \vec{v}_\sigma \frac{\partial \vec{r}_\sigma}{\partial q_\lambda} \right] - m_\sigma \vec{v}_\sigma \frac{d}{dt} \left( \frac{\partial \vec{r}_\sigma}{\partial q_\lambda} \right). \quad (2.12)$$

Since

$$\vec{v}_\sigma = \frac{d\vec{r}_\sigma}{dt} = \frac{\partial \vec{r}_\sigma}{\partial q_\lambda} \dot{q}_\lambda + \frac{\partial \vec{r}_\sigma}{\partial t}, \quad (2.13)$$

hence, differentiating (2.13) with respect to  $\dot{q}_\lambda$  we obtain the equality

$$\frac{\partial \vec{r}_\sigma}{\partial q_\lambda} = \frac{\partial \vec{v}_\sigma}{\partial \dot{q}_\lambda}. \quad (2.14)$$

Differentiating (2.13) with respect to  $q_\nu$  we obtain

$$\frac{\partial \vec{v}_\sigma}{\partial q_\nu} = \frac{\partial^2 \vec{r}_\sigma}{\partial q_\nu \partial q_\lambda} \dot{q}_\lambda + \frac{\partial^2 \vec{r}_\sigma}{\partial t \partial q_\nu}. \quad (2.15)$$

But, on the other hand, we have

$$\frac{d}{dt} \left( \frac{\partial \vec{r}_\sigma}{\partial q_\nu} \right) = \frac{\partial^2 \vec{r}_\sigma}{\partial q_\nu \partial q_\lambda} \dot{q}_\lambda + \frac{\partial^2 \vec{r}_\sigma}{\partial t \partial q_\nu}. \quad (2.16)$$

Comparing (2.15) and (2.16) we find

$$\frac{d}{dt} \left( \frac{\partial \vec{r}_\sigma}{\partial q_\nu} \right) = \frac{\partial \vec{v}_\sigma}{\partial q_\nu}. \quad (2.17)$$

In formulae (2.13), (2.15) and (2.16) summation is performed over identical indices  $\lambda$ .

Making use of equalities (2.14) and (2.17) we represent expression (2.12) in the form

$$\frac{d}{dt} \left[ \frac{\partial}{\partial \dot{q}_\lambda} \left( \frac{m_\sigma v_\sigma^2}{2} \right) \right] - \frac{\partial}{\partial q_\lambda} \left( \frac{m_\sigma v_\sigma^2}{2} \right). \quad (2.18)$$

Since (2.18) is the left-hand part of equations (2.11) we obtain Lagrangian equations

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_\lambda} \right) - \frac{\partial T}{\partial q_\lambda} = - \frac{\partial U}{\partial q_\lambda}, \quad \lambda = 1, 2, \dots, n. \quad (2.19)$$

Here  $T$  is the kinetic energy of the system of material points

$$T = \frac{m_\sigma v_\sigma^2}{2}, \quad (2.20)$$

summation is performed over identical indices  $\sigma$ . If one introduces the Lagrangian function  $L$  as follows

$$L = T - U, \quad (2.21)$$

then the Lagrangian equations assume the form

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_\lambda} \right) - \frac{\partial L}{\partial q_\lambda} = 0, \quad \lambda = 1, 2, \dots, n. \quad (2.22)$$

The state of a mechanical system is fully determined by the generalized coordinates and velocities. The form of Lagrangian equations (2.22) is independent of the choice of **generalized coordinates**. Although these equations are totally equivalent to the set of equations (2.6), this form of the equations of classical mechanics, however, turns out to be extremely fruitful, since it opens up the possibility of its generalization to phenomena which lie far beyond the limits of classical mechanics.

The most general formulation of the law of motion of a mechanical system is given by the **principle of least action** (or the principle of stationary action). The action is composed as follows

$$S = \int_{t_1}^{t_2} L(q, \dot{q}) dt. \quad (2.23)$$

The integral (functional) (2.23) depends on the behaviour of functions  $q$  and  $\dot{q}$  within the given limits. Thus, these functions are functional arguments of the integral (2.23). The least action principle is written in the form

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}) dt = 0. \quad (2.24)$$

The equations of motion of mechanics are obtained from (2.24) by varying the integrand expression

$$\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = 0. \quad (2.25)$$

Here  $\delta q$  and  $\delta \dot{q}$  represent infinitesimal variations in the form of the functions. The variation commutes with differentiation, so

$$\delta \dot{q} = \frac{d}{dt}(\delta q). \quad (2.26)$$

Integrating by parts in the second term of (2.25) we obtain

$$\delta S = \frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \cdot \frac{\partial L}{\partial \dot{q}} \right) \delta q dt = 0. \quad (2.27)$$

Since the variations  $\delta q$  at points  $t_1$  and  $t_2$  are zero, expression (2.27) assumes the form

$$\delta S = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \cdot \frac{\partial L}{\partial \dot{q}} \right) \delta q dt = 0. \quad (2.28)$$

The variation  $\delta q$  is arbitrary within the interval of integration, so, by virtue of the main lemma of variational calculus, from here the **necessary condition for an extremum** follows in the form of the equality to zero of the variational derivative

$$\frac{\delta L}{\delta q} = \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0. \quad (2.29)$$

Such equations were obtained by Leonard Euler in the course of development of variational calculus. For our choice of function  $L$ ,

these equations in accordance with (2.21) coincide with the Lagrangian equations.

From the above consideration it is evident that mechanical motion satisfying the Lagrangian equations provides for extremum of the integral (2.23), and, consequently, the action has a stationary value.

The application of the Lagrangian function for describing a mechanical system with a finite number of degrees of freedom turned out to be fruitful, also, in describing a physical field possessing an infinite number of degrees of freedom. In the case of a field, the function  $\psi$  describing it depends not only on time, but also on the space coordinates. This means that, instead of the variables  $q_\sigma, \dot{q}_\sigma$  of a mechanical system, it is necessary to introduce the variables  $\psi(x^\nu), \frac{\partial\psi}{\partial x^\lambda}$ . Thus, the field is considered as a mechanical system with an infinite number of degrees of freedom. We shall see further (Sections 10 and 15) how the principle of stationary action is applied in electrodynamics and classical field theory.

The formulation of classical mechanics within the framework of Hamiltonian approach has become very important. Consider a certain quantity determined as follows

$$H = p_\sigma \dot{q}_\sigma - L, \quad (2.30)$$

and termed the Hamiltonian. In (2.30) summation is performed over identical indices  $\sigma$ . We define the **generalized momentum** as follows:

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}. \quad (2.31)$$

Find the differential of expression (2.30)

$$dH = p_\sigma dq_\sigma + \dot{q}_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} d\dot{q}_\sigma - \frac{\partial L}{\partial t} dt. \quad (2.32)$$

Making use of (2.31) we obtain

$$dH = \dot{q}_\sigma dp_\sigma - \frac{\partial L}{\partial q_\sigma} dq_\sigma - \frac{\partial L}{\partial t} dt. \quad (2.33)$$

On the other hand,  $H$  is a function of the independent variables  $q_\sigma$ ,  $p_\sigma$  and  $t$ , and therefore

$$dH = \frac{\partial H}{\partial q_\sigma} dq_\sigma + \frac{\partial H}{\partial p_\sigma} dp_\sigma + \frac{\partial H}{\partial t} dt. \quad (2.34)$$

Comparing (2.33) and (2.34) we obtain

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}, \quad \frac{\partial L}{\partial q_\sigma} = -\frac{\partial H}{\partial q_\sigma}, \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}. \quad (2.35)$$

These relations were obtained by transition from independent variables  $q_\sigma$ ,  $\dot{q}_\sigma$  and  $t$  to independent variables  $q_\sigma$ ,  $p_\sigma$  and  $t$ .

Now, we take into account the Lagrangian equations (2.22) in relations (2.35) and obtain the Hamiltonian equations

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}, \quad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}. \quad (2.36)$$

When the Hamiltonian  $H$  does not depend explicitly on time,

$$\frac{\partial H}{\partial t} = 0, \quad (2.37)$$

we have

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} \dot{p}_\sigma. \quad (2.38)$$

Taking into account equations (2.36) in the above expression, we obtain

$$\frac{dH}{dt} = 0; \quad (2.39)$$

this means that the Hamiltonian remains constant during the motion.

We have obtained the Hamiltonian equations (2.36) making use of the Lagrangian equations. But they can be found also directly with the aid of the least action principle (2.24), if, as  $L$ , we take, in accordance with (2.30), the expression

$$L = p_\sigma \dot{q}_\sigma - H,$$

$$\begin{aligned} \delta S = & \int_{t_1}^{t_2} \delta p_\sigma \left( dq_\sigma - \frac{\partial H}{\partial p_\sigma} dt \right) - \\ & - \int_{t_1}^{t_2} \delta q_\sigma \left( dp_\sigma + \frac{\partial H}{\partial q_\sigma} dt \right) + p_\sigma \delta q_\sigma \Big|_{t_1}^{t_2} = 0. \end{aligned}$$

Since variations  $\delta q_\sigma$  at the points  $t_1$  and  $t_2$  are zero, while inside the interval of integration variations  $\delta q_\sigma$ ,  $\delta p_\sigma$  are arbitrary, then, by virtue of the main lemma of variational calculus, we obtain the Hamiltonian equations

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}, \quad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}.$$

If during the motion the value of a certain function remains constant

$$f(q, p, t) = \text{const}, \quad (2.40)$$

then it is called as integral of motion. Let us find the equation of motion for function  $f$ .

Now we take the total derivative with respect to time of expression (2.40):

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial f}{\partial p_\sigma} \dot{p}_\sigma = 0. \quad (2.41)$$

Substituting the Hamiltonian equations (2.36) into (2.41), we obtain

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_\sigma} \cdot \frac{\partial H}{\partial p_\sigma} - \frac{\partial f}{\partial p_\sigma} \cdot \frac{\partial H}{\partial q_\sigma} = 0. \quad (2.42)$$

The expression

$$(f, g) = \left| \begin{array}{cc} \frac{\partial f}{\partial q_\sigma} & \frac{\partial f}{\partial p_\sigma} \\ \frac{\partial g}{\partial q_\sigma} & \frac{\partial g}{\partial p_\sigma} \end{array} \right| = \frac{\partial f}{\partial q_\sigma} \cdot \frac{\partial g}{\partial p_\sigma} - \frac{\partial f}{\partial p_\sigma} \cdot \frac{\partial g}{\partial q_\sigma} \quad (2.43)$$

has been termed the Poisson bracket. In (2.43) summation is performed over the index  $\sigma$ .

On the basis of (2.43), Eq. (2.42) for function  $f$  can be written in the form

$$\frac{\partial f}{\partial t} + (f, H) = 0. \quad (2.44)$$

Poisson brackets have the following properties

$$\begin{aligned} (f, g) &= -(g, f), \\ (f_1 + f_2, g) &= (f_1, g) + (f_2, g), \\ (f_1 f_2, g) &= f_1 (f_2, g) + f_2 (f_1, g), \end{aligned} \quad (2.45)$$

$$(f, (g, h)) + (g, (h, f)) + (h, (f, g)) \equiv 0. \quad (2.46)$$

Relation (2.46) is called the **Jacobi identity**. On the basis of (2.43)

$$(f, q_\sigma) = -\frac{\partial f}{\partial p_\sigma}, \quad (f, p_\sigma) = \frac{\partial f}{\partial q_\sigma}. \quad (2.47)$$

Hence we find

$$(q_\lambda, q_\sigma) = 0, \quad (p_\lambda, p_\sigma) = 0, \quad (q_\lambda, p_\sigma) = \delta_{\lambda\sigma}. \quad (2.48)$$

In the course of development of the quantum mechanics, by analogy with the classical Poisson brackets (2.43), there originated quantum Poisson brackets, which also satisfy all the conditions (2.45), (2.46). The application of relations (2.48) for quantum Poisson brackets has permitted to establish the commutation relations between coordinates and momenta.

The discovery of the Lagrangian and Hamiltonian methods in classical mechanics permitted, at the time, to generalize and extend them to other physical phenomena. The search for various representations of the physical theory is always extremely important, since on their basis the possibility may arise of their generalization for describing new physical phenomena. Within the depths of the theory created there may be found formal sprouts of the future theory. The experience of classical and quantum mechanics bears witness to this assertion.

### 3. Electrodynamics. Space-time geometry

Following the discoveries made by Faraday in electromagnetism, Maxwell combined magnetic, electric and optical phenomena and, thus, completed the construction of electrodynamics by writing out his famous equations.

H. Poincaré in the book “The Value of Science“ wrote the following about Maxwell’s studies:

*“At the time, when Maxwell initiated his studies, the laws of electrodynamics adopted before him explained all known phenomena. He started his work not because some new experiment limited the importance of these laws. But, considering them from a new standpoint, Maxwell noticed that the equations became more symmetric, when a certain term was introduced into them, although, on the other hand, this term was too small to give rise to phenomena, that could be estimated by the previous methods.*

*A priori ideas of Maxwell are known to have waited for their experimental confirmation for twenty years; if you prefer another expression, — Maxwell anticipated the experiment by twenty years. How did he achieve such triumph?*

*This happened because Maxwell was always full of a sense of mathematical symmetry . . . ”*

According to Maxwell **there exist no currents, except closed currents**. He achieved this by introducing a small term — **a displacement current**, which resulted in the law of electric charge conservation following from the new equations.

In formulating the equations of electrodynamics, Maxwell applied the Euclidean geometry of three-dimensional space and absolute time, which is identical for all points of this space. Guided

by a profound sense of symmetry, he supplemented the equations of electrodynamics in such a way that, in the same time explaining available experimental facts, they were the equations of electromagnetic waves. He, naturally, did not suspect that the information on the geometry of space-time was concealed in the equations. But his supplement of the equations of electrodynamics turned out to be so indispensable and precise, that it clearly led H. Poincaré, who relied on the work of H. Lorentz, to the discovery of the pseudo-Euclidean geometry of space-time. Below, we shall briefly describe, how this came about.

In the same time we will show that the striking desire of some authors to prove that H. Poincaré “**has not made the decisive step**” to create the theory of relativity is based upon both misunderstanding of the essence of the theory of relativity and the shallow knowledge of Poincaré works. We will show this below in our comments to such statements. Just for this reason in this book I present results, first discovered and elucidated by the light of consciousness by H. Poincaré, minutely enough. Here the need to compare the content of A. Einstein’s work of 1905 both with results of publications [2, 3] by H. Poincaré, and with his earlier works naturally arises. After such a comparison it becomes clear what **new** each of them has produced.

**How could it be happened that the outstanding research of Twentieth Century — works [2,3] by H. Poincaré — were used in many ways at in the same time were industriously consigned to oblivion?** It is high time at least now, a hundred years later, to return everyone his property. It is also our duty.

Studies of the properties of the equations of electrodynamics revealed them not to retain their form under the Galilean transformations (2.5), i. e. not to be form-invariant with respect to Galilean transformations. Hence the conclusion follows that the Galilean relativity principle is violated, and, consequently, the ex-

perimental possibility arises to distinguish between one inertial reference system and another with the aid of electromagnetic or optical phenomena. However, various experiments performed, especially Michelson's experiments, showed that it is impossible to find out even by electromagnetic (optical) experiments, with a precision up to  $(v/c)^2$ , whether one is in a state of rest or of uniform and rectilinear motion. H. Lorentz found an explanation for the results of these experiments, as H. Poincaré noted, "**only by piling up hypotheses**".

In his book "Science and Hypothesis" (1902) H. Poincaré noted:

*"And now allow me to make a digression; I must explain why I do not believe, in spite of Lorentz, that more exact observations will ever make evident anything else but the relative displacements of material bodies. Experiments have been made that should have disclosed the terms of the first order; the results were nugatory. Could that have been by chance? No one has admitted this; general explanation was sought, and Lorentz found it. He showed that the terms of the first order should cancel each other, but not the terms of the second order. Then more exact experiments were made, which were also negative; neither could this be the result of chance. An explanation was necessary, and was forthcoming; they always are; hypotheses are what we lack the least. But this is not enough. Who is there who does not think that this leaves to chance far too important role? Would it not also be chance that this singular concurrence should cause certain circumstance to destroy the terms of the first order, and that totally different but very opportune circumstance should cause those of the second*

*order to vanish? No; the same explanation must be found for the two cases, and everything tends to show that this explanation would serve equity well for the terms of the higher order, and that the mutual destruction of these terms will be rigorous and absolute”.*

In 1904, on the basis of experimental facts, Henri Poincaré generalized the Galilean relativity principle to all natural phenomena. He wrote [1]:

*“The principle of relativity, according to which the laws of physical phenomena should be the same, whether to an observer fixed, or for an observer carried along in a uniform motion of translation, so that we have not and could not have any means of discovering whether or not we are carried along in such a motion”.*

Just this **principle has become the key one** for the subsequent development of both electrodynamics and the theory of relativity. It can be formulated as follows. **The principle of relativity is the preservation of form by all physical equations in any inertial reference system.**

But if this formulation uses the notion of the *inertial reference system* then it means that the physical law of inertia by Galilei is already incorporated into this formulation of the relativity principle. This is just the difference between this formulation and formulations given by Poincaré and Einstein.

Declaring this principle Poincaré precisely knew that one of its consequences was the impossibility of **absolute motion**, because **all inertial reference systems were equitable**. It follows from here that the principle of relativity by Poincaré does not require a denial of **ether** in general, it only deprives ether of relation to any system of reference. In other words, it removes the **ether** in Lorentz sense. Poincaré does not exclude the concept

of **ether** because it is difficult to imagine more absurd thing than empty space. Therefore the word **ether**, which can be found in the Poincaré articles even after his formulation of the relativity principle, has another meaning, different of the **Lorentz ether**. Just this **Poincaré's ether** has to satisfy the relativity principle. Also Einstein has come to the idea of ether in 1920.

In our time such a role is played by physical vacuum. Namely this point is up to now not understood by some physicists (we keep silence about philosophers and historians of science). So they erroneously attribute to Poincaré the interpretation of relativity principle as impossibility to register the translational uniform motion relative to ether. Though, as the reader can see, there is no the word "ether" in the formulation of the relativity principle.

One must distinguish between the **Galilean relativity principle** and **Galilean transformations**. While Poincaré extended the **Galilean relativity principle** to all physical phenomena **without altering its physical essence**, the **Galilean transformations** turned out to hold valid only when the velocities of bodies are small as compared to the velocity of light.

Applying this relativity principle to electrodynamical phenomena in ref.[3], H. Poincaré wrote:

*"This impossibility of revealing experimentally the Earth's motion seems to represent a general law of Nature; we naturally come to accept this law, which we shall term the **relativity postulate**, and to accept it without reservations. It is irrelevant, whether this postulate, that till now is consistent with experiments, will or will not later be confirmed by more precise measurements, at present, at any rate, it is interesting to see, what consequences can be deduced from it".*

In 1904, after the critical remarks made by Poincaré, H. Lorentz made a most important step by attempting again to write

electrodynamics equations in a moving reference system and showing that the **wave equation of electrodynamics** remained **unaltered** (form-invariant) under the following transformations of the coordinates and time:

$$X' = \gamma(X - vT), \quad T' = \gamma\left(T - \frac{v}{c^2}X\right), \quad Y' = Y, \quad Z' = Z, \quad (3.1)$$

Lorentz named  $T'$  as the **modified local time** in contrast to **local time**  $\tau = T'/\gamma$  introduced earlier in 1895;

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (3.2)$$

where  $c$  is the electrodynamic constant.

H. Poincaré termed these transformations the Lorentz transformations. The Lorentz transformations, as it is evident from (3.1), are related to two inertial reference systems. H. Lorentz did not establish the relativity principle for electromagnetic phenomena, since he did not succeed in demonstrating the form-invariance of all the Maxwell-Lorentz equations under these transformations.

From formulae (3.1) it follows that the wave equation being independent of translational uniform motion of the reference system is achieved only by changing the time. Hence, the conclusion arises, naturally, that for each inertial reference system it is necessary to introduce its own physical time.

In 1907, A. Einstein wrote on this:

*«Surprisingly, however, it turned out that a sufficiently sharpened conception of time was all that was needed to overcome the difficulty discussed. One had only to realize that an auxiliary quantity introduced by H. A. Lorentz, and named by him “local time”, could*

*be defined as “time” in general. If one adheres to this definition of time, the basic equations of Lorentz’s, theory correspond to the principle of relativity ... »*

Or, speaking more precisely, instead of the **true time** there arose the **modified local time** by Lorentz different for each inertial reference system.

But H. Lorentz did not notice this, and in 1914 he wrote on that in detailed article “The two papers by Henri Poincaré on mathematical physics”:

*“These considerations published by myself in 1904, have stimulated Poincaré to write his article on the dynamics of electron where he has given my name to the just mentioned transformation. I have to note as regards this that a similar transformation have been already given in an article by Voigt published in 1887 and I have not taken all possible benefit from it. Indeed I have not given the most appropriate transformation for some physical quantities encountered in the formulae. This was done by Poincaré and later by Einstein and Minkowski. ... I had not thought of the straight path leading to them, since I considered there was an essential difference between the reference systems  $x, y, z, t$  and  $x', y', z', t'$ . In one of them were used — such was my reasoning — coordinate axes with a definite position in ether and what could be termed **true time**; in the other, on the contrary, one simply dealt with subsidiary quantities introduced with the aid of a mathematical trick. Thus, for instance, the variable  $t'$  could not be called **time** in the same sense as the variable  $t$ . Given such reasoning, I did not think of describing phenomena in the reference system  $x', y', z', t'$  **in precisely the same way, as***

*in the reference system  $x, y, z, t$  ... I later saw from the article by Poincaré that, if I had acted in a more systematic manner, I could have achieved an even more significant simplification. Having not noticed this, I was not able to achieve total invariance of the equations; my formulae remained cluttered up with excess terms, that should have vanished. These terms were too small to influence phenomena noticeably, and by this fact I could explain their independence of the Earth's motion, revealed by observations, but I did not establish the relativity principle as a rigorous and universal truth. On the contrary, Poincaré achieved total invariance of the equations of electrodynamics and formulated the **relativity postulate** — a term first introduced by him ... I may add that, while thus correcting the defects of my work, he never reproached me for them.*

*I am unable to present here all the beautiful results obtained by Poincaré. Nevertheless let me stress some of them. First, he did not restrict himself by demonstration that the relativistic transformations left the form of electromagnetic equations unchangeable. He explained this success of transformations by the opportunity to present these equations as a consequence of the least action principle and by the fact that the fundamental equation expressing this principle and the operations used in derivation of the field equations are identical in systems  $x, y, z, t$  and  $x', y', z', t'$  ... There are some new notions in this part of the article, I should especially mark them. Poincaré notes, for example, that in consideration of quantities  $x, y, z, t\sqrt{-1}$  as coordinates of a point in four-dimensional space*

*the relativistic transformations reduces to rotations in this space. He also comes to idea to add to the three components  $X, Y, Z$  of the force a quantity*

$$T = X\xi + Y\eta + Z\zeta,$$

*which is nothing more than the work of the force at a unit of time, and which may be treated as a fourth component of the force in some sense. When dealing with the force acting at a unit of volume of a body the relativistic transformations change quantities  $X, Y, Z, T\sqrt{-1}$  in a similar way to quantities  $x, y, z, t\sqrt{-1}$ . I remind on these ideas by Poincaré because they are closed to methods later used by Minkowski and other scientists to easing mathematical actions in the theory of relativity.”*

As one can see, in the course of studying the article by Poincaré, H. Lorentz sees and accepts the possibility of **describing phenomena in the reference system  $x', y', z', t'$  in exactly the same way as in the reference system  $x, y, z, t$** , and that all this fully complies with the relativity principle, formulated by Poincaré. Hence it follows that **physical phenomena are identical**, if they take place in identical conditions in inertial reference systems  $(x, y, z, t)$  and  $(x', y', z', t')$ , moving with respect to each other with a velocity  $v$ . All this was a direct consequence of the **physical equations not altering** under the Lorentz transformations, that together with space rotations form a group. Precisely all this is contained, also, in articles by Poincaré [2, 3].

H. Lorentz writes in 1915 in a new edition of his book “Theory of electrons”, in comment 72\*:

*“The main reason of my failure was I always thought that only quantity  $t$  could be treated as a true time and*

*that my local time  $t'$  was considered only as an auxiliary mathematical value. In the Einstein theory, just opposite,  $t'$  is playing the same role as  $t$ . If we want to describe phenomena as dependent on  $x', y', z', t'$ , then we should operate with these variables in just the same way as with  $x, y, z, t$  ”.*

Compare this quotation with the detailed analysis of the Poincaré article given by Lorentz in 1914.

Further he demonstrates in this comment the derivation of velocity composition formulae, just in the same form as it is done in article [3] by Poincaré. In comment 75\* he discusses the transformation of forces, exploits invariant (3.22) in the same way as it is done by Poincaré. The Poincaré work is cited only in connection with a particular point. It is surprising but Lorentz in his dealing with the theory of relativity even does not cite Poincaré articles [2; 3]. What may happen with Lorentz in the period after 1914? How we can explain this? To say the truth, we are to mention that because of the war the Lorentz article written in 1914 has appeared in print only in 1921. But it was printed in the same form as Lorentz wrote it in 1914. In fact he seems to confirm by this that his opinion has not been changed. But **all this** in the long run **does not mean nothing substantial, because now we can ourselves examine deeper and in more detail who has done the work, what has been done and what is the level of this work, being informed on the modern state of the theory** and comparing article of 1905 by Einstein to articles by Poincaré.

**The scale of works can be better estimated from the time distance.** Recollections of contemporaries are valuable for us as a testimony on how new ideas have been admitted by the physical community of that time. But moreover one may obtain some knowledge on the ethic of science for some scientists, on group interests, and maybe even something more, which is absolutely unknown to us.

It is necessary to mention that Lorentz in his article of 1904 in calculating his transformations has made an error and as a result Maxwell-Lorentz equations in a moving reference frame have become different than electrodynamics equations in the rest frame. These equations were overloaded by **superfluous** terms. But Lorentz has not been troubled by this. He would easily see the error if he were not **keep away of the relativity principle**. After all, just the relativity principle requires that equations have to be the same in both two reference frames. But he singled out **one** reference frame directly connected with the ether.

Now, following the early works of H. Poincaré we shall deal with the definition of simultaneity, on the synchronization of clocks occupying different points of space, and we shall clarify the physical sense of **local time**, introduced by Lorentz. In the article “**Measurement of time**”, published in 1898, Poincaré discusses the issue of time measurement in detail. This article was especially noted in the book “**Science and hypothesis**” by Poincaré, and, therefore, it is quite comprehensible to an inquisitive reader.

In this article, for instance the following was said:

*“ But let us pass to examples less artificial; to understand the definition implicitly supposed by the savants, let us watch them at work and look for the rules by which they investigate simultaneity. ...*

*When an astronomer tells me that some stellar phenomenon, which his telescope reveals to him at this moment, happened, nevertheless, fifty years ago, I seek his meaning, and to that end I shall ask him first how he knows it, that is, how he has measured the velocity of light.*

*He has begun by **supposing** that light has a constant velocity, and in particular that its velocity is the same in all directions. That is a postulate with-*

*out which no measurement of this velocity could be attempted. This postulate could never be verified directly by experiment; it might be contradicted by it if the results of different measurements were not concordant. We should think ourselves fortunate that this contradiction has not happened and that the slight discordances which may happen can be readily explained.*

*The postulate, et all events, resembling the principle of sufficient reason, has been accepted by everybody; what I wish to emphasize is that **it furnishes us with a new rule for the investigation of simultaneity**, (singled out by me. — A.L.) entirely different from that which we have enunciated above”.*

**It follows from this postulate that the value of light velocity does not depend on velocity of the source of this light.** This statement is also a straightforward consequence of Maxwell electrodynamics. **The above postulate together with the relativity principle** formulated by H. Poincaré in 1904 for all physical phenomena precisely **become the initial statements** in Einstein work of 1905.

Lorentz dealt with the Maxwell-Lorentz equations in a “motionless” reference system related to the ether. He considered the coordinates  $X, Y, Z$  to be **absolute**, and the time  $T$  to be the **true time**.

In a reference system moving along the  $X$  axis with a velocity  $v$  relative to a reference system “at rest”, the coordinates with respect to the axes moving together with the reference system have the values

$$x = X - vT, \quad y = Y, \quad z = Z, \quad (3.3)$$

while the time in the moving reference system was termed by Lo-

rentz **local time** (1895) and defined as follows:

$$\tau = T - \frac{v}{c^2}X. \quad (3.4)$$

He introduced this time so as to be able, in agreement with experimental data, to exclude from the theory the influence of the Earth's motion on optical phenomena in the first order over  $v/c$ .

This time, as he noted, “**was introduced with the aid of a mathematical trick**”. The physical meaning of **local time** was uncovered by H. Poincaré.

In the article “The theory of Lorentz and the principle of equal action and reaction“, published in 1900, he wrote about the **local time**  $\tau$ , defined as follows (Translation from French by V. A. Petrov):

*“I assume observers, situated at different points, to compare their clocks with the aid of light signals; they correct these signals for the transmission time, but, without knowing the relative motion they are undergoing and, consequently, considering the signals to propagate with the same velocity in both directions, they limit themselves to performing observations by sending signals from A to B and, then, from B to A. The **local time**  $\tau$  is the time read from the clocks thus controlled. Then, if  $c$  is the velocity of light, and  $v$  is the velocity of the Earth's motion, which I assume to be parallel to the positive X axis, we will have:*

$$\tau = T - \frac{v}{c^2}X \text{ ”}. \quad (3.5)$$

Taking into account (3.3) in (3.5) we obtain

$$\tau = T \left( 1 - \frac{v^2}{c^2} \right) - \frac{v}{c^2}x. \quad (3.6)$$

The velocity of light in a reference system “at rest“ is  $c$ . In a moving reference system, in the variables  $x, T$ , it will be equal, in the direction parallel to the  $X$  axis, to

$$c - v \quad (3.7)$$

in the positive, and

$$c + v \quad (3.8)$$

— in the negative direction.

This is readily verified, if one recalls that the velocity of light in a reference system “at rest“ is, in all directions, equal to  $c$ , i. e.

$$c^2 = \left(\frac{dX}{dT}\right)^2 + \left(\frac{dY}{dT}\right)^2 + \left(\frac{dZ}{dT}\right)^2. \quad (\lambda)$$

In a moving reference system  $x = X - vT$  the upper expression assumes, in the variables  $x, T$ , the form

$$c^2 = \left(\frac{dx}{dT} + v\right)^2 + \left(\frac{dY}{dT}\right)^2 + \left(\frac{dZ}{dT}\right)^2.$$

Hence it is evident that in a moving reference system the coordinate velocity of a light signal parallel to the  $X$  axis  $\frac{dx}{dT}$  is given as follows

$$\frac{dx}{dT} = c - v$$

in the positive direction,

$$\frac{dx}{dT} = c + v$$

— in the negative direction.

The coordinate velocity of light in a moving reference system along the  $Y$  or  $Z$  axis equals the quantity

$$\sqrt{c^2 - v^2}.$$

We note that if we had made use of the Lorentz transformations inverse to (3.1), then taking into account the equality

$$c^2\gamma^2 \left( dT' + \frac{v}{c^2}dX' \right)^2 - \gamma^2 (dX' + vdT')^2 = c^2(dT')^2 - (dX')^2,$$

we would have obtained from Eq. (λ) the expression

$$c^2 = \left( \frac{dX'}{dT'} \right)^2 + \left( \frac{dY'}{dT'} \right)^2 + \left( \frac{dZ'}{dT'} \right)^2, \quad (\rho)$$

which would signify that the velocity of light equals  $c$  in all directions in a moving reference system, too. Let us mention also that the light cone equation remains the same after multiplying r.h.s. of Eqs. (3.1) (Lorentz transformations) by arbitrary function  $\phi(x)$ . The light cone equation preserves its form under conformal transformations.

Following Poincaré, we shall perform synchronization of the clocks in a moving reference system with the aid of Lorentz's **local time**. Consider a light signal leaving point  $A$  with coordinates  $(0, 0, 0)$  at the moment of time  $\tau_a$ :

$$\tau_a = T \left( 1 - \frac{v^2}{c^2} \right). \quad (3.9)$$

This signal will arrive at point  $B$  with coordinates  $(x, 0, 0)$  at the moment of time  $\tau_b$

$$\tau_b = \left( T + \frac{x}{c-v} \right) \left( 1 - \frac{v^2}{c^2} \right) - \frac{v}{c^2}x = \tau_a + \frac{x}{c}. \quad (3.10)$$

Here, we have taken into account the transmission time of the signal from  $A$  to  $B$ . The signal was reflected at point  $B$  and arrived at point  $A$  at the moment of time  $\tau'_a$

$$\tau'_a = \left( T + \frac{x}{c-v} + \frac{x}{c+v} \right) \left( 1 - \frac{v^2}{c^2} \right) = \tau_b + \frac{x}{c}. \quad (3.11)$$

On the basis of (3.9), (3.11) and (3.10) we have

$$\frac{\tau_a + \tau'_a}{2} = \tau_b. \quad (3.12)$$

Thus the definition of simultaneity has been introduced, which was later applied by A. Einstein for deriving the Lorentz transformations. We have verified that the Lorentz “**local time**” (3.6) satisfies condition (3.12). Making use of (3.12) as the initial equation for defining time in a moving reference system, Einstein arrived at the same Lorentz “**local time**” (3.6) multiplied by an arbitrary function depending only on the velocity  $v$ . From (3.10), (3.11) we see that in a reference system moving along the  $X$  axis with the **local time**  $\tau$  the light signal has velocity  $c$  along any direction parallel to the  $X$  axis. The transformations, inverse to (3.3) and (3.4), will be as follows

$$T = \frac{\tau + \frac{v}{c^2}x}{1 - \frac{v^2}{c^2}}, \quad X = \frac{x + v\tau}{1 - \frac{v^2}{c^2}}, \quad Y = y, \quad Z = z. \quad (3.13)$$

Since the velocity of light in a reference system “at rest” is  $c$ , in the new variables  $\tau, x, y, z$  we find from Eqs. (3.1) and (3.13)

$$\gamma^2 \left( \frac{dx}{d\tau} \right)^2 + \left( \frac{dy}{d\tau} \right)^2 + \left( \frac{dz}{d\tau} \right)^2 = \gamma^2 c^2. \quad (3.14)$$

We can see from the above that to have the velocity of light equal to  $c$  in any direction in the moving reference system, also, it is necessary to multiply the right-hand sides of transformations (3.3) and (3.4) for  $x$  and  $\tau$  by  $\gamma$  and to divide the right-hand sides in transformations (3.13) for  $T$  and  $X$  by  $\gamma$ . Thus, this requirement leads to appearance of the Lorentz transformations here.

H. Lorentz in 1899 used transformation of the following form

$$X' = \gamma(X - vT), \quad Y' = Y, \quad Z' = Z, \quad T' = \gamma^2 \left( T - \frac{v}{c^2}X \right),$$

to explain the Michelson experiment. The inverse transformations are

$$X = \gamma X' + vT', \quad Y = Y', \quad Z = Z', \quad T = T' + \frac{v}{c^2} \gamma X'.$$

If H. Lorentz would proposed the relativity principle for all physical phenomena and required in this connection that a spherical wave should have the same form in unprimed and primed systems of reference, then he would come to Lorentz transformations. Let we have in unprimed system of reference

$$c^2 T^2 - X^2 - Y^2 - Z^2 = 0,$$

then according to his formulae this expression in new variables is as follows

$$c^2 \left( T' + \frac{v}{c^2} \gamma X' \right)^2 - (\gamma X' + vT')^2 - Y'^2 - Z'^2 = 0,$$

and after some simplifications we obtain

$$c^2 T'^2 \left( 1 - \frac{v^2}{c^2} \right) - X'^2 - Y'^2 - Z'^2 = 0.$$

We see that to guarantee the same form of a spherical wave in new variables as in the old ones it is necessary to change variable  $T'$  replacing it by new variable  $\tau$

$$\frac{1}{\gamma} T' = \tau.$$

After transition to the new variable we obtain Lorentz transformations

$$X' = \gamma(X - vT), \quad Y' = Y, \quad Z' = Z, \quad \tau = \gamma \left( T - \frac{v}{c^2} X \right),$$

and the inverse transformations

$$X = \gamma(X' + v\tau), \quad Y = Y', \quad Z = Z', \quad T = \gamma\left(\tau + \frac{v}{c^2}X'\right).$$

But H. Lorentz has not seen this in 1899. He obtained these transformations in 1904 only, then he also came closely to the theory of relativity, but did not make the decisive step. Lorentz transformations (3.1) were obtained in 1900 by Larmor. But he also did not propose the principle of relativity for all physical phenomena and did not require form-invariance of Maxwell equations under these transformations. Therefore Larmor also has not made a decisive step to construct the theory of relativity.

Precisely the constancy of the velocity of light in any inertial reference system is what A. Einstein chose to underlie his approach to the electrodynamics of moving bodies. But it is provided for not by transformations (3.3) and (3.4), but by the Lorentz transformations.

A. Einstein started from the relativity principle and from the principle of constancy of the light velocity. Both principles were formulated as follows:

« 1. The laws governing the changes of the state of any physical system do not depend on which one of two coordinate systems in uniform translational motion relative to each other these changes of the state are referred to.

2. Each ray of light moves in the coordinate system “at rest” with the definite velocity  $V$  independent of whether this ray of light is emitted by a body at rest or a body in motion».

Let us note that **Galilean principle of relativity** is not included into these principles.

It is necessary to specially emphasize that the **principle of constancy of velocity of light**, suggested by A. Einstein as the **second independent postulate**, is really a special consequence of requirements of the relativity principle by H. Poincaré. This principle was extended by him on all physical phenomena. To be convinced in this it is sufficient to consider requirements of the relativity principle for an elementary process — propagation of the electromagnetic spherical wave. We will discuss this later.

In 1904, in the article “The present and future of mathematical physics”, H. Poincaré formulates the relativity principle for all natural phenomena, and in the same article he again returns to Lorentz’s idea of **local time**. He writes:

*«Let us imagine two observers, who wish to regulate their watches by means of optical signals; they exchange signals, but as they know that the transmission of light is not instantaneous, they are careful to cross them. When station B sees the signal from station A, its timepiece should not mark the same hour as that of station A at the moment the signal was sent, but this hour increased by constant representing the time of transmission. Let us suppose, for example, that station A sends its signal at the moment when its time-piece marks the hour zero, and that station B receives it when its time-piece marks the hour  $t$ . The watches will be set, if the time  $t$  is the time of transmission, and in order to verify it, station in turn sends signal at the instant when its time-piece is at zero; station must then see it when its time-piece is at  $t$ . Then the watches are regulated.*

*And, indeed, they mark the same hour at the same physical instant, but under one condition, namely, that the two stations are stationary. Otherwise, the time*

*of transmission will not be the same in the two directions, since the station B, for example, goes to meet the disturbance emanating from A, whereas station A flees before the disturbance emanating from B. Watches regulated in this way, therefore, will not mark the true time; (the time in the reference system “at rest” — A.L.) they will mark what might be called the local time, so that one will gain on the other. It matters little, since we have means of perceiving it. All the phenomena which take place at B, for example, will be behind time, but all just the same amount, and the observer will not notice it since his watch is also behind time; thus, in accordance with the principle of relativity he will have means of ascertaining whether he is at rest or in absolute motion. Unfortunately this is not sufficient; additional hypotheses are necessary. We must admit that the moving bodies undergo a uniform contraction in the direction of motion».*

Such was the situation before the work of Lorentz, which also appeared in 1904. Here Lorentz presents again the transformations connecting a reference system “at rest” with a reference system moving with a velocity  $v$  relative to the one “at rest”, which were termed by Poincaré the Lorentz transformations. In this work, Lorentz, instead of the **local time** (3.4) introduced the time  $T'$ , equal to

$$T' = \gamma\tau. \quad (3.15)$$

Lorentz called time  $T'$  as the **modified local time**. Precisely this time will be present in any inertial reference system in Galilean coordinates. It does not violate the condition of synchronization (3.12)

Below we shall see following Lorentz that the wave equation does indeed not alter its form under the Lorentz transformations

(3.1). Let us check this. The wave equation of electrodynamics has the form:

$$\square\phi = \left( \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \phi = 0. \quad (3.16)$$

Here  $\phi$  is a scalar function in four-dimensional space, which changes under coordinate-time transformations according to the rule  $\phi'(x') = \phi(x)$ ,  $c$  is the **electrodynamic constant**, that has the dimension of velocity.

Let us establish the form-invariance of the operator  $\square$  with respect to transformations (3.1). We represent part of the operator  $\square$  in the form

$$\frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left( \frac{1}{c} \cdot \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) \left( \frac{1}{c} \cdot \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right). \quad (3.17)$$

We calculate the derivatives in the new coordinates, applying formulae (3.1)

$$\frac{1}{c} \cdot \frac{\partial}{\partial t} = \frac{1}{c} \cdot \frac{\partial t'}{\partial t} \cdot \frac{\partial}{\partial t'} + \frac{1}{c} \cdot \frac{\partial x'}{\partial t} \cdot \frac{\partial}{\partial x'} = \gamma \left( \frac{1}{c} \cdot \frac{\partial}{\partial t'} - \frac{v}{c} \cdot \frac{\partial}{\partial x'} \right),$$

$$\frac{\partial}{\partial x} = \frac{\partial t'}{\partial x} \cdot \frac{\partial}{\partial t'} + \frac{\partial x'}{\partial x} \cdot \frac{\partial}{\partial x'} = -\gamma \left( \frac{v}{c^2} \cdot \frac{\partial}{\partial t'} - \frac{\partial}{\partial x'} \right).$$

Hence we find

$$\frac{1}{c} \cdot \frac{\partial}{\partial t} - \frac{\partial}{\partial x} = \gamma \left( 1 + \frac{v}{c} \right) \left( \frac{1}{c} \cdot \frac{\partial}{\partial t'} - \frac{\partial}{\partial x'} \right), \quad (3.18)$$

$$\frac{1}{c} \cdot \frac{\partial}{\partial t} + \frac{\partial}{\partial x} = \gamma \left( 1 - \frac{v}{c} \right) \left( \frac{1}{c} \cdot \frac{\partial}{\partial t'} + \frac{\partial}{\partial x'} \right). \quad (3.19)$$

Substituting these expressions into (3.17) we obtain

$$\frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2}. \quad (3.20)$$

Taking into account that the variables  $y$  and  $z$  in accordance with (3.1) do not change, on the basis of (3.20) we have

$$\begin{aligned} & \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \\ & = \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial y'^2} - \frac{\partial^2}{\partial z'^2}. \end{aligned} \quad (3.21)$$

This means that the **wave equation** (3.16) **remains form-invariant with respect to the Lorentz transformations** (3.1). In other words, it is the same in both inertial reference systems. Hence, for instance, it follows that the velocity of a light wave equals  $c$ , both in a reference system “at rest” and in any other reference system moving relative to the one “at rest” with a velocity  $v$ .

We have shown that the Lorentz transformations leave the operator  $\square$  unaltered, i. e. they conserve the form-invariance of the wave equation. On the other hand, this computation can be considered as an exact derivation of the Lorentz transformations based on the form-invariance of the operator  $\square$ .

In electrodynamics, the wave equation holds valid outside the source both for the scalar and vector potentials,  $\varphi$  and  $\vec{A}$ , respectively. In this case,  $\varphi$  is defined as a scalar with respect to three-dimensional coordinate transformations, and  $\vec{A}$  is defined as a vector with respect to the same transformations. For the wave equation to be form-invariant under the Lorentz transformations it is necessary to consider the quantities  $\varphi$  and  $\vec{A}$  as components of the four-dimensional vector  $A^\nu = (\varphi, \vec{A})$

$$\square A^\nu = 0, \quad \nu = 0, 1, 2, 3.$$

In 1905 Henri Poincaré first established [2, 3] the invariance of the Maxwell-Lorentz equations and of the equations of motion of charged particles under the action of the Lorentz force with respect to the Lorentz transformations (3.1) on the basis of the 1904 work by Lorentz, in which the Lorentz transformations were discovered, and on the relativity principle, formulated by Poincaré in the same year for all natural phenomena. All the above will be demonstrated in detail in Sections 8 and 9.

**H. Poincaré discovered that these transformations, together with spatial rotations form a group. He was the first to introduce the notion of four-dimensionality of a number of physical quantities.** The discovery of this group together with quantum ideas created the foundation of modern theoretical physics.

Poincaré established that the scalar and vector potential  $(\varphi, \vec{A})$ , the charge density and current  $(c\rho, \rho\vec{v})$ , the four-velocity  $(\gamma, \gamma\vec{v}/c)$ , the work per unit time and force normalized to unit volume,  $(\vec{f}, \vec{v}/c, \vec{f})$ , as well as the four-force transform like the quantities  $(ct, \vec{x})$ . The existence of the Lorentz group signifies that in all inertial reference systems the Maxwell-Lorentz equations in Galilean coordinates remain form-invariant, i. e. the relativity principle is satisfied. Hence it directly follows that the descriptions of phenomena are the same both in the reference system  $x, y, z, t$  and in the reference system  $x', y', z', t'$ , so, consequently, time  $t$ , like the other variables  $x, y, z$ , is relative. Thus, time being relative is a direct consequence of the existence of the group, which itself arises as a consequence of the requirement to fulfil the relativity principle for electromagnetic phenomena. The existence of this group led to the discovery of the geometry of space-time.

**H. Poincaré discovered a number of invariants of the group and among these — the fundamental invariant**

$$J = c^2T^2 - X^2 - Y^2 - Z^2, \quad (3.22)$$

which arose in exploiting the Lorentz transformation. **It testifies that space and time form a unique four-dimensional continuum of events with metric properties determined by the invariant (3.22).** The four-dimensional space-time discovered by **H. Poincaré**, and defined by invariant (3.22), was later called the **Minkowski space**. Precisely this is the essence of special relativity theory. This is why it is related to all physical phenomena. It is space-time determined by the invariant (3.22) that provides for the existence of physically equal inertial reference systems in Nature. However, as earlier in classical mechanics, it remains unclear, how the inertial reference systems are related to the distribution of matter in the Universe. From expression (3.22) it follows that in any inertial reference system a given quantity  $J$  in Galilean (Cartesian) coordinates remains unaltered (form-invariant), while its projections onto the axes change. Thus, depending on the choice of inertial reference system the projections  $X, Y, Z, T$  **are relative quantities**, while the **quantity  $J$**  for any given  $X, Y, Z, T$  **has an absolute value**. A positive interval  $J$  can be measured by a clock whereas a negative one — by a rod. According to (3.22), in differential form we have

$$(d\sigma)^2 = c^2(dT)^2 - (dX)^2 - (dY)^2 - (dZ)^2. \quad (3.23)$$

The quantity  $d\sigma$  is called an **interval**.

**The geometry of space-time, i. e. the space of events (the Minkowski space) with the measure (3.23) has been termed pseudo-Euclidean geometry.**

As it could be seen from the structure of invariant  $J$ , written in orthogonal (Galilean) coordinates, it is always possible to introduce a unique time  $T$  for all points of the three-dimensional space. This means that the three-dimensional space of a given inertial reference system is orthogonal to the lines of time. Since, as we shall see below, the invariant  $J$  in another inertial reference system

assumes the form (3.27), it hence follows that in this reference system the unique time will already be different, it is determined by the variable  $T'$ . But length will change simultaneously. Thus, the possibility to introduce simultaneity for all the points of three-dimensional space is a direct consequence of the pseudo-Euclidean geometry of the four-dimensional space of events.

Drawing a conclusion to all the above, we see that H. Lorentz found the transformations (3.1), which conserve the form of the wave equation (3.16). On the basis of the relativity principle for all physical phenomena formulated by him in 1904 and of the Lorentz transformations, Henri Poincaré established form-invariance of the Maxwell-Lorentz equations and discovered the pseudo-Euclidean geometry of space-time, determined by the invariant (3.22) or (3.23).

A short exposition of the detailed article [3] was given by H. Poincaré in the reports to the French academy of sciences [2] and published even before the work by Einstein was submitted for publication. This paper contained a precise and rigorous description of the solution to the problem of the electrodynamics of moving bodies and, at the same time, an extension of the Lorentz transformations to all natural forces, independently of their origin. In this publication H. Poincaré discovered Lorentz group with accordance to that a whole set of four-dimensional physical values transforming similar to  $t, x, y, z$  arose. The presence of Lorentz group automatically provides the synchronization of clocks in any inertial reference system. So the proper physical time arises in any inertial system of reference — the **modified local time** by Lorentz. In paper [2] relativistic formulae for adding velocities and the transformation law for forces arose for the first time. There existence of **gravitational waves** propagating with light velocity was **predicted**.

It should be emphasized that just the discovery of Lorentz

group provided the uniformity of description of all physical effects in all the inertial reference systems in full accordance with the relativity principle. Just all this automatically provided the relativity of time and length.

H. Poincaré discovered the invariant (3.22) on the basis of the Lorentz transformations (3.1). On the other hand, applying the invariant (3.22) it is easy to derive the actual Lorentz transformations (3.1). Let the invariant  $J$  in an inertial reference system have the form (3.22) in Galilean coordinates. Now, we pass to another inertial reference system

$$x = X - vT, \quad Y' = Y, \quad Z' = Z, \quad (3.24)$$

then the invariant  $J$  assumes the form

$$J = c^2 \left(1 - \frac{v^2}{c^2}\right) T^2 - 2xvT - x^2 - Y'^2 - Z'^2. \quad (3.25)$$

Hence we have

$$J = c^2 \left[ \sqrt{1 - \frac{v^2}{c^2}} T - \frac{xv}{c^2 \sqrt{1 - \frac{v^2}{c^2}}} \right]^2 - x^2 \left[ 1 + \frac{v^2}{c^2 - v^2} \right] - Y'^2 - Z'^2. \quad (3.26)$$

Expression (3.26) can be written in the form

$$J = c^2 T'^2 - X'^2 - Y'^2 - Z'^2, \quad (3.27)$$

where

$$T' = \sqrt{1 - \frac{v^2}{c^2}} T - \frac{xv}{c^2 \sqrt{1 - \frac{v^2}{c^2}}} = \frac{T - \frac{v}{c^2} X}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (3.28)$$

$$X' = \frac{x}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{X - vT}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (3.29)$$

We see from expression (3.27) that the form-invariance of the invariant  $J$  is provided for by the Lorentz transformations (3.28) and (3.29). In deriving the Lorentz transformations from the expression for the invariant (3.22) we took advantage of the fact that the invariant  $J$  may assume an arbitrary real value. Precisely this circumstance has permitted us to consider quantities  $T$  and  $X$  as independent variables, that can assume any real values. If we, following Einstein, knew only one value of  $J$ , equal to zero, we could not, in principle, obtain Lorentz transformations of the general form, since the space variables would be related to the time variable.

In this case the following **heuristic** approach can be realized. The equation of spherical electromagnetic wave having its center in the origin of the coordinate system has the following form

$$c^2T^2 - X^2 - Y^2 - Z^2 = 0,$$

where  $c$  is the electrodynamic constant, if we use Galilean coordinates of the “rest” system of reference  $K$ . This fact follows from the Maxwell-Lorentz equations.

Let us consider two inertial reference systems  $K$  and  $K'$  with Galilean coordinates moving relative to each other with velocity  $v$  along axis  $X$ . Let their origins coincide at the moment  $T = 0$  and let a spherical electromagnetic wave is emitted just at this moment from their common origin. In reference system  $K$  it is given by equation

$$c^2T^2 - X^2 - Y^2 - Z^2 = 0.$$

As system of reference  $K'$  is moving with velocity  $v$ , we can use

Galilean transformations

$$x = X - vT, \quad Y = Y', \quad Z = Z'$$

and rewrite the preceding equation of spherical wave in the following form

$$c^2T^2 \left(1 - \frac{v^2}{c^2}\right) - 2xvT - x^2 - Y'^2 - Z'^2 = 0.$$

The requirement of relativity principle here is reduced to necessity that the electromagnetic wave in a new inertial reference system  $K'$  has to be also spherical having its center at the origin of this reference system.

Having this in mind we transform the above equation (as done before) to the following form

$$c^2T'^2 - X'^2 - Y'^2 - Z'^2 = 0.$$

So, we derive the Lorentz transformations.

$$T' = \frac{T - \frac{v}{c^2}X}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad X' = \frac{X - vT}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad Y' = Y, \quad Z' = Z,$$

but at the light cone only.

Now, we go to the most important stuff. Let us treat variables  $T, X, Y, Z$ , appearing in the derived transformations as **independent**. Then after inserting these expressions into r.h.s. of equation (3.27) we can see that they leave quantity

$$c^2T^2 - X^2 - Y^2 - Z^2$$

unchanged due to the linear character of transformations. Therefore we come to the fundamental invariant  $J$ , and so to pseudo-Euclidean geometry of space-time. It follows from the above, in

particular, that velocity of light both in system  $K$ , and in system  $K'$  is the same and therefore the principle of constancy of velocity of light is a particular consequence of the relativity principle. Precisely this circumstance remained unnoticed by A. Einstein in his 1905 work, in which the Lorentz transformations were derived.

Earlier we have shown, following Poincaré, that Lorentz's "**local time**" permits to perform synchronization of clocks in a moving reference system at different spatial points with the aid of a light signal. Precisely expression (3.12) is the condition for the synchronization of clocks in a moving reference system. It introduces the definition of simultaneity of events at different points of space. Poincaré established that Lorentz's "**local time**" satisfies this condition.

**So, the definition of simultaneity of events in different spatial points by means of a light signal as well as the definition of time in a moving reference system by means of light signal both were considered by Poincaré in his papers of 1898, 1900 and 1904. Therefore nobody has any ground to believe that these ideas have been first treated by A. Einstein in 1905.**

But let us see, for example, what is written by Academician L. I. Mandel'stam in his lectures [8]:

*"So, the great achievement of Einstein consists in discovering that the concept of simultaneity is a concept ... that we have to define. People had the knowledge of space, the knowledge of time, had this knowledge many centuries, but nobody guessed that idea."*

And the following was written by H. Weyl:

*"... we are to discard our belief in the objective meaning of simultaneity; it was the great achievement of Einstein in the field of the theory of knowledge that he banished this dogma from our minds,*

*and this is what leads us to rank his name with that of Copernicus”.*

Is it possible that L. I. Mandel'stam and H. Weyl have not read articles and books by Poincaré?

Academician V. L. Ginzburg in his book “On physics and astrophysics” (Moscow: Nauka, 1985) in the article “How and who created Special Relativity Theory?”<sup>1</sup> wrote:

*“From the other side, in earlier works, in articles and reports by Poincaré there are a set of comments which sound almost prophetic. I mean both the necessity to define a concept of simultaneity, and an opportunity to use light signals for this purpose, and on the principle of relativity. But Poincaré have not developed these ideas and followed Lorentz in his works of 1905-1906”.*

Let us give some comments to this citation.

To be precise it should be said that **Poincaré was the first who formulated the relativity principle for all physical processes.** He also **defined the concept of simultaneity at different spatial points by means of the light signal** in his papers 1898, 1900 and 1904. In Poincaré works [2; 3] **these concepts** have been adequately realized in the language of Lorentz group which provides fulfilment both the requirement of relativity principle, and the introduction of his own **modified local** Lorentz time in every inertial system of reference. All that automatically provided a unique synchronization of clocks by means of the light signal in every inertial reference system. Just due to this not any **further development of these concepts** were required after H. A. Lorentz work of 1904.

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<sup>1</sup>All the citations of Academician V. L. Ginzburg presented here and below are taken from this article. — A. L.

It was necessary only to introduce these concepts into the bosom of the theory. It was precisely realized in works [2; 3] by means of the Lorentz group, discovered by H. Poincaré. Poincaré does not follow Lorentz, he develops **his own ideas** by using Lorentz achievements and he completes the creation of the theory of relativity in this way. Exactly in papers [2; 3] he extends Lorentz invariance on all the forces of nature, including gravitational; he discovers equations of the relativistic mechanics; he discovers fundamental invariant

$$c^2t^2 - x^2 - y^2 - z^2,$$

which determines the geometry of space-time.

H. Poincaré approach is transparent and contemporary though it is realized almost one hundred years ago. How is it possible not to understand this after reading Poincaré works [2; 3]?

In the article (1905) "On the electrodynamics of moving bodies" (§3) A. Einstein took the relation (3.12) as the initial equation in searching for the function  $\tau$ . But hence one can naturally obtain nothing, but Lorentz's "**local time**". We write the equation obtained by him in the form

$$\tau = \frac{a}{1 - \frac{v^2}{c^2}} \left[ \left( 1 - \frac{v^2}{c^2} \right) T - \frac{v}{c^2} x \right],$$

Where  $a$  is an unknown function depending only on the velocity  $v$ .

Hence it is seen that this expression differs from the Lorentz "**local time**" (3.6) only by a factor depending on the velocity  $v$  and which is not determined by condition (3.12). It is strange to see that A. Einstein knows that this is Lorentz "local time", but he does not refer to the author. Such a treatment is not an exception for him.

Further, for a beam of light leaving the source at the time moment  $\tau = 0$  in the direction of increasing  $\xi$  values, Einstein writes:

$$\xi = c\tau$$

or

$$\xi = \frac{ac}{1 - \frac{v^2}{c^2}} \left[ \left( 1 - \frac{v^2}{c^2} \right) T - \frac{v}{c^2} x \right]. \quad (\beta)$$

He further finds

$$x = (c - v)T. \quad (\delta)$$

Substituting this value of  $T$  into the equation for  $\xi$ , Einstein obtains

$$\xi = \frac{a}{1 - \frac{v^2}{c^2}} x.$$

Since, as it will be seen further from Einstein's article, the quantity  $a$  is given as follows

$$a = \sqrt{1 - \frac{v^2}{c^2}},$$

then, with account of this expression, we obtain:

$$\xi = \frac{x}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Substituting, instead of  $x$ , its value (3.3),

$$x = X - vT, \quad (\nu)$$

Einstein obtains for  $\xi$  an expression of the form:

$$\xi = \frac{X - vT}{\sqrt{1 - \frac{v^2}{c^2}}},$$

which he namely considers as the Lorentz transformation for  $\xi$ , **implying that  $X$  and  $T$  are arbitrary and independent**. However, this is not so. He does not take into account, that according to  $(\delta)$  and  $(\nu)$ , there exists the equality

$$X - vT = (c - v)T,$$

hence it follows that

$$X = cT.$$

Hence it follows, that Einstein obtained the Lorentz transformations for  $\xi$  only for the partial case of  $X = cT$ :

$$\xi = X \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}.$$

This can be directly verified, if in formula  $(\beta)$  for  $\xi$  one substitutes, instead of the value of  $T$  from formula  $(\delta)$ , as done by Einstein, the value of  $x$  from the same formula. Then we obtain:

$$\xi = \frac{a}{1 + \frac{v}{c}} X, \quad X = cT.$$

Taking into account the expression for  $a$ , we again arrive at the formula found earlier,

$$\xi = X \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}, \quad X = cT.$$

But further in the text of the article A. Einstein exploits the general form of Lorentz transformations without any comments. A. Einstein has not observed that the principle of relativity together

with electrodynamics obligatory requires a construction of four-dimensional physical quantities, in accordance with the Lorentz group. As a result this requires presence of the group invariants testifying to the pseudo-Euclidean geometry of space-time. Just due to this Einstein has not succeeded in finding relativistic equations of mechanics, because he has not discovered the law of transformation for Lorentz force. He also has not understood that energy and momentum of a particle constitute a unified quantity and that they transform under Lorentz transformations in the same way as  $ct, x, y, z$ . It should be especially emphasized that Einstein, in his work of 1905, in contrast to Poincaré, has not extended Lorentz transformations onto all forces of nature, for example, onto gravitation. He wrote later that “*in the framework of special relativity theory there is no place for a satisfactory theory of gravitation*”. But as it is shown in [5] this statement is not correct.

Owing to the Maxwell-Lorentz equations, the relativity principle for inertial reference systems led Poincaré [3] and, subsequently, Minkowski [4] to discovering the pseudo-Euclidean geometry of space-time. Precisely for this, we indebted to Poincaré and Minkowski. In 1908 H. Minkowski, addressing the 80-th meeting of German naturalists and doctors in Cologne, noted [4]:

*“The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality”.*

Therefore, the essence of special relativity theory consists in the following (**it is a postulate**): **all physical processes proceed in four-dimensional space-time  $(ct, \vec{x})$ , the geometry of which is pseudo-Euclidean and is determined by the interval (3.23).**

The consequences of this postulate are energy-momentum and angular momentum conservation laws, the existence of inertial reference systems, the relativity principle for all physical phenomena, the Lorentz transformations, the constancy of the velocity of light in Galilean coordinates of an inertial system, the retardation of time, the Lorentz contraction, the opportunity to use non-inertial reference systems, the “clock paradox”, the Thomas precession, the Sagnac effect and so on. On the base of this postulate and the quantum ideas a set of fundamental conclusions was obtained and the quantum field theory was constructed.

By centennial of the theory of relativity it is high time to make clear that constancy of the light velocity in all inertial systems of reference is not a fundamental statement of the theory of relativity.

Thus, investigation of electromagnetic phenomena together with Poincaré’s relativity principle resulted in the unification of space and time in a unique four-dimensional continuum of events and permitted to establish the pseudo-Euclidean geometry of this continuum. Such a four-dimensional space-time is homogeneous and isotropic.

These properties of space-time provide validity of fundamental conservation laws of energy, momentum and angular momentum in a closed physical system. **The pseudo-Euclidean geometry of space-time reflects the general dynamical properties of matter, which make it universal.** Investigation of various forms of matter, of its laws of motion is at the same time investigation of space and time. Although the actual structure of space-time has been revealed to us as a result of studying matter (electrodynamics), we sometimes speak of space as of an arena, in which some or other phenomena take place. Here, we will make no mistake, if we remember that this arena does not exist by itself, without matter. Sometimes it is said that space-time (Minkowski space) is given a priori, since its structure does not change under the influ-

ence of matter. Such an invariability of Minkowski space arises owing to its **universality** for all physical fields, so the impression is thus created that it exists as if independently of matter. Probably just due to a vagueness of the essence of special relativity theory for him A. Einstein arrived at the conclusion that “*within special relativity theory there is no place for a satisfactory theory of gravity*”.

In Einstein’s general relativity theory, special relativity theory is certainly not satisfied, it is considered a limit case. In 1955 A. Einstein wrote:

*«An essential achievement of general relativity theory consists in that it has saved physics from the necessity of introducing an “inertial reference system” (or “inertial reference systems”)>>.*

However, even now, there exists absolutely no experimental or observational fact that could testify to the violation of special relativity theory. For this reason no renunciation, to whatever extent, of its rigorous and precise application in studies of gravitational phenomena, also, can be justified. Especially taking into account that all known gravitational effects are explained precisely within the framework of special relativity theory [5]. **Renunciation of special relativity theory leads to renunciation of the fundamental conservation laws of energy, momentum and angular momentum.** Thus, having adopted the hypothesis that all natural phenomena proceed in pseudo-Euclidean space-time, we automatically comply with all the requirements of fundamental conservation laws and confirm the **existence of inertial reference systems.**

The space-time continuum, determined by the interval (3.23) can be described in arbitrary coordinates, also. **In transition to arbitrary coordinates, the geometry of four-dimensional space-time does not change. However, three-dimensional space will no longer be Euclidean in arbitrary coordinates.** To simplify

our writing we shall, instead of variables  $T, X, Y, Z$ , introduce the variables  $X^\nu$ ,  $\nu = 0, 1, 2, 3$ ,  $X^0 = cT$ . We now perform transition from the variables  $X^\nu$  to the arbitrary variables  $x^\nu$  with the aid of the transformations

$$X^\nu = f^\nu(x^\sigma). \quad (3.30)$$

These transformation generally lead to a non-inertial reference system. Calculating the differentials

$$dX^\nu = \frac{\partial f^\nu}{\partial x^\lambda} dx^\lambda \quad (3.31)$$

(here and further summation is performed from 0 to 3 over identical indices  $\lambda$ ) and substituting them into (3.23) we obtain an expression for the interval in the non-inertial reference system

$$(d\sigma)^2 = \gamma_{\mu\lambda}(x) dx^\mu dx^\lambda. \quad (3.32)$$

Here,  $\gamma_{\mu\lambda}(x)$  is the metric tensor of four-dimensional space-time, it is given as follows

$$\gamma_{\mu\lambda}(x) = \sum_{\nu=0}^3 \varepsilon^\nu \frac{\partial f^\nu}{\partial x^\mu} \cdot \frac{\partial f^\nu}{\partial x^\lambda}, \quad \varepsilon^\nu = (1, -1, -1, -1). \quad (3.33)$$

Expression (3.32) is invariant with respect to arbitrary coordinate transformations. Expression (3.33) represents the general form of the pseudo-Euclidean metric.

The difference between a metric of the form (3.23) from the metric (3.32) is usually, in accordance with Einstein's ideas, attributed to the existence of the gravitational field. But this is incorrect. No gravitational field is present in a metric of the form (3.32). Ideas of accelerated reference systems in Minkowski space have played an important heuristic role in Einstein reflections on the problem of gravitation. They contributed to his arriving at the idea of describing the gravitational field with the aid of the

metric tensor of Riemannian space, and for this reason Einstein tried to retain them, although they have nothing to do with the gravitational field. Precisely such circumstances prevented him from revealing the essence of special relativity theory. From a formal, mathematical, point of view Einstein highly appreciated Minkowski's work, but he never penetrated the profound physical essence of Minkowski's work, even though the article dealt with a most important discovery in physics — the **discovery of the pseudo-Euclidean structure of space and time**.

Einstein considered special relativity theory only related to an interval of the form (3.23), while ascribing (3.32) to general relativity theory. Regretfully, such a point of view still prevails in textbooks and monographs expounding relativity theory.

Consider a certain non-inertial reference system where the metric tensor of space-time is given as  $\gamma_{\mu\lambda}(x)$ . It is, then, readily shown that there exists an infinite number of reference systems, in which the interval (3.32) is as follows

$$(d\sigma)^2 = \gamma_{\mu\lambda}(x') dx'^{\mu} dx'^{\lambda}. \quad (3.34)$$

A partial case of such transformations is represented by the Lorentz transformations, which relate one inertial reference system to another. We see that the transformations of coordinates, which leave the metric form-invariant, result in that physical phenomena proceeding in such reference systems at identical conditions can never permit to distinguish one reference system from another. Hence, one can give a more general formulation of the **relativity principle**, which not only concerns **inertial reference systems**, but **non-inertial ones** [6], as well:

*“Whatever physical reference system (inertial or non-inertial) we choose, it is always possible to point to an infinite set of other reference systems, such as all physical phenomena proceed there exactly like in*

*the initial reference system, so we have no, and cannot have any, experimental means to distinguish, namely in which reference system of this infinite set we are”.*

It must be noted that, though the metric tensor  $\gamma_{\mu\lambda}(x)$  in (3.33) depends on coordinates, nevertheless the space remains pseudo-Euclidean. Although this is evident, it must be pointed out, since even in 1933 A. Einstein wrote the absolute opposite [7]:

*«In the special theory of relativity, as Minkowski had shown, this metric was quasi-Euclidean one, i. e., the square of the “length”  $ds$  of line element was a certain quadratic function of the differentials of the coordinates.*

*If other coordinates are introduced by means of non-linear transformation,  $ds^2$  remains a homogeneous function of the differentials of the coordinates, but the coefficients of this function ( $g_{\mu\nu}$ ) cease to be constant and become certain functions of the coordinates. In mathematical terms this means that physical (four-dimensional) space has a Riemannian metric».*

This is, naturally, **wrong**, since it is impossible to transform pseudo-Euclidean geometry into Riemannian geometry by applying the transformations of coordinates (3.30). Such a statement by A. Einstein had profound physical roots. Einstein was convinced that the pseudo-Euclidean metric in arbitrary coordinates,  $\gamma_{\mu\lambda}(x)$ , describes the gravitational field, also. These ideas, put forward by Einstein, restricted the framework of special relativity theory and in such form became part of the material expounded in textbooks and monographs, which had hindered comprehension of the essence of relativity theory.

Thus, for example, Academician L.I. Mandel'stam, in his lectures on relativity theory [8], especially noted:

*“What actually happens, how an accelerated moving clock shows time and why it slows down or does the opposite cannot be answered by special relativity theory, because it absolutely does not deal with the issue of accelerated moving reference systems”.*

The physical sources of such a limited understanding of special relativity theory origin from A. Einstein. Let us present a number of his statements concerning special relativity theory. In 1913 he wrote [9]:

*“In the case of the customary theory of relativity only linear orthogonal substitutions are permissible”.*

In the next article of the same year he writes [10]:

*“While in the original theory of relativity the independent of the physical equations from the special choice of the reference system is based on the postulation of the fundamental invariant  $ds^2 = \sum_i dx_i^2$ , we are concerned with constructing a theory in which the most general line element of the form*

$$ds^2 = \sum_{i,k} g_{ik} dx^i dx^k$$

*plays the role of the fundamental invariant”.*

Later, in 1930, A. Einstein wrote [11]:

*“In the special theory of relativity those coordinate changes (by transformation) are permitted for which also in new coordinate system the quantity  $ds^2$  (fundamental invariant) equals the sum of squares of the coordinate. Such transformations are called Lorentz transformation”.*

Although Einstein, here, takes advantage of the invariant (interval) discovered by Poincaré, he understands it only in a limited (strictly diagonal) sense. For A. Einstein it was difficult to see that the Lorentz transformations and the relativity of time concealed a fundamental fact: space and time form a unique four-dimensional continuum with pseudo-Euclidean geometry, determined by the interval

$$ds^2 = \gamma_{\mu\nu}(x)dx^\mu dx^\nu, \quad \det(\gamma_{\mu\nu}) = \gamma < 0, \quad (3.35)$$

with the metric tensor  $\gamma_{\mu\nu}(x)$ , for which the Riemannian curvature tensor equals zero. But, precisely, the existence of the four-dimensional space of events with a pseudo-Euclidean metric permitted to establish that a number of vector quantities in Euclidean three-dimensional space are at the same time components of four-dimensional quantities together with certain scalars in Euclidean space.

This was performed by H. Poincaré and further developed by H. Minkowski. Very often, without understanding the essence of theory, some people write that Minkowski allegedly gave the geometrical interpretation of relativity theory. This is not true. **On the basis of the group discovered by Poincaré, H. Poincaré and H. Minkowski revealed the pseudo-Euclidean geometry of space-time, which is precisely the essence of special relativity theory.**

In 1909 H. Minkowski wrote about this in the article "Space and time":

*"Neither Einstein, nor Lorentz dealt with the concept of space, maybe because in the case of the aforementioned special transformation, under which the  $x', t'$  plane coincides with the  $x, t$  plane, it may be understood that the  $x$  axis of space retains its position. The attempt to thus evade the concept of space could have indeed been regarded as a certain impudence of*

*the mathematical thought. But after making this step, surely unavoidable for true comprehension of the  $G_c$  group (the Lorentz group. — A.L.), the term “relativity postulate” for requiring invariance with respect to the  $G_c$  group seems to me too insipid. Since the meaning of the postulate reduces to that in phenomena we only have the four-dimensional world in space and time, but that the projections of this world onto space and time can be taken with a certain arbitrariness, I would rather give this statement the title “postulate of the absolute world” or, to be short, world postulate”.*

It is surprising, but in H. Minkowski’s work there is no reference to the articles [2] and [3] by H. Poincaré, although it just gives the details of what had already been presented in refs. [2] and [3]. However, by the brilliant exposition before a broad audience of naturalists it attracted general attention. In 1913, in Germany, a collection of articles on relativity by H. A. Lorentz, A. Einstein, H. Minkowski was published. The fundamental works [2] and [3] by H. Poincaré were not included in the collection. In the comments by A. Sommerfeld to Minkowski’s work Poincaré is only mentioned in relation to particulars. Such hushing up of the fundamental works of H. Poincaré in relativity theory is difficult to understand.

E. Whittaker was the first who came to the conclusion of the decisive contribution of H. Poincaré to this problem when studying the history of creation of the special relativity theory, 50 years ago. His monograph caused a remarkably angry reaction of some authors. **But E. Whittaker was mainly right. H. Poincaré really created the special theory of relativity grounding upon the Lorentz work of 1904 and gave to this theory a general character by extending it onto all physical phenomena.** Instead of

a more thorough study and comparison of Einstein's 1905 work and Poincaré's papers (it is the only way of objective study of the problem) the way of complete rejection of Whittaker's conclusions was chosen. So, the idea that the theory of relativity was created independently and exclusively by A. Einstein was propagated in literature without detail investigations. This was also my view up to the middle of 80-s until I had read the articles by H. Poincaré and A. Einstein.

#### 4. The relativity of time and the contraction of length

Consider the course of time in two inertial reference systems, one of which will be considered to be at rest, while another one will move with respect to the first one with a velocity  $v$ . According to the relativity principle, the change in time shown by the clocks (for a given time scale) in both reference systems is the same. Therefore, the both count **their own physical** time in the same manner. If the clock in the moving reference system is at rest, then its interval in this system of reference is

$$d\sigma^2 = c^2 dt'^2, \quad (4.1)$$

$t'$  is the time shown by the clock in this reference system.

Since this clock moves relative to the other reference system with the velocity  $v$ , the same interval, but now in the reference system at rest will be

$$d\sigma^2 = c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right), \quad (4.2)$$

here  $t$  is the time shown by the clock at rest in this reference system, and

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2. \quad (4.3)$$

From relations (4.1) and (4.2) we find the relationship between the **time durations** in these inertial reference systems in the description of the **physical phenomenon**

$$dt' = dt \sqrt{1 - \frac{v^2}{c^2}}. \quad (4.4)$$

One often reads that a retardation of moving clocks takes place. It is wrong, because such a statement contradicts the principle of relativity. The clock rate in all inertial reference systems does not change. The clocks equally measure physical time of their own inertial system of reference. This is not a change of the clock rate but a change of a physical process duration. The duration of a local physical process according to the clock of this inertial system or a clock in other inertial system is in general different. It is minimal in the system where the process is localized in one spatial point. Precisely this meaning is implied in saying about the **retardation of time**.

Integrating this relation, we obtain

$$\Delta t' = \Delta t \sqrt{1 - \frac{v^2}{c^2}}. \quad (4.5)$$

This expression is a consequence of the existence of the fundamental invariant (3.22). The “time dilation” (4.5) was considered as early as in 1900 by J. Larmor. As noted by W. Pauli, “**it received its first clear statement only by Einstein**” from the “Lorentz transformation”.

We shall apply this equality to an elementary particle with a lifetime at rest equal to  $\tau_0$ . From (4.5) after setting  $\Delta t' = \tau_0$ , we find the lifetime of the moving particle

$$\Delta t = \frac{\tau_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (4.6)$$

Precisely owing to this effect it turns out to be possible to transport beams of high energy particles in vacuum over quite large distances from the accelerator to the experimental devices, although their lifetime in the state of rest is very small.

In the case considered above we dealt with a time-like interval  $d\sigma^2 > 0$ . We shall now consider another example, when the inter-

val between the events is space-like,  $d\sigma^2 < 0$ . Again we consider two such inertial reference systems. Consider measurement, in a moving reference system, of the length of a rod that is at rest in another reference system. We first determine the method for measuring the length of a moving rod. Consider an observer in the moving reference system, who records the ends of the rod,  $X'_1$  and  $X'_2$ , at the **same moment of time**

$$T'_1 = T'_2, \quad (4.7)$$

this permits to reduce the interval  $S_{12}^2$  in the moving reference system to the spatial part only

$$S_{12}^2 = -(X'_2 - X'_1)^2 = -\ell^2. \quad (4.8)$$

Thus, in our method of determining the length of a moving rod, it is rather natural to consider the quantity  $\ell$  as its length.

The same interval in the reference system at rest, where the rod is in the state of rest, is given as follows

$$S_{12}^2 = c^2(T_2 - T_1)^2 - (X_2 - X_1)^2. \quad (4.9)$$

But, in accordance with the Lorentz transformations we have

$$T'_2 - T'_1 = \gamma \left[ (T_2 - T_1) - \frac{v}{c^2} (X_2 - X_1) \right], \quad (4.10)$$

whence for our case (4.7) we find

$$T_2 - T_1 = \frac{v}{c^2} (X_2 - X_1) = \frac{v}{c^2} \ell_0, \quad (4.11)$$

$\ell_0$  is length of the rod in the reference system at rest. Substituting this expression into (4.9) we obtain

$$S_{12}^2 = -\ell_0^2 \left( 1 - \frac{v^2}{c^2} \right). \quad (4.12)$$

Comparing (4.8) and (4.12) we find

$$\ell = \ell_0 \sqrt{1 - \frac{v^2}{c^2}}. \quad (4.13)$$

From relations (4.7) and (4.11) we see, **that events that are simultaneous in one inertial reference system will not be simultaneous in another inertial reference system, so the notion of simultaneity is relative. Relativity of time** is a straightforward consequence of the **definition of simultaneity** for different spatial points of inertial reference system by means of a light signal. The contraction (4.13) is a consequence of the relative nature of simultaneity, or to be more precise, of the **existence of the fundamental invariant** (3.22).

Length contraction (4.13), as a hypothesis to explain the negative result of the Michelson-Morley experiment, was initially suggested by G.F. FitzGerald in 1889. Later, in 1892, the same hypothesis was formulated by H.A. Lorentz.

Thus, we have established that, in accordance with special relativity theory, the time interval between events for a local object and the length of a rod, given the method of measurement of (4.7), are relative. They depend on the choice of the inertial reference system. Only the interval between events has an absolute sense. It must be especially noted that contraction of the length of a rod (4.13) is determined not only by the pseudo-Euclidean structure of space-time, but also by our **method of measuring length**, so contraction, unlike the slowing down of time (4.5), does not have such essential physical significance. This is due to the slowing down of time being related to a **local object**, and such objects exist in Nature, they are described by the time-like interval  $d\sigma^2 > 0$ ; consequently, a causal relationship is realized, here. Contraction of length is related to different points in space and is, therefore, described by the space-like interval  $d\sigma^2 < 0$ , when no causal relationship is present.

Let us return to the issue of Lorentz contraction, determined by formula (4.13). We saw that in the case considered above, when the rod is at rest in the unprimed inertial reference system and has a length  $\ell_0$ , for all observers in other inertial reference systems there occurs, given the adopted method of measuring length (4.7), contraction, and the length will be determined by formula (4.13). It is quite evident that here nothing happens to the rod. Some authors call this contraction effect kinematical, since the rod undergoes no deformation, here. And they are right in this case, and there is no reason for criticizing them. However, it must be noted that this kinematics is a consequence of the pseudo-Euclidean structure of space, which reflects the general dynamic properties of matter — the conservation laws.

Back in 1905 H. Poincaré wrote the following about this situation:

*“If we were to accept the relativity principle, then we would find a common constant in the law of gravity and in electromagnetic laws, the velocity of light. Precisely in the same way, we would also encounter it in all the other forces of whatever origin, which can be only explained from two points of view: either everything existing in the world is of electromagnetic origin, or this property, that is, so to say, common to all physical phenomena, is nothing more, than an external appearance, something related to the methods of our measurements. How do we perform our measurements? Earlier we would have answered as follows: by carrying bodies, considered solid and unchangeable, one to the place of the other; but in modern theory, taking into account the Lorentz contraction, this is no longer correct. According to this theory, two segments are, by definition, equal, if light covers*

*them in the same time* (singled out by me. — A.L.)”.

A totally different situation arises in the case of motion with acceleration. If, for instance, the rod, that is at rest in the unprimed inertial reference system and has a length  $\ell_0$ , starts moving with acceleration along its length so that **both of its ends start moving simultaneously**, then in the reference system, related to the rod, its length will increase according to the law

$$L = \frac{\ell_0}{\sqrt{1 - \frac{v^2(t)}{c^2}}},$$

or, if formula (12.3) is taken into account, then one can express the velocity  $v(t)$  via the acceleration  $a$  and obtain the expression

$$L = \ell_0 \sqrt{1 + \frac{a^2 t^2}{c^2}}.$$

of events being the same in the unprimed inertial reference system and in the reference system moving with acceleration  $a$ , in which the rod is at rest. This means that the rod undergoes rupture strains.

Earlier we found the Lorentz transformations for the case, when the motion of one reference system with respect to another inertial reference system proceeded with a constant velocity along the  $X$  axis. Now, consider the general case, when the motion takes place with a velocity  $\vec{v}$  in an arbitrary direction

$$\vec{r} = \vec{R} - \vec{v}T. \quad (4.14)$$

Transformation (4.14) provides the transition to the inertial reference system the origin of which moves with a constant velocity  $\vec{v}$  related to the initial reference system.

Let us decompose vectors  $\vec{R}, \vec{R}'$  in the initial Galilean reference system along the direction of the velocity  $\vec{v}$  and along the

direction perpendicular to the velocity  $\vec{v}$ :

$$\vec{R} = \frac{\vec{v}}{|\vec{v}|} R_{\parallel} + \vec{R}_{\perp}, \quad \vec{R}' = \frac{\vec{v}}{|\vec{v}|} R'_{\parallel} + \vec{R}'_{\perp}. \quad (4.15)$$

On the basis of the Lorentz transformations (3.1), one may expect that only the longitudinal quantities will be changed, while the transverse quantities remain without change.

$$R'_{\parallel} = \gamma(R_{\parallel} - vT), \quad T' = \gamma\left(T - \frac{v}{c^2}R_{\parallel}\right), \quad \vec{R}'_{\perp} = \vec{R}_{\perp}. \quad (4.16)$$

In accordance with (4.15) we find

$$R_{\parallel} = \frac{(\vec{v}\vec{R})}{v}, \quad \vec{R}_{\perp} = \vec{R} - \frac{\vec{v}(\vec{v}\vec{R})}{v^2}. \quad (4.17)$$

Substituting (4.17) into (4.16) and, then, into (4.15) we obtain

$$\vec{R}' = \vec{R} + (\gamma - 1)\frac{(\vec{v}\vec{R})}{v^2}\vec{v} - \gamma\vec{v}T, \quad (4.18)$$

$$T' = \gamma\left(T - \frac{(\vec{v}\vec{R})}{c^2}\right). \quad (4.19)$$

In obtaining formulae (4.18) and (4.19) we have considered that under the general transformation (4.14) only the component  $\vec{R}$  of the vector along the velocity  $\vec{v}$  changes, in accordance with the Lorentz transformations (3.1), while the transverse component remains unchanged.

Let us verify that this assertion is correct. To this end we shall take as a starting point the invariant (3.22). Substituting (4.14) into (3.22) we obtain

$$J = c^2T^2 - (\vec{r} + \vec{v}T)^2 = c^2T^2\left(1 - \frac{v^2}{c^2}\right) - 2(\vec{v}\vec{r})T - \vec{r}^2. \quad (4.20)$$

In the invariant  $J$  we single out the time-like part

$$J = c^2 \left[ \frac{T}{\gamma} - \gamma \frac{(\vec{v} \vec{r})}{c^2} \right]^2 - \vec{r}^2 - \frac{\gamma^2}{c^2} (\vec{v} \vec{r})^2. \quad (4.21)$$

Our goal is to find such new variables  $T'$  and  $\vec{R}'$ , in which this expression can be written in a diagonal form

$$J = c^2 T'^2 - \vec{R}'^2. \quad (4.22)$$

From comparison of (4.21) and (4.22) we find time  $T'$  in the moving reference system:

$$T' = \frac{T}{\gamma} - \frac{\gamma}{c^2} (\vec{v} \vec{r}). \quad (4.23)$$

Expressing the right-hand part in (4.23) via the variables  $T$ ,  $\vec{R}$ , we obtain

$$T' = \gamma \left( T - \frac{\vec{v} \vec{R}}{c^2} \right). \quad (4.24)$$

We also express the space-like part of the invariant  $J$  in terms of variables  $T$ ,  $\vec{R}$

$$\vec{r}^2 + \frac{\gamma^2}{c^2} (\vec{v} \vec{r})^2 = \vec{R}^2 + \frac{\gamma^2}{c^2} (\vec{v} \vec{R})^2 - 2\gamma^2 (\vec{v} \vec{R}) T + \gamma^2 v^2 T^2. \quad (4.25)$$

One can readily verify that the first two terms in (4.25) can be written in the form

$$\vec{R}^2 + \frac{\gamma^2}{c^2} (\vec{v} \vec{R})^2 = \left[ \vec{R} + (\gamma - 1) \frac{\vec{v}(\vec{v} \vec{R})}{v^2} \right]^2, \quad (4.26)$$

and, consequently, expression (4.25) assumes the form

$$\begin{aligned} \vec{r}^2 + \frac{\gamma^2}{c^2} (\vec{v} \vec{r})^2 &= \left[ \vec{R} + (\gamma - 1) \frac{\vec{v}(\vec{v} \vec{R})}{v^2} \right]^2 - \\ &- 2\gamma^2 (\vec{v} \vec{R}) T + v^2 \gamma^2 T^2. \end{aligned} \quad (4.27)$$

The right-hand part of (4.27) can be written as

$$\vec{r}^2 + \frac{\gamma^2}{c^2}(\vec{v}\vec{r})^2 = \left[ \vec{R} + (\gamma - 1)\frac{\vec{v}(\vec{v}\vec{R})}{v^2} - \gamma\vec{v}T \right]^2. \quad (4.28)$$

Thus, the space-like part of the invariant  $J$  assume a diagonal form

$$\vec{r}^2 + \frac{\gamma^2}{c^2}(\vec{v}\vec{r})^2 = (\vec{R}')^2, \quad (4.29)$$

where

$$\vec{R}' = \vec{R} + (\gamma - 1)\frac{\vec{v}(\vec{v}\vec{R})}{v^2} - \gamma\vec{v}T. \quad (4.30)$$

Formulae (4.24) and (4.30) coincide with formulae (4.18) and (4.19); this testifies to our assumption, made earlier in the course of their derivation, being correct.

It should be especially emphasized that we have derived above the **general formulae** relating coordinates  $(T, \vec{R})$  of the initial inertial reference system to coordinates  $(T', \vec{R}')$  of the reference system moving with constant velocity  $\vec{v}$  relative to the first system. We have used the form-invariance of invariant (3.22) and **identical transformations** only. If by means of transformations (4.24) and (4.30) we go from an inertial reference system  $S$  to a system  $S'$  and later to a system  $S''$ , then after these two subsequent transformations we will get transformation which will be different from transformations (4.24) and (4.30) by a rotation in 3-dimensional space. It means that transformations (4.24) and (4.30) do not form a subgroup of the Lorentz group. The rotation mentioned above reduces the axes of the reference system  $S$  to the same orientation as axes of the system  $S''$ . Thomas' effect which will be considered in Section 14 is caused just by this circumstance.

The general derivation of Lorentz transformations from the relativity principle, Galilean principle of inertia and the wavefront equation was done by Academician V.A. Fock in his monograph

[12, Appendix A]. His analysis demonstrates that it is impossible to derive Lorentz transformations from the two Einstein postulates only (see p. 39 of this book).

## 5. Adding velocities

Differentiating the Lorentz transformations (3.1) with respect to the variable  $T$ , we obtain the formulae relating velocities

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}, \quad u'_y = u_y \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{u_x v}{c^2}}, \quad u'_z = u_z \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{u_x v}{c^2}}. \quad (5.1)$$

Here

$$u_x = \frac{dX}{dT}, \quad u_y = \frac{dY}{dT}, \quad u_z = \frac{dZ}{dT}, \quad (5.2)$$

$$u'_x = \frac{dX'}{dT'}, \quad u'_y = \frac{dY'}{dT'}, \quad u'_z = \frac{dZ'}{dT'}. \quad (5.3)$$

In deriving (5.1) we made use of the formula

$$\frac{dT'}{dT} = \gamma \left( 1 - \frac{v}{c^2} u_x \right). \quad (5.4)$$

It is possible, in a similar way, to obtain general formulae, also, if one takes advantage of expressions (4.18), (4.19):

$$\vec{u}' = \frac{\vec{u} + (\gamma - 1) \frac{(\vec{v}\vec{u})}{v^2} \vec{v} - \gamma \vec{v}}{\gamma \left[ 1 - \frac{(\vec{v}\vec{u})}{c^2} \right]}, \quad \frac{dT'}{dT} = \gamma \left[ 1 - \frac{(\vec{v}\vec{u})}{c^2} \right]. \quad (5.5)$$

We will further (Section 16) see, that the velocity space is the Lobachevsky space.

## 6. Elements of vector and tensor analysis in Minkowski space

All physical quantities must be defined in a way, to have their physical meaning independent on the choice of reference system.

Consider a certain reference system  $x^\nu$ ,  $\nu = 0, 1, 2, 3$  given in four-dimensional Minkowski space. Instead of this reference system, it is possible to choose another, defined by the expression

$$x'^\nu = f^\nu(x^\sigma). \quad (6.1)$$

We shall consider functions  $f^\nu$  as continuous and differentiable.

If at any point Jacobian of the transformation

$$J = \det \left| \frac{\partial f^\nu}{\partial x^\sigma} \right| \quad (6.2)$$

differs from zero, then under this condition the variables  $x'^\nu$  will be independent, and, consequently, the initial variables  $x^\nu$  may unambiguously be expressed in terms of the new ones  $x'^\nu$

$$x^\alpha = \varphi^\alpha(x'^\sigma). \quad (6.3)$$

Physical quantities must not depend on the choice of reference system, therefore, it should be possible to express them in terms of geometrical objects. The simplest geometrical object is scalar  $\phi(x)$ , which under transition to new variables transforms as follows:

$$\phi'(x') = \phi[x(x')]. \quad (6.4)$$

The gradient of scalar function  $\phi(x)$  transforms by the rule for differentiating composite functions

$$\frac{\partial \phi'(x')}{\partial x'^\nu} = \frac{\partial \phi}{\partial x^\sigma} \cdot \frac{\partial x^\sigma}{\partial x'^\nu}, \quad (6.5)$$

here, and below, summation is performed over identical indices  $\sigma$  from 0 to 3.

A set of functions that transforms, under coordinate transformation, by the rule (6.5) is called a **covariant vector**

$$A'_\nu(x') = \frac{\partial x^\sigma}{\partial x'^\nu} A_\sigma(x). \quad (6.6)$$

Accordingly, a quantity  $B_{\mu\nu}$ , that transforms by the rule

$$B'_{\mu\nu}(x') = \frac{\partial x^\sigma}{\partial x'^\mu} \cdot \frac{\partial x^\lambda}{\partial x'^\nu} B_{\sigma\lambda}, \quad (6.7)$$

is called a covariant tensor of the second rank, and so on.

Let us now pass to another group of geometrical objects. Consider the transformation of the differentials of coordinates:

$$dx'^\nu = \frac{\partial x'^\nu}{\partial x^\sigma} dx^\sigma. \quad (6.8)$$

A set of functions that transforms, under coordinate transformations, by the rule (6.8) is called a **contravariant vector**:

$$A'^\nu(x') = \frac{\partial x'^\nu}{\partial x^\sigma} A^\sigma(x). \quad (6.9)$$

Accordingly, a quantity  $B^{\mu\nu}$ , that transforms by the rule

$$B'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\sigma} \cdot \frac{\partial x'^\nu}{\partial x^\lambda} B^{\sigma\lambda}(x), \quad (6.10)$$

has been termed a contravariant tensor of the second rank, and so on.

From the transformational properties of a vector or a tensor it follows, that, if all its components are zero in one reference system, then they are zero, also, in any other reference system. Note, that **coordinates  $x^\nu$  do not form a vector**, while the differential

$dx^\nu$  is a vector. The coordinates  $x^\nu$  only form a vector with respect to linear transformations.

Now, we calculate the quantity  $A'_\sigma(x')B'^\sigma(x')$

$$A'_\sigma(x')B'^\sigma(x') = \frac{\partial x^\mu}{\partial x'^\sigma} \cdot \frac{\partial x'^\sigma}{\partial x^\lambda} A_\mu(x)B^\lambda(x), \quad (6.11)$$

but, it is easy to see that

$$\frac{\partial x^\mu}{\partial x'^\sigma} \cdot \frac{\partial x'^\sigma}{\partial x^\lambda} = \delta_\lambda^\mu = \begin{cases} 0, & \text{for } \mu \neq \lambda \\ 1, & \text{for } \mu = \lambda. \end{cases} \quad (6.12)$$

The symbol  $\delta_\lambda^\mu$  is a mixed tensor of the second rank and is known as the Kronecker symbol.

Taking into account (6.12) in expression (6.11) we find

$$A'_\sigma(x')B'^\sigma(x') = A_\lambda(x)B^\lambda(x). \quad (6.13)$$

Hence it is evident that this **quantity is a scalar, it is usually called an invariant.**

In writing expression (3.32) we actually dealt with the fundamental invariant

$$d\sigma^2 = \gamma_{\mu\lambda}(x)dx^\mu dx^\lambda, \quad \det(\gamma_{\mu\nu}) = \gamma < 0. \quad (6.14)$$

The existence of the metric tensor of Minkowski space, that has the general form (3.33), permits to raise and to lower indices of vector and tensor quantities, for example:

$$A_\nu = \gamma_{\nu\lambda}(x)A^\lambda, \quad A^\lambda = \gamma^{\lambda\sigma}A_\sigma, \quad A_\nu A^\nu = \gamma_{\lambda\sigma}A^\lambda A^\sigma. \quad (6.15)$$

$$\gamma_{\mu\lambda} \gamma^{\lambda\nu} = \delta_\mu^\nu. \quad (6.16)$$

Tensors can be added and subtracted, for example,

$$C_{\mu\nu\sigma}^{\alpha\beta} = A_{\mu\nu\sigma}^{\alpha\beta} \pm B_{\mu\nu\sigma}^{\alpha\beta}. \quad (6.17)$$

They can also be multiplied, independently of their structure

$$C_{\mu\nu\sigma\rho}^{\alpha\beta\lambda} = A_{\mu\nu\sigma}^{\alpha\beta} \cdot B_{\rho}^{\lambda}. \quad (6.18)$$

Here it is necessary to observe both the order of multipliers and the order of indices.

Transformations (6.9) form a group. Consider

$$A^{\nu}(x') = \frac{\partial x^{\nu}}{\partial x^{\sigma}} A^{\sigma}(x), \quad A''^{\mu} = \frac{\partial x''^{\mu}}{\partial x'^{\lambda}} A'^{\lambda}(x'), \quad (6.19)$$

hence we have

$$A''^{\mu}(x'') = \frac{\partial x''^{\mu}}{\partial x'^{\lambda}} \cdot \frac{\partial x'^{\lambda}}{\partial x^{\sigma}} A^{\sigma}(x) = \frac{\partial x''^{\mu}}{\partial x^{\sigma}} A^{\sigma}(x). \quad (6.20)$$

Note that tensor calculus does not depend on the metric properties of space. It is persisted, for example, in Riemannian geometry, where the group of motion of space-time is absent, in the general case. On the other hand, the group of general coordinate transformations (6.19–6.20) is fully persisted, since it is independent of the metric properties of space, but it has no any physical meaning.

## 7. Lorentz group

**H.Poincaré discovered that the Lorentz transformations, together with all space rotations, form a group.** Consider, for example,

$$x' = \gamma_1(x - v_1t), \quad t' = \gamma_1 \left( t - \frac{v_1}{c^2}x \right), \quad (7.1)$$

$$x'' = \gamma_2(x' - v_2t'), \quad t'' = \gamma_2 \left( t' - \frac{v_2}{c^2}x' \right). \quad (7.2)$$

Substituting (7.1) into (7.2) we obtain

$$x'' = \gamma_1\gamma_2 \left( 1 + \frac{v_1v_2}{c^2} \right) x - \gamma_1\gamma_2(v_1 + v_2)t, \quad (7.3)$$

$$t'' = \gamma_1\gamma_2 \left( 1 + \frac{v_1v_2}{c^2} \right) t - \gamma_1\gamma_2 \left( \frac{v_1 + v_2}{c^2} \right) x, \quad (7.4)$$

But since

$$x'' = \gamma(x - vt), \quad t'' = \gamma \left( t - \frac{v}{c^2}x \right). \quad (7.5)$$

From comparison of (7.3) and (7.4) with (7.5) we obtain

$$\gamma = \gamma_1\gamma_2 \left( 1 + \frac{v_1v_2}{c^2} \right), \quad \gamma v = \gamma_1\gamma_2(v_1 + v_2). \quad (7.6)$$

From relations (7.6) we find

$$v = \frac{v_1 + v_2}{1 + \frac{v_1v_2}{c^2}}. \quad (7.7)$$

It is readily verified that

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma_1\gamma_2 \left( 1 + \frac{v_1v_2}{c^2} \right). \quad (7.8)$$

Thus, we have established that transition from the reference system  $x^\nu$  to the reference system  $x'^\nu$  and, subsequently, to the reference system  $x''^\nu$  is equivalent to direct transition from the reference system  $x^\nu$  to the reference system  $x''^\nu$ . Precisely in this case, it can be said that the Lorentz transformations form a group. **Poincaré discovered [2] this group and named it the Lorentz group. He found the group generators and constructed the Lie algebra of the Lorentz group. Poincaré was the first to establish that, for universal invariance of the laws of Nature with respect to the Lorentz transformations to hold valid, it is necessary for the physical fields and for other dynamical and kinematical characteristics to form a set of quantities transforming under the Lorentz transformations in accordance with the group, or, to be more precise, in accordance with one of the representations of the Lorentz group.**

Several general words about a group. A group is a set of elements  $A, B, C \dots$  for which the operation of multiplication is defined. Elements may be of any nature. The product of any two elements of a group yields an element of the same group. In the case of a group, multiplication must have the following properties.

1. The law of associativity

$$(AB)C = A(BC).$$

2. A group contains a unit element  $E$

$$AE = A.$$

3. Each element of a group has its inverse element

$$AB = E, \quad B = A^{-1}.$$

Transformations of the Lorentz group can be given in matrix form

$$X' = AX,$$

where

$$A = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma \\ -\frac{v}{c}\gamma & \gamma \end{pmatrix}, \quad X = \begin{pmatrix} x \\ x_0 \end{pmatrix}, \quad X' = \begin{pmatrix} x' \\ x'_0 \end{pmatrix},$$
$$x'_0 = ct', \quad x_0 = ct.$$

It is readily verified, that the set of all Lorentz transformations satisfies all the listed requirements of a group.

Coordinate transformations which preserve the form of the metric tensor form the **group of motions of the space**. In particular the Lorentz group is such a group.

## 8. Invariance of Maxwell-Lorentz equations

The Maxwell-Lorentz equations in an inertial reference system, said to be “at rest“, have the form

$$\begin{aligned} \operatorname{rot} \vec{H} &= \frac{4\pi}{c} \rho \vec{v} + \frac{1}{c} \cdot \frac{\partial \vec{E}}{\partial t}, \quad \operatorname{rot} \vec{E} = -\frac{1}{c} \cdot \frac{\partial \vec{H}}{\partial t}, \\ \operatorname{div} \vec{E} &= 4\pi \rho, \quad \operatorname{div} \vec{H} = 0, \end{aligned} \quad (8.1)$$

$$\vec{f} = \rho \vec{E} + \frac{1}{c} \rho [\vec{v}, \vec{H}]. \quad (8.2)$$

The second term in the right-hand part of the first equation of (8.1) is precisely that small term — the **displacement current**, introduced by Maxwell in the equations of electrodynamics. Namely it was mentioned in Section 3. Since the divergence of a curl is zero, from the first and third equations of (8.1) follows the conservation law of current

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0, \quad \vec{j} = \rho \vec{v}. \quad (8.3)$$

As one sees from (8.3), the displacement current permitted to achieve accordance between the equations of electrodynamics and the conservation law of electric charge. To make the fourth equation from (8.1) be satisfied identically we represent  $\vec{H}$  in the form

$$\vec{H} = \operatorname{rot} \vec{A}. \quad (8.4)$$

Thus, we have introduced the vector potential  $\vec{A}$ . Substituting expression (8.4) into the second equation from (8.1) we obtain

$$\operatorname{rot} \left( \vec{E} + \frac{1}{c} \cdot \frac{\partial \vec{A}}{\partial t} \right) = 0. \quad (8.5)$$

To have equation (8.5) satisfied identically the expression in brackets must be gradient of some function  $\phi$

$$\vec{E} = -\frac{1}{c} \cdot \frac{\partial \vec{A}}{\partial t} - \text{grad } \phi. \quad (8.6)$$

Thus, we have introduced the notion of scalar potential  $\phi$ . For given values of  $\vec{E}$  and  $\vec{H}$  the potentials  $\phi$  and  $\vec{A}$ , as we shall see below (Section 10), are determined ambiguously. So by choosing them to provide for the validity of the L. Lorenz condition

$$\frac{1}{c} \cdot \frac{\partial \phi}{\partial t} + \text{div } \vec{A} = 0, \quad (8.7)$$

from equations (8.1), with account of formulae

$$\text{div grad } \phi = \nabla^2 \phi, \quad \text{rot rot } \vec{A} = \text{grad div } \vec{A} - \nabla^2 \vec{A},$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2},$$

and relation (8.7) as well, we find equations for potentials  $\phi$  and  $\vec{A}$  in the following form:

$$\square \vec{A} = \frac{4\pi}{c} \vec{j}, \quad \square \phi = 4\pi \rho. \quad (8.8)$$

For the equation of charge conservation to be form-invariant with respect to the Lorentz transformations it is necessary that the density  $\rho$  and current be components of the contravariant vector  $S^\nu$

$$S^\nu = (c\rho, \vec{j}) = (S^0, \vec{S}), \quad \vec{j} = \rho \vec{v}. \quad (8.9)$$

The contravariant vector  $S^\nu$  transforms under the Lorentz transformations in the same way as  $(ct, \vec{x})$ . The equation (8.3) of charge conservation assume the form

$$\frac{\partial S^\nu}{\partial x^\nu} = 0, \quad (8.10)$$

summation is performed over identical indices  $\nu$ . Taking into account Eq. (8.9) we rewrite Eqs. (8.8) as follows

$$\square \vec{A} = \frac{4\pi}{c} \vec{S}, \quad \square \phi = \frac{4\pi}{c} S^0. \quad (8.11)$$

For these equations not to alter their form under Lorentz transformations, it is necessary that the scalar and vector potentials be components of a contravariant vector  $A^\nu$

$$A^\nu = (A^0, \vec{A}) = (\phi, \vec{A}). \quad (8.12)$$

Since, as we showed earlier, the operator  $\square$  does not alter its form under the Lorentz transformations, Eqs. (8.11) at any inertial system of reference will have the following form

$$\square A^\nu = \frac{4\pi}{c} S^\nu, \quad \nu = 0, 1, 2, 3. \quad (8.13)$$

**The vectors  $S^\nu$  and  $A^\nu$  were first introduced by Henri Poincaré [3].**

Unification of  $\phi$  and  $\vec{A}$  into the four-vector  $A^\nu$  is necessary, since, as the right-hand part of (8.13) represents the vector  $S^\nu$ , then the left-hand part must also transform like a vector. Hence, it directly follows that, if in a certain inertial reference system only an electric field exists, then in any other reference system there will be found, together with the electric field, a magnetic field, also, owing to  $A^\nu$  transforming like a vector. This is an immediate consequence of validity of the relativity principle for electromagnetic phenomena.

The Lorentz transformations for the vector  $S^\nu$  have the same form, as in the case of the vector  $(ct, \vec{x})$

$$S'_x = \gamma \left( S_x - \frac{u}{c} S_0 \right), \quad S'^0 = \gamma \left( S^0 - \frac{u}{c} S_x \right). \quad (8.14)$$

Taking into account the components of the vector  $S^\nu$  (8.9), we find

$$\rho' = \gamma\rho \left(1 - \frac{u}{c^2}v_x\right), \quad \rho'v'_x = \gamma\rho(v_x - u). \quad (8.15)$$

Here

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad (8.16)$$

where  $u$  is the velocity of the reference system.

The transformations for the components  $S_y, S_z$  have the form

$$\rho'v'_y = \rho v_y, \quad \rho'v'_z = \rho v_z. \quad (8.17)$$

**All these formulae were first obtained by H. Poincaré [2].** From these the formulae for velocity addition follow

$$v'_x = \frac{v_x - u}{1 - \frac{uv_x}{c^2}}, \quad v'_y = v_y \frac{\sqrt{1 - \frac{u^2}{c^2}}}{1 - \frac{uv_x}{c^2}}, \quad v'_z = v_z \frac{\sqrt{1 - \frac{u^2}{c^2}}}{1 - \frac{uv_x}{c^2}}. \quad (8.18)$$

We now introduce the covariant vector  $S_\nu$

$$S_\nu = \gamma_{\nu\lambda} S^\lambda. \quad (8.19)$$

Taking into account that  $\gamma_{\nu\lambda} = (1, -1, -1, -1)$ , we obtain from (8.19)

$$S_0 = S^0, \quad S_i = -S^i, \quad i = 1, 2, 3. \quad (8.20)$$

Now compose the invariant

$$S_\nu S^\nu = c^2 \rho^2 \left(1 - \frac{v^2}{c^2}\right) = c^2 \rho_0^2, \quad (8.21)$$

here  $\rho_0$  is the charge density in the reference system, where the charge is at rest. Hence we have

$$\rho_0 = \rho \sqrt{1 - \frac{v^2}{c^2}}. \quad (8.22)$$

**H. Minkowski introduced antisymmetric tensor  $F_{\mu\nu}$** 

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}, \quad \nu = 0, 1, 2, 3, \quad (8.23)$$

which automatically satisfies the equation

$$\frac{\partial F_{\mu\nu}}{\partial x^\sigma} + \frac{\partial F_{\nu\sigma}}{\partial x^\mu} + \frac{\partial F_{\sigma\mu}}{\partial x^\nu} = 0. \quad (8.24)$$

Since  $\vec{H} = \text{rot } \vec{A}$ ,  $\vec{E} = -\text{grad } \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$ , the following equations are easily verified

$$-H_x = F_{23}, \quad -H_y = F_{31}, \quad -H_z = F_{12}, \quad (8.25)$$

$$-E_x = F_{10}, \quad -E_y = F_{20}, \quad -E_z = F_{30}.$$

The set of equations (8.24) is equivalent to the set of Maxwell equations

$$\text{rot } \vec{E} = -\frac{1}{c} \cdot \frac{\partial \vec{H}}{\partial t}, \quad \text{div } \vec{H} = 0. \quad (8.26)$$

With the aid of the tensor  $F^{\mu\nu}$ , the set of equations (8.13) can be written as follows:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = -\frac{4\pi}{c} S^\mu. \quad (8.27)$$

The tensor  $F^{\mu\nu}$  is related to the field components  $\vec{E}$  and  $\vec{H}$  by the following relations:

$$-H_x = F^{23}, \quad -H_y = F^{31}, \quad -H_z = F^{12}, \quad (8.28)$$

$$E_x = F^{10}, \quad E_y = F^{20}, \quad E_z = F^{30}.$$

All this can be presented in the following table form, where first index  $\mu = 0, 1, 2, 3$  numerates lines, and second  $\nu$  — columns

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix},$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix}.$$

Hence it is seen that the quantities  $\vec{E}$  and  $\vec{H}$  change under the Lorentz transformations like individual components of the tensor  $F^{\mu\nu}$ . Neither Lorentz, nor Einstein established this, so they, did not succeed in demonstrating the invariance of the Maxwell-Lorentz equations with respect to the Lorentz transformations neither in space without charges, nor in space with charges.

**We emphasize that the identical appearance of equations in two systems of coordinates under Lorentz transformations still does not mean their form-invariance under these transformations. To prove the form-invariance of equations we are to ascertain that Lorentz transformations form a group and field variables (for example,  $\vec{E}$  and  $\vec{H}$ ) transform according to some representation of this group.**

Taking into account the relationship between the components of the tensor  $F^{\mu\nu}$  and the components of the electric and magnetic fields, it is possible to obtain the transformation law for the components of the electric field

$$E'_x = E_x, \quad E'_y = \gamma \left( E_y - \frac{u}{c} H_z \right),$$

$$E'_z = \gamma \left( E_z + \frac{u}{c} H_y \right),$$
(8.29)

and for the components of the magnetic field

$$\begin{aligned} H'_x &= H_x, \quad H'_y = \gamma \left( H_y + \frac{u}{c} E_z \right), \\ H'_z &= \gamma \left( H_z - \frac{u}{c} E_y \right). \end{aligned} \quad (8.30)$$

**These formulae were first discovered by Lorentz, however, neither he, nor, later, Einstein established their group nature. This was first done by H. Poincaré, who discovered the transformation law for the scalar and vector potentials [3].** Since  $\phi$  and  $\vec{A}$  transform like  $(ct, \vec{x})$ , H. Poincaré has found, with the aid of formulae (8.4) and (8.6), the procedure of calculation for the quantities  $\vec{E}$  and  $\vec{H}$  under transition to any other inertial reference system.

From the formulae for transforming the electric and magnetic fields it follows that, if, for example, in a reference system  $K'$  the magnetic field is zero, then in another reference system it already differs from zero and equals

$$H_y = -\frac{u}{c} E_z, \quad H_z = \frac{u}{c} E_y, \quad \text{or } \vec{H} = \frac{1}{c} [\vec{u}, \vec{E}]. \quad (8.31)$$

From the field components it is possible to construct two invariants with respect to the Lorentz transformations.

$$E^2 - H^2, \quad (\vec{E}\vec{H}). \quad (8.32)$$

**These invariants of the electromagnetic field were first discovered by H. Poincaré [3].**

The invariants (8.32) can be expressed via antisymmetric tensor of the electromagnetic field  $F^{\mu\nu}$

$$E^2 - H^2 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}, \quad \vec{E}\vec{H} = -\frac{1}{4} F_{\mu\nu} F^{*\mu\nu} \quad (8.33)$$

here

$$\vec{F}^{\mu\nu} = -\frac{1}{2}\varepsilon^{\mu\nu\sigma\lambda}F_{\sigma\lambda}, \quad (8.34)$$

$\varepsilon^{\mu\nu\sigma\lambda}$  is the Levi-Civita tensor,  $\varepsilon^{0123} = 1$ , transposition of any two indices alters the sign of the Levi-Civita tensor.

In accordance with the second invariant (8.32), the fields  $\vec{E}$  and  $\vec{H}$ , that are reciprocally orthogonal in one reference system, persist this property in any other reference system. If in reference system  $K$  the fields  $\vec{E}$  and  $\vec{H}$  are orthogonal, but not equal, it is always possible to find such a reference system, in which the field is either purely electric or purely magnetic, depending on the sign of the first invariant from (8.33).

Now let us consider the derivation of the **Poynting equation** (1884). To do so we multiply both parts of first equation from Eqs. (8.1) by vector  $\vec{E}$ , and both parts of second equation from Eqs. (8.1) — by vector  $\vec{H}$ ; then we subtract the results and obtain

$$\frac{1}{4\pi} \left( \vec{E} \frac{\partial \vec{E}}{\partial t} + \vec{H} \frac{\partial \vec{H}}{\partial t} \right) = -\rho \vec{v} \vec{E} - \frac{c}{4\pi} \left( \vec{H} \text{rot } \vec{E} - \vec{E} \text{rot } \vec{H} \right).$$

By using the following formula from vector analysis

$$\text{div}[\vec{a}, \vec{b}] = \vec{b} \text{rot } \vec{a} - \vec{a} \text{rot } \vec{b},$$

we obtain the **Poynting equation**

$$\frac{\partial}{\partial t} \left( \frac{E^2 + H^2}{8\pi} \right) = -\rho \vec{v} \vec{E} - \text{div } \vec{S},$$

where

$$\vec{S} = \frac{c}{4\pi} [\vec{E} \vec{H}]$$

is called the **Poynting vector**. After integration of the Poynting equation over volume  $V$  and using Gauss theorem we get

$$\frac{\partial}{\partial t} \int_V \frac{E^2 + H^2}{8\pi} dV = - \int_V \rho \vec{v} \vec{E} dV - \oint_{\Sigma} \vec{S} d\vec{\sigma}.$$

The term standing in l.h.s. determines a change of electromagnetic energy in volume  $V$  at a unit of time. The first term in r.h.s. characterizes work done by electric field on charges in volume  $V$ . The second term in r.h.s. determines the **energy flow of electromagnetic field** through surface  $\Sigma$ , bounding volume  $V$ .

Formulation of the energy conservation law with help of the **notion of the energy flow** was first proposed by N. A. Umov. The notion of the **energy flow** has become one of the most important in physics. With help of the **Poynting equation** it is possible to prove the **uniqueness theorem** in the following formulation (see: *I. E. Tamm Foundations of the theory of electricity*. Moscow: “Nauka”, 1976 (in Russian), pp. 428-429):

*“...electromagnetic field at any instant of time  $t_1 > 0$  and at any point of volume  $V$ , bounded by an arbitrary closed surface  $S$  is uniquely determined by Maxwell equations, if initial values for electromagnetic vectors  $\vec{E}$  and  $\vec{H}$  are prescribed in all this part of space at time  $t = 0$  and if also **for one of these vectors (for example,  $\vec{E}$ ) boundary values of its tangential components on surface  $S$  are given during the whole time interval from  $t = 0$  to  $t = t_1$ .***

*Let us suppose the opposite, i. e. suppose there are two different systems of solutions of Maxwell equations  $\vec{E}', \vec{H}'$  and  $\vec{E}'', \vec{H}''$ , satisfying the same initial and boundary conditions. Due to linear character of the field equations the difference of these solutions  $\vec{E}''' = \vec{E}' - \vec{E}''$  and  $\vec{H}''' = \vec{H}' - \vec{H}''$  should also satisfy Maxwell equations under the following additional conditions:*

- a)  $\vec{E}^{\text{extra}} = 0$ ,*
- b) at time  $t = 0$  in each point of the volume  $V$ :  $\vec{E}''' = 0, \vec{H}''' = 0$  (because at  $t = 0$   $\vec{E}', \vec{E}''$  and  $\vec{H}', \vec{H}''$*

have, as supposed, equal given values),

c) during the whole time interval from  $t = 0$  to  $t = t_1$  in all points of the surface  $S$  tangential components of vector  $\vec{E}'''$  or vector  $\vec{H}'''$  are equal to zero (by the same reason).

Let us apply the Poynting theorem (which is a consequence of Maxwell equations) to this field  $\vec{E}'''$ ,  $\vec{H}'''$  and put work of extraneous forces  $P$  equal to zero. The surface integral which enters the Poynting equation is equal to zero during the whole time interval from  $t = 0$  to  $t = t_1$ , because from Eq. (c) it follows that on surface  $S$

$$S_n = [\vec{E}''' \vec{H}''']_n = 0;$$

therefore, at any time during this interval we get

$$\frac{\partial W'''}{\partial t} = - \int_V \frac{\vec{j}'''^2}{\lambda} dV.^2$$

As the integrand is strictly positive, we have

$$\frac{\partial W'''}{\partial t} \leq 0,$$

i. e. field energy  $W'''$  may decrease or stay constant. But at  $t = 0$ , according to Eq. (b), energy  $W'''$  of field  $\vec{E}'''$ ,  $\vec{H}'''$  is equal to zero. It also can not become negative, therefore during the whole interval considered  $0 \leq t \leq t_1$  the energy

$$W''' = \frac{1}{8\pi} \int_V (\vec{E}'''^2 + \vec{H}'''^2) dV$$

---

<sup>2</sup> $\vec{j}''' = \lambda \vec{E}'''$

should stay equal to zero. This may take place only if  $\vec{E}'''$  and  $\vec{H}'''$  stay equal to zero at all points of the volume  $V$ . Therefore, the two systems of solutions of the initial problem  $\vec{E}', \vec{H}'$  and  $\vec{E}'', \vec{H}''$ , supposed by us to be different, are necessarily identical. So the uniqueness theorem is proved.

It is easy to get convinced that in case of the whole infinite space the fixing of field vectors values on the bounding surface  $S$  may be replaced by putting the following conditions at infinity:

$$ER^2 \text{ and } HR^2 \text{ at } R \rightarrow \infty \text{ stay finite.}$$

Indeed, it follows from these conditions that the integral of the Poynting vector over an infinitely distant surface is occurred to be zero. This fact enables us to prove applicability of the above inequality to the whole infinite space, starting from the Poynting equation. Also uniqueness of solutions for the field equations follows from this inequality”.

**For consistency with the relativity principle for all electromagnetic phenomena, besides the requirement that the Maxwell-Lorentz equations remain unaltered under the Lorentz transformations, it is necessary that the equations of motion of charged particles under the influence of the Lorentz force remain unaltered, also.**

All the aforementioned was only performed in works [2, 3] by H. Poincaré. The invariability of physical equations in all inertial reference systems is just what signifies the identity of physical phenomena in these reference systems under identical conditions. Precisely for this reason, all **natural standards** are **identical** in all inertial reference systems. Hence, for instance, follows the **equality** of the *NaCl* crystal lattice constants taken to be at rest in two

inertial reference systems moving with respect to each other. This is just the essence of the relativity principle. The relativity principle was understood exactly in this way in classical mechanics, also. Therefore, one can only be surprised at what Academician V. L. Ginzburg writes in the same article (see this edition, the footnote on page 51):

*“I add that, having reread now (70 years after they were published!) the works of Lorentz and Poincaré, I have been only able with difficulty and knowing the result beforehand (which is known to extremely facilitate apprehension) to understand why invariance of the equations of electrodynamics with respect to the Lorentz transformations, demonstrated in those works, could at the time be considered as evidence for validity of the relativity principle”.*

Though A. Einstein wrote in 1948

*“With the aid of the Lorentz transformation the special relativity principle can be formulated as follows: the laws of Nature are invariant with respect to the Lorentz transformation (i. e. a law of Nature must not change, if it would be referred to a new inertial reference system obtained with the aid of Lorentz transformation for  $x, y, z, t$ )”.*

Now, compare the above with that written by H. Poincaré in 1905:

*“... If it is possible to give general translational motion to a whole system without any visible changes taking place in phenomena, this means that the equations of the electromagnetic field will not change as a result of certain transformations, which we shall call*

*Lorentz transformations; two systems, one at rest, and another undergoing translational motion represent, therefore, an exact image of each other”.*

We see that classical works require attentive reading, not to mention contemplation.

We shall now establish the law for transformation of the Lorentz force under transformation from one inertial reference system to another. The equations of motion will be established in Section 9. The expression for the Lorentz force, referred to unit volume, will, in reference system  $K$ , have the form (8.2)

$$\vec{f} = \rho \vec{E} + \rho \frac{1}{c} [\vec{v}, \vec{H}]. \quad (8.35)$$

Then, in reference system  $K'$  we must have a similar expression

$$\vec{f}' = \rho' \vec{E}' + \rho' \frac{1}{c} [\vec{u}', \vec{H}']. \quad (8.36)$$

Replacing all the quantities by their values (8.15), (8.17), (8.29), (8.30) and (8.35), we obtain

$$f'_x = \gamma \left( f_x - \frac{u}{c} f \right), \quad f' = \gamma \left( f - \frac{u}{c} f_x \right), \quad (8.37)$$

$$f'_y = f_y, \quad f'_z = f_z, \quad (8.38)$$

here by  $f$  we denote the expression

$$f = \frac{1}{c} (\vec{v} \vec{f}). \quad (8.39)$$

**These formulae were first found by Poincaré.** We see that scalar  $f$  and vector  $\vec{f}$  transform like components of  $(x^0, \vec{x})$ . Now

let us establish the law for the transformation of a force referred to unit charge

$$\vec{F} = \vec{E} + \frac{1}{c} [\vec{v}, \vec{H}], \quad \vec{F} = \frac{\vec{f}}{\rho}, \quad F = \frac{f}{\rho}. \quad (8.40)$$

Making use of (8.37), (8.38) and (8.39), we find

$$F'_x = \gamma \frac{\rho}{\rho'} \left( F_x - \frac{u}{c} F \right), \quad F' = \gamma \frac{\rho}{\rho'} \left( F - \frac{u}{c} F_x \right), \quad (8.41)$$

$$F'_y = \frac{\rho}{\rho'} F_y, \quad F'_z = \frac{\rho}{\rho'} F_z. \quad (8.42)$$

On the basis of (8.15) we have

$$\frac{\rho}{\rho'} = \frac{\sqrt{1 - \frac{u^2}{c^2}}}{1 - \frac{u}{c^2} v_x}. \quad (8.43)$$

To simplify (8.43) we shall derive an identity. Consider a certain inertial reference system  $K$ , in which there are two bodies with four-velocities  $U_1^\nu$  and  $U_2^\nu$  (see (9.1)) respectively

$$U_1^\nu = \left( \gamma_1, \frac{\vec{v}_1}{c} \gamma_1 \right), \quad U_2^\nu = \left( \gamma_2, \frac{\vec{v}_2}{c} \gamma_2 \right); \quad (8.44)$$

then, in the reference system  $K'$ , in which the first body is at rest, their four-velocities will be

$$U_1^{\nu'} = (1, 0), \quad U_2^{\nu'} = \left( \gamma', \frac{\vec{v}'}{c} \gamma' \right). \quad (8.45)$$

Since the product of four-vectors is an invariant, we obtain

$$\gamma' = \gamma_1 \gamma_2 \left( 1 - \frac{\vec{v}_1 \vec{v}_2}{c^2} \right). \quad (8.46)$$

Setting in this expression  $\vec{v}_2 = \vec{v}$ , and  $\vec{v}_1 = \vec{u}$  (velocity along the  $x$  axis), we find

$$\frac{\sqrt{1 - \frac{u^2}{c^2}} \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{uv_x}{c^2}} = \sqrt{1 - \frac{(v')^2}{c^2}}. \quad (8.47)$$

On the basis of (8.43) and (8.47) we obtain

$$\frac{\rho}{\rho'} = \frac{\sqrt{1 - \frac{(v')^2}{c^2}}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (8.48)$$

where  $\vec{v}'$  is the charge's velocity in the reference system  $K'$ . Substituting (8.48) into (8.41) and (8.42) we obtain the four-force  $R^\nu$  determined by the expression

$$R = \frac{F}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \vec{R} = \frac{\vec{F}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (8.49)$$

which transforms under the Lorentz transformations like  $(ct, \vec{x})$

$$R'_x = \gamma \left( R_x - \frac{u}{c} R \right), \quad R' = \gamma \left( R - \frac{u}{c} R_x \right), \quad (8.50)$$

$$R'_y = R_y, \quad R'_z = R_z.$$

**Such a four-vector of force was first introduced by Poincaré [2, 3].**

With the aid of formulae (8.28) and (8.9) the Lorentz force (8.35) can be written as

$$f^\nu = \frac{1}{c} F^{\nu\mu} S_\mu; \quad (8.51)$$

similarly, for the four-vector of force  $R^\nu$  we have

$$R^\nu = F^{\nu\mu}U_\mu. \quad (8.52)$$

Now let us calculate the energy-momentum tensor of the electromagnetic field. By means of Eqs. (8.51) and (8.27) we obtain

$$f_\nu = -\frac{1}{4\pi}[\partial_\alpha(F_{\nu\mu}F^{\mu\alpha}) - F^{\mu\alpha}\partial_\alpha F_{\nu\mu}]. \quad (8.53)$$

With the help of identity (8.24) the second term may be written as follows

$$F^{\mu\alpha}\partial_\alpha F_{\nu\mu} = -\frac{1}{4}\partial_\nu F^{\mu\beta}F_{\mu\beta}.$$

Taking into account this equation we get

$$f_\nu = -\partial_\alpha T_\nu^\alpha, \quad (8.54)$$

where  $T_\nu^\alpha$  is energy-momentum tensor of the electromagnetic field

$$T_\nu^\alpha = \frac{1}{4\pi}F_{\nu\mu}F^{\mu\alpha} + \frac{1}{16\pi}\delta_\nu^\alpha F^{\mu\beta}F_{\mu\beta},$$

or in symmetric form

$$T^{\alpha\sigma} = -\frac{1}{4\pi}F^{\alpha\mu}F^{\sigma\rho}\gamma_{\mu\rho} + \frac{1}{16\pi}\gamma^{\alpha\sigma}F^{\mu\beta}F_{\mu\beta}. \quad (8.55)$$

For more details see Section 15 p. 217.

The components of energy-momentum tensor may be expressed through  $\vec{E}$  and  $\vec{H}$  as follows

$$\begin{aligned} T^{00} &= \frac{1}{8\pi}(E^2 + H^2), \\ cT^{0i} &= S^i = \frac{c}{4\pi}[\vec{E}\vec{H}]_i, \\ T^{ik} &= -\frac{1}{4\pi}\left(E_i E_k + H_i H_k - \frac{1}{2}\delta_{ik}(E^2 + H^2)\right). \end{aligned}$$

From Eq. (8.54) by integrating it over the whole space we get

$$\vec{F} = \int dV \vec{f} = -\frac{d}{dt} \int \frac{1}{4\pi c} [\vec{E}\vec{H}] dV.$$

This result coincides with the expression obtained by H. Poincaré (see Section 9, p. 116)

Thus, the entire set of Maxwell-Lorentz equations is written via vectors and tensors of four-dimensional space-time. **The Lorentz group, that was discovered on the basis of studies of electromagnetic phenomena, was extended by H. Poincaré [2, 3] to all physical phenomena.**

In ref. [3] developing Lorentz ideas he wrote:

*“... All forces, of whatever origin they may be, behave, owing to the Lorentz transformations (and, consequently, owing to translational motion) precisely like electromagnetic forces”.*

H. Poincaré wrote:

*“The principle of physical relativity may serve us in defining space. It gives us, so to say, a new instrument for measurement. Let me explain. How can a solid body serve us in measuring or, to be more correct, in constructing space? The point is the following: by transferring a solid body from one place to another, we thus notice that it can, first, be applied to one figure and, then, to another, and we agree to consider these figures equal. This convention gave rise to geometry... Geometry is nothing, but a doctrine on the reciprocal relationships between these transformations or, to use mathematical language, a doctrine on the structure of the group composed by these transformations, i. e. the group of motions of solid bodies.*

*Now, take another group, the group of transformations, that do not alter our differential equations. We obtain a new way of determining the equality of two figures. We no longer say: two figures are equal, when one and the same solid body can be applied both to one figure and to the other. We will say: two figures are equal, when one and the same mechanical system, sufficiently distant from neighbouring that it can be considered isolated, being first accommodated so that its material points reproduce the first figure, and then so that they reproduce the second figure, behaves in the second case like in the first. Do these two views differ from each other in essence? No. . .*

*A solid body is much the same mechanical system as any other. The only difference between our previous and new definitions of space consists in that the latter is broader, allowing the solid body to be replaced by any other mechanical system. Moreover, our new conventional agreement not only defines space, but time, also. It explains to us, what are two simultaneous moments, what are two equal intervals of time, or what is an interval of time twice greater than another interval”.*

Further he notes:

*“Just transformations of the “Lorentz group” do not alter differential equations of dynamics. If we suppose that our system is referred not to axes at rest, but to axes in translational motion, then we have to admit, that all bodies are deformed. For example, a sphere is transformed to an ellipsoid which smallest axis coincides with the direction of translational motion of*

*coordinate axes. In this case the time itself is experienced profound changes. Let us consider two observers, the first is connected to axes at rest, the second — to moving axes, but both consider themselves at rest. We observe that not only the geometric object treated as a sphere by first observer will be looked liked an ellipsoid for the second observer, but also two events treated as simultaneous by the first will not be simultaneous for the second.”*

All the above formulated by H. Poincaré’s (not mentioning the content of his articles [2, 3]) completely contradicts to A. Einstein words written in his letter to professor Zangger (Director of Law Medicine Institute of Zurich University) 16.11.1911, that H. Poincaré “*has taken up a position of unfounded denial (of the theory of relativity) and has revealed insufficient understanding of the new situation at all*”. (B. Hoffmann “. Einstein”, Moscow: Progress, 1984, p. 84 (in Russian)).

If one reflects upon H. Poincaré’s words, one can immediately perceive the depth of his penetration into the essence of physical relativity and the relationship between geometry and group. Precisely in this way, starting from the invariability of the Maxwell-Lorentz equations under the Lorentz group transformations, which provided for consistency with the principle of physical relativity, H. Poincaré discovered the geometry of space-time, determined by the invariant (3.22).

Such space-time possesses the properties of homogeneity and isotropy. It reflects the existence in Nature of the fundamental conservation laws of energy, momentum and angular momentum for a closed system. Thus, the “new convention” is not arbitrary, it is based on the fundamental laws of Nature.

Now let us quote one striking statement by Hermann Weyl. It is written in his book “Raum. Zeit. Materie” appeared in 1918:

*“The solution of Einstein (here is the reference to the 1905 paper by A. Einstein — A.L.), which at one stroke overcomes all difficulties, is then this: **the world is a four-dimensional affine space whose metrical structure is determined by a non-definite quadratic form***

$$Q(\vec{x}) = (\vec{x}\vec{x})$$

*with has one negative and three positive dimensions”.*

Then he writes:

*“(* $\vec{OA}$ *,*  $\vec{OA}$ *) = -* $x_0^2$  *+*  $x_1^2$  *+*  $x_2^2$  *+*  $x_3^2$ *,*  
*in which the*  $x_i$ *'s are the co-ordinates of*  $A$ *”.*

But all this mentioned by H. Weyl was discovered by H. Poincaré (see articles [2, 3]), and not by A. Einstein. Nonetheless H. Weyl does not see this and even more, he writes in his footnote:

*“Two almost simultaneously appeared works by **H. Lorentz** and **H. Poincaré**, are closely related to it (the article by A. Einstein of 1905 — A.L.). They are not so clear and complete in presenting principal issues as Einstein’s article is.*

Then references to works by Lorentz and Poincaré are given. Very strange logic. H. Weyl has exactly formulated the solution, “*which at one stroke overcomes all difficulties*”, but namely this **is contained in articles by H. Poincaré [2, 3], and not in Einstein’s** ones. It is surprising how he has not seen this during his reading the Poincaré articles, because, as he mentions correctly, the essence of the theory of relativity is namely this. All the main consequences of it follow trivially from this, including the definition of the simultaneity concept for different space points by means of

the light signal, introduced by H. Poincaré in his articles published in 1898, 1900 and 1904.

What a clearness and completeness of presentation of the principal issues is additionally necessary for Weyl when he himself has demonstrated what “*at one stroke overcomes all difficulties.*” H. Weyl should better be more attentive in reading and more accurate in citing literature.

Above we have convinced ourselves that the symmetric set of equations of electrodynamics, (8.1), (8.2), which is invariant with respect to coordinate three-dimensional orthogonal transformations, at the same time turned out to be invariant, also, with respect to Lorentz transformations in four-dimensional space-time. This became possible due to a number of vector quantities of Euclidean space become, together with certain scalar quantities of the same space, components of four-dimensional quantities. At the same time, some vector quantities, such as, for example,  $\vec{E}$ ,  $\vec{H}$ , are derivatives of the components of four-dimensional quantities, which is the evidence that they are components of a tensor of the second rank in Minkowski space. The latter leads to the result that such concepts as *electric and magnetic field strengths* are not absolute.

## 9. Poincaré's relativistic mechanics

In Section 3 we saw that the requirement of fulfilment of the relativity principle for electrodynamics leads to transition from one inertial reference system to another, moving with respect to the first along the  $x$  axis with a velocity  $v$ , being realized not by Galilean transformations (2.5), but by Lorentz transformations (3.1). Hence it follows, of necessity, that the equations of mechanics must be changed to make them form-invariant with respect to the Lorentz transformations. Since space and time are four-dimensional, the physical quantities described by vectors will have four components. The sole four-vector describing a point-like body has the form

$$U^\nu = \frac{dx^\nu}{d\sigma}. \quad (9.1)$$

Here the interval  $d\sigma$  in Galilean coordinates is as follows

$$(d\sigma)^2 = c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right). \quad (9.2)$$

Substituting the expression for  $d\sigma$  into (9.1) we obtain

$$U^0 = \gamma, \quad U^i = \gamma \frac{v^i}{c}, \quad v^i = \frac{dx^i}{dt}, \quad i = 1, 2, 3. \quad (9.3)$$

**This four-vector of velocity was first introduced by Poincaré [3].**

We now introduce the four-vector of momentum

$$P^\nu = mcU^\nu \quad (9.4)$$

where  $m$  is rest mass of a point-like body.

The relativistic equations of mechanics can intuitively be written in the form

$$mc^2 \frac{dU^\nu}{d\sigma} = F^\nu, \quad (9.5)$$

here  $F^\nu$  is the four-vector of force, which is still to be expressed via the ordinary Newtonian force  $\vec{f}$ . It is readily verified that the four-force is orthogonal to the four-velocity, i. e.

$$F^\nu U_\nu = 0.$$

On the basis of (9.2) and (9.3) equation (9.5) can be written in the form

$$\frac{d}{dt} \left( \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \vec{F} \sqrt{1 - \frac{v^2}{c^2}}, \quad (9.6)$$

$$\frac{d}{dt} \left( \frac{mc}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = F^0 \sqrt{1 - \frac{v^2}{c^2}}. \quad (9.7)$$

Since from the correspondence principle at small velocities equation (9.6) should coincide with Newton's equation, it is natural to define  $\vec{F}$  as follows:

$$\vec{F} = \frac{\vec{f}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (9.8)$$

here  $\vec{f}$  is the usual three-dimensional force.

Now let us verify, that equation (9.7) is a consequence of equation (9.6). Multiplying equation (9.6) by the velocity  $\vec{v}$  and differentiating with respect to time, we obtain

$$\frac{m}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \cdot \left( \vec{v} \frac{d\vec{v}}{dt} \right) = \vec{f} \vec{v}. \quad (9.9)$$

On the other hand, upon differentiation with respect to time, equation (9.7) assumes the form

$$\frac{m}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \cdot \left(\vec{v} \frac{d\vec{v}}{dt}\right) = cF^0 \sqrt{1 - \frac{v^2}{c^2}}. \quad (9.10)$$

Comparing (9.9) and (9.10), we find

$$F^0 = \frac{\left(\frac{\vec{v}}{c} \vec{f}\right)}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (9.11)$$

On the basis of relations (9.8) and (9.11) the equations of relativistic mechanics assume the form

$$\frac{d}{dt} \left( \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \vec{f}, \quad (9.12)$$

$$\frac{d}{dt} \left( \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \vec{f}\vec{v}. \quad (9.13)$$

**These equations were first obtained by H. Poincaré [3].** Equation (9.13) relates the change in particle energy and the work done per unit time.

Having obtained these equations, Poincaré applied them for explaining the anomalies in the movement of Mercury. In this connection he wrote:

*“Thus, the **new mechanics** is still on unsteady soil. So we are to wish it new confirmations. Let us see what astronomical observations give us in this connection. The velocities of planets are, doubtless, relatively very small, but, on the other hand, astronomical observations exhibit a high degree of precision and extend over long intervals of time. Small actions can, apparently, add up to such an extent, that they acquire values permitting to be estimated. The only effect, with respect to which one could expect it to be noticeable is the one we actually see: I mean the perturbations of the fastest of all planets — Mercury. It indeed shows such anomalies in its motion that can still not be explained by celestial mechanics. The shift of its perihelion is much more significant than calculated on the basis of classical theory. Much effort has been applied with the aim of explaining these deviations . . . The new mechanics somewhat corrects the error in the theory of Mercury's motion lowering it to 32", but does not achieve total accordance between the observation and calculation. This result, is, thus, not in favour of the **new mechanics**, but at any rate, it also is not against it. The new doctrine does not contradict astronomical observations directly”.*

One can see here, how careful H. Poincaré was in his estimation of results. This was quite understandable, since the theory was still under development, and therefore attentive and multiple experimental tests of its conclusions were required. It turned out that these equations were valid only when gravity was neglected. Later A. Einstein explained the anomaly in the motion of Mercury on the basis of general relativity theory, in which gravity is a consequence of the curvature of space-time. But to explain the anomaly in the

motion of Mercury Einstein actually had to renounce special relativity theory and, as a consequence, the fundamental conservation laws of energy-momentum and of angular momentum.

From equations (9.12) it follows that the equations of classical mechanics are valid only when the velocity  $v$  is small as compared with the velocity of light. It is just the approximate character of the equations of classical mechanics that has led to the origination of the Galilean transformations, that leave the equations of mechanics unchanged in all inertial reference systems.

In three-dimensional form the momentum and energy have the form

$$\vec{p} = \gamma m \vec{v}, \quad E = p^0 c = \gamma m c^2. \quad (9.14)$$

From (9.12) and (9.13) it follows that for a closed system energy and momentum are conserved. **As we see from formula (9.14), energy  $E$  is not an invariant. It has been and remains to be an invariant only with respect to three-dimensional coordinate transformations, and at the same time it is the zeroth component of the four-dimensional momentum vector in Minkowski.**

As an example let us calculate the energy of a system of two particles  $a$  and  $b$  in two different systems of reference. To proceed so let us consider the invariant

$$\mathcal{V} = (p_a + p_b)^2.$$

In the system of reference where one particle is at rest,

$$\vec{p}_a = 0,$$

we have

$$\mathcal{V} = 2mE + 2m^2c^2.$$

Here we take masses for particle  $a$  and for particle  $b$  as equal. The same invariant is

$$\mathcal{V} = (p_a + p_b)^2 = 4 \frac{\mathcal{E}^2}{c^2},$$

when estimated at the reference system where the center of mass is at rest

$$\vec{p}_a + \vec{p}_b = 0,$$

and  $\mathcal{E}$  is a particle energy calculated in this system of reference.

When comparing these expressions we get a connection between the energies in these two reference systems:

$$E = 2 \frac{\mathcal{E}^2}{mc^2} - mc^2.$$

The collision energy of two particles is used with most efficiency in case when the center of mass of the two particles is at rest in laboratory system of reference. Just this situation is realized in colliders. There is no loss of energy for the center of mass motion.

One who has felt the four-dimensionality of space-time, could have seen immediately that energy and momentum are combined in the four-momentum. Moreover, he would have understood that in the case of a closed system they obey the energy and momentum conservation law.

In 1905 A. Einstein has proposed really existent quanta of the light energy  $\hbar\omega$  to explain the photo-effect. If he would understand in deep the existence and meaning of the group, and so the requirement of relativity principle that physical quantities should be four-dimensional, then he could introduce for light the quantum of momentum in line with the quantum of energy. Moreover that time it was already proved experimentally (P. N. Lebedev, 1901) that the light was carrying not only the energy, but also the momentum and so it was exerting pressure on solid bodies. But A. Einstein has not done this. The momentum of the quantum of light has been introduced by J. Stark in 1909. He took it into account in the momentum conservation law. So the quantum of light, the **photon**, has appeared (as a particle).

Energy and momentum according to (9.4) transform as follows under Lorentz transformations

$$p'_x = \gamma \left( p_x - v \frac{E}{c^2} \right), \quad p'_y = p_y, \quad p'_z = p_z, \quad E' = \gamma(E - vp_x).$$

A monochromatic plane light wave is characterized by frequency  $\omega$  and wave vector  $\vec{K} = \frac{\omega}{c} \vec{n}$ . Together they are components of four-dimensional wave vector

$$K^\nu = \left( \frac{\omega}{c}, \frac{\omega}{c} \vec{n} \right).$$

Square of this four-dimensional wave vector is zero due to the wave equation

$$K^\nu K_\nu = 0.$$

The meaning of this fact is that the rest mass is zero.

The frequency  $\omega$  and the wave vector  $\vec{K}$  transform under Lorentz transformations in the same way as  $ct, \vec{x}$ , i. e. as follows

$$\begin{aligned} \omega' &= \omega \gamma \left( 1 - \frac{v}{c} n_x \right), \\ \omega' n'_x &= \omega \gamma \left( n_x - \frac{v}{c} \right), \\ \omega' n'_y &= \omega n_y, \quad \omega' n'_z = \omega n_z. \end{aligned}$$

Just the same formulae stay valid for **photon** which rest mass is zero. The vector of four-momentum of photon is as follows

$$p^\nu = \left( \frac{\hbar \omega}{c}, \hbar \vec{K} \right),$$

where  $\hbar$  is the Planck constant.

It follows from the above that energy  $E$  and frequency  $\omega$  transforms in the same way. Formulae given above explain **Doppler effect**, i. e. the change of the light frequency when it is emitted by

a moving source. The **Doppler effect** takes place also when the direction of movement of the light source is perpendicular to the direction of observation ( $n'_x = 0$ ). So far as

$$\omega = \omega' \gamma \left( 1 + \frac{v}{c} n'_x \right),$$

we obtain for the transverse **Doppler effect** the following result

$$\omega' = \omega \sqrt{1 - v/c^2}.$$

This effect is small enough in comparison with the longitudinal one. From the above formulae it is also possible to determine how the direction of light beam changes under transformation to another inertial reference system

$$n'_x = \frac{n_x - \frac{v}{c}}{1 - \frac{v}{c} n_x}.$$

This formula shows the effect of **aberration**. We will return to this subject in Section 16.

The covariant vector of four-velocity is  $U_\nu = U^\sigma \gamma_{\sigma\nu}$ , but since in Galilean coordinates  $\gamma_{\sigma\nu} = (1, -1, -1, -1)$ , we obtain

$$U_\nu = (U^0, -U^i). \quad (9.15)$$

Taking into account (9.1) and (9.15) it is possible to compose the invariant

$$U_\nu U^\nu = (U^0)^2 - (\vec{U})^2 = 1, \quad (9.16)$$

which by virtue of the definition of the four-vector  $U^\nu$  will be unity. This is readily verified, if the values determined by formulae (9.3) are substituted into (9.16). Thus, we have

$$p_\nu p^\nu = (mc)^2, \text{ or } E = c \sqrt{\vec{p}^2 + m^2 c^2}. \quad (9.17)$$

In formula (9.17) we have retained for energy only the positive sign, however the negative sign of energy also has sense. It turns out to be significant in the case of unification of relativity theory and quantum ideas. This led Dirac to predicting the particle (positron) with the mass of the electron and positive charge, equal to the electron charge. Then the ideas arose of “elementary” particles creation in the process of interaction, of the physical vacuum, of the antiparticles (V. Ambartsumyan, D. Ivanenko, E. Fermi ). It has opened the possibility of transformation of the colliding particles **kinetic energy** to the **material substance** possessing **rest mass**. So the need to construct accelerators for high energies to study microcosm's mysteries has arisen.

On the basis of (9.14) equation (9.12) assumes the form

$$\frac{d}{dt} \left( \frac{E}{c^2} \vec{v} \right) = \vec{f}, \text{ or } \frac{E}{c^2} \cdot \frac{d\vec{v}}{dt} = \vec{f} - \frac{\vec{v}}{c^2} \cdot \frac{dE}{dt}. \quad (9.18)$$

From (9.18) it follows that the acceleration of a body, determined by the expression  $\frac{d\vec{v}}{dt}$  does not coincide in direction with the acting force  $\vec{f}$ . From the equations of Poincaré's relativistic mechanics we have on the basis of (9.17), for a body in a state of rest

$$E_0 = mc^2,$$

where  $E_0$  is the energy,  $m$  is the mass of the body at rest.

From (9.17) it is evident, that mass  $m$  is an invariant. This relation is a direct consequence of pseudo-Euclidean structure of the space-time geometry. The connection between energy and mass first arose in relation to the inert property of the electromagnetic radiation. Formula  $E = mc^2$  for radiation had been found for the first time in the article by H. Poincaré in 1900 in clear and exact form.

Let us quote some extractions from the article by H. Poincaré published in 1900 “Lorentz theory and principle of equality of action and reaction” (put into modern notations by V. A. Petrov):

*“First of all let us shortly remind the derivation proving that the principle of equality of action and reaction is no more valid in the Lorentz theory, at least when it is applied to the matter.*

*We shall search for the resultant of all ponderomotive forces applied to all electrons located inside a definite volume. This resultant is given by the following integral*

$$\vec{F} = \int \rho dV \left( \frac{1}{c} [\vec{v}, \vec{H}] + \vec{E} \right),$$

*where integration is over elements  $dV$  of the considered volume, and  $\vec{v}$  is the electron velocity.*

*Due to the following equations*

$$\begin{aligned} \frac{4\pi}{c} \rho \vec{v} &= -\frac{1}{c} \cdot \frac{\partial \vec{E}}{\partial t} + \text{rot } \vec{H}, \\ 4\pi \rho &= \text{div } \vec{E}, \end{aligned}$$

*and by adding and subtracting the expression  $\frac{1}{8\pi} \nabla H^2$ , I can write the following formula*

$$\vec{F} = \sum_1^4 \vec{F}_i,$$

where

$$\vec{F}_1 = \frac{1}{4\pi c} \int dV \left[ \vec{H} \frac{\partial \vec{E}}{\partial t} \right],$$

$$\vec{F}_2 = \frac{1}{4\pi} \int dV (\vec{H} \nabla) \vec{H},$$

$$\vec{F}_3 = -\frac{1}{8\pi} \int dV \nabla H^2,$$

$$\vec{F}_4 = \frac{1}{4\pi} \int dV \vec{E} (\operatorname{div} \vec{E}).$$

Integration by parts gives the following

$$\vec{F}_2 = \frac{1}{4\pi} \int d\sigma \vec{H} (\vec{n} \vec{H}) - \frac{1}{4\pi} \int dV \vec{H} (\operatorname{div} \vec{H}),$$

$$\vec{F}_3 = -\frac{1}{8\pi} \int d\sigma \vec{n} H^2,$$

where integrals are taken over all elements  $d\sigma$  of the surface bounding the volume considered, and where  $\vec{n}$  denotes the normal vector to this element. Taking into account

$$\operatorname{div} \vec{H} = 0,$$

it is possible to write the following

$$\vec{F}_2 + \vec{F}_3 = \frac{1}{8\pi} \int d\sigma \left( 2\vec{H} (\vec{n} \vec{H}) - \vec{n} H^2 \right). \quad (A)$$

Now let us transform expression  $\vec{F}_4$ . Integration by parts gives the following

$$\vec{F}_4 = \frac{1}{4\pi} \int d\sigma \vec{E} (\vec{n} \vec{E}) - \frac{1}{4\pi} \int dV (\vec{E} \nabla) \vec{E}.$$

Let us denote two integrals from r.h.s. as  $\vec{F}'_4$  and  $\vec{F}''_4$ , then

$$\vec{F}_4 = \vec{F}'_4 - \vec{F}''_4.$$

Accounting for the following equations

$$[\nabla \vec{E}] = -\frac{1}{c} \cdot \frac{\partial \vec{H}}{\partial t},$$

we can obtain the following formula

$$\vec{F}_4'' = \vec{Y} + \vec{Z},$$

where

$$\begin{aligned} \vec{Y} &= \frac{1}{8\pi} \int dV \nabla E^2, \\ \vec{Z} &= \frac{1}{4\pi c} \int dV \left[ \vec{E} \frac{\partial \vec{H}}{\partial t} \right]. \end{aligned}$$

As a result we find that

$$\begin{aligned} \vec{Y} &= \frac{1}{8\pi} \int d\sigma \vec{n} E^2, \\ \vec{F}_1 - \vec{Z} &= \frac{d}{dt} \int \frac{dV}{4\pi c} [\vec{H} \vec{E}]. \end{aligned}$$

At last we get the following

$$\vec{F} = \frac{d}{dt} \int \frac{dV}{4\pi c} [\vec{H} \vec{E}] + (\vec{F}_2 + \vec{F}_3) + (\vec{F}_4' - \vec{Y}),$$

where  $(\vec{F}_2 + \vec{F}_3)$  is given by Eq. (A), whereas

$$\vec{F}_4' - \vec{Y} = \frac{1}{8\pi} \int d\sigma \left( 2\vec{E}(\vec{n}\vec{E}) - \vec{n}E^2 \right).$$

Term  $(\vec{F}_2 + \vec{F}_3)$  represents the pressure experienced by different elements  $d\sigma$  of the surface bounding the volume considered. It is straightforward to see that this

pressure is nothing else, but the Maxwell **magnetic pressure** introduced by this scientist in well-known theory. Similarly, term  $(\vec{F}'_4 - \vec{Y})$  represents action of the Maxwell electrostatic pressure. In the absence of the first term,

$$\frac{d}{dt} \int dV \frac{1}{4\pi c} [\vec{H} \vec{E}],$$

the ponderomotive force would be nothing else, but a result of the Maxwell pressures. If our integrals are extended on the whole space, then forces  $\vec{F}_2, \vec{F}_3, \vec{F}'_4$  and  $\vec{Y}$  disappear, and the rest is simply

$$\vec{F} = \frac{d}{dt} \int \frac{dV}{4\pi c} [\vec{H} \vec{E}].$$

If we denote as  $M$  the mass of one of particles considered, and as  $\vec{v}$  its velocity, then we will have in case when the principle of equality of action and reaction is valid the following:

$$\sum M\vec{v} = \text{const.}^3$$

Just the opposite, we will have:

$$\sum M\vec{v} - \int \frac{dV}{4\pi c} [\vec{H} \vec{E}] = \text{const.}$$

Let us notice that

$$-\frac{c}{4\pi} [\vec{H} \vec{E}]$$

is the Poynting vector of radiation.

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<sup>3</sup>The matter only is considered here. — A. L.

If we put

$$J = \frac{1}{8\pi}(H^2 + E^2),$$

then the Poynting equation gives the following

$$\int \frac{dJ}{dt} dV = - \int d\sigma \frac{c}{4\pi} \vec{n} [\vec{H} \vec{E}] - \int dV \rho (\vec{v} \vec{E}). \quad (B)$$

The first integral in the r.h.s., as well known, is the amount of electromagnetic energy flowing into the considered volume through the surface and the second term is the amount of electromagnetic energy created in the volume by means of transformation from other species of energy.

We may treat the electromagnetic energy as a fictitious fluid with density  $J$  which is distributed in space according to the Poynting laws. It is only necessary to admit that this fluid is not indestructible, and it is decreasing over value  $\rho dV \vec{E} \vec{v}$  in volume element  $dV$  in a unit of time (or that an equal and opposite in sign amount of it is created, if this expression is negative). This does not allow us to get a full analogy with the real fluid for our fictitious one. The amount of this fluid which flows through a unit square surface oriented perpendicular to the axis  $i$ , at a unit of time is equal to the following

$$JU_i,$$

where  $U_i$  are corresponding components of the fluid velocity.

Comparing this to the Poynting formulae, we obtain

$$J\vec{U} = \frac{c}{4\pi} [\vec{E} \vec{H}];$$

so our formulae take the following form

$$\sum M\vec{v} + \int dV \frac{J\vec{U}}{c^2} = \text{const.}^4 \quad (C)$$

They demonstrate that the momentum of substance plus the momentum of our fictitious fluid is given by a constant vector.

In standard mechanics one concludes from the constancy of the momentum that the motion of the mass center is rectilinear and uniform. But here we have no right to conclude that the center of mass of the system composed of the substance and our fictitious fluid is moving rectilinearly and uniformly. This is due to the

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<sup>4</sup>In Eq. (C) the second term in the l.h.s. determines the total momentum of the electromagnetic radiation. Just here the concept of *radiation momentum density* arises

$$\vec{g} = \frac{J}{c^2} \vec{U},$$

and also the concept of *mass density of the electromagnetic field*

$$m = \frac{J}{c^2},$$

where  $J$  is the electromagnetic energy density. It is also easy to see from here that radiation energy density

$$\vec{S} = \frac{c}{4\pi} [\vec{E} \vec{H}]$$

is related to the momentum density

$$\vec{g} = \frac{\vec{S}}{c^2}.$$

So the notions of local *energy* and *momentum* appeared. All this was firstly obtained by H. Poincaré. Later these items were discussed in the Planck work (Phys. Zeitschr. 1908. **9**. S. 828) — A. L.

*fact that this fluid is not indestructible.*

*The position of the mass center depends on value of the following integral*

$$\int \vec{x} J dV,$$

*which is taken over the whole space. The derivative of this integral is as follows*

$$\int \vec{x} \frac{dJ}{dt} dV = - \int \vec{x} \operatorname{div}(J\vec{U}) dV - \int \rho \vec{x} (\vec{E}\vec{v}) dV.$$

*But the first integral of the r.h.s. after integration transforms to the following expression*

$$\int J\vec{U} dV$$

*or*

$$\left( \vec{C} - \sum M\vec{v} \right) c^2,$$

*when we denote by  $\vec{C}$  the constant sum of vectors from Eq. (C).*

*Let us denote by  $M_0$  the total mass of substance, by  $\vec{R}_0$  the coordinates of its center of mass, by  $M_1$  the total mass of fictitious fluid, by  $\vec{R}_1$  its center of mass, by  $M_2$  the total mass of the system (substance + fictitious fluid), by  $\vec{R}_2$  its center of mass, then we have*

$$M_2 = M_0 + M_1, \quad M_2 \vec{R}_2 = M_0 \vec{R}_0 + M_1 \vec{R}_1,$$

$$\int \vec{x} \frac{J}{c^2} dV = M_1 \vec{R}_1.^5$$

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<sup>5</sup>H. Poincaré also exploits in this formula the concept of the *mass density of the electromagnetic field* introduced by him earlier. — A. L.

Then we come to the following equation

$$\frac{d}{dt}(M_2 \vec{R}_2) = \vec{C} - \int \vec{x} \frac{\rho(\vec{v} \vec{E})}{c^2} dV. \quad (D)$$

Eq. (D) may be expressed in standard terms as follows. If the electromagnetic energy is created or annihilated nowhere, then the last term disappears, whereas the center of mass of the system formed of the matter and electromagnetic energy (treated as a fictitious fluid) has a rectilinear and uniform motion”.

Then H. Poincaré writes:

“So, the electromagnetic energy behaves as a fluid having inertia from our point of view. And we have to conclude that if some device producing electromagnetic energy will send it by means of radiation in a definite direction, then this device must experience a recoil, as a cannon which fire a shot. Of course, this recoil will be absent if the device radiates energy isotropically in all directions; just opposite, it will be present when this symmetry is absent and when the energy is emitted in a single direction. This is just the same as this proceeds, for example, for the H. Hertz emitter situated in a parabolic mirror. It is easy to estimate numerically the value of this recoil. If the device has mass 1 kg, and if it sends three billion Joules in a single direction with the light velocity, then the velocity due to recoil is equal to 1 sm/sec”.

When determining the velocity of recoil H. Poincaré again exploits the formula

$$M = \frac{E}{c^2}.$$

In §7 of article [3] H. Poincaré derives equations of relativistic mechanics. If we change the system of units in this paragraph from  $M = 1, c = 1$  to Gaussian system of units, then it is easy to see that **inert mass of a body** is also determined by formula:

$$M = \frac{E}{c^2}.$$

Therefore, it follows from works by H. Poincaré that the **inert mass** both of **substance**, and of **radiation** is determined by their energy. All this has been a consequence of the electrodynamics and the relativistic mechanics.

In 1905 Einstein has published the article "Does the inertia of a body depend on the energy contained in it?". Max Jammer wrote on this article in his book "The concept of mass in classical and modern physics" (Harvard University Press, 1961.):

*«It is generally said that "the theorem of inertia of energy in its full generality was stated by Einstein (1905)" (Max Born. "Atomic physics". Blackie, London, Glasgow ed. 6, p. 55). The article referred to is Einstein's paper, "Does the inertia of a body depend upon its energy content?". On the basis of the Maxwell-Hertz equations of the electromagnetic field Einstein contended that "if a body gives off the energy  $E$  in the form of radiation, its mass diminishes by  $E/c^2$ ". Generalizing this result for all energy transformations Einstein concludes: "The mass of a body is a measure of its energy content".*

*It is a curious incident in the history of scientific thought that Einstein's own derivation of formula  $E = mc^2$ , as published in his article in the "Annalen der Physik", was basically fallacious. In fact, what for the layman is known as "the most famous mathematical formula ever projected" in science (William Cahn.*

"Einstein, a pictorial biography". New York: Citadel. 1955. P. 26) *was but the result of petitio principii, the conclusion of begging the question*».

*"The logical illegitimacy of Einstein's derivation has been shown by Ives (Journal of the Optical Society of America. 1952. 42, pp. 540-543)"*.

Let us consider shortly Einstein's article of 1905 "Does the inertia of a body depend on the energy contained in it?" Einstein writes:

*"Let there be a body at rest in the system  $(x, y, z)$ , whose energy, referred to the system  $(x, y, z)$ , is  $E_0$ . The energy of the body with respect to the system  $(\zeta, \eta, \varsigma)$ , which is moving with velocity  $v$  as above, shall be  $H_0$ .*

*Let this body simultaneously emit plane waves of light of energy  $L/2$  (measured relative to  $(x, y, z)$ ) in a direction forming an angle  $\varphi$  with the  $x$ -axis and an equal amount of light in the opposite direction. All the while, the body shall stay at rest with respect to the system  $(x, y, z)$ . This process must satisfy the energy principle, and this must be true (according to the principle of relativity) with respect to both coordinate systems. If  $E_1$  and  $H_1$  denote the energy of the body after the emission of light, as measured relative to systems  $(x, y, z)$  and  $(\zeta, \eta, \varsigma)$ , respectively, we obtain, using the relation indicated above,*

$$E_0 = E_1 + \left[ \frac{L}{2} + \frac{L}{2} \right],$$

$$\begin{aligned}
H_0 &= H_1 + \left[ \frac{L}{2} \cdot \frac{1 - \frac{v}{V} \cos \varphi}{\sqrt{1 - \left[\frac{v}{V}\right]^2}} + \frac{L}{2} \cdot \frac{1 + \frac{v}{V} \cos \varphi}{\sqrt{1 - \left[\frac{v}{V}\right]^2}} \right] = \\
&= H_1 + \frac{L}{\sqrt{1 - \left[\frac{v}{V}\right]^2}}.
\end{aligned}$$

*Subtracting, we get from these equations*

$$(H_0 - E_0) - (H_1 - E_1) = L \left\{ \frac{1}{\sqrt{1 - \left[\frac{v}{V}\right]^2}} - 1 \right\}. \quad (N)$$

A. Einstein tries to get all the following just from this relation. Let us make an elementary analysis of the equation derived by him. According to the theory of relativity

$$H_0 = \frac{E_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad H_1 = \frac{E_1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Einstein seemingly did not take into account such formulae. It follows then that

$$H_0 - E_0 = E_0 \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right), \quad H_1 - E_1 = E_1 \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right),$$

and consequently the l.h.s. of the Einstein equation is equal to the following

$$(H_0 - E_0) - (H_1 - E_1) = (E_0 - E_1) \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right);$$

then Eq. (N) takes an apparent form

$$E_0 - E_1 = L.$$

Therefore, it is impossible to get something more substantial from the initial Einstein equation (N). In this work A. Einstein has not succeeded in discovering neither physical arguments, nor a method of calculation to prove that formula

$$M = \frac{E}{c^2}$$

is valid at least for radiation. So, the critics given by Ives on the A. Einstein work is correct. In 1906 Einstein once more returns to this subject, but his work reproduces the Poincaré results of 1900, as he notes himself.

Later, Planck in 1907 and Langevin in 1913 revealed, on this basis, the role of internal interaction energy (binding energy), which **led to the mass defect, providing conditions for possible energy release, for example, in fission and fusion of atomic nuclei.** The relativistic mechanics has become an engineering discipline. Accelerators of elementary particles are constructed with the help of it.

“Disproofs” of the special theory of relativity appearing sometimes are related to unclear and inexact presentation of its basics in many textbooks. Often its meaning is deeply hidden by plenty of minor or even needless details presented. The special theory of relativity is strikingly simple in its basics, almost as Euclidean geometry.

*On the transformations of force*

According to (9.8) and (9.11) the four-force is

$$F^\nu = \left( \gamma \frac{\vec{v}\vec{f}}{c}, \gamma\vec{f} \right), \quad (9.19)$$

$$F_\nu = \left( \gamma \frac{\vec{v} \cdot \vec{f}}{c}, -\gamma \vec{f} \right). \quad (9.20)$$

As we noted above, the force as a four-vector transforms like the quantities  $ct$  and  $x$ , so,

$$f'_x = \frac{\gamma}{\gamma'} \gamma_1 \left( f_x - \beta \frac{\vec{v} \cdot \vec{f}}{c} \right), \quad (\vec{v}' \cdot \vec{f}') = \frac{\gamma}{\gamma'} \gamma_1 (\vec{v} \cdot \vec{f} - c\beta f_x), \quad (9.21)$$

$$f'_y = \frac{\gamma}{\gamma'} f_y, \quad f'_z = \frac{\gamma}{\gamma'} f_z. \quad (9.22)$$

Here

$$\beta = \frac{u}{c}, \quad \gamma_1 = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad (9.23)$$

$u$  is the velocity along the  $x$  axis.

Consider two particles in the unprimed inertial reference system with the four-velocities

$$U_1^\nu = \left( \gamma, \gamma \frac{\vec{v}}{c} \right), \quad U_2^\nu = \left( \gamma_1, \gamma_1 \frac{\vec{u}}{c} \right). \quad (9.24)$$

Then, in the inertial reference system, in which the second particle is at rest, we have the following expressions for the respective four-vectors:

$$U_1^{\nu'} = \left( \gamma', \gamma' \frac{\vec{v}'}{c} \right), \quad U_2^{\nu'} = (1, 0).$$

Hence, on the basis of invariance of expression  $U_{1\sigma} U_2^\sigma$  we have the following equality:

$$\gamma' = \gamma \gamma_1 \left( 1 - \frac{\vec{v} \cdot \vec{u}}{c^2} \right). \quad (9.25)$$

Thus, we obtain

$$\frac{\gamma}{\gamma'} = \frac{1}{\gamma_1 \left(1 - \frac{\vec{v}\vec{u}}{c^2}\right)}. \quad (9.26)$$

In our case, when the velocity  $\vec{u}$  is directed along the  $x$  axis, we have

$$\frac{\gamma}{\gamma'} = \frac{1}{\gamma_1 \left(1 - \frac{v_x u}{c^2}\right)}. \quad (9.27)$$

Substituting (9.27) into (9.21) and (9.22) we obtain

$$f'_x = \frac{f_x - \beta \frac{\vec{v}}{c} \vec{f}}{1 - \beta \frac{v_x}{c}}, \quad f'_y = \frac{f_y \sqrt{1 - \beta^2}}{1 - \beta \frac{v_x}{c}}, \quad f'_z = \frac{f_z \sqrt{1 - \beta^2}}{1 - \beta \frac{v_x}{c}}, \quad (9.28)$$

$$\vec{f}'\vec{v}' = \frac{(\vec{f}\vec{v}) - \beta c f_x}{1 - \beta \frac{v_x}{c}}. \quad (9.29)$$

Hence it is evident that, if the force  $\vec{f}$  in a certain inertial Galilean reference system is zero, it is, then, zero in any other inertial reference system, also. This means that, if the law of inertia is valid in one inertial reference system, then it is also obeyed in any other inertial reference system. Moreover, the conclusion concerning the force is not only valid for an inertial reference system, but also for any accelerated (non-inertial) reference system. **Force cannot arise as a result of coordinate transformations.** If motion by inertia in an inertial reference system proceeds along a straight line, then in a non-inertial reference system free motion will proceed along the geodesic line, which in these coordinates will no longer be a straight line.

In classical mechanics the force  $\vec{f}$  is the same in all inertial reference systems, in relativistic mechanics this is no longer so, the components of force, in this case, vary in accordance with (9.28).

Let us, now, dwell upon a general comment concerning inertial reference systems. Inertial reference systems being equitable signifies that, if we create in each reference system identical conditions for the evolution of matter, then we, naturally, should have the same description of a phenomenon in each reference system, in other words, we will not be able to single out any one of the inertial reference systems. But, if we have provided some conditions for the motion of matter in one inertial reference system, then, in describing what goes on in this reference system by observing from any other inertial reference system, we will already obtain another picture. This does not violate the equality of inertial reference systems, since in this case the initial reference system has been singled out by the actual formulation of the problem. **Precisely such a situation arises, when we consider the Universe. In this case, there exists a unique, physically singled out inertial reference system in the Universe, which is determined by the distribution of matter. Time in this reference system will have a special status as compared with other inertial reference systems. This time could be termed the “true time” of the Universe. As an example of such a special reference system one could choose a reference system, in which the relict electromagnetic radiation is homogeneous and isotropic** (see ref.[5]).

From the above exposition, especially from Sections 3, 5, 7, 8, and 9 it is evident that Henri Poincaré discovered all the essentials that make up the content of the special theory of relativity. Any person, who has graduated from a University in theoretical physics and who has attentively read at least two of his articles “On the dynamics of the electron”, may verify this.

There exist, also, other points of view: “*Poincaré did not make the decisive step*” (de Broglie), “*Poincaré was, most likely, quite close to creating the STR, but did not arrive at the end. One can only guess why this happened.*” (V.L. Ginzburg). But these state-

ments characterize their authors' own level of understanding the problem, instead of H. Poincaré's outstanding achievements in the theory of relativity. What is surprising is that the authors show no trace of doubt in considering their own incomprehension, or the difficulty they had in understanding, as a criterion in evaluating the outstanding studies performed by Poincaré. In this case there is no need to "guess". It is only necessary to read the works by Poincaré [2, 3] and to think.

Professor A. Pais wrote the following in his book "Subtle is the Lord: the science and the life of Albert Einstein", Oxford University Press, 1982:

*"It is evident that as late as 1909 Poincaré did not know **that the contraction of rods is a consequence of the two Einstein's postulates.** (singled out by me — A. L.) Poincaré therefore did not understand one of the most basic traits of special relativity".*

We right away note that the underlined statement is wrong. But about this later.

From everything that A. Pais has written it clearly follows that he himself **did not understand** the fundamentals of special relativity. Let me explain. Poincaré demonstrated the invariability of the Maxwell-Lorentz equations with respect to the Lorentz transformations, which was consistent with the relativity principle, formulated by Poincaré in 1904 for all natural physical phenomena. As we already noted, H. Poincaré discovered the fundamental invariant (3.22)

$$J = c^2T^2 - X^2 - Y^2 - Z^2 ,$$

that establishes the geometry of space-time. Namely hence it follows, that the light velocity being constant is a particular consequence of this formula, when the invariant  $J$  is zero. A. Pais had

to understand that the Lorentz contraction is related to negative  $J$ , i. e. to a space-like value of  $J$ , not equal to zero. As to the slowing down of time, it is related to positive  $J$ , i.e time-like  $J$ , but certainly not equal to zero. Thus, from the above it is clear, **that contraction of the dimensions of rods is not a consequence of the two Einstein's postulates only.** Such is the result of a superficial knowledge of the relativity theory foundations.

So with such a knowledge of material A. Pais had tried to prove on the pages of his book that H. Poincaré had not made the decisive step to create the theory of relativity! He, a physicist, "reinforced" his view on the contribution of H. Poincaré by the decision of the Paris Session of the French Philosophical Society in 1922.

So simple it is! The philosophers have met and made a decision whereas they probably have not studied works by Poincaré on the theory of relativity at all. But such a study required a corresponding professional level. I doubt whether their professional level had been higher than one by A. Pais in this field. We should say that A. Pais was an outstanding scientist irrespective to this criticism and he made a lot of remarkable investigations.

As to the Lorentz contraction, in the article [3] (§ 6 "The contraction of electrons") H. Poincaré deals with this issue in detail, making use of the Lorentz transformations. All this is clearly presented in article [3]. Precisely unification of relativity and the Maxwell-Lorentz electrodynamics permitted Poincaré to formulate in articles [2] and [3] the foundation of the theory of relativity. As to the postulate concerning the constancy of the velocity of light, it proved to be just a simple heuristic device, but not a fundamental of the theory. It is a consequence of the requirement that electrodynamical phenomena, described by the Maxwell-Lorentz equations in Galilean coordinates, be consistent with the relativity principle.

A. Pais, mentioning the group character of Lorentz transforma-

tions, writes (see p. 130 of the book cited above):

*“He did, of course, not know that a few weeks earlier someone (A. Einstein is understood. — A.L.) else had independently noted the group properties of Lorentz transformations . . .”*

But all this is **absolutely incorrect**. Article [2] by H. Poincaré, appeared in “Comptes Rendus” on June, 5, 1905, whereas the article by A. Einstein had been sent to publisher on June, 30, 1905.

H. Poincaré, discovered the group and named it as **Lorentz group**. He wrote in article [2]:

*“All these transformations together with all rotations should form a group”.*

In articles [2; 3] by H. Poincaré, the group properties are widely used for constructing four-dimensional physical quantities, providing the invariance of electrodynamics equations under the Lorentz group. While in the article by A. Einstein only the following is told:

*“. . . from this we see that such parallel transformations form a group — as they indeed must”.*

There is no any other word on the group in the Einstein article. From here his misunderstanding that electrodynamic quantities should be transformed according to the group in order to provide the invariance of equations required by relativity principle follows naturally. But all this leads to the consequence that some physical quantities become four-dimensional, for example, current density, potentials, momentum, and force.

Striking “discoveries” are made by certain historians near science. Here, follows, for example, one “masterpiece” of such a creative activity. S. Goldberg wrote the following in his article

(“The British Journal for the History of Science”. 1970. Vol. V, No. 17, p. 73):

*“Poincaré had retained the notion of absolute space in his work, whether or not that space was accessible to observation”.*

*“There was in Poincaré's mind a **preferred** frame of reference in which the velocity of light was **really** a constant, and only one such frame”.*

S. Goldberg attributes all this to Poincaré without any grounds whatsoever. Thus, back in 1902, in the book “Science and Hypothesis”, Poincaré wrote:

*“Absolute space does not exist. We only perceive relative motions”.*

*“Absolute time does not exist”.*

In 1904 Poincaré formulated the principle of relativity for all physical phenomena (see Section 3 p. 25) and in 1905 established that, in accordance with the relativity principle, the equations of the electromagnetic field remain the same in all inertial reference systems, owing to the Lorentz transformations.

Thus the equality and constancy of the velocity of light is provided for any inertial reference system. All this is expounded in the articles by H. Poincaré [2, 3], which should have been studied carefully by S. Goldberg before writing about an opinion of Poincaré.

**In evaluating works [2] and [3], as well as the early works of H. Poincaré in physics**, it is necessary to proceed only from their content, comparing it with contemporary ideas, but not to be guided by outside statements on the issue, even made by well-known scientists, contemporaries of Poincaré, since the level of

many of them was insufficient to fully apprehend what Poincaré has written. At the time his personality was especially manifest in that for him physical problems and their adequate mathematical formulation joined naturally and composed a single whole. Namely for this reason, his creations are exact and modern even after a hundred years. H. Poincaré was one of those rare researchers, to whom natural sciences and mathematics are their proper surroundings. The young people of today, who are prepared in theoretical physics, can readily perceive this, if only they, at least, read Poincaré's works [2] and [3]. What concerns the statements by Professor A. Pais and Doctor S. Goldberg, we once more encounter, what we saw earlier is a clear attempt to attribute their own incomprehension to the author.

Some authors wishing to stress the preceding character of H. Poincaré's articles [2], [3] on relativity give two following quotes from the book of W. Pauli "Theory of Relativity" written by him in young age in 1921:

*"Is was Einstein, finally, who in a way completed the basic formulation of this new discipline "*.

*"It includes not only all the essential results contained in the other two papers, but shows an entirely novel, and much more profound, understanding of the whole problem"*.

Below we will give a quotation from W. Pauli related to the same subject, but written later, in 1955.

To the first Pauli quotation it should be said that no further completion of works [2], [3] by H. Poincaré is required. All the main results which contain the full content of the theory of relativity are formulated there and in the most definite form.

What about the second statement by Pauli, the case is just opposite. It is sufficient to compare the content of the Poincaré and

Einstein works to conclude that articles [2], [3] by Poincaré contain not only all the main content of the article by Einstein of 1905 (moreover Poincaré has formulated everything definitely in contrast to Einstein), but also contain main parts of the later work by Minkowski. What about words by Pauli on “*deep understanding of the whole problem*”, it is just present in articles [2], [3] by Poincaré. For example:

*“All forces behave in the same way as electromagnetic forces irrespective of their origin. This is due to Lorentz transformations (and consequently due to translational motion)”.*

In other words Lorentz invariance is universal. All the above in full can be said about gravitational forces.

Further, Poincaré discovered pseudo-Euclidean geometry of space-time, revealed the four-dimensionality of physical quantities. He constructed the equations of relativistic mechanics, predicted existence of the gravitational waves, propagating with the velocity of light. Then, what else “*deep understanding of the whole problem*” may be spoken about?

There is a surprising statement by L. de Broglie made in 1954:

*“A bit more and it would be H. Poincaré, and not A. Einstein, who first built the theory of relativity in its full generality and that would deliver to French science the honor of this discovery... But Poincaré has not made the decisive step and left to Einstein the honor to uncover all the consequences following from the principle of relativity, and in particular, by means of a deep analysis of measurements of length and time, to discover the real physical nature of relation between space and time maintained by the principle of relativity”.*

**In fact all is just opposite** to the L. de Broglie writings. H. Poincaré gave detailed analysis of time measurements already in his article of 1898 "The measurement of time", in particular, by means of a light signal. Later, in articles of 1900 and 1904 he describes a procedure for **determination of simultaneity** at different points of space by means of a light signal in a moving inertial system of reference, and therefore reveals the physical meaning of **local time** by Lorentz. In 1904 in article [1] he was the first who formulated the principle of relativity for all physical phenomena. In 1905 being based on the Lorentz paper H. Poincaré has discovered the Lorentz group in articles [2; 3] and on this ground proved invariance of Maxwell-Lorentz equations under Lorentz transformations in full agreement with the relativity principle. H. Poincaré extrapolated the Lorentz group on all physical forces. Therefore the Lorentz invariance became universal and valid also for gravitational phenomena. In article [3], being based on the Lorentz group H. Poincaré introduced pseudo-Euclidean space-time geometry. So, the homogeneous and isotropic space-time arose which was defined by the **invariant**

$$c^2t^2 - x^2 - y^2 - z^2.$$

It was developed in relativity of *time* and *length* concepts, in symmetry of physical laws, in conservation laws, in existence of the limiting velocity for material bodies, in four-dimensionality of physical quantities. The connection between space and time was determined in full by the structure of geometry. There is no such a deep insight into the essence of the problem in the article by A. Einstein. Following these ideas H. Poincaré discovered equations of relativistic mechanics and predicted existence of gravitational waves propagating with velocity of light. Therefore H. Poincaré deduced all the most general consequences from the principle of relativity. **There is no an idea from the 1905 work by**

**A. Einstein which has not been present in articles by H. Poincaré.** The work by A. Einstein is rather elementary in realization of ideas. Though in fact the realization of ideas required high level of analysis. In H. Poincaré's works [2; 3] there is not only a high level analysis and realization, but they contain also much new which is not contained in the article by A. Einstein and which has determined further development of the theory of relativity. How Louis de Broglie has not seen all this when reading the Poincaré articles? Compare writings by Louis de Broglie to writings by W. Pauli of 1955 (see present edition, p. 136).

It is quite evident, **that Louis de Broglie has not gained an understanding of the essence of the problem as a matter of fact.** Though being the Director of the Henri Poincaré Institute he had to do so.

Being based upon opinions by Louis de Broglie Academician V. L. Ginzburg writes:

*“As we see, the position of L. de Broglie, referring to the memory of H. Poincaré with a deep respect and with a maximal kindness, should be considered as one more testimony that the main author of the SRT is A. Einstein”.*

All this is strange. One would think everything is simple here: if your qualification admits you, then take the article by A. Einstein of 1905 and the articles by H. Poincaré, compare them and **all will be clear.** Just this will be considered in details in further Sections. What about the quotation of L. de Broglie, it clearly demonstrates his superficial knowledge of the works by H. Poincaré.

P. A. M. Dirac wrote in 1979 (Proceedings of the 1979 Einstein Centennial Symposium: Some Strangeness in the Proportion. Addison-Wesley MA 1980. P. 111.):

*“In one respect Einstein went far beyond Lorentz*

*and Poincaré and the others, and that was in asserting that the Lorentz transformation would apply to the whole of physics and not merely to phenomena based on electrodynamics. Any other physical forces that may be introduced in the future will have to conform to Lorentz transformations, which is going far beyond what the people who were working with electrodynamics were thinking about”.*

But just relating to this H. Poincaré wrote in 1905-1906 in articles [2; 3]:

*“... All forces, despite of the nature they may have, behave according to Lorentz transformations (and consequently, according to translational motion) just in the same way as electromagnetic forces”.*

Comparing the quotation from Poincaré with the words by Dirac, it is easy to get convinced, that all this considered by Dirac as the achievement by Einstein is contained in full in article [2] by Poincaré. Therefore the quoted statement by Dirac: *“In one respect Einstein went far beyond ... Poincaré”* is simply incorrect. Poincaré was the first who extrapolated Lorentz transformations onto any forces of nature, including gravitational ones.

The following, for example, is what Richard Feynman wrote (see his book *The Character of Physical law*. BBC, 1965):

*“It was Poincaré’s suggestion to make this analysis of what you can do to the equations and leave them alone. It was Poincaré’s attitude to pay attention to the symmetries of physical laws”.*

In 1955, in connection with the 50-th anniversary of relativity theory W. Pauli wrote:

*“ Both Einstein and Poincaré, took their stand on the preparatory work of H.A. Lorentz, who had already come quite close to the result, without however quite reaching it. In the agreement between the results of the methods followed independently of each other by Einstein and Poincaré I discern a deeper significance of a harmony between the mathematical method and analysis by means of conceptual experiments (Gedankenexperimente), which rests on general features of physical experience”.*

Compare this quotation from W. Pauli with words by L. de Broglie of 1954.

The articles [2, 3] by Henri Poincaré are extremely modern both in content and form and in the exactness of exposition. Truly, they are pearls of theoretical physics.

Now let us return to words by Academician V.L. Ginzburg (see this edition, p. 94), further he says about the principle of relativity:

*“... Moreover, Lorentz and Poincaré interpreted this principle only as a statement on impossibility to register the uniform motion of a body relative to ether”.*

This is absolutely incorrect in relation to Poincaré. Let me explain. This principle in Poincaré formulation is as follows [1]:

*“The principle of relativity, according to which the laws for physical phenomena should be the same both for observer at rest and for observer in uniform motion, i. e. we have no any method to determine whether we participate in such motion or not and we cannot have such a method in principle.”.*

There is no term “ether” in this formulation of the relativity principle. Therefore the statement by V.L. Ginzburg is a simple misunderstanding. Let us present some trivial explanations in this

connection. It follows from the formulation of the relativity principle that an observer performing a translational uniform motion can move with any constant velocity and so there is an infinite set of equitable reference systems with the same laws for physical phenomena. This set of equitable reference systems includes also a system of reference taken as a system of rest.

Then V. L. Ginzburg continues:

*“...It is possible to go from above to consideration of all inertial systems of reference as completely equitable (this is the modern treatment of the relativity principle) without special efforts only in case **if we understand Lorentz transformations as transformations corresponding to transition to the moving reference system** (emphasized by me. — A. L.)”.*

To have in mind that Poincaré has not understood that Lorentz transformations correspond to transition from the “rest” system of reference to the moving one is also a misunderstanding. This trivially follows from the Lorentz transformations.

From the Lorentz transformations

$$x' = \gamma(x - \varepsilon t)$$

it follows that the origin of the new system of reference

$$x' = 0, y' = 0, z' = 0$$

moves along axis  $x$  with velocity  $\varepsilon$ :

$$x = \varepsilon t$$

in relation to another system of reference. Therefore, Lorentz

transformations connect variables  $(t, x, y, z)$  referring to one system of reference with variables  $(t', x', y', z')$  referring to another system moving uniformly and straightforwardly with velocity  $\varepsilon$  along axis  $x$  relatively to the first system. The Lorentz transformations has taken place of the Galilean transformations speaking figuratively.

Let us consider in more detail the statement by V. L. Ginzburg. He notes that “*if one understands Lorentz transformations as transformations corresponding to transition to a moving system of reference*”, then “*it is possible without special efforts*” to go on to “*the treatment of all inertial systems of reference as completely equitable (this is the modern treatment of the relativity principle)*”.

But it is not so. This is not enough for the fulfilment of requirements of the principle of relativity. It is necessary to prove (and this is the most important) that the Lorentz transformations together with the spatial rotations form **the group**. But we are obliged for this solely to Poincaré. Only after discovering **the group** it is possible to say that all physical equations stay untouched at any inertial reference system. Then all the corresponding physical characteristics transform exactly according to **the group**. Just this provides the fulfilment of requirements of the relativity principle.

In connection with the quotation from Ginzburg (see this edition, p. 137) we will give some comments. Let us admit that the principle of relativity is treated as a statement of impossibility to register a uniform translational motion of a body relative to the ether. What follows from here? First, from here it follows directly that the physical equations are the same, both in the ether system of reference and in any other reference system, moving with constant velocity relative to the ether system. The invariableness of equations is provided by the Lorentz transformations. Second, as the Lorentz transformations **form a group**, it is impossible to

prefer one system of reference to another. The ether system of reference will be a member of this totality of equitable inertial systems. Therefore it will lose the meaning of the fixed system of reference. But this leads to the fact that **the ether** in the Lorentz sense **disappears**.

Very often in order to stress that Poincaré has not created the theory of relativity one cites his words:

*“The importance of this subject ought me to return to this again; the results obtained by me are in correspondence with those of Lorentz in all the most important points. I only tried to modify slightly and enlarge them.”.*

One usually concludes from this that Poincaré has exactly followed Lorentz views. But Lorentz, as he notes himself, has not established the relativity principle for electrodynamics. So, one concludes that also Poincaré has not made this decisive step. **But this is incorrect**. Those authors who write so have not read Poincaré articles [2, 3] carefully. Let us give some more explanations. H. Poincaré writes in his article [2]:

*“The idea by Lorentz is that electromagnetic field equations are invariant under some transformations (which I will call by name of H.A. Lorentz) of the following form. . .”.*

Poincaré writes: **“the idea by Lorentz”**, but Lorentz never wrote so before Poincaré. Here Poincaré has formulated his own fundamental idea, but ascribed it to Lorentz. He always appreciated and celebrated extremely high anybody who gave a stimulus to his thought, a joy of creation, probably as nobody else. He was absolutely deprived of personal priority sentiments. But descendants are obliged to restore truth and pay duty to the creator.

In the same article (see this edition, the footnote on p. 51) Academician V.L. Ginzburg writes:

*“One can suspect that Poincaré has not estimated the Einstein contribution as a very substantial one, and maybe he even has believed that he “has made everything himself”. But that’s just the point that we are trying to guess about the Poincaré feelings from his silence and not from some claims told by him. ”.*

One may readily find out what Poincaré has done in the theory of relativity: for a theoretical physicist it is enough to read his articles [2, 3]. Therefore it is not necessary “to guess” about the Poincaré feelings in order to answer the question: what he really has done. Academician V. L. Ginzburg usually cites writings by W. Pauli of 1921, but surprisingly does not cite writings by W. Pauli of 1955. Some people for some reason want to see only A. Einstein treated as the creator of special theory of relativity. But we should follow facts and only them.

Now let us consider words by professor Pais written in the same book at p. 169.

*“Why did Poincaré not mention Einstein in his Göttingen lectures? Why is there no paper by Poincaré in which Einstein and relativity are linked? It is inconceivable that Poincaré would have studied Einstein’s papers of 1905 without understanding them. It is impossible that in 1909 (the year he spoke at Göttingen) he would never have heard of Einstein’s activities in this area. Shall I write of petulance or professional envy?”.*

There is a unique answer to these questions. After reading the articles and books published by Poincaré up to 1905 it is easy

to get convinced that there has been nothing new for Poincaré in the Einstein article. Being based on his own previous works and on Lorentz investigations Poincaré formulated all the main content of the special theory of relativity, discovered the laws of relativistic mechanics, extended Lorentz transformations to all the forces of nature. But all this he ascribed to **the Great destructor** H.A. Lorentz, because just his article of 1904 provided a stimulus for Poincaré thought. This was his usual practice. It is strange that professor Pais addresses questions only to Poincaré, and not to Einstein. How Einstein decided to submit his paper on electrodynamics of moving body if he knew papers by Lorentz of ten years ago only and papers by Poincaré of five years ago only? What prevented Einstein from acquaintance with reviews published in the journal "Beiblätter Annalen der Physik", if he himself prepared many reviews for this journal? 21 reviews by Einstein were published there in 1905.

The journal "Beiblätter Annalen der Physik" was printed in Leipzig in separate issues. 24 issues were published in a year. The review of the Lorentz article which appeared in the journal "Verh. K. Ak. van Wet." (1904. **12** (8). S. 986–1009) was published in 4th issue of 1905. This review contained Lorentz transformations also.

A review by Einstein on the article by M. Ponsot from the May issue of the French journal "Comptes Rendus" 1905. **140**. S. 1176–1179 was published in the 18th issue of 1905. The same issue (S. 1171–1173) contains article by P. Langevin "On impossibility to register the translational motion of Earth by physical experiments". In this article P. Langevin refers to the articles by Lorentz of 1904 and Larmor of 1900.

Why Einstein never refers to articles [2, 3] by Poincaré? By the way, he wrote a lot of articles on the theory of relativity during the next 50 years. What personal qualities explain this? How is it

possible not to refer to articles, if they are published earlier and if you exploit ideas and concepts from them?

Academicians V. L. Ginzburg and Ya. B. Zel'dovich wrote in 1967 (see "Zel'dovich — known and unknown (in the recollections of his friends, colleagues, students). Moscow: "Nauka", 1993, p. 88):

*"For example, despite how much a person would do himself, he could not pretend to have a priority, if later it will be clear that the same result has been obtained earlier by other persons".*

This is a quite right view. We are to follow it. Ideas and results should be referred to that person who has discovered them first.

How strange the fate happened to be, if one can say that, of the works by Henri Poincaré, "On the dynamics of the electron", published in 1905-1906. These outstanding papers by H. Poincaré have become a peculiar source from which ideas and methods were drawn and then published without references to the author. When references to these articles were done, they always had nothing to do with the essence. All those discovered and introduced by Poincaré, in articles [2; 3] can be easily found in one or another form in articles by other authors published later.

M. Planck wrote in article of 1906 "The relativity principle and the general equations of mechanics":

*"The relativity principle, suggested by Lorentz and in more general formulation by Einstein means... "*

But after all this is incorrect. The relativity principle was first formulated in general form by Poincaré, in 1904. Then M. Planck derives equations of relativistic mechanics, but there are no references to the Poincaré article [3], though the equations of relativistic mechanics have been derived in it earlier. If ever M. Planck has

not been informed on the Poincaré work that time, he could refer to it later. But such a reference to article [3] did not appear also later. Articles by Poincaré, [2; 3] did not appear also in the German collection devoted to the theory of relativity. How one could explain all this?

B. Hoffmann (Proceedings of the 1979 Einstein Centennial Symposium: Some Strangeness in the Proportion. Addison-Wesley MA 1980. P. 111):

*“I am wondering whether people would have discovered the special theory of relativity without Einstein. It is true that Poincaré had all the mathematics and somewhat more than Einstein had in his 1905 paper, but in Poincaré's work there was always the implication that there was a rest system - something at rest in the ether — and so you get the impression that Poincaré and any followers would have said, yes, if something is moving relative to the ether, it is contracted. But, of course, people who believe this would think that our stationary rods were expanded, instead of contracted, and Poincaré would have had one clock going slower, but the other going faster. This reciprocity was a very subtle point, and it is quite likely that people might never have realized that it was a reciprocal relationship”.*

All this is inaccurate or follows from misunderstanding of the SRT basics. First, the SRT has already been discovered by Poincaré in articles [2; 3] according to the principle of relativity formulated by Poincaré in 1904 for all physical phenomena. In accordance with the principle of relativity physical equations are the same in all inertial reference systems. All inertial reference systems are equitable, and so the existence of a rest system of reference is excluded. From this it follows that the reversibility is realized here.

Second, Poincaré discovered the Lorentz group and the existence of the inverse element follows from here, consequently, the reversibility follows from existence of the group. Third, in the SRT constructed by Poincaré really this fact — “the reversible nature of this connection is a very subtle point” — is a trivial consequence, so writing “that people would never recognize this” is an intention of the author to see the problem there where it is absent. Moreover, it is absurdly to ascribe his own misunderstanding to Poincaré.

It is surprising to read a quotation from A. Einstein given by G. Holton (Proceedings of the 1979 Einstein Centennial Symposium: Some Strangeness in the Proportion. Addison-Wesley MA 1980. P. 111):

*“Einstein himself said that not Poincaré or Lorentz but Langevin might have developed the special theory of relativity”.*

If we trust G. Holton, then we see that A. Einstein without any doubt thought that it was exclusively he who discovered the special theory of relativity. Was it possible that he did not read the Poincaré papers [2; 3] where all the main content of the special theory of relativity was given in the extremely definite and general form? Therefore it is rather strange even such an appearance of this statement from A. Einstein. But if we admit that A. Einstein really has not read Poincaré articles [2; 3] during next fifty years, then this is also surprising. How this could be connected with the “punctilious honesty of Einstein” as a scientist which is expressively described by G. Holton?

**The suppression of Poincaré articles [2; 3] continued all the twentieth century. The opinion was created that the special theory of relativity is created by A. Einstein alone.** This is written in textbooks, including those used at school, in monographs, in science popular books, in encyclopedia. German physicists as distinct from French physicists have made a lot of efforts in

order to arrange the situation when A. Einstein alone was considered as the creator of the special theory of relativity, and this scientific achievement as a fruit of German science. But fortunately “manuscripts do not burn”. Articles [2; 3] clearly demonstrate the fundamental contribution by Poincaré to the discovering of the special theory of relativity. All the following done in this direction are applications and developments of his ideas and methods.

In 1913, a collection of the works of Lorentz, Einstein and Minkowski in the special relativity theory was published in Germany. But the fundamental works by H. Poincaré were not included in the collection. How this could be explained?

In 1911 the French physicist Paul Langevin published two articles on the relativity theory: “Evolution of the concept of space and time”; “Time, space and causality in modern physics”. But in these articles H. Poincaré is not even mentioned, although they deal with the relativity principle, the Lorentz group, space and time, determined by the interval. In 1920 in the article by P. Langevin “The historical development of the relativity principle” H. Poincaré is also not mentioned. How could P. Langevin do that?

In 1935 a collection “The relativity principle”, edited by professors V. K. Frederix and D. D. Ivanenko was published, which for the first time contained works in the relativity theory of Lorentz, Poincaré, Einstein and Minkowski. However, the first work by H. Poincaré, “On the dynamics of the electron” happened not to be included. And only in 1973, in the collection “The relativity principle” (with an introductory article by corresponding member of the USSR Academy of Sciences Professor D. I. Blokhintsev; the collection was compiled by Professor A. A. Tyapkin), the works of H. Poincaré in relativity theory were presented most completely, which permitted many people to appreciate the decisive contribution made by Poincaré in the creation of special relativity theory.

Somewhat later, Academician V. A. Matveev and me decided to rewrite the formulae in the articles by H. Poincaré "On the dynamics of the electron" in modern notations, so as to facilitate studying these articles.

In 1984, to the 130-th anniversary of H. Poincaré his articles "On the dynamics of the electron" together with comments were published by the Publishing Department of the Joint Institute for Nuclear Research (Dubna), and later, in 1987 they were published by the Publishing Department of the M.V. Lomonosov Moscow State University.

Henri Poincaré is one of the most rare personalities in the history of science. A greatest mathematician, specialist in mechanics, theoretical physicist; his fundamental works have left a most brilliant imprint in many fields of modern science. He, moreover, possessed the rare gift of profound vision of science as a whole. In the beginning of the past century (1902–1912) several books by Poincaré were published: "Science and hypothesis"; "The value of science"; "Science and method"; "Recent thoughts". Some of them were nearly at once translated into Russian. These books are marvellous both in content and in the free, extremely brilliant and illustrative manner of presentation. They have not become obsolete, and for everyone, who studies mathematics, physics, mechanics, philosophy, it would be extremely useful to become familiarized with them. It is quite regretful that for various reasons they were not republished for a long time. And only owing to the persistent efforts of Academician L. S. Pontryagin they have been republished and become available to present-day readers in Russia.

We also would like to note that some interesting books devoted to various aspects and "non-orthodoxal" views of the history of the relativity theory were published recently in the West [12].

## 10. The principle of stationary action in electrodynamics

Many equations of theoretical physics are obtained from requiring the functional, termed action, to achieve an extremum. Earlier (Section 2), the principle of least action was applied in mechanics, resulting in the Lagrange equations. We must in the case of electrodynamics, also, compose action so as to have its variation with respect to the fields leading to the Maxwell-Lorentz equations.

Action is constructed with the aid of scalars composed of functions of the field and current. We introduce tensor of the electromagnetic field

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}, \quad (10.1)$$

which by construction satisfies the equation

$$\frac{\partial F_{\mu\nu}}{\partial x^\sigma} + \frac{\partial F_{\nu\sigma}}{\partial x^\mu} + \frac{\partial F_{\sigma\mu}}{\partial x^\nu} = 0, \quad (10.2)$$

that is equivalent to the Maxwell-Lorentz equations (8.26). We need further the two simplest invariants only

$$A_\nu S^\nu, \quad F_{\lambda\nu} F^{\lambda\nu}. \quad (10.3)$$

Here  $S^\nu$  is the four-vector of current (8.9).

The sought action will have the form

$$S = \frac{1}{c} \int L d\Omega, \quad (10.4)$$

$L$  is the density of the Lagrangian function, equal to

$$L = -\frac{1}{c} A_\nu S^\nu - \frac{1}{16\pi} F_{\lambda\sigma} F^{\lambda\sigma}, \quad d\Omega = dx^0 dx^1 dx^2 dx^3. \quad (10.5)$$

In seeking for the field equations we shall only vary the field potentials in the action functional, considering the field sources  $S^\nu$  as given.

Then

$$\delta S = -\frac{1}{c} \int \left[ \frac{1}{c} S^\nu \delta A_\nu + \frac{1}{8\pi} F^{\lambda\sigma} \delta F_{\lambda\sigma} \right] d\Omega = 0. \quad (10.6)$$

Since the variations commute with differentiation, we obtain

$$F^{\lambda\sigma} \left( \frac{\partial}{\partial x^\lambda} \delta A_\sigma - \frac{\partial}{\partial x^\sigma} \delta A_\lambda \right) = -2F^{\sigma\lambda} \frac{\partial}{\partial x^\lambda} \delta A_\sigma. \quad (10.7)$$

Substituting (10.7) into (10.6) we find

$$\delta S = -\frac{1}{c} \int \left[ \frac{1}{c} S^\nu \delta A_\nu - \frac{1}{4\pi} F^{\sigma\lambda} \frac{\partial}{\partial x^\lambda} \delta A_\sigma \right] d\Omega = 0. \quad (10.8)$$

Integrating in the second term by parts and taking into account that the variations of potentials at the initial and final moments of time are zero, while the field vanishes at infinity, we obtain

$$\delta S = -\frac{1}{c} \int \left[ \frac{1}{c} S^\sigma + \frac{1}{4\pi} \cdot \frac{\partial F^{\sigma\lambda}}{\partial x^\lambda} \right] \delta A_\sigma d\Omega = 0. \quad (10.9)$$

Hence, owing to the arbitrariness of  $\delta A_\sigma$ , we find

$$\frac{\partial F^{\sigma\lambda}}{\partial x^\lambda} = -\frac{4\pi}{c} S^\sigma. \quad (10.10)$$

Thus, our choice of density of the Lagrangian function (10.5) is justified, since we have obtained exactly the second pair of Maxwell-Lorentz equations

$$\text{rot } \vec{H} = \frac{4\pi}{c} \vec{S} + \frac{1}{c} \cdot \frac{\partial \vec{E}}{\partial t}, \quad \text{div } \vec{E} = 4\pi\rho. \quad (10.11)$$

One must bear in mind, that the choice of density of the Lagrangian function in the action functional is not unambiguous, however, it is readily verified that adding to the density of the Lagrangian function an additional term in the form of the four-dimensional divergence of a vector does not influence the form of the field equations. The Maxwell-Lorentz equations (10.2), (10.10) are invariant with respect to gauge transformations of the potentials,

$$A'_\sigma = A_\sigma + \frac{\partial f}{\partial x^\sigma}, \quad (10.12)$$

here  $f$  is an arbitrary function.

The density of the Lagrangian (10.5) we have constructed is not invariant under transformations (10.12). On the basis of the conservation law of current  $S^\nu$  (8.10), it only varies by a divergence,

$$L' = L - \frac{1}{c} \cdot \frac{\partial}{\partial x^\nu} (f S^\nu), \quad (10.13)$$

which has not effect on the field equations.

From the point of view of classical electrodynamics the potential  $A^\nu$  has no physical sense, since only the Lorentz force acts on the charge, and it is expressed via the field strength  $\vec{E}$ ,  $\vec{H}$ . However, in quantum mechanics this is no longer so. It turns out to be that the vector-potential does act on the electron in a certain situation. This is the Aharonov-Bohm effect. It was observed in 1960. The experiment was carried out as follows: a long narrow solenoid was used, the magnetic field outside the solenoid was zero, nevertheless, the motion of electrons outside the solenoid was influenced. The effect is explained by the solenoid violating the simple connectedness of space-time, which gave rise to the influence of the potential  $A^\nu$ , as it should be in quantum gauge theory.

We shall now find the equations of motion for charged particles in an electromagnetic field. To obtain them it is necessary to compose an action with a part related to the particles and, also, the

already known to us part containing the field interaction with particles. Since for a particle having charge  $e$  the following equations are valid

$$\rho = e\delta(\vec{r} - \vec{r}_0), \quad j^i = e\frac{dx^i}{dt}\delta(\vec{r} - \vec{r}_0), \quad (10.14)$$

we have

$$-\frac{1}{c^2} \int S^\nu A_\nu d\Omega = -\frac{e}{c} \int A_\nu dx^\nu. \quad (10.15)$$

The action for a particle in an electromagnetic field is

$$S = -mc \int d\sigma - \frac{e}{c} \int A_\nu dx^\nu. \quad (10.16)$$

Varying over the particle coordinates, we obtain

$$\delta S = - \int \left( mcU_\nu d\delta x^\nu + \frac{e}{c} A_\nu d\delta x^\nu + \frac{e}{c} \delta A_\nu dx^\nu \right) = 0. \quad (10.17)$$

Integrating by parts in the first two terms and setting the variations of coordinates to zero at the ends, we obtain

$$\delta S = \int \left( mcdU_\nu \delta x^\nu + \frac{e}{c} dA_\nu \delta x^\nu - \frac{e}{c} \delta A_\nu dx^\nu \right) = 0. \quad (10.18)$$

With account of the obvious relations

$$dA_\nu = \frac{\partial A_\nu}{\partial x^\lambda} dx^\lambda, \quad \delta A_\nu = \frac{\partial A_\nu}{\partial x^\lambda} \delta x^\lambda, \quad (10.19)$$

expression (10.18) assumes the form

$$\delta S = \int \left[ mc \frac{dU_\nu}{d\sigma} - \frac{e}{c} \left( \frac{\partial A_\lambda}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\lambda} \right) U^\lambda \right] d\sigma \delta x^\nu = 0, \quad (10.20)$$

hence, due to arbitrariness of variation  $\delta x^\nu$  being arbitrary, we have

$$mc^2 \frac{dU_\nu}{d\sigma} = eF_{\nu\lambda}U^\lambda, \quad (10.21)$$

or

$$mc^2 \frac{dU^\nu}{d\sigma} = eF^{\nu\lambda}U_\lambda. \quad (10.22)$$

In three-dimensional form (10.22) assumes the form

$$mc \frac{d}{dt} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{e}{c} (\vec{v} \cdot \vec{E}), \quad (10.23)$$

$$\frac{d}{dt} \left( \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = e\vec{E} + \frac{e}{c} [\vec{v}, \vec{H}]. \quad (10.24)$$

Let us calculate the energy loss for an electron moving with acceleration. In case of electron velocity small in comparison to the velocity of light the radiation energy loss is given by the following formula due to Larmor:

$$-\frac{\partial E}{\partial t} = \frac{2e^2}{3c^3} \left( \frac{d\vec{v}}{dt} \right)^2. \quad (10.25)$$

In the system of reference where the electron is at rest this formula takes the form

$$-\frac{\partial E}{\partial t} = \frac{2e^2}{3c^3} \left( \frac{d\vec{v}}{dt} \right)_0^2, \quad (10.26)$$

where acceleration is calculated in the given system of reference.

In the given system of reference the total momentum radiated is zero due to the symmetry of radiation:

$$-\frac{\partial p^i}{\partial t} = 0. \quad (10.27)$$

In order to find the formula for radiation energy loss of a charge with high velocity it is necessary to apply the Lorentz group, according to which it is easy to make the transition from one system of reference to another. To do so we consider the acceleration four-vector, which is as follows according to (9.5)

$$a^\nu = c^2 \frac{dU^\nu}{d\sigma} = c \frac{dU^\nu}{d\tau}. \quad (10.28)$$

By means of this relation and also formulae (9.3) we get

$$a^0 = \gamma^4 \left( \frac{\vec{v} d\vec{v}}{c dt} \right), \quad \vec{a} = \gamma^2 \frac{d\vec{v}}{dt} + \vec{v} \frac{\gamma^4}{c^2} \left( \vec{v} \frac{d\vec{v}}{dt} \right). \quad (10.29)$$

Using (10.29) we find invariant

$$(a^0)^2 - (\vec{a})^2 = -\gamma^6 \left\{ \left( \frac{d\vec{v}}{dt} \right)^2 - \left[ \frac{\vec{v}}{c}, \frac{d\vec{v}}{dt} \right]^2 \right\}. \quad (10.30)$$

In the rest system of reference we have

$$\gamma^6 \left\{ \left( \frac{d\vec{v}}{dt} \right)^2 - \left[ \frac{\vec{v}}{c}, \frac{d\vec{v}}{dt} \right]^2 \right\} = \left( \frac{d\vec{v}}{dt} \right)_0^2. \quad (10.31)$$

Let us now write formulae (10.26) and (10.27) in the covariant form

$$-\frac{\partial p^\nu}{\partial \tau} = \frac{2e^2}{3c^4} \left( \frac{d\vec{v}}{dt} \right)_0^2 U^\nu. \quad (10.32)$$

Substituting now (10.31) into this relation we obtain

$$-\frac{\partial E}{\partial t} = \frac{2e^2}{3c^3} \gamma^6 \left\{ \left( \frac{d\vec{v}}{dt} \right)^2 - \left[ \frac{\vec{v}}{c}, \frac{d\vec{v}}{dt} \right]^2 \right\}, \quad (10.33)$$

$$-\frac{\partial \vec{p}}{\partial t} = \frac{2e^2}{3c^5} \vec{v} \gamma^6 \left\{ \left( \frac{d\vec{v}}{dt} \right)^2 - \left[ \frac{\vec{v}}{c}, \frac{d\vec{v}}{dt} \right]^2 \right\}. \quad (10.34)$$

Formula (10.33) has been derived first by Liénard in 1898.

The equations of motion (10.22) in an external electromagnetic field do not account for the reaction of radiation. Therefore, these equations are valid only for the motion of a charged particle in weak fields. In 1938 Dirac took into account the reaction forces, and this led to the equation

$$\begin{aligned}
 mc^2 \frac{dU^\nu}{d\sigma} &= eF^{\nu\lambda}U_\lambda + \\
 &+ \frac{2}{3}e^2 \left[ \frac{d^2U^\nu}{d\sigma^2} + U^\nu \cdot \left( \frac{dU_\mu}{d\sigma} \cdot \frac{dU^\mu}{d\sigma} \right) \right],
 \end{aligned} \tag{10.35}$$

called the Dirac-Lorentz equation.

Let us apply these formulae to motion of an ultra-relativistic charge with mass  $m$  in a strong constant uniform magnetic field  $H$ . We admit that the circular charge motion is determined by the Lorentz force only. So we neglect by influence of the force of reaction on the motion. Let us write equations (10.35) in the form of Eqs. (9.12), (9.13):

$$\frac{d}{dt}(m\gamma\vec{v}) = \frac{e}{c} \cdot [\vec{v}, \vec{H}] + \vec{f}_R, \tag{10.36}$$

$$\vec{f}_R = \frac{2e^2}{3\gamma} \cdot \left[ \frac{d^2\vec{U}}{d\sigma^2} + \vec{U} \cdot \left( \frac{dU_\nu}{d\sigma} \cdot \frac{dU^\nu}{d\sigma} \right) \right], \tag{10.37}$$

$$\frac{dE}{dt} = \frac{2e^2c}{3\gamma^2} \left[ \vec{U} \cdot \frac{d^2\vec{U}}{d\sigma^2} + U^2 \cdot \left( \frac{dU_\nu}{d\sigma} \cdot \frac{dU^\nu}{d\sigma} \right) \right], \tag{10.38}$$

where  $E$  is the energy of the particle.

As in our approximation the equations of motion are the following

$$mc \cdot \frac{d\vec{U}}{d\sigma} = \frac{e}{c} \cdot [\vec{U}, \vec{H}], \quad (\vec{U} \cdot \vec{H}) = 0, \tag{10.39}$$

it follows from here that

$$U^2 \cdot \left( \frac{d\vec{U}}{d\sigma} \right)^2 = \left( \frac{eH}{mc^2} \right)^2 \cdot U^4, \quad (10.40)$$

$$\vec{U} \cdot \frac{d^2\vec{U}}{d\sigma^2} = - \left( \frac{eH}{mc^2} \right)^2 \cdot U^2, \quad (10.41)$$

where  $U$  is the length of vector  $\vec{U}$ . For ultra-relativistic particles

$$U \simeq \frac{E}{mc^2}. \quad (10.42)$$

As  $U^2 \gg 1$  we can neglect first term (10.41) in comparison to second one (10.40) in expression (10.38). In our approximation we can also neglect by the following term

$$U^2 \cdot \left( \frac{dU^0}{d\sigma} \right)^2, \quad (10.43)$$

in second term (10.38) due to its smallness in comparison with (10.40). Expression (10.38) after taking into account (10.40) and (10.42) is as follows

$$-\frac{dE}{dt} = \frac{2}{3} \cdot \frac{e^4 H^2 E^2}{m^4 c^7}. \quad (10.44)$$

With regard to the fact that for the motion of a charge over circle of radius  $R$  the following equation takes place

$$H = \frac{E}{eR}; \quad (10.45)$$

we can rewrite formula (10.44) in the following form

$$-\frac{dE}{dt} = \frac{2e^2 c}{3R^2} \cdot \left( \frac{E}{mc^2} \right)^4. \quad (10.46)$$

If the energy of electrons and the value of the magnetic field are large enough, then energy losses for synchrotron radiation become rather substantial. Synchrotron radiation is widely used in biology and medicine, in production of integral schemes and so on. Special storage rings for generation of the intense X-rays are constructed (see more details in: Ya. P. Terletsky, Yu. P. Rybakov "Electrodynamics". Moscow: "Vysshaja Shkola", 1980 (in Russian)).

## 11. Inertial motion of a test body. Covariant differentiation

In an arbitrary reference system the interval is known to have the form

$$d\sigma^2 = \gamma_{\mu\nu}(x)dx^\mu dx^\nu, \quad \det(\gamma_{\mu\nu}) = \gamma < 0. \quad (11.1)$$

The pseudo-Euclidean metric  $\gamma_{\mu\nu}$  is determined by expression (3.33). Precisely for this metric the Riemannian curvature tensor is zero. The action for a free moving point-like body of mass  $m$  has the form

$$S = -mc \int d\sigma. \quad (11.2)$$

Owing to the principle of stationary action, we have

$$\delta S = -mc \int \delta(d\sigma) = 0, \quad (11.3)$$

$$\begin{aligned} \delta(d\sigma^2) &= 2d\sigma\delta(d\sigma) = \delta(\gamma_{\mu\nu}(x)dx^\mu dx^\nu) = \\ &= \frac{\partial\gamma_{\mu\nu}}{\partial x^\lambda} \delta x^\lambda dx^\mu dx^\nu + 2\gamma_{\mu\nu} dx^\mu \delta(dx^\nu). \end{aligned} \quad (11.4)$$

Since

$$\delta(dx^\nu) = d(\delta x^\nu), \quad (11.5)$$

from expression (11.4) we find

$$\delta(d\sigma) = \frac{1}{2} \cdot \frac{\partial\gamma_{\mu\nu}}{\partial x^\lambda} U^\mu dx^\nu \delta x^\lambda + \gamma_{\mu\nu} U^\mu d(\delta x^\nu), \quad (11.6)$$

here

$$U^\mu = \frac{dx^\mu}{d\sigma}. \quad (11.7)$$

Substituting (11.6) into (11.3) we obtain

$$\delta S = -mc \int \left[ \frac{1}{2} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x^\lambda} U^\mu U^\nu \delta x^\lambda + \gamma_{\mu\nu} U^\mu \frac{d(\delta x^\nu)}{d\sigma} \right] d\sigma = 0. \quad (11.8)$$

Since

$$\gamma_{\mu\nu} U^\mu \frac{d(\delta x^\nu)}{d\sigma} = \frac{d}{d\sigma} (\gamma_{\mu\nu} U^\mu \delta x^\nu) - \delta x^\nu \frac{d}{d\sigma} (\gamma_{\mu\nu} U^\mu), \quad (11.9)$$

then, with account of the variations at the boundary of the region being zero, we find

$$\begin{aligned} \delta S = -mc \int \left[ \frac{1}{2} \cdot \frac{\partial \gamma_{\mu\nu}}{\partial x^\lambda} U^\mu U^\nu - \gamma_{\mu\lambda} \frac{dU^\mu}{d\sigma} - \right. \\ \left. - \frac{\partial \gamma_{\mu\lambda}}{\partial x^\alpha} U^\mu U^\alpha \right] d\sigma \delta x^\lambda = 0. \end{aligned} \quad (11.10)$$

We represent the last term in (11.10) as

$$\frac{\partial \gamma_{\mu\lambda}}{\partial x^\alpha} U^\mu U^\alpha = \frac{1}{2} \left( \frac{\partial \gamma_{\mu\lambda}}{\partial x^\alpha} + \frac{\partial \gamma_{\alpha\lambda}}{\partial x^\mu} \right) U^\mu U^\alpha. \quad (11.11)$$

With account of (11.11), expression (11.10) assumes the form

$$\begin{aligned} \int \left[ \frac{1}{2} \left( \frac{\partial \gamma_{\mu\lambda}}{\partial x^\alpha} + \frac{\partial \gamma_{\alpha\lambda}}{\partial x^\mu} - \frac{\partial \gamma_{\mu\alpha}}{\partial x^\lambda} \right) U^\mu U^\alpha + \right. \\ \left. + \gamma_{\mu\lambda} \frac{dU^\mu}{d\sigma} \right] d\sigma \delta x^\lambda = 0. \end{aligned} \quad (11.12)$$

Since the factors  $\delta x^\lambda$  are arbitrary, we find

$$\gamma_{\mu\lambda} \frac{dU^\mu}{d\sigma} + \frac{1}{2} \left( \frac{\partial \gamma_{\mu\lambda}}{\partial x^\alpha} + \frac{\partial \gamma_{\alpha\lambda}}{\partial x^\mu} - \frac{\partial \gamma_{\mu\alpha}}{\partial x^\lambda} \right) U^\mu U^\alpha = 0. \quad (11.13)$$

Multiplying (11.13) by  $\gamma^{\lambda\nu}$ , we obtain

$$\frac{dU^\nu}{d\sigma} + \Gamma_{\mu\alpha}^\nu U^\mu U^\alpha = 0, \quad (11.14)$$

here  $\Gamma_{\mu\alpha}^\nu$  is the Christoffel symbol

$$\Gamma_{\mu\alpha}^\nu = \frac{1}{2} \gamma^{\nu\lambda} (\partial_\alpha \gamma_{\mu\lambda} + \partial_\mu \gamma_{\alpha\lambda} - \partial_\lambda \gamma_{\mu\alpha}). \quad (11.15)$$

We see that inertial motion of any test body, independently of its mass, proceeds along the geodesic line, determined by equation (11.14). It is absolutely evident that in arbitrary coordinates the geodesic lines could not be treated as direct lines, this is confirmed by nonlinear dependence of spatial coordinates  $x^i$  ( $i = 1, 2, 3$ ) on time variable  $x^0$ . Motion along a geodesic line (11.14) in Minkowski space is a free motion. **Thus, forces of inertia cannot cause any deformation by themselves. Under their influence free motion takes place. The situation changes, when there are forces of reaction, which counteract the forces of inertia. In this case deformation is unavoidable.** In weightlessness, in a satellite, deformation does not exist, because, owing to the gravitational field being homogeneous, in each element of the volume of a body compensation of the force of gravity by the forces of inertia takes place. The forces of gravity and the forces of inertia are volume forces.

Physical forces are four-vectors in Minkowski space. But the forces of inertia are not, since they can be rendered equal to zero by transition to an inertial reference system in Minkowski space.

Now we shall dwell upon the issue of covariant differentiation. In Cartesian coordinates  $x^\lambda$  ordinary differentiation, for example, of a vector  $A^\nu$  results in a tensor quantity

$$\frac{\partial A^\nu}{\partial x^\lambda} = B_\lambda^\nu \quad (11.16)$$

with respect to linear transformations. In arbitrary coordinates  $y^\lambda$  this property is not conserved, and, therefore, the quantity  $\partial A^\nu / \partial y^\lambda$  will no longer be a tensor.

It is necessary to introduce the covariant derivative, which will provide for differentiation of a tensor yielding a tensor, again. This will permit us to easily render covariant any physical equations. **Covariance is not a physical, but a mathematical, requirement.**

Earlier (see 6.13) we saw that of two vectors  $A^\nu, B_\nu$  it is possible to construct an invariant

$$A^\nu(x)B_\nu(x). \quad (11.17)$$

We shall consider an invariant of a particular form

$$A_\lambda(x)U^\lambda(x), \quad (11.18)$$

where

$$U^\lambda = \frac{dx^\lambda}{d\sigma}, \quad (11.19)$$

fulfils Eq. (11.14).

Differentiating (11.18) with respect to  $d\sigma$ , we also obtain an invariant (a scalar)

$$\frac{d}{d\sigma}(A_\lambda U^\lambda) = \frac{dA_\lambda}{d\sigma}U^\lambda + A_\nu \frac{dU^\nu}{d\sigma}.$$

Substituting expression (11.14) into the right-hand part, we find

$$\begin{aligned} \frac{d}{d\sigma}(A_\lambda U^\lambda) &= \frac{\partial A_\lambda}{\partial x^\alpha} U^\alpha U^\lambda - \Gamma_{\alpha\lambda}^\nu U^\alpha U^\lambda A_\nu, \text{ i. e.} \\ \frac{d}{d\sigma}(A_\lambda U^\lambda) &= \left( \frac{\partial A_\lambda}{\partial x^\alpha} - \Gamma_{\alpha\lambda}^\nu A_\nu \right) U^\alpha U^\lambda. \end{aligned} \quad (11.20)$$

Since (11.20) is an invariant,  $U^\lambda$  is a vector, hence it follows that the quantity

$$\frac{\partial A_\lambda}{\partial x^\alpha} - \Gamma_{\alpha\lambda}^\nu A_\nu$$

is a covariant tensor of the second rank  $A_{\lambda;\alpha}$

$$A_{\lambda;\alpha} = \frac{DA_\lambda}{\partial x^\alpha} = \frac{\partial A_\lambda}{\partial x^\alpha} - \Gamma_{\alpha\lambda}^\nu A_\nu. \quad (11.21)$$

Here and further the semicolon denotes covariant differentiation.

Thus, we have defined the covariant derivative of the covariant vector  $A_\nu$ . Now, we shall define the covariant derivative of the contravariant vector  $A^\nu$ . To this end we write the same invariant as

$$\begin{aligned} \frac{d}{d\sigma}(A^\mu U^\nu \gamma_{\mu\nu}) &= \frac{\partial A^\mu}{\partial x^\lambda} U^\nu U^\lambda \gamma_{\mu\nu} + \\ &+ A^\mu \gamma_{\mu\nu} \frac{dU^\nu}{d\sigma} + A^\mu U^\nu U^\lambda \frac{\partial \gamma_{\mu\nu}}{\partial x^\lambda}. \end{aligned}$$

Substituting expression (11.14) into the right-hand part, we obtain

$$\begin{aligned} \frac{d}{d\sigma}(A_\nu U^\nu) &= \\ &= \left[ \gamma_{\mu\nu} \frac{\partial A^\mu}{\partial x^\lambda} - A^\mu \gamma_{\mu\alpha} \Gamma_{\nu\lambda}^\alpha + A^\mu \frac{\partial \gamma_{\mu\nu}}{\partial x^\lambda} \right] U^\nu U^\lambda. \end{aligned} \quad (11.22)$$

Taking into account definition (11.15) we find

$$A^\mu \partial_\lambda \gamma_{\mu\nu} - A^\mu \gamma_{\mu\alpha} \Gamma_{\nu\lambda}^\alpha = \frac{1}{2} A^\mu (\partial_\lambda \gamma_{\mu\nu} + \partial_\mu \gamma_{\nu\lambda} - \partial_\nu \gamma_{\mu\lambda}). \quad (11.23)$$

Substituting this expression into (11.22), and applying expression  $U_\alpha \gamma^{\alpha\nu}$ , instead of  $U^\nu$ , we obtain

$$\frac{d}{d\sigma}(A_\nu U^\nu) = \left[ \frac{\partial A^\alpha}{\partial x^\lambda} + \Gamma_{\mu\lambda}^\alpha A^\mu \right] U^\lambda U_\alpha. \quad (11.24)$$

Since (11.24) is an invariant (a scalar), and  $U^\nu$  is a vector, from (11.24) it follows that the first factor in the right-hand part is a tensor.

Hence it follows, **that the covariant derivative of the contravariant vector  $A^\nu$  is**

$$A^\alpha_{;\lambda} = \frac{DA^\alpha}{\partial x^\lambda} = \frac{\partial A^\alpha}{\partial x^\lambda} + \Gamma_{\mu\lambda}^\alpha A^\mu. \quad (11.25)$$

Making use of formulae (11.21) and (11.25) one can also obtain covariant derivatives of a tensor of the second rank.

$$A_{\mu\nu;\lambda} = \frac{\partial A_{\mu\nu}}{\partial x^\lambda} - \Gamma_{\lambda\mu}^\alpha A_{\alpha\nu} - \Gamma_{\lambda\nu}^\alpha A_{\mu\alpha}, \quad (11.26)$$

$$A^{\mu\nu}_{;\lambda} = \frac{\partial A^{\mu\nu}}{\partial x^\lambda} + \Gamma_{\lambda\alpha}^\mu A^{\alpha\nu} + \Gamma_{\lambda\alpha}^\nu A^{\mu\alpha}, \quad (11.27)$$

$$A^\nu_{\rho;\sigma} = \frac{\partial A^\nu_\rho}{\partial x^\sigma} - \Gamma_{\rho\sigma}^\lambda A^\nu_\lambda + \Gamma_{\sigma\lambda}^\nu A^\lambda_\rho. \quad (11.28)$$

We see, that the rules established for (11.21) and (11.25) are applied independently for each index of the tensor. Precisely in this way, one can obtain the covariant derivative of a tensor of any rank.

With the aid of expression (11.26) it is easy to show that the covariant derivative of a metric tensor is zero,

$$\gamma_{\mu\nu;\rho} \equiv 0. \quad (11.29)$$

Applying the technique of covariant differentiation, one can readily write the equations of relativistic mechanics and of electrodynamics in arbitrary coordinates of Minkowski space.

Thus, substituting the covariant derivative for the ordinary one in (9.5) we find the equation of relativistic mechanics in arbitrary coordinates

$$mc^2 \frac{DU^\nu}{d\sigma} = mc^2 \left( \frac{dU^\nu}{d\sigma} + \Gamma_{\mu\lambda}^\nu U^\mu U^\lambda \right) = F^\nu, \quad (11.30)$$

here

$$\Gamma_{\mu\lambda}^{\nu} = \frac{1}{2}\gamma^{\nu\rho}(\partial_{\mu}\gamma_{\lambda\rho} + \partial_{\lambda}\gamma_{\mu\rho} - \partial_{\rho}\gamma_{\mu\lambda}). \quad (11.31)$$

In a similar way, it is also possible to write the Maxwell-Lorentz equations in arbitrary coordinates. To this end, it is necessary to substitute covariant derivatives for ordinary derivatives in equations (8.24) and (8.27),

$$D_{\sigma}F_{\mu\nu} + D_{\mu}F_{\nu\sigma} + D_{\nu}F_{\sigma\mu} = 0, \quad (11.32)$$

$$D_{\nu}F^{\mu\nu} = -\frac{4\pi}{c}S^{\mu}. \quad (11.33)$$

One can readily verify, that the following equalities hold valid:

$$F_{\mu\nu} = D_{\mu}A_{\nu} - D_{\nu}A_{\mu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}, \quad (11.34)$$

$$D_{\sigma}F_{\mu\nu} + D_{\mu}F_{\nu\sigma} + D_{\nu}F_{\sigma\mu} = \partial_{\sigma}F_{\mu\nu} + \partial_{\mu}F_{\nu\sigma} + \partial_{\nu}F_{\sigma\mu}. \quad (11.35)$$

On the basis of (11.31) we find

$$\Gamma_{\mu\nu}^{\nu} = \frac{1}{2}\gamma^{\nu\rho}\partial_{\mu}\gamma_{\nu\rho}. \quad (11.36)$$

But, since the following equalities hold valid:

$$\frac{1}{\gamma} \cdot \frac{\partial\gamma}{\partial x^{\mu}} = \gamma^{\nu\rho}\partial_{\mu}\gamma_{\rho\nu}, \quad \frac{\partial\sqrt{-\gamma}}{\partial\gamma_{\mu\nu}} = \frac{1}{2}\sqrt{-\gamma}\gamma^{\mu\nu}, \quad (11.37)$$

[here  $\gamma = \det(\gamma_{\mu\nu}) < 0$ ], we obtain

$$\Gamma_{\mu\nu}^{\nu} = \frac{1}{2\gamma} \cdot \frac{\partial\gamma}{\partial x^{\mu}} = \partial_{\mu} \ln \sqrt{-\gamma}. \quad (11.38)$$

Making use of (11.27), we find

$$D_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} + \Gamma_{\nu\alpha}^\mu F^{\alpha\nu} + \Gamma_{\alpha\nu}^\nu F^{\mu\alpha}. \quad (11.39)$$

The second term in (11.39) equals zero, owing to the tensor  $F^{\alpha\nu}$  being antisymmetric. On the basis of (11.38), expression (11.39) can be written as

$$D_\nu(\sqrt{-\gamma} F^{\mu\nu}) = \partial_\nu(\sqrt{-\gamma} F^{\mu\nu}). \quad (11.40)$$

Thus, equation (11.33) assumes the form

$$\frac{1}{\sqrt{-\gamma}} \partial_\nu(\sqrt{-\gamma} F^{\mu\nu}) = -\frac{4\pi}{c} S^\mu. \quad (11.41)$$

The equations of motion of charged particles can be obtained by substituting covariant derivatives for the ordinary derivatives in (10.22)

$$mc^2 \frac{DU^\nu}{d\sigma} = eF^{\nu\lambda} U_\lambda. \quad (11.42)$$

Thus, we have established that transition in Minkowski space from Galilean coordinates in an inertial reference system to arbitrary coordinates is a simple mathematical procedure, if covariant differentiation has been defined.

**The property of covariance of the equations has nothing to do with the relativity principle. This has long ago been clarified by V. A. Fock [13].**

**Therefore, no “general relativity principle”, as a physical principle, exists.**

## 12. Relativistic motion with constant acceleration. The clock paradox. Sagnac effect

Relativistic motion with constant acceleration is a motion under the influence of a force  $\vec{f}$ , that is constant in value and direction. According to (9.12) we have

$$\frac{d}{dt} \left( \frac{\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \frac{\vec{f}}{m} = \vec{a}. \quad (12.1)$$

Integrating equation (12.1) over time, we obtain

$$\frac{\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} = \vec{a}t + \vec{v}_0. \quad (12.2)$$

Setting the constant  $\vec{v}_0$  to zero, which corresponds to zero initial velocity, we find after squaring

$$\frac{1}{1 - \frac{v^2}{c^2}} = 1 + \frac{a^2 t^2}{c^2}. \quad (12.3)$$

Taking into account this expression in (12.2), we obtain

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{\vec{a}t}{\sqrt{1 + \frac{a^2 t^2}{c^2}}}. \quad (12.4)$$

Integrating this equation, we find

$$\vec{r} = \vec{r}_0 + \frac{\vec{a}c^2}{a^2} \left[ \sqrt{1 + \frac{a^2 t^2}{c^2}} - 1 \right]. \quad (12.5)$$

Since the interval  $ds$  is

$$ds = cdt\sqrt{1 - \frac{v^2}{c^2}}, \quad (12.6)$$

the proper time  $d\tau$  for a moving test body is

$$d\tau = \frac{ds}{c} = dt\sqrt{1 - \frac{v^2}{c^2}}. \quad (12.7)$$

Taking account of (12.3), from equation (12.7) we find the total proper time  $\tau$

$$\tau = t_0 + \frac{c}{a} \ln \left[ \frac{at}{c} + \sqrt{1 + \frac{a^2 t^2}{c^2}} \right]. \quad (12.8)$$

From this formula it follows that, as time  $t$  increases in an inertial reference system, the proper time for a moving body flows slowly, according to a logarithmic law. We considered the motion of a body with acceleration  $\vec{a}$  with respect to an inertial reference system in Galilean coordinates.

Now consider a reference system moving with constant acceleration. Let the inertial and moving reference systems have coordinate axes oriented in the same way, and let one of them be moving with respect to the other along the  $x$  axis. Then, if one considers their origins to have coincided at  $t = 0$ , from expression (12.5) one obtains the law of motion of the origin of the reference system moving relativistically with constant acceleration,

$$x_0 = \frac{c^2}{a} \left[ \sqrt{1 + \frac{a^2 t^2}{c^2}} - 1 \right]. \quad (12.9)$$

Therefore, the formula for coordinate transformation, when transition is performed from the inertial reference system  $(X, T)$  to

the reference system  $(x, t)$  moving relativistically with constant acceleration, will have the form

$$x = X - x_0 = X - \frac{c^2}{a} \left[ \sqrt{1 + \frac{a^2 T^2}{c^2}} - 1 \right]. \quad (12.10)$$

The transformation of time can be set arbitrarily. Let it be the same in both reference systems

$$t = T. \quad (12.11)$$

In the case of transformations (12.10) and (12.11) the interval  $d\sigma$  assumes the form

$$d\sigma^2 = \frac{c^2 dt^2}{1 + \frac{a^2 t^2}{c^2}} - \frac{2a t dt dx}{\sqrt{1 + \frac{a^2 t^2}{c^2}}} - dx^2 - dY^2 - dZ^2. \quad (12.12)$$

*We shall now proceed to deal with the “clock paradox”.*

Consider two reference systems. If two observers, who are in these reference systems, compared their clocks at moment  $t = 0$ , and then departed from each other, and after some period of time they again met at one point in space, what time will their clocks show? The answer to this question is the solution of the so-called “clock paradox”. However, two observers, who are in different inertial reference systems, after they have compared their clocks at one and the same point of space, will never be able to meet in the future at any other point of space, because to do so, at least, one of them would have to interrupt his inertial motion and for some time go over to a non-inertial reference system. In scientific literature, and in textbooks, as well, it is often written that the answer to this question cannot be given within the framework of special relativity theory.

This is, naturally, wrong, the issue is resolved precisely within the framework of special relativity theory. The point is that reference systems moving with acceleration in pseudo-Euclidean geometry, contrary to A. Einstein's point of view, have nothing to do with the gravitational field, and for this reason general relativity theory is not required for explaining the "clock paradox".

Let us illustrate this statement by a concrete computation. Suppose we have two identical (ideal) clocks at one and the same point of an inertial reference system. Consider their readings to coincide at the initial moment  $T = 0$ . Let one of these clocks always be at rest at the initial point and, thus, be inertial. Under the influence of an applied force, at moment  $t = 0$ , the other clock starts to move relativistically with a constant acceleration  $a$  along the  $x$  axis, and continues moving thus till the moment of time  $t = T_1$ , shown by the clock at rest. Further, the influence of the force on the second clock ceases, and during the time interval  $T_1 \leq t \leq T_1 + T_2$  it moves with constant velocity. After that a decelerating force is applied to it, and under the influence of this force it starts moving relativistically with constant acceleration  $-a$  and continues to move thus till the moment of time  $t = 2T_1 + T_2$ , as a result of which its velocity with respect to the first clock turns zero. Then, the entire cycle is reversed, and the second clock arrives at the same point, at which the first clock is.

We shall calculate the difference in the readings of these clocks in the inertial reference system, in which the first clock is at rest. By virtue of the symmetry of the problem (four segments of motion with constant acceleration and two segments of uniform rectilinear motion), the reading of the clock at rest, by the moment the two clocks meet, will be

$$T = 4T_1 + 2T_2. \quad (12.13)$$

For the second clock

$$T' = 4T'_1 + 2T'_2. \quad (12.14)$$

Here  $T'_1$  is the time interval between the moment when the second clock started to accelerate and the moment when the acceleration ceased, measured by the moving clock.  $T'_2$  is the interval of the second clock's proper time between the first and second accelerations, during which second clock's motion is uniform and rectilinear.

In an inertial reference system, the interval for a moving body is

$$ds = cdt\sqrt{1 - \frac{v^2(t)}{c^2}}. \quad (12.15)$$

Therefore

$$T'_1 = \int_0^{T_1} \sqrt{1 - \frac{v^2(t)}{c^2}} dt. \quad (12.16)$$

On the basis of (12.3) we obtain

$$T'_1 = \int_0^{T_1} \frac{dt}{\sqrt{1 + \frac{a^2 t^2}{c^2}}}. \quad (12.17)$$

Hence we find

$$T'_1 = \frac{c}{a} \ln \left( \frac{aT_1}{c} + \sqrt{1 + \frac{a^2 T_1^2}{c^2}} \right). \quad (12.18)$$

The motion of the second clock during the interval of time

$$T_1 \leq t \leq T_1 + T_2$$

due to Eq. (12.4) proceeds with the velocity

$$v = \frac{aT_1}{\sqrt{1 + \frac{a^2 T_1^2}{c^2}}}, \quad (12.19)$$

and, therefore, in accordance with (12.15) we obtain

$$T'_2 = \frac{T_2}{\sqrt{1 + \frac{a^2 T_1^2}{c^2}}}. \quad (12.20)$$

Consequently, by the moment the two clocks meet the reading of the second clock will be

$$T' = \frac{4c}{a} \ln \left( \frac{aT_1}{c} + \sqrt{1 + \frac{a^2 T_1^2}{c^2}} \right) + \frac{2T_2}{\sqrt{1 + \frac{a^2 T_1^2}{c^2}}}. \quad (12.21)$$

Subtracting (12.13) from (12.21), we find

$$\begin{aligned} \Delta T = T' - T &= \frac{4c}{a} \ln \left( \frac{aT_1}{c} + \sqrt{1 + \frac{a^2 T_1^2}{c^2}} \right) - \\ &- 4T_1 + 2T_2 \left[ \frac{1}{\sqrt{1 + \frac{a^2 T_1^2}{c^2}}} - 1 \right]. \end{aligned} \quad (12.22)$$

It can be verified that for any  $a > 0, T_1 > 0, T_2 > 0$  the quantity  $\Delta T$  is negative. This means that at the moment the clocks meet the reading of the second clock will be less than the reading of the first clock.

Now consider the same process in the reference system, where the second clock is always at rest. This reference system is not

inertial, since part of the time the second clock moves with a constant acceleration with respect to the inertial reference system related to the first clock, while the remaining part of time its motion is uniform. At the first stage the second clock moves with constant acceleration, according to the law (12.9)

$$x_0 = \frac{c^2}{a} \left[ \sqrt{1 + \frac{a^2 t^2}{c^2}} - 1 \right].$$

Therefore, at this segment of the journey, the interval in the non-inertial reference system, according to (12.12) has the form

$$d\sigma^2 = \frac{c^2 dt^2}{1 + \frac{a^2 t^2}{c^2}} - \frac{2at dx dt}{\sqrt{1 + \frac{a^2 t^2}{c^2}}} - dx^2 - dY^2 - dZ^2. \quad (12.23)$$

In this reference system the second clock is at rest at point  $x = 0$ , while the first clock moves along the geodesic line determined by Eqs. (11.14)

$$\frac{dU^\nu}{d\sigma} + \Gamma_{\alpha\beta}^\nu U^\alpha U^\beta = 0, \quad \nu = 0, 1, 2, 3. \quad (12.24)$$

Of these four equations only three are independent, since the following relation is always valid:

$$\gamma_{\mu\nu} U^\mu U^\nu = 1, \quad U^\nu = \frac{dx^\nu}{d\sigma}. \quad (12.25)$$

From expression (12.23) we find

$$\gamma_{00} = \frac{1}{1 + \frac{a^2 t^2}{c^2}}, \quad \gamma_{01} = -\frac{at}{c \sqrt{1 + \frac{a^2 t^2}{c^2}}}, \quad \gamma_{11} = -1. \quad (12.26)$$

From Eq. (12.26) and the following equation

$$\gamma^{\mu\nu} \cdot \gamma_{\nu\lambda} = \delta_\lambda^\mu,$$

we find

$$\gamma^{00} = 1, \quad \gamma^{01} = -\frac{at}{c\sqrt{1 + \frac{a^2t^2}{c^2}}}, \quad \gamma^{11} = -\frac{1}{1 + \frac{a^2t^2}{c^2}}.$$

By means of these formulae and also Eqs. (11.31) and (12.26) it is easy to see that there is only one nonzero Christoffel symbol

$$\Gamma_{00}^1 = \frac{1}{c^2 \left(1 + \frac{a^2t^2}{c^2}\right)^{3/2}}.$$

We do not have to resolve equation (12.24), we shall only take advantage of relation (12.25)

$$\gamma_{00}(U^0)^2 + 2\gamma_{01}U^0U^1 - (U^1)^2 = 1. \quad (12.27)$$

Taking into account (12.26), from equation (12.27) we find a partial solution

$$U^1 = -\frac{at}{c\sqrt{1 + \frac{a^2t^2}{c^2}}}, \quad U^0 = 1, \quad (12.28)$$

which as easy to check satisfies also Eqs. (12.24). From (12.28) it follows

$$\frac{dx^1}{dt} = -\frac{at}{\sqrt{1 + \frac{a^2t^2}{c^2}}}. \quad (12.29)$$

Resolving this equation with the initial conditions  $x(0) = 0$ ,  $\dot{x}(0) = 0$ , we obtain

$$x = \frac{c^2}{a} \left[ 1 - \sqrt{1 + \frac{a^2t^2}{c^2}} \right]. \quad (12.30)$$

Thus, we have everything necessary for determining the readings of both clocks by the final moment of the first stage in their motion. The proper time  $d\tau$  of the first clock at this stage of motion, by virtue of (12.29), coincides with the time  $dT$  of the inertial reference system

$$d\tau = \frac{ds}{c} = dT, \quad (12.31)$$

therefore, by the end of this stage of the journey the reading  $\tau_1$  of the first clock will be  $T_1$

$$\tau_1 = T_1. \quad (12.32)$$

Since the second clock is at rest with respect to the non-inertial reference system, its proper time can be determined from expression

$$d\tau' = \sqrt{\gamma_{00}} dt. \quad (12.33)$$

Since the first stage of the journey occupies the interval  $0 \leq t \leq T_1$  of inertial time, then at the end of this segment the reading  $\tau'_1$  of the second clock will be

$$\tau'_1 = \int_0^{T_1} \sqrt{\gamma_{00}} dt = \frac{c}{a} \ln \left[ \frac{aT_1}{c} + \sqrt{1 + \frac{a^2 T_1^2}{c^2}} \right]. \quad (12.34)$$

At the end of the first stage of the journey, upon reaching the velocity

$$v = \frac{aT_1}{\sqrt{1 + \frac{a^2 T_1^2}{c^2}}}, \quad (12.35)$$

the action of the accelerating force ceases, this means that the reference system related to the second clock will be inertial. The interval in this reference system, in accordance with (12.23) will,

by the moment  $T_1$ , have the form

$$d\sigma^2 = c^2 \left(1 - \frac{v^2}{c^2}\right) dt^2 - 2v dx dt - dx^2 - dY^2 - dZ^2, \quad (12.36)$$

here

$$v = \frac{aT_1}{\sqrt{1 + \frac{a^2 T_1^2}{c^2}}}. \quad (12.37)$$

Taking advantage, for the metric (12.36), of the identity

$$\gamma_{\mu\nu} U^\mu U^\nu = 1, \quad U^\nu = \frac{dx^\nu}{d\sigma}, \quad (12.38)$$

we find

$$\frac{dx^1}{d\sigma} = -v. \quad (12.39)$$

Taking into account (12.39) in (12.36), we obtain

$$d\tau = \frac{d\sigma}{c} = dt, \quad (12.40)$$

i. e. the time, shown by the first clock at this stage, coincides with the time  $T_2$

$$\tau_2 = T_2. \quad (12.41)$$

Since the second clock is at rest, its reading of its proper time is

$$d\tau' = \sqrt{\gamma_{00}} dt. \quad (12.42)$$

Hence follows

$$\tau'_2 = \int_{T_1}^{T_1+T_2} \sqrt{\gamma_{00}} dt = \frac{T_2}{\sqrt{1 + \frac{a^2 T_1^2}{c^2}}}. \quad (12.43)$$

Owing to the symmetry of the problem, the information obtained is sufficient for determining the readings of the clocks at the moment they meet. Indeed, the reading of the first clock  $\tau$ , determined in the reference system, related to the second clock, is

$$\tau = 4\tau_1 + 2\tau_2, \quad (12.44)$$

which on the basis of (12.32) and (12.41) gives

$$\tau = 4T_1 + 2T_2. \quad (12.45)$$

The reading of the second clock  $\tau'$ , determined in the same reference system, where the second clock is at rest, is

$$\tau' = 4\tau'_1 + 2\tau'_2, \quad (12.46)$$

which on the basis of (13.34) and (13.43) gives

$$\tau' = \frac{4c}{a} \ln \left[ \frac{aT_1}{c} + \sqrt{1 + \frac{a^2 T_1^2}{c^2}} \right] + \frac{2T_2}{\sqrt{1 + \frac{a^2 T_1^2}{c^2}}}. \quad (12.47)$$

Subtracting from (12.47) expression (12.45), we obtain

$$\begin{aligned} \Delta\tau = \tau' - \tau = \frac{4c}{a} \ln \left[ \frac{aT_1}{c} + \sqrt{1 + \frac{a^2 T_1^2}{c^2}} \right] - \\ - 4T_1 + 2T_2 \left[ \frac{1}{\sqrt{1 + \frac{a^2 T_1^2}{c^2}}} - 1 \right]. \end{aligned} \quad (12.48)$$

Comparing (12.22) and (12.48) we see, that the computation performed in the inertial reference system, where the first clock is at

rest, yields the same result as the computation performed in the non-inertial reference system related to the second clock.

Thus,

$$\Delta\tau = \Delta T < 0. \quad (12.49)$$

Hence it follows that no paradox exists, since the reference system related to the first clock is inertial, while the reference system, in which the second clock is at rest, is non-inertial.

Precisely for this reason, the slowing down of the second clock, as compared to the first clock, is an absolute effect and does not depend on the choice of reference system, in which this effect is computed.

The arguments concerning the relativity of motion, which were used previously, in this case cannot be applied, since the reference systems are not equitable. Qualitatively, the slowing down of the second clock, as compared to the first, can be explained as follows. It is known, that in arbitrary coordinates the free motion of a test body proceeds along a geodesic line, i. e. the extremal line, which in pseudo-Euclidean space is the maximum distance between two points, if on the entire line, joining these points, the quantity  $d\sigma^2$  is positive. In the case, when we choose an inertial reference system in Galilean coordinates, related to the first clock, this means that the first clock describes a geodesic line, while the second clock, owing to the influence of the force, moves along a line differing from the geodesic, and, therefore, slows down. The same happens, also, when the reference system is related to the second clock. In the case of transition to this reference system, the interval somewhat changes its form. In this case the first clock again describes a geodesic line in an altered metric, while the second clock is at rest, and, consequently, do not describe a geodesic line and, therefore, slow down.

We have considered the influence of accelerated motion on the readings of clocks and have showed their slowing down. But this

effect concerns not only clocks, but all physical, or, to be more general, all natural phenomena. On this basis, interstellar flights become fantastically fascinating. Back in 1911, Paul Langevin discussed in an article [14] the voyage of a human being at high velocities, close to the velocity of light, subsequently returning to the Earth. In principle, this is possible, but it still remains only a fantasy.

Let us now pay attention to the Sagnac effect (see more details in: *Uspekhi Fiz. Nauk.* 1988. Vol. 156, issue 1, pp. 137-143. In collaboration with *Yu. V. Chugreev*). As is well known, the Sagnac effect in line with the Michelson experiment is one of the basic experiments of the theory of relativity. But till now it is possible to read incorrect explanations of this effect with the help of signals propagating faster than light or with the help of general relativity (see in more detail below). So we consider it as necessary to stress once more purely special relativistic nature of Sagnac effect.

Let us at first describe the Sagnac experiment. There are mirrors situated at the angles of a quadrangle on a disk. The angles of their reciprocal disposition are such that the beam from a monochromatic source after reflections over these mirrors passes a closed circle and returns to the source. With the help of a semi-transparent plate it is possible to divide the beam coming from a source into two beams moving in opposite directions over this closed circle.

Sagnac has discovered that if the disk is subjected to rotation, then the beam with the direction of its round coinciding with the direction of rotation will come back to the source later than the beam with opposite round, resulting in a shift of the interference picture on the photographic plate. After interchanging the direction of rotation the interference bands shift in opposite direction.

What explanation was given to this effect? Sagnac himself has obtained a theoretical value for the magnitude of the effect by purely classical addition of the light velocity with the linear

velocity of rotation for the beam moving oppositely to rotation and corresponding subtraction for the beam moving in the direction of rotation. The discrepancy of this result with the experiment was of percent order.

This explanation of the experimental results remained later more or less invariable or even became obscure. As a typical example we present a related quotation from “Optics” by A. Sommerfeld:

*“The negative result of Michelson’s experiment has, of course, no bearing on the problem of the propagation of light in **rotating** media. To discuss this problem one must use not the special but rather the general theory of relativity with its additional terms which correspond to the mechanical centrifugal centrifugal forces. However, in view of the fact that in the following experiments (by Sagnac and others. — A.L.) only velocities  $v \ll c$  occur and only first order effects in  $v/c$  are important, relativity theory can be dispensed with entirely and the computations can be carried out classically”.*

We will see below that the explanation of the Sagnac effect lies in full competence of the special theory of relativity and neither general theory of relativity nor super-luminal velocities are not required as well as any other additional postulates. We will consider in detail how to calculate the time difference between arrivals of the two beams to the source in the inertial rest system of reference. We will also do that in the rotating with the disk non-inertial reference system. The results of calculations will coincide as should be expected. For simplicity of calculations we will consider the motion of light in a light guide over circular trajectory which corresponds to the case of infinite number of mirrors in the Sagnac experiment.

We begin with the case of inertial system of reference. Let us express the interval in cylindrical coordinates:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2. \quad (12.50)$$

Let as it has been told before light beams move in plane  $z = 0$  over circle of radius  $r = r_0 = \text{const}$ . The interval is exactly equal to zero for light, so we obtain the following

$$\frac{d\phi_{\pm}(t)}{dt} = \pm \frac{c}{r_0}. \quad (12.51)$$

The beam moving in the direction of rotation is marked by index “+”, and the beam moving in opposite direction is marked by “-”.

With account for the initial conditions  $\phi_+(0) = 0$ ,  $\phi_-(0) = 2\pi$  we find the law of angle  $\phi_{\pm}$  dependence of the two beams on time  $t$ :

$$\begin{aligned} \phi_+(t) &= \frac{c}{r_0} t, \\ \phi_-(t) &= 2\pi - \frac{c}{r_0} t. \end{aligned} \quad (12.52)$$

The beams will meet at time  $t_1$ , when  $\phi_+(t_1) = \phi_-(t_1)$ . Substituting (12.52) we obtain

$$\phi_+(t_1) = \phi_-(t_1) = \pi.$$

Then taking time  $t_1$  as the initial time and repeating our argumentation we will find that the next meeting of beams will take place just at that spatial point where they have been emitted, i. e. at point with coordinates  $\phi = 0$ ,  $r = r_0$ ,  $z = 0$ .

We emphasize that this result does not depend on the angular velocity of rotation of the system of reference which is the rest system for the source and mirrors.

The law of dependence of the angular coordinate of the source by definition is as follows (for initial condition  $\phi_s(0) = 0$ ):

$$\phi_s(t) = \omega t. \quad (12.53)$$

Therefore, the meeting of the source with “+”-beam will take place at time moment  $t_+$  determined by condition  $\phi_s(t_+) = \phi_+(t_+) - 2\pi$ , i. e.

$$t_+ = \frac{2\pi}{(c/r_0) - \omega}, \quad (12.54)$$

and with “-”-beam — at time moment  $t_-$  determined by condition  $\phi_s(t_-) = \phi_-(t_-)$ :

$$t_- = \frac{2\pi}{(c/r_0) + \omega}. \quad (12.55)$$

It may seem from the form of Eqs. (12.54), (12.55) that the velocity of light is here anisotropic and is different from  $c$ . But this is incorrect. The light velocity is the same for both beams and it is equal to  $c$ , and the different time of return to the source is explained by the fact that the source has moved over some distance during the time of beams propagation (“+”-beam has travelled over larger distance).

Let us now find the interval of proper time between arrivals of the two beams for an observer sitting on the source. By definition it is equal to

$$\Delta = \frac{1}{c} \int_{s(t_-)}^{s(t_+)} ds = \frac{1}{c} \int_{t_-}^{t_+} \frac{ds}{dt} dt, \quad (12.56)$$

where  $s$  is the interval. As a value of interval after using (12.53) we get

$$ds^2 = c^2 dt^2 - r_0^2 d\phi^2 = c^2 dt^2 \left( 1 - \frac{r_0^2 \omega^2}{c^2} \right),$$

where  $\omega^2 r_0^2 / c^2 < 1$ .

Substituting this into Eq. (12.56) we will find exact value of the Sagnac effect <sup>6</sup>:

$$\Delta = \left(1 - \frac{r_0^2 \omega^2}{c^2}\right)^{1/2} \cdot (t_+ - t_-) = \frac{4\pi\omega r_0^2}{c^2 [1 - (r_0^2 \omega^2 / c^2)]^{1/2}}. \quad (12.57)$$

Let us note that in deriving Eq. (12.57) we used only absolute concepts of events of beams meeting (with each other and with the source), and not the concept of the light velocity relative to the rotating reference system.

Let us consider now the same physical process of propagation of beams over circle towards each other in rotating with angular velocity  $\omega$  non-inertial system of reference. In order to find out the form of interval in this system we will make a coordinate transformation:

$$\begin{aligned} \phi_{new} &= \phi_{old} - \omega t_{old}, \\ t_{new} &= t_{old}, \\ r_{new} &= r_{old}, \\ z_{new} &= z_{old}. \end{aligned} \quad (12.58)$$

In new coordinates  $t_{new}$ ,  $r_{new}$ ,  $\phi_{new}$ ,  $z_{new}$  we obtain (after lowering index “new” for simplicity) interval in the following form

$$\begin{aligned} ds^2 &= \left(1 - \frac{\omega^2 r^2}{c^2}\right) c^2 dt^2 - \frac{2\omega r^2}{c} d\phi c dt - \\ &- dr^2 - r^2 d\phi^2 - dz^2. \end{aligned} \quad (12.59)$$

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<sup>6</sup>In calculation for the realistic Sagnac effect, when the light beam trajectory is a polygonal line, it is necessary to take into account the centrifuge deformation due to centrifugal forces.

Let us note that time  $t$  in this expression is the coordinate time for the rotating system of reference.

After accounting for initial conditions  $\phi_+(0) = 0$ ,  $\phi_-(0) = 2\pi$  we get:

$$\phi_+(t) = \frac{ct}{r_0} \left( 1 - \frac{\omega r_0}{c} \right), \quad (12.60)$$

$$\phi_-(t) = 2\pi - \frac{ct}{r_0} \left( 1 + \frac{\omega r_0}{c} \right).$$

the first meeting of beams will happen at time  $t_1$ , when  $\phi_+(t_1) = \phi_-(t_1)$ , i.e. when angular variable will be equal to  $\phi_1 = \pi[1 - (\omega r_0/c)]$ . After analogous reasoning we conclude that the second meeting of beams will happen “at angle”

$$\phi_2 = 2\pi \left( 1 - \frac{\omega r_0}{c} \right), \quad (12.61)$$

i.e. at angular distance  $2\pi r_0 \omega/c$  from the source. The dependence of source angular coordinate is trivial  $\phi_s = \text{const} = 0$ .

The moment of coordinate time  $t_+$  corresponding to meeting of “+”-beam with the source could be found, as before, from relation  $\phi_s(t_+) = 0 = \phi_+(t_+) - 2\pi$ :

$$t_+ = \frac{2\pi r_0}{c - \omega r_0}, \quad (12.62)$$

and similarly we find moment  $t_-$ :

$$t_- = \frac{2\pi r_0}{c + \omega r_0}. \quad (12.63)$$

The proper time interval between two events of coming the beams into the point where the source is disposed can be calculated

with the help of definition (12.56) and interval (12.59):

$$\begin{aligned}\Delta &= \frac{1}{c} \int_{t_-}^{t_+} \frac{ds}{dt} dt = \left(1 - \frac{\omega^2 r_0^2}{c^2}\right)^{1/2} \cdot (t_+ - t_-) = \\ &= \frac{4\pi\omega r_0^2}{c^2 [1 - (r_0^2 \omega^2 / c^2)]^{1/2}},\end{aligned}$$

i. e. we come to the same expression (12.57).

Therefore we demonstrated that for explanation of the Sagnac effect one does not need neither modify the special theory of relativity, nor use super-luminal velocities, nor apply to the general theory of relativity. One only has to strictly follow the special theory of relativity.

### 13. Concerning the limiting velocity

The interval for pseudo-Euclidean geometry, in arbitrary coordinates, has, in accordance with (3.32) and (3.33) the following general form:

$$d\sigma^2 = \gamma_{\mu\nu}(x)dx^\mu dx^\nu, \quad \gamma = \det(\gamma_{\mu\nu}) < 0. \quad (13.1)$$

The metric tensor  $\gamma_{\mu\nu}$  equals

$$\gamma_{\mu\lambda}(x) = \sum_{\nu=0}^3 \varepsilon^\nu \frac{\partial f^\nu}{\partial x^\mu} \cdot \frac{\partial f^\nu}{\partial x^\lambda}, \quad \varepsilon^\nu = (1, -1, -1, -1). \quad (13.2)$$

Here  $f^\nu$  are four arbitrary continuous functions with continuous derivatives, that relate Galilean coordinates with the arbitrary  $x^\lambda$ .

Depending on the sign of  $d\sigma^2$ , events can be identified as time-like

$$d\sigma^2 > 0, \quad (13.3)$$

space-like

$$d\sigma^2 < 0, \quad (13.4)$$

and isotropic

$$d\sigma^2 = 0. \quad (13.5)$$

Such a division of intervals is absolute, it does not depend on the choice of reference system.

For a time-like interval  $d\sigma^2 > 0$  there always exists an inertial reference system, in which it is only determined by time

$$d\sigma^2 = c^2 dT^2.$$

For a space-like interval  $d\sigma^2 < 0$  there can always be found an inertial reference system, in which it is determined by the distance between infinitesimally close points

$$d\sigma^2 = -d\ell^2, \quad d\ell^2 = dx^2 + dy^2 + dz^2.$$

These assertions are also valid in the case of a finite interval  $\sigma$ .

Any two events, related to a given body, are described by a time-like interval. An isotropic interval corresponds to a field without rest mass. Let us see, what conclusions result from an isotropic interval

$$\gamma_{\mu\nu}dx^\mu dx^\nu = \gamma_{00}(dx^0)^2 + 2\gamma_{0i}dx^0 dx^i + \gamma_{ik}dx^i dx^k = 0. \quad (13.6)$$

We single out in (14.6) the time-like part

$$c^2 \left[ \sqrt{\gamma_{00}} dt + \frac{\gamma_{0i}dx^i}{c\sqrt{\gamma_{00}}} \right]^2 - \left[ -\gamma_{ik} + \frac{\gamma_{0i}\gamma_{0k}}{\gamma_{00}} \right] dx^i dx^k = 0. \quad (13.7)$$

The quantity

$$d\tau = \sqrt{\gamma_{00}} dt + \frac{\gamma_{0i}dx^i}{c\sqrt{\gamma_{00}}} = \frac{1}{c} \left( \frac{\gamma_{0\lambda}dx^\lambda}{\sqrt{\gamma_{00}}} \right) \quad (13.8)$$

is to be considered as physical time, which, as we shall see below, is independent of the choice of time variable. In the general case (non-inertial reference systems) the quantity  $d\tau$  is not a total differential, since the following conditions will not be satisfied:

$$\frac{\partial}{\partial x^i}(\sqrt{\gamma_{00}}) = \frac{1}{c} \frac{\partial}{\partial t} \left( \frac{\gamma_{0i}}{\sqrt{\gamma_{00}}} \right), \quad (13.9)$$

$$\frac{\partial}{\partial x^k} \left( \frac{\gamma_{0i}}{\sqrt{\gamma_{00}}} \right) = \frac{\partial}{\partial x^i} \left( \frac{\gamma_{0k}}{\sqrt{\gamma_{00}}} \right).$$

The second term in (13.7) is nothing, but the square distance between two infinitesimally close points of **three-dimensional space**, which is independent of the choice of coordinates in this space:

$$d\ell^2 = \chi_{ik}dx^i dx^k, \quad (13.10)$$

here the metric tensor of three-dimensional space,  $\chi_{ik}$ , is

$$\chi_{ik} = -\gamma_{ik} + \frac{\gamma_{0i}\gamma_{0k}}{\gamma_{00}}. \quad (13.11)$$

With account of (13.8) and (13.10), from expression (13.7) we find

$$\frac{d\ell}{d\tau} = c. \quad (13.12)$$

The quantities  $d\ell$  and  $d\tau$  are of local character. In this case the **concept of simultaneity** loses **sense** for events at different sites, because it is impossible to **synchronize clocks** with the aid of a light signal, since it depends on the synchronization path. From (13.12) it follows, that the field at each point of Minkowski space, in accordance with the local characteristics of  $d\ell$  and  $d\tau$ , have a velocity equal to the electrodynamic constant  $c$ . This is the limiting velocity, that is not achievable for particles with rest mass, since for them

$$d\sigma^2 > 0.$$

This inequality is the causality condition. The causality principle is not contained in the Maxwell-Lorentz equations. It is imposed as a natural complementary condition. In 1909 H. Minkowski formulated it as the principal axiom as follows:

*“A substance, found at any world point, given the appropriate definition of space and time (i. e. given the corresponding choice of reference system in Minkowski space. — A.L.) can be considered to be at rest. The axiom expresses the idea, that at each world point the expression*

$$c^2 dt^2 - dx^2 - dy^2 - dz^2$$

*is always positive or, in other words, that any velocity  $v$  is always less than  $c$ ”.*

H. Poincaré has demonstrated the deep physical meaning of the limiting velocity in his article [1] published in 1904 even before his fundamental works [2; 3]. He wrote:

*«If all these results would be confirmed there will arise an absolutely new mechanics. It will be characterized mainly by the fact that neither velocity could exceed the velocity of light,<sup>7</sup> as the temperature could not drop below the absolute zero. Also no any observable velocity could exceed the light velocity for any observer performing a translational motion but not suspecting about it. There would be a contradiction here if we will not remember that this observer uses another clock than the observer at rest. Really he uses the clock showing “the local time”».*

Just these thoughts by H. Poincaré and his principle of relativity were reported by him in a talk given at The Congress of Art and Science in Sent-Louis (in September of 1904) and they found their realization in articles [2; 3]. They underlie the work by A. Einstein of 1905.

Signal from one object to another can only be transferred by means of a material substance; from the aforementioned it is clear, that  $c$  is the **maximum velocity for transferring interaction or information**. Since particles, corresponding to the electromagnetic field, — photons — are usually considered to be massless, the quantity  $c$  is identified with the velocity of light. The existence of a maximum velocity is a direct consequence of the pseudo-Euclidean geometry of space-time.

If we choose the function  $f^\nu$  in (13.2) by a special way as follows

$$f^0(x^\lambda), f^i(x^k), \quad (13.13)$$

then, owing such transformation, we do not leave the inertial reference system.

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<sup>7</sup>*Because bodies would oppose to the forces trying to accelerate their motion by means of the increasing inertia, and this inertia would become infinite in approaching the velocity of light.*

In this case the metric tensor  $\gamma_{\mu\nu}$ , in accordance with (13.2) and (13.13), assumes the form

$$\gamma_{00} = \left( \frac{\partial f^0}{\partial x^0} \right)^2, \quad \gamma_{0i} = \frac{\partial f^0}{\partial x^0} \cdot \frac{\partial f^0}{\partial x^i}, \quad (13.14)$$

$$\gamma_{ik} = \frac{\partial f^0}{\partial x^i} \cdot \frac{\partial f^0}{\partial x^k} - \sum_{\ell=1}^3 \frac{\partial f^\ell}{\partial x^i} \cdot \frac{\partial f^\ell}{\partial x^k}. \quad (13.15)$$

Substituting the values for the metric coefficients  $\gamma_{00}$ ,  $\gamma_{0i}$  from (13.14) into (13.8) we obtain, with account for (3.30) and (13.13),

$$d\tau = \frac{1}{c} \left( \frac{\partial f^0}{\partial x^\nu} dx^\nu \right) = \frac{1}{c} df^0 = \frac{1}{c} dX^0. \quad (13.16)$$

We see, that proper time, in this case, is a total differential, since our reference system is inertial. Substituting (13.14) and (13.15) into (13.11), we obtain

$$\chi_{ik} = \sum_{n=1}^3 \frac{\partial f^n}{\partial x^i} \cdot \frac{\partial f^n}{\partial x^k}. \quad (13.17)$$

Hence, with account for (3.30) and (13.13), we find

$$d\ell^2 = \chi_{ik} dx^i dx^k = \sum_{n=1}^3 (df^n)^2 = \sum_{n=1}^3 (dX^n)^2. \quad (13.18)$$

In an inertial reference system, ambiguity exists in the coordinate description of Minkowski space, depending on the choice of functions (13.13). This is the reason for arbitrariness in adopting an **agreement** concerning simultaneity at different points of space. All such agreements are conventional. However, this ambiguity and, consequently, the arbitrariness in reaching an agreement do not influence the physical quantities. Eqs. (13.16) and (13.18)

show that in an inertial reference system the physical quantities of time (13.8) and distance (13.10) do not depend on the choice of agreement concerning simultaneity. Let me clarify. In formulae (13.16) and (13.18), given any choice of functions (13.13), there only arise **Galilean coordinates**  $X^0, X^n$  of Minkowski space, that correspond to the invariant (3.22). This is precisely what removes, in the physical quantities of time (13.8) and distance (13.10), arbitrariness in the choice of a conventional agreement concerning simultaneity. Moreover, no physical quantities can, in principle, depend on the choice of this agreement on simultaneity. And if someone has written, or writes, the opposite, this only testifies to that person's **incomprehension** of the essence of relativity theory. One must distinguish between coordinate quantities and physical quantities. For details concerning this issue see ref. [6].

Let us demonstrate a particular special example of the simultaneity convention. Let the synchronization of clocks in different spatial points is provided by the light signal having velocity  $c_1$  in the direction parallel to the positive semi-axis  $X$ , and having velocity  $c_2$  in the direction of the negative semi-axis  $X$ . Then the signal sent from point  $A$  at the moment of time  $t_A$  will arrive to point  $B$  at time  $t_B$  which is given as follows

$$t_B = t_A + \frac{X_{AB}}{c_1}. \quad (M)$$

The reflected signal will arrive at point  $A$  at time  $t'_A$

$$t'_A = t_B + \frac{X_{AB}}{c_2}.$$

After substituting into this expression value  $t_B$ , determined by formula (M) we get

$$t'_A - t_A = X_{AB} \left( \frac{1}{c_1} + \frac{1}{c_2} \right).$$

From here it follows

$$X_{AB} = \frac{c_1 c_2}{c_1 + c_2} (t'_A - t_A).$$

Applying this expression to Eq. (M) we find

$$t_B = t_A + \frac{c_2}{c_1 + c_2} (t'_A - t_A).$$

So we come to the synchronization proposed by Reichenbach (see his book: "The philosophy of space & time". Dover Publications, Inc. New York. 1958, p. 127):

$$t_B = t_A + \varepsilon(t'_A - t_A), \quad 0 < \varepsilon < 1.$$

The conditional convention on the synchronization of clocks and therefore on simultaneity at different spatial points accepted by us corresponds to the choice of interval in inertial reference system in the following form:

$$\begin{aligned} d\sigma^2 = & (dx^0)^2 - \frac{c(c_2 - c_1)}{c_1 c_2} dx^0 dx - \\ & - \frac{c^2}{c_1 c_2} (dx)^2 - (dy)^2 - (dz)^2. \end{aligned} \quad (K)$$

Here we deal with coordinate time  $t = x^0/c$  and other coordinate values.

Metric coefficients of interval (K) are as follows:

$$\begin{aligned} \gamma_{00} = 1, \quad \gamma_{01} = & -\frac{c(c_2 - c_1)}{2c_1 c_2}, \\ \gamma_{11} = & -\frac{c^2}{c_1 c_2}, \quad \gamma_{22} = -1, \quad \gamma_{33} = -1. \end{aligned} \quad (L)$$

With the help of Eqs. (13.14), (13.15), and also (L), we obtain transformation functions (13.13) for our case:

$$\begin{aligned} f^0 = X^0 &= x^0 - \frac{x}{2} \cdot \frac{c(c_2 - c_1)}{c_1 c_2}, \\ f^1 = X &= x \frac{c(c_1 + c_2)}{2c_1 c_2}, \\ f^2 = Y &= y, \quad f^3 = Z = z. \end{aligned}$$

Deriving from the above the inverse transformation functions, calculating with them differentials  $dx^0$ ,  $dx$  and then substituting them into (K), we find

$$d\sigma^2 = (dX^0)^2 - (dX)^2 - (dY)^2 - (dZ)^2. \quad (H)$$

Therefore, **the physical time**  $d\tau$  in our example is given as follows:

$$\begin{aligned} d\tau &= dt - \frac{dx}{2} \cdot \frac{c_2 - c_1}{c_1 c_2}, \\ dX^0 &= cd\tau, \end{aligned}$$

and it does not depend on the choice of functions (13.13), because it is completely determined by interval (H) only. Any change in coordinate values like (13.13) leads only to changing of the connection between **the physical time** and coordinate values.

To any conditional convention on the simultaneity there will correspond a definite choice of the coordinate system in an inertial system of reference of the Minkowski space. Therefore **a conditional convention on the simultaneity** is nothing more than **a definite choice of the coordinate system** in an inertial system of reference of the Minkowski space.

An important contribution into understanding of some fundamental questions of the theory of relativity related to the definition of simultaneity in different spatial points was provided by Professor A. A. Tyapkin (Uspekhi Fiz. Nauk. 1972. Vol. 106, issue 4.)

Now let us return to the analysis of physical time  $d\tau$ . Quantity  $d\tau$  characterizes physical time, which is independent on the choice of coordinate time. Indeed, let us introduce new variable  $x'^0$ , such that

$$x'^0 = x'^0(x^0, x^i), \quad x'^i = x'^i(x^k). \quad (13.19)$$

Then due to tensorial character of  $\gamma_{\mu\nu}$  transformation

$$\gamma'_{\mu\nu} = \gamma_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \cdot \frac{\partial x^\beta}{\partial x'^\nu},$$

we will obtain for our case

$$\gamma'_{00} = \gamma_{00} \left( \frac{\partial x^0}{\partial x'^0} \right)^2, \quad \gamma'_{0\lambda} = \gamma_{0\beta} \frac{\partial x^0}{\partial x'^0} \cdot \frac{\partial x^\beta}{\partial x'^\lambda}; \quad (13.20)$$

similarly

$$dx'^\lambda = \frac{\partial x'^\lambda}{\partial x^\sigma} dx^\sigma. \quad (13.21)$$

Exploiting the Kronecker delta symbol

$$\frac{\partial x^\beta}{\partial x'^\lambda} \cdot \frac{\partial x'^\lambda}{\partial x^\sigma} = \delta_\sigma^\beta, \quad (13.22)$$

we get

$$cd\tau = \frac{\gamma'_{0\lambda} dx'^\lambda}{\sqrt{\gamma'_{00}}} = \frac{\gamma_{0\sigma} dx^\sigma}{\sqrt{\gamma_{00}}}. \quad (13.23)$$

We can see that physical time  $d\tau$  does not depend on the choice of the coordinate system in an inertial system of reference of the Minkowski space.

Physical time determines the flow of time in a physical process, however, the quantity  $d\tau$  exhibits local character in a non-inertial reference system, since it is not a total differential and therefore **no variable  $\tau$  exists**.

In this case, there exists no unique physical time with lines orthogonal to three-dimensional space. In a non-inertial reference system the interval  $d\sigma$  is expressed via the physical quantities  $d\tau, d\ell$  as follows:

$$d\sigma^2 = c^2 d\tau^2 - d\ell^2.$$

There **exist no** variables  $\tau, \ell$  in this case. Here **coordinate quantities** arise which permit to describe any effects in space and time in non-inertial system of reference.

In an inertial reference system  $d\tau$  coincides, in Galilean coordinates, with the differential  $dt$ , so in Minkowski space one can introduce unique time  $t$ . It will be physical. Introduction of simultaneity for all the points of three-dimensional space is a consequence of the pseudo-Euclidean geometry of the four-dimensional space of events.

One can only speak of the velocity of light being constant, the same in all directions, and identical with the electrodynamic constant  $c$  in an **inertial reference system in Galilean coordinates**. In an inertial reference system, in any other admissible coordinates, the velocity of light will be the same, if time is defined in accordance with formula (13.8) and distance by formula (13.10). In a non-inertial reference system the electrodynamic constant  $c$  is only expressed via the local quantities  $d\tau, d\ell$ . There exist no variables  $\tau, \ell$  in this case.

It is often written that the principle of constancy of the velocity of light underlies special relativity theory. This is wrong. **No principle of constancy of the velocity of light exists as a first physical principle**, because this principle is a simple consequence of the Poincaré relativity principle for all the nature phenomena. It is enough to apply it to the emission of a spherical electromagnetic wave to get convinced that the velocity of light at any inertial reference system is equal to electrodynamic constant  $c$ . Therefore, this proposition, having only secondary role, as we already noted (see

Sections 3 and 9), does not underlie relativity theory. Precisely in the same way, the synchronization of clocks at different points of space, also, has a limited sense, since it is possible only in inertial reference systems. One cannot perform transition to accelerated reference systems on the basis of the principle of the constancy of the velocity of light, because the concept of simultaneity loses sense, since the synchronization of clocks at different points in space depends on the synchronization path. The need to describe effects by means of coordinate quantities arises.

We, now, define the coordinate velocity of light

$$v^i = \frac{dx^i}{dt} = v\ell^i, \quad (13.24)$$

here  $\ell^i$  is a unit vector satisfying the condition

$$\chi_{ik}\ell^i\ell^k = 1. \quad (13.25)$$

With account for formulae (13.8), (13.10) and (13.25) expression (13.12) assumes the following form

$$\frac{v}{\sqrt{\gamma_{00}} + \frac{v}{c} \cdot \frac{\gamma_{0i}\ell^i}{\sqrt{\gamma_{00}}}} = c. \quad (13.26)$$

Hence one finds the coordinate velocity

$$v = c \cdot \frac{\sqrt{\gamma_{00}}}{1 - \frac{\gamma_{0i}\ell^i}{\sqrt{\gamma_{00}}}}. \quad (13.27)$$

In the general case, the coordinate velocity varies, both in value and in direction. It can take any value satisfying the condition

$$0 < v < \infty. \quad (13.28)$$

In Galilean coordinates of an inertial reference system coordinate velocity coincides with physical velocity.

In an arbitrary non-inertial reference system, for describing physical processes it is possible to introduce unique coordinate time throughout space in many ways. In this case, the synchronization of clocks at different points in space must be performed with the aid of coordinate velocity. In non-inertial systems it is necessary to use **coordinate quantities** in order to describe physical processes because in this case **physical quantities are determined only locally**.

## 14. Thomas precession

Consider a particle with its own angular momentum (spin)  $S^\nu$ . In a reference system, where the particle is at rest, its four-vector of angular momentum (spin) has the components  $(0, \vec{J})$ . In any arbitrary inertial reference system we have the relation

$$S^\nu U_\nu = 0. \quad (14.1)$$

When a force  $\vec{f}$  without torque acts on the particle, the following relation should be valid

$$\frac{dS^\nu}{d\tau} = ZU^\nu, \quad (14.2)$$

here  $U^\nu$  is the four-vector of velocity;  $\tau$  is proper time,

$$d\tau = dt \frac{1}{\gamma}. \quad (14.3)$$

If the velocity  $U^i$  is not zero, then the quantity  $Z$  can be determined from the relation

$$\frac{d}{d\tau}(S^\nu U_\nu) = \frac{dS^\nu}{d\tau} U_\nu + \frac{dU_\nu}{d\tau} S^\nu = 0. \quad (14.4)$$

Substituting (14.2) into (14.4), we obtain

$$Z = - \left( S_\mu \frac{dU^\mu}{d\tau} \right), \quad (14.5)$$

the covariant vector  $S_\mu$  has the components

$$S_\mu = (S^0, -S^1, -S^2, -S^3). \quad (14.6)$$

With account for Eq. (14.5) the equation of motion for the spin vector (14.2) assumes the form

$$\frac{dS^\nu}{d\tau} = - \left( S_\mu \frac{dU^\mu}{d\tau} \right) U^\nu. \quad (14.7)$$

Our further goal will be to try to provide the details of these equations making use of the Lorentz transformations. Consider a particle of spin  $\vec{J}$  moving with a velocity  $\vec{v}$  in a laboratory inertial reference system. In this case, the inertial laboratory reference system will move with respect to the inertial reference system, in which the particle is at rest, with a velocity  $-\vec{v}$ . Applying the Lorentz transformations (4.18) and (4.19) and taking the sign of the velocity into account, we obtain

$$S^0 = \gamma \frac{(\vec{v}\vec{J})}{c}, \quad \vec{S} = \vec{J} + \frac{\gamma - 1}{v^2} \vec{v}(\vec{v}\vec{J}). \quad (14.8)$$

The four-vectors  $U^\mu, \frac{dU^\mu}{d\tau}$  have the following components:

$$U^\mu = \left( \gamma, \gamma \frac{\vec{v}}{c} \right), \quad \frac{dU^\mu}{d\tau} = \left( \frac{d\gamma}{d\tau}, \frac{\gamma}{c} \cdot \frac{d\vec{v}}{d\tau} + \frac{\vec{v}}{c} \cdot \frac{d\gamma}{d\tau} \right). \quad (14.9)$$

Applying (14.6), (14.8) and (14.9), we obtain

$$\begin{aligned} \left( S_\mu \frac{dU^\mu}{d\tau} \right) &= \gamma \frac{(\vec{v}\vec{J})}{c} \cdot \frac{d\gamma}{d\tau} - \\ &- \left( \frac{\gamma}{c} \cdot \frac{d\vec{v}}{d\tau} + \frac{\vec{v}}{c} \cdot \frac{d\gamma}{d\tau} \right) \cdot \left( \vec{J} + \frac{\gamma - 1}{v^2} \vec{v}(\vec{v}\vec{J}) \right). \end{aligned} \quad (14.10)$$

Computations in the right-hand part of expression (14.10) will only leave terms obtained by multiplication of the first term in brackets and the two terms in the second pair of brackets, while all other terms mutually cancel out

$$\left( S_\mu \frac{dU^\mu}{d\tau} \right) = -\frac{\gamma}{c} \left\{ \left( \vec{J} \frac{d\vec{v}}{d\tau} \right) + \frac{\gamma - 1}{v^2} (\vec{v}\vec{J}) \left( \vec{v} \frac{d\vec{v}}{d\tau} \right) \right\}. \quad (14.11)$$

Making use of (14.8) and (14.11), we write equation (14.7) separately for the zeroth component of the four-vector of spin  $S^\nu$  and for its vector part,

$$\frac{d}{d\tau} \left\{ \gamma(\vec{v}\vec{J}) \right\} = \gamma^2 \left\{ \left( \vec{J} \frac{d\vec{v}}{d\tau} \right) + \frac{\gamma-1}{v^2} (\vec{v}\vec{J}) \left( \vec{v} \frac{d\vec{v}}{d\tau} \right) \right\}, \quad (14.12)$$

$$\begin{aligned} \frac{d}{d\tau} \left\{ \vec{J} + \frac{\gamma-1}{v^2} \vec{v}(\vec{v}\vec{J}) \right\} &= \\ &= \frac{\gamma^2}{c^2} \vec{v} \left\{ \left( \vec{J} \frac{d\vec{v}}{d\tau} \right) + \frac{\gamma-1}{v^2} (\vec{v}\vec{J}) \left( \vec{v} \frac{d\vec{v}}{d\tau} \right) \right\}. \end{aligned} \quad (14.13)$$

From equations (14.12) and (14.13) we find

$$\frac{d}{d\tau} \left\{ \vec{J} + \frac{\gamma-1}{v^2} \vec{v}(\vec{v}\vec{J}) \right\} - \frac{\vec{v}}{c^2} \frac{d}{d\tau} \left\{ \gamma(\vec{v}\vec{J}) \right\} = 0. \quad (14.14)$$

From equation (14.12) we find

$$\gamma^2 \frac{(\gamma-1)}{v^2} (\vec{v}\vec{J}) \left( \vec{v} \frac{d\vec{v}}{d\tau} \right) = \frac{d}{d\tau} \left\{ \gamma(\vec{v}\vec{J}) \right\} - \gamma^2 \left( \vec{J} \frac{d\vec{v}}{d\tau} \right). \quad (14.15)$$

Now we write the first term of equation (14.14) in expanded form

$$\begin{aligned} \frac{d\vec{J}}{d\tau} + \frac{\gamma^4}{c^4(1+\gamma)^2} \vec{v}(\vec{v}\vec{J}) \left( \vec{v} \frac{d\vec{v}}{d\tau} \right) + \\ + \frac{\gamma\vec{v}}{c^2(1+\gamma)} \frac{d}{d\tau} \left\{ \gamma(\vec{v}\vec{J}) \right\} + \frac{\gamma^2}{c^2(1+\gamma)} (\vec{v}\vec{J}) \frac{d\vec{v}}{d\tau}. \end{aligned} \quad (14.16)$$

In computation we took into account the equalities

$$\frac{\gamma-1}{v^2} = \frac{\gamma^2}{c^2(1+\gamma)}, \quad \frac{d\gamma}{d\tau} = \frac{\gamma^3}{c^2} \left( \vec{v} \frac{d\vec{v}}{d\tau} \right). \quad (14.17)$$

The second term in (14.16) can be transformed, taking advantage of (14.15), to the form

$$\begin{aligned} & \frac{\gamma^4}{c^4(1+\gamma)^2} \vec{v}(\vec{v}\vec{J}) \left( \vec{v} \frac{d\vec{v}}{d\tau} \right) = \\ & = \frac{\vec{v}}{c^2(1+\gamma)} \left[ \frac{d}{d\tau} \left\{ \gamma(\vec{v}\vec{J}) \right\} - \gamma^2 \left( \vec{J} \frac{d\vec{v}}{d\tau} \right) \right]. \end{aligned} \quad (14.18)$$

Applying (14.18) we see, that the second term together with the third term in (14.16) can be reduced to the form

$$\frac{\vec{v}}{c^2} \cdot \frac{d}{d\tau} \left\{ \gamma(\vec{v}\vec{J}) \right\} - \frac{\gamma^2}{c^2(1+\gamma)} \vec{v} \left( \vec{J} \frac{d\vec{v}}{d\tau} \right). \quad (14.19)$$

With account of (14.16) and (14.19) equation (14.14) is reduced to the following form:

$$\frac{d\vec{J}}{d\tau} + \frac{\gamma^2}{c^2(1+\gamma)} \left\{ \frac{d\vec{v}}{d\tau}(\vec{v}\vec{J}) - \vec{v} \left( \vec{J} \frac{d\vec{v}}{d\tau} \right) \right\} = 0. \quad (14.20)$$

Using the formula

$$\left[ \vec{a} [\vec{b}, \vec{c}] \right] = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b}), \quad (14.21)$$

and choosing the vectors

$$\vec{a} = \vec{J}, \quad \vec{b} = \frac{d\vec{v}}{d\tau}, \quad \vec{c} = \vec{v}, \quad (14.22)$$

equation (14.20) is reduced to the form

$$\frac{d\vec{J}}{d\tau} = \left[ \vec{\Omega} \vec{J} \right], \quad (14.23)$$

here

$$\vec{\Omega} = -\frac{\gamma-1}{v^2} \left[ \vec{v}, \frac{d\vec{v}}{d\tau} \right]. \quad (14.24)$$

**When the particle moves along a curvilinear trajectory, the spin vector  $\vec{J}$  undergoes precession around the direction  $\vec{\Omega}$  with angular velocity  $|\vec{\Omega}|$ . This effect was first discovered by Thomas [15].**

The equation of relativistic mechanics (9.12) can be written in the form

$$m \frac{d\vec{v}}{d\tau} = \vec{f} - \frac{\vec{v}}{c^2} (\vec{v} \vec{f}). \quad (14.25)$$

With account of this equation, expression (14.24) assumes the form

$$\vec{\Omega} = -\frac{\gamma - 1}{mv^2} [\vec{v}, \vec{f}]. \quad (14.26)$$

Thus, a force without torque, by virtue of the pseudo-Euclidean structure of space-time, gives rise to the precession of spin, if its action results in curvilinear motion in the given inertial reference system. In the case, when the force is directed, in a certain reference system, along the velocity of the particle, no precession of the spin occurs. But parallelism of the vectors of force  $\vec{f}$  and of velocity  $\vec{v}$  is violated, when transition is performed from one inertial reference system to another. Therefore, the effect of precession, equal to zero for an observer in one inertial reference system, will differ from zero for an observer in some other inertial reference system.

## 15. The equations of motion and conservation laws in classical field theory

Earlier we saw that, with the aid of the Lagrangian approach, it is possible to construct all the Maxwell-Lorentz equations. This approach possesses an explicit general covariant character. It permits to obtain field equations of motion and conservation laws in a general form without explicit concretization of the Lagrangian density function. In this approach each physical field is described by a one- or multi-component function of coordinates and time, called the field function (or field variable). As field variables, quantities are chosen that transform with respect to one of the linear representations of the Lorentz group, for example, scalar, spinor, vector, or even tensor. Apart the field variables, an important role is attributed, also, to the metric tensor of space-time, which determines the geometry for the physical field, as well as the choice of one or another coordinate system, in which the description of physical processes is performed. The choice of coordinate system is, at the same time, a choice of reference system. Naturally, not every choice of coordinate system alters the reference system. Any transformations in a given reference system of the form

$$\begin{aligned}x'^0 &= f^0(x^0, x^1, x^2, x^3), \\x'^i &= f^i(x^1, x^2, x^3),\end{aligned}\tag{15.1}$$

always leave us in this reference system. Any other choice of coordinate system will necessarily lead to a change in reference system. The choice of coordinate system is made from the class of admissible coordinates,

$$\gamma_{00} > 0, \quad \gamma_{ik} dx^i dx^k < 0, \quad \det |\gamma_{\mu\nu}| = \gamma < 0.\tag{15.2}$$

The starting-point of the Lagrangian formalism is construction of the action function. Usually, the expression determining the action function is written as follows

$$S = \frac{1}{c} \int_{\Omega} L(x^0, x^1, x^2, x^3) dx^0 dx^1 dx^2 dx^3, \quad (15.3)$$

where integration is performed over a certain arbitrary four-dimensional region of space-time. Since the action must be invariant, the Lagrangian density function is the density of a scalar of weight +1. The density of a scalar of weight +1 is the product of a scalar function and the quantity  $\sqrt{-\gamma}$ . The choice of Lagrangian density is performed in accordance with a number of requirements. One of them is that the lagrangian density must be real.

Thus, the Lagrangian density may be constructed with the aid of the fields studied,  $\varphi$ , the metric tensor  $\gamma_{\mu\nu}$ , and partial derivatives with respect to the coordinates,

$$L = L(\varphi_A, \partial_{\mu}\varphi_A, \dots, \gamma_{\mu\nu}, \partial_{\lambda}\gamma_{\mu\nu}). \quad (15.4)$$

For simplicity we shall assume, that the system we are dealing with consists of a real vector field. We shall consider the field Lagrangian not to contain derivatives of orders higher, than the first. This restriction results in all our field equations being equations of the second order,

$$L = L(A^{\nu}, \partial_{\lambda}A^{\nu}, \gamma_{\mu\nu}, \partial_{\lambda}\gamma_{\mu\nu}). \quad (15.5)$$

Note, **that, if the Lagrangian has been constructed, the theory is defined.** We find the field equations from the least action principle.

$$\delta S = \frac{1}{c} \int_{\Omega} d^4x \delta L = 0. \quad (15.6)$$

The variation  $\delta L$  is

$$\delta L = \frac{\partial L}{\partial A_\lambda} \delta A_\lambda + \frac{\partial L}{\partial(\partial_\nu A_\lambda)} \delta(\partial_\nu A_\lambda), \text{ or} \quad (15.7)$$

$$\delta L = \frac{\delta L}{\delta A_\lambda} \delta A_\lambda + \partial_\nu \left[ \frac{\partial L}{\partial(\partial_\nu A_\lambda)} \delta A_\lambda \right]. \quad (15.8)$$

Here we have denoted Euler's variational derivative by

$$\frac{\delta L}{\delta A_\lambda} = \frac{\partial L}{\partial A_\lambda} - \partial_\nu \left( \frac{\partial L}{\partial(\partial_\nu A_\lambda)} \right). \quad (15.9)$$

In obtaining expression (15.8) we took into account, that

$$\delta(\partial_\nu A_\lambda) = \partial_\nu(\delta A_\lambda). \quad (15.10)$$

Substituting (15.8) into (15.6) and applying the Gauss theorem we obtain

$$\delta S = \frac{1}{c} \int_{\Omega} d\Omega d^4x \left( \frac{\delta L}{\delta A_\lambda} \right) \delta A_\lambda + \frac{1}{c} \int_{\Sigma} ds_\nu \left[ \frac{\partial L}{\partial(\partial_\nu A_\lambda)} \delta A_\lambda \right].$$

Since the field variation at the boundary  $\Sigma$  is zero, we have

$$\delta S = \frac{1}{c} \int_{\Omega} d\Omega d^4x \left( \frac{\delta L}{\delta A_\lambda} \right) \delta A_\lambda = 0. \quad (15.11)$$

Owing to the variations  $\delta A_\nu$  being arbitrary, we obtain, with the aid of the main lemma of variational calculus, the equation for the field

$$\frac{\delta L}{\delta A_\lambda} = \frac{\partial L}{\partial A_\lambda} - \partial_\nu \left( \frac{\partial L}{\partial(\partial_\nu A_\lambda)} \right) = 0. \quad (15.12)$$

**We see, that if the Lagrangian has been found, then the theory has been defined.** Besides field equations, the Lagrangian

method provides the possibility, also, to obtain differential conservation laws: **strong** and **weak**. A **strong conservation law** is a differential relation, that holds valid by virtue of the invariance of action under the transformation of coordinates. Weak conservation laws are obtained from strong laws, if the field equation (15.12) is taken into account in them.

It must be especially stressed that, in the general case, strong differential conservation laws do not establish the conservation of anything, neither local, nor global. For our case the action has the form

$$S = \frac{1}{c} \int_{\Omega} d^4x L(A_\lambda, \partial_\nu A_\lambda, \gamma_{\mu\nu}, \partial_\lambda \gamma_{\mu\nu}). \quad (15.13)$$

Now, we shall perform infinitesimal transformation of the coordinates,

$$x'^\nu = x^\nu + \delta x^\nu, \quad (15.14)$$

here  $\delta x^\nu$  is an infinitesimal four-vector.

Since action is a scalar, then in this transformation it remains unaltered, and, consequently,

$$\delta_c S = \frac{1}{c} \int_{\Omega'} d^4x' L'(x') - \frac{1}{c} \int_{\Omega} d^4x L(x) = 0, \quad (15.15)$$

where

$$L'(x') = L'(A'_\lambda, \partial'_\nu A'_\lambda(x'), \gamma'_{\mu\nu}(x'), \partial'_\lambda \gamma'_{\mu\nu}(x')).$$

The first term in (15.15) can be written as

$$\int_{\Omega'} d^4x' L'(x') = \int_{\Omega} J d^4x L'(x'), \quad (15.16)$$

where the Jacobian of the transformation

$$J = \frac{\partial(x'^0, x'^1, x'^2, x'^3)}{\partial(x^0, x^1, x^2, x^3)} = \det \left| \frac{\partial x'^\nu}{\partial x^\lambda} \right|. \quad (15.17)$$

In the case of transformation (15.14) the Jacobian has the form

$$J = 1 + \partial_\lambda \delta x^\lambda. \quad (15.18)$$

Expanding  $L'(x')$  into a Taylor series, we have

$$L'(x') = L(x) + \delta x^\lambda \frac{\partial L}{\partial x^\lambda}. \quad (15.19)$$

Taking (15.16), (15.18) and (15.19) into account, we rewrite variation (15.15) as

$$\delta_c S = \frac{1}{c} \int_{\Omega} d^4 x \left[ \delta_L L(x) + \frac{\partial}{\partial x^\lambda} \left( \delta x^\lambda L(x) \right) \right] = 0; \quad (15.20)$$

here we have denoted

$$\delta_L L(x) = L'(x) - L(x).$$

This variation is usually called the Lie variation. It commutes with partial differentiation

$$\delta_L \partial_\nu = \partial_\nu \delta_L. \quad (15.21)$$

The Lie variation of the Lagrangian density function is

$$\begin{aligned} \delta_L L(x) &= \frac{\partial L}{\partial A_\lambda} \delta_L A_\lambda + \frac{\partial L}{\partial (\partial_\nu A_\lambda)} \delta_L \partial_\nu A_\lambda + \\ &+ \frac{\partial L}{\partial \gamma_{\mu\nu}} \delta_L \gamma_{\mu\nu} + \frac{\partial L}{\partial (\partial_\lambda \gamma_{\mu\nu})} \delta_L \partial_\lambda \gamma_{\mu\nu}. \end{aligned} \quad (15.22)$$

The following identity

$$\delta_L L(x) + \frac{\partial}{\partial x^\lambda} (\delta x^\lambda L(x)) = 0, \quad (15.23)$$

is a consequence of Eq. (15.20) due to arbitrariness of volume  $\Omega$ . It was obtained by D. Hilbert in 1915.

Upon performing elementary transformations, we obtain

$$\begin{aligned} \delta_c S &= \\ &= \frac{1}{c} \int_{\Omega} d^4x \left[ \frac{\delta L}{\delta A_\lambda} \delta_L A_\lambda + \frac{\delta L}{\delta \gamma_{\mu\nu}} \delta_L \gamma_{\mu\nu} + D_\lambda J^\lambda \right] = 0, \end{aligned} \quad (15.24)$$

here

$$\begin{aligned} \frac{\delta L}{\delta \gamma_{\mu\nu}} &= \frac{\partial L}{\partial \gamma_{\mu\nu}} - \partial_\sigma \left( \frac{\partial L}{\partial (\partial_\sigma \gamma_{\mu\nu})} \right), \\ J^\nu &= L \delta x^\nu + \frac{\partial L}{\partial (\partial_\nu A_\lambda)} \delta_L A_\lambda + \frac{\partial L}{\partial (\partial_\nu \gamma_{\lambda\mu})} \delta_L \gamma_{\lambda\mu}. \end{aligned} \quad (15.25)$$

Since  $J^\nu$  is the density of a vector of weight +1, then, in accordance with (11.25) and (11.28), we find

$$\partial_\nu J^\nu = D_\nu J^\nu, \quad (15.26)$$

where  $D^\nu$  is a covariant derivative in pseudo-Euclidean spacetime. It must be pointed out, that the variations  $\delta_L A_\lambda$ ,  $\delta_L \gamma_{\mu\nu}$  originate from the coordinate transformation (15.14), so they can, therefore, be expressed via the components  $\delta x^\lambda$ .

Let us find the Lie variation of field variables, that is due to coordinate transformation. According to the transformation law of the vector  $A_\lambda$

$$A'_\lambda(x') = A_\nu(x) \frac{\partial x^\nu}{\partial x'^\lambda},$$

we have

$$A'_\lambda(x + \delta x) = A_\lambda(x) - A_\nu(x) \frac{\partial \delta x^\nu}{\partial x^\lambda}. \quad (15.27)$$

Expanding the quantity  $A'_\lambda(x + \delta x)$  in a Taylor series, we find

$$A'_\lambda(x + \delta x) = A'_\lambda(x) + \frac{\partial A_\lambda}{\partial x^\nu} \delta x^\nu. \quad (15.28)$$

Substituting (15.28) into (15.27) we obtain

$$\delta_L A_\lambda(x) = -\delta x^\nu \frac{\partial A_\lambda}{\partial x^\nu} - A_\nu(x) \frac{\partial \delta x^\nu}{\partial x^\lambda}, \quad (15.29)$$

or, in covariant form

$$\delta_L A_\lambda(x) = -\delta x^\nu D_\nu A_\lambda - A_\nu D_\lambda \delta x^\nu. \quad (15.30)$$

Now let us find the Lie variation of the metric tensor  $\gamma_{\mu\nu}$  from the transformation law

$$\gamma'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu} \cdot \frac{\partial x^\sigma}{\partial x'^\nu} \gamma_{\lambda\sigma}(x)$$

we obtain

$$\gamma'_{\mu\nu}(x + \delta x) = \gamma_{\mu\nu} - \gamma_{\mu\sigma} \partial_\nu \delta x^\sigma - \gamma_{\nu\sigma} \partial_\mu \delta x^\sigma, \quad (15.31)$$

hence we find

$$\delta_L \gamma_{\mu\nu} = -\gamma_{\mu\sigma} \partial_\nu \delta x^\sigma - \gamma_{\nu\sigma} \partial_\mu \delta x^\sigma - \delta x^\sigma \partial_\sigma \gamma_{\mu\nu}. \quad (15.32)$$

Taking into account the equality

$$\partial_\sigma \gamma_{\mu\nu} = \gamma_{\mu\lambda} \Gamma_{\nu\sigma}^\lambda + \gamma_{\nu\lambda} \Gamma_{\mu\sigma}^\lambda, \quad (15.33)$$

we write expression (15.32) through covariant derivatives,

$$\delta_L \gamma_{\mu\nu} = -\gamma_{\mu\sigma} D_\nu \delta x^\sigma - \gamma_{\nu\sigma} D_\mu \delta x^\sigma. \quad (15.34)$$

Substituting expressions (15.30) and (15.34) into action (15.24), we obtain

$$\begin{aligned} \delta_c S = \frac{1}{c} \int_{\Omega} d^4x \left[ -\delta x^\lambda \frac{\delta L}{\delta A_\nu} D_\lambda A_\nu - \frac{\delta L}{\delta A_\lambda} A_\nu D_\lambda \delta x^\nu - \right. \\ \left. -(\gamma_{\mu\sigma} D_\nu \delta x^\sigma + \gamma_{\nu\sigma} D_\mu \delta x^\sigma) \frac{\delta L}{\delta \gamma_{\mu\nu}} + D_\nu J^\nu \right] = 0. \end{aligned} \quad (15.35)$$

We introduce the following notation:

$$T^{\mu\nu} = -2 \frac{\delta L}{\delta \gamma_{\mu\nu}}. \quad (15.36)$$

As we will further see, **this quantity, first introduced by Hilbert, is the tensor density of the field energy-momentum.**

Integrating by parts in expression (15.35) we obtain

$$\begin{aligned} \delta_c S = \frac{1}{c} \int_{\Omega} d^4x \left\{ -\delta x^\lambda \left[ \frac{\delta L}{\delta A_\nu} D_\lambda A_\nu - \right. \right. \\ \left. -D_\nu \left( \frac{\delta L}{\delta A_\nu} A_\lambda \right) + D_\nu (T^{\mu\nu} \gamma_{\mu\lambda}) \right] + \\ \left. + D_\nu \left( J^\nu - \frac{\delta L}{\delta A_\nu} A_\lambda \delta x^\lambda + T^{\mu\nu} \gamma_{\mu\sigma} \delta x^\sigma \right) \right\} = 0. \end{aligned} \quad (15.37)$$

Substituting into expression (15.25) for the density of vector  $J^\nu$  the values of variations  $\delta_L A_\lambda(x)$ ,  $\delta_L \gamma_{\mu\nu}(x)$ , in accordance with formulae (15.30) and (15.34), and grouping the terms at  $\delta x^\nu$  and  $D_\lambda \delta x^\nu$ , we obtain

$$J^\nu - \frac{\delta L}{\delta A_\nu} A_\lambda \delta x^\lambda = -\tau_\sigma^\nu \delta x^\sigma - \sigma_\mu^{\nu\lambda} D_\lambda \delta x^\mu, \quad (15.38)$$

here we denote

$$\tau_\sigma^\nu = -L\delta_\sigma^\nu + \frac{\partial L}{\partial(\partial_\nu A_\lambda)} D_\sigma A_\lambda + \frac{\delta L}{\delta A_\nu} A_\sigma. \quad (15.39)$$

This quantity is usually **called the density of the canonical energy-momentum tensor**, while the quantity

$$\sigma_\mu^{\nu\lambda} = 2\frac{\partial L}{\partial(\partial_\nu \gamma_{\sigma\lambda})} \gamma_{\sigma\mu} + \frac{\partial L}{\partial(\partial_\nu A_\lambda)} A_\mu \quad (15.40)$$

is called the spin tensor density.

If function  $L$  depends only on  $\gamma^{\mu\nu}$ ,  $A_\mu$ ,  $\partial_\nu A_\mu$ , then quantity  $\sigma_\mu^{\nu\lambda}$  according to Eq. (15.40) may be written as follows

$$\sigma_\mu^{\nu\lambda} = \left( \frac{\partial L}{\partial(\partial_\nu A_\lambda)} \right) A_\mu. \quad (15.40a)$$

On the basis of (15.38) we represent the covariant divergence in (15.37) as

$$\begin{aligned} D_\nu \left( J^\nu - \frac{\delta L}{\delta A_\nu} A_\lambda \delta x^\lambda + T_\sigma^\nu \delta x^\sigma \right) = \\ -\delta x^\sigma \left[ D_\nu T_\sigma^\nu - D_\nu \tau_\sigma^\nu \right] + D_\nu (\delta x^\lambda) \times \\ \times \left[ T_\lambda^\nu - \tau_\lambda^\nu - D_\mu \sigma_\lambda^{\mu\nu} \right] - \sigma_\mu^{\nu\lambda} D_\nu D_\lambda \delta x^\mu. \end{aligned} \quad (15.41)$$

Taking advantage of this expression, the variation of action (15.37) can be written in the form

$$\begin{aligned} \delta_c S = \frac{1}{c} \int_\Omega d^4x \left[ -\delta x^\lambda \left( \frac{\delta L}{\delta A_\nu} D_\lambda A_\nu - \right. \right. \\ \left. \left. - D_\nu \left( \frac{\delta L}{\delta A_\nu} A_\lambda \right) + D_\nu \tau_\lambda^\nu \right) + D_\nu (\delta x^\lambda) \times \right. \\ \left. \times (T_\lambda^\nu - \tau_\lambda^\nu - D_\mu \sigma_\lambda^{\mu\nu}) - \sigma_\mu^{\nu\lambda} D_\nu D_\lambda \delta x^\mu \right] = 0. \end{aligned} \quad (15.42)$$

Since the integration volume  $\Omega$  is arbitrary, it hence follows that the integrand function is zero.

$$\begin{aligned}
 & -\delta x^\lambda \left( \frac{\delta L}{\delta A_\nu} D_\lambda A_\nu - D_\nu \left( \frac{\delta L}{\delta A_\nu} A_\lambda \right) + D_\nu \tau_\lambda^\nu \right) + \\
 & + \left( T_\lambda^\nu - \tau_\lambda^\nu - D_\mu \sigma_\lambda^{\mu\nu} \right) D_\nu \delta x^\lambda - \sigma_\mu^{\nu\lambda} D_\nu D_\lambda \delta x^\mu = 0.
 \end{aligned} \tag{15.43}$$

This expression turns to zero for arbitrary  $\delta x^\lambda$  independently of the choice of coordinate system. Precisely this permits to readily establish that the tensor  $\sigma_\mu^{\nu\lambda}$  is antisymmetric with respect to  $\nu, \lambda$ . Due to antisymmetry of quantity  $\sigma_\mu^{\nu\lambda}$  in upper indices  $\nu, \lambda$  we get from Eq. (15.40) the following

$$\left( \frac{\partial L}{\partial(\partial_\nu A_\lambda)} + \frac{\partial L}{\partial(\partial_\lambda A_\nu)} \right) = 0.$$

It follows from the above that function  $L$  depends on derivatives in this case as follows

$$L(F_{\nu\lambda}),$$

$$F_{\nu\lambda} = D_\nu A_\lambda - D_\lambda A_\nu.$$

This result was obtained by D. Hilbert in 1915. Of course, this does not exclude an explicit dependence of  $L$  on variable  $A_\nu$ .

By virtue of the tensor transformation law, if it becomes zero in one coordinate system, then it is equal to zero in any other coordinate system. Hence the identities follow:

$$D_\nu \tau_\lambda^\nu + \frac{\delta L}{\delta A_\nu} D_\lambda A_\nu - D_\nu \left( \frac{\delta L}{\delta A_\nu} A_\lambda \right) = 0. \tag{15.44}$$

$$T_\lambda^\nu - \tau_\lambda^\nu - D_\mu \sigma_\lambda^{\mu\nu} = 0, \quad \sigma_\mu^{\nu\lambda} = -\sigma_\mu^{\lambda\nu}. \tag{15.45}$$

As to the last term in (15.43), it should become zero owing to the quantities  $\sigma_\mu^{\nu\lambda}$  being antisymmetric with respect to the upper indices. From the antisymmetry of the spin tensor follows

$$D_\nu T_\lambda^\nu = D_\nu \tau_\lambda^\nu. \quad (15.46)$$

Identities (15.44) and (15.45) are called strong conservation laws, they are obeyed by virtue of action being invariant under coordinate transformations. Applying relation (15.46), expression (15.44) can be written in the form

$$D_\nu T_\lambda^\nu + \frac{\delta L}{\delta A_\nu} F_{\lambda\nu} - A_\lambda D_\nu \left( \frac{\delta L}{\delta A_\nu} \right) = 0, \quad (15.47)$$

$$F_{\lambda\nu} = D_\lambda A_\nu - D_\nu A_\lambda.$$

If we take into account the field equations (15.12), we will obtain

$$D_\nu T_\lambda^\nu = 0, \quad T_\lambda^\nu - \tau_\lambda^\nu = D_\mu \sigma_\lambda^{\mu\nu}, \quad (15.48)$$

here the quantity  $\tau_\lambda^\nu$  equals

$$\tau_\lambda^\nu = -L\delta_\lambda^\nu + \frac{\partial L}{\partial(\partial_\nu A_\mu)} D_\lambda A_\mu. \quad (15.49)$$

The existence of a weak conservation law of the symmetric energy-momentum tensor provides for conservation of the field angular momentum tensor. By defining the angular momentum tensor in Galilean coordinates of an inertial reference system

$$M^{\mu\nu\lambda} = x^\nu T^{\mu\lambda} - x^\mu T^{\nu\lambda}, \quad (15.50)$$

it is easy, with the aid of (15.48), to establish that

$$\partial_\lambda M^{\mu\nu\lambda} = 0. \quad (15.51)$$

The weak conservation laws we have obtained for the energy-momentum tensor and for the angular momentum tensor do not yet testify in favour of the conservation of energy-momentum or angular momentum for a closed system.

**The existence of integral conservation laws for a closed system is due to the properties of space-time, namely, to the existence of the group of space-time motions. The existence of the Poincaré group (the Lorentz group together with the group of translations) for pseudo-Euclidean space provides for the existence of the conservation laws of energy, momentum and angular momentum for a closed system [6].** The group of space-time motion provides form-invariance of the metric tensor  $\gamma_{\mu\nu}$  of Minkowski space.

Let us consider this in more detail. The density of substance energy-momentum tensor according to Eq. (15.36) is the following

$$T^{\mu\nu} = -2 \frac{\delta L}{\delta \gamma_{\mu\nu}}, \quad (15.52)$$

$$\frac{\delta L}{\delta \gamma_{\mu\nu}} = \frac{\partial L}{\partial \gamma_{\mu\nu}} - \partial_\sigma \left( \frac{\partial L}{\partial \gamma_{\mu\nu,\sigma}} \right).$$

This tensor density satisfies Eq. (15.48)

$$D_\nu T^{\mu\nu} = 0, \quad (15.53)$$

that may be written as follows

$$\partial_\nu T_\mu^\nu + \frac{1}{2} T_{\sigma\nu} \partial_\mu g^{\sigma\nu} = 0. \quad (15.54)$$

In general case Eq. (15.53) could not be written as an equality of an ordinary divergence to zero, and so it does not demonstrate any conservation law. But an expression of the form

$$D_\nu A^\nu, \quad (15.55)$$

where  $A^\nu$  is an arbitrary vector, is easy to convert into a divergence form even in the Riemannian space.

From Eq. (11.25) one has

$$D_\lambda A^\lambda = \partial_\lambda A^\lambda + \Gamma_{\mu\lambda}^\lambda A^\mu. \quad (15.56)$$

By means of Eq. (11.38) one obtains

$$D_\lambda(\sqrt{-\gamma} A^\lambda) = \partial_\lambda(\sqrt{-\gamma} A^\lambda). \quad (15.57)$$

Let us exploit this below. Multiply the energy-momentum density onto vector  $\eta_\nu$

$$T^{\mu\nu}\eta_\nu. \quad (15.58)$$

According to Eq. (15.57) we obtain

$$D_\mu(T^{\mu\nu}\eta_\nu) = \partial_\mu(T^{\mu\nu}\eta_\nu). \quad (15.59)$$

Quantity (15.58) already is a vector density in our case. Therefore we should not substitute  $\sqrt{-\gamma}$  into Eq. (15.59). We rewrite Eq. (15.59) in the following form

$$\frac{1}{2}T^{\mu\nu}(D_\mu\eta_\nu + D_\nu\eta_\mu) = \partial_\mu(T^{\mu\nu}\eta_\nu). \quad (15.60)$$

After integration of Eq. (15.60) over volume containing the substance we get

$$\frac{1}{2} \int_V dV T^{\mu\nu}(D_\mu\eta_\nu + D_\nu\eta_\mu) = \frac{\partial}{\partial x^0} \int_V (T^{\nu 0}\eta_\nu) dV. \quad (15.61)$$

If vector  $\eta_\nu$  fulfils the Killing equation

$$D_\mu\eta_\nu + D_\nu\eta_\mu = 0, \quad (15.62)$$

then we have integral of motion

$$\int_V T^{\nu 0}\eta_\nu dV = \text{const.} \quad (15.63)$$

We have already derived Eq. (15.34):

$$\delta_L \gamma_{\mu\nu} = -(D_\nu \delta x_\mu + D_\mu \delta x_\nu). \quad (15.64)$$

From Eqs. (15.62) it follows that if they are fulfilled, then the metric is form-invariant

$$\delta_L \gamma_{\mu\nu} = 0. \quad (15.65)$$

In case of pseudo-Euclidean (Minkowski space) geometry Eqs. (15.62) may be written in a Galilean (Cartesian) coordinate system:

$$\partial_\mu \eta_\nu + \partial_\nu \eta_\mu = 0. \quad (15.66)$$

This equation has the following general solution

$$\eta_\nu = a_\nu + \omega_{\nu\sigma} x^\sigma, \quad \omega_{\nu\sigma} = -\omega_{\sigma\nu}, \quad (15.67)$$

containing ten arbitrary parameters  $a_\nu, \omega_{\mu\nu}$ . This means that there are ten independent Killing vectors, and so there are ten integrals of motion. Taking

$$\eta_\nu = a_\nu \quad (15.68)$$

and substituting this to Eq. (15.63), one finds four integrals of motion:

$$P^\nu = \frac{1}{c} \int_V T^{\nu 0} dV = \text{const.} \quad (15.69)$$

Here  $P^0$  is the system energy, and  $P^i$  is the momentum of the system. Taking Killing vector in the following form

$$\eta_\nu = \omega_{\nu\sigma} x^\sigma \quad (15.70)$$

and substituting it in the initial expression (15.63), one gets the following expression for the angular momentum tensor:

$$P^{\sigma\nu} = \frac{1}{c} \int (T^{\nu 0} x^\sigma - T^{\sigma 0} x^\nu) dV. \quad (15.71)$$

Quantities  $P^{i0}$  are center of mass integrals of motion, and  $P^{ik}$  are angular momentum integrals of motion.

In correspondence with Eq. (15.50) we introduce the following quantity

$$P^{\sigma\nu\lambda} = \frac{1}{c} \int (T^{\nu\lambda}x^\sigma - T^{\sigma\lambda}x^\nu)dV, \quad (15.72)$$

where

$$M^{\sigma\nu\lambda} = T^{\nu\lambda}x^\sigma - T^{\sigma\lambda}x^\nu \quad (15.73)$$

is tensor density, satisfying the following condition

$$\partial_\lambda M^{\sigma\nu\lambda} = 0. \quad (15.74)$$

Therefore, we have been convinced, by deriving Eqs. (15.69) and (15.71), that all these ten integrals of motion arise on the base of pseudo-Euclidean geometry of space-time. Namely this geometry possesses ten independent Killing vectors. There may be also ten Killing vectors in a Riemannian space, but only in case of a constant curvature space [6].

Note that conservation laws are automatically satisfied for an arbitrary scalar (Lagrangian) density of the form  $L(\psi_\lambda, \partial_\sigma\psi_\mu)$  in Minkowski space, that provides for the field energy being positive, if we only consider second-order field equations. I especially recall this here, since from discussions with certain Academicians working in theoretical physics, I have seen, that this is unknown even to them.

Now let us find, as an example, the symmetric tensor of the electromagnetic field energy-momentum. According to (10.5) the Lagrangian density for this field is

$$L_f = -\frac{1}{16\pi}\sqrt{-\gamma} F_{\alpha\beta}F^{\alpha\beta}. \quad (15.75)$$

We write it in terms of the variables  $F_{\mu\nu}$  and the metric coefficients

$$L_f = -\frac{1}{16\pi}\sqrt{-\gamma} F_{\alpha\beta}F_{\mu\nu}\gamma^{\alpha\mu}\gamma^{\beta\nu}. \quad (15.76)$$

According to (11.37) we have

$$\frac{\partial\sqrt{-\gamma}}{\partial\gamma_{\mu\nu}} = \frac{1}{2}\sqrt{-\gamma}\gamma^{\mu\nu}. \quad (15.77)$$

With the aid of (15.77) we obtain

$$\frac{\partial^*L}{\partial\gamma_{\mu\nu}} = -\frac{1}{32\pi}\sqrt{-\gamma}\gamma^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \quad (15.78)$$

\* indicates that differentiation is performed with respect to  $\gamma_{\mu\nu}$ , present in expression (15.76).

Similarly

$$\begin{aligned} \frac{\partial^*L}{\partial\gamma^{\mu\nu}} &= -\frac{1}{16\pi}\sqrt{-\gamma}F_{\alpha\beta}F_{\sigma\lambda} \times \\ &\times \left[ \frac{\partial\gamma^{\alpha\sigma}}{\partial\gamma^{\mu\nu}} \cdot \gamma^{\beta\lambda} + \gamma^{\alpha\sigma} \cdot \frac{\partial\gamma^{\beta\lambda}}{\partial\gamma^{\mu\nu}} \right]. \end{aligned} \quad (15.79)$$

Since

$$\frac{\partial\gamma^{\alpha\sigma}}{\partial\gamma^{\mu\nu}} = \frac{1}{2}(\delta_\mu^\alpha\delta_\nu^\sigma + \delta_\nu^\alpha\delta_\mu^\sigma),$$

then using the antisymmetry properties of the tensor  $F_{\alpha\beta} = -F_{\beta\alpha}$ , we obtain

$$\frac{\partial^*L}{\partial\gamma^{\mu\nu}} = -\frac{1}{8\pi}\sqrt{-\gamma}F_{\mu\lambda}F_{\nu\sigma}\gamma^{\lambda\sigma}. \quad (15.80)$$

In obtaining (15.78) and (15.80) we considered quantities  $\gamma_{\mu\nu}, \gamma^{\lambda\sigma}$  as independent.

Since no derivatives of the metric tensor are present in the density of the electromagnetic field Lagrangian, the density of the symmetric energy-momentum tensor will be

$$T^{\mu\nu} = -2\frac{\partial L}{\partial\gamma_{\mu\nu}} = -2\left[\frac{\partial^*L}{\partial\gamma_{\mu\nu}} + \frac{\partial^*L}{\partial\gamma^{\alpha\beta}} \cdot \frac{\partial\gamma^{\alpha\beta}}{\partial\gamma_{\mu\nu}}\right]. \quad (15.81)$$

From the relation

$$\gamma^{\alpha\beta}\gamma_{\beta\nu} = \delta_\nu^\alpha, \quad (15.82)$$

we find

$$\frac{\partial\gamma^{\alpha\beta}}{\partial\gamma_{\mu\nu}} = -\frac{1}{2}(\gamma^{\alpha\mu}\gamma^{\beta\nu} + \gamma^{\alpha\nu}\gamma^{\beta\mu}). \quad (15.83)$$

Substituting this expression into (15.81), we obtain

$$T^{\mu\nu} = -2\frac{\partial L}{\partial\gamma_{\mu\nu}} = -2\left[\frac{\partial^* L}{\partial\gamma_{\mu\nu}} - \frac{\partial^* L}{\partial\gamma^{\alpha\beta}}\gamma^{\alpha\mu}\gamma^{\beta\nu}\right]. \quad (15.84)$$

Using expressions (15.78) and (15.80) we find the density of the energy-momentum tensor of the electromagnetic field

$$T^{\mu\nu} = \frac{\sqrt{-\gamma}}{4\pi} \left[ -F^{\mu\sigma}F^{\nu\lambda}\gamma_{\sigma\lambda} + \frac{1}{4}\gamma^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \right]. \quad (15.85)$$

Hence it is readily verified, that the trace of the electromagnetic field energy-momentum tensor turns to zero, i. e.

$$T = \gamma_{\mu\nu}T^{\mu\nu} = 0.$$

We shall now construct the energy-momentum tensor of substance. The density of the conserved mass or charge is

$$\mu = \sqrt{-\gamma}\mu_0U^0, \quad \partial_\nu(\sqrt{-\gamma}\mu_0U^\nu) = 0, \quad (15.86)$$

due to Eq. (11.41), where  $\mu_0$  is the density in the rest reference system. The four-dimensional velocity  $U^\nu$  is defined by the expression

$$U^\nu = \frac{v^\nu}{\sqrt{\gamma_{\alpha\beta}v^\alpha v^\beta}}, \quad v^\nu = \frac{dx^\nu}{dt}, \quad v^0 \equiv c. \quad (15.87)$$

Hence, it is clear that

$$U^\nu U^\lambda \gamma_{\nu\lambda} = 1.$$

Take the variation of expression (15.86) with respect to the metric tensor. The quantity  $\mu$  is independent of the metric tensor, therefore,

$$\delta\mu = U^0 \delta(\sqrt{-\gamma} \mu_0) + \sqrt{-\gamma} \mu_0 \delta U^0 = 0, \quad (15.88)$$

here

$$\delta U^0 = -\frac{c}{2} \frac{v^\alpha v^\beta \delta\gamma_{\alpha\beta}}{(\gamma_{\alpha\beta} v^\alpha v^\beta)^{3/2}}. \quad (15.89)$$

From expression (15.88) and (15.89) we find

$$\delta(\sqrt{-\gamma} \mu_0) = \sqrt{-\gamma} \mu_0 \frac{1}{2} U^\alpha U^\beta \delta\gamma_{\alpha\beta}. \quad (15.90)$$

Since the density of the Lagrangian of substance has the form

$$L = -\sqrt{-\gamma} \mu_0 c^2, \quad (15.91)$$

the density of the energy-momentum tensor of substance can be determined as

$$t^{\mu\nu} = -2 \frac{\partial L}{\partial \gamma_{\mu\nu}}. \quad (15.92)$$

On the basis of (15.90) we obtain

$$t^{\mu\nu} = \mu_0 c^2 U^\mu U^\nu. \quad (15.93)$$

Taking into account Eq. (15.86) we obtain in Cartesian coordinate system:

$$\partial_\nu t^{\mu\nu} = \mu_0 c^2 \frac{\partial U^\mu}{\partial x^\nu} \cdot \frac{dx^\nu}{ds} = \mu_0 c^2 \frac{dU^\mu}{ds}. \quad (15.94)$$

Let us rewrite Eq. (10.22) for mass and charge densities:

$$\mu_0 c^2 \frac{dU^\mu}{ds} = \rho_0 F^{\mu\lambda} U_\lambda = f^\mu. \quad (15.95)$$

After comparing Eqs. (15.94) and (15.95) we have

$$f_\nu = \partial_\alpha t_\nu^\alpha. \quad (15.96)$$

From Eqs. (8.54) and (15.96) we can see that the law of energy-momentum tensor conservation for electromagnetic field and sources of charge taken together takes place:

$$\partial_\alpha (T_\nu^\alpha + t_\nu^\alpha) = 0. \quad (15.97)$$

As we noted above, addition to the Lagrangian density of a covariant divergence does not alter the field equations. It is also possible to show [6], that it does not alter the density of the Hilbert energy-momentum tensor, as well. On the contrary, the density of the canonical tensor (15.49) does change. But at the same time the divergence of the spin tensor density changes with it, also. The sum of the canonical tensor density and of the divergence of the spin density remains intact.

## 16. Lobachevsky velocity space

Let us remind that the relativistic law of composition of velocities (see Eq. (9.26)) has the following form:

$$1 - \frac{v'^2}{c^2} = \frac{\left(1 - \frac{v^2}{c^2}\right) \left(1 - \frac{u^2}{c^2}\right)}{\left(1 - \frac{\vec{v}\vec{u}}{c^2}\right)^2}. \quad (16.1)$$

Note that this expression is a direct consequence of the existence of the following invariant

$$\gamma_u \gamma_v (1 - \vec{u}\vec{v}) = \text{inv.}$$

Here  $\gamma_u = (1 - u^2)^{-1/2}$ ,  $\gamma_v = (1 - v^2)^{-1/2}$ .

This invariant has been demonstrated first in the H. Poincaré article [3] (see §9, Eq. (5)), where the system of units is taken so that velocity of light is equal to 1.

It follows just from here that in pseudo-Euclidean space-time the velocity space follows the Lobachevsky geometry.

For the further presentation it will be more convenient to introduce the following notation:

$$v' = v_a, \quad v = v_b, \quad u = v_c, \quad (16.2)$$

$$\cosh a = \frac{1}{\sqrt{1 - \frac{v_a^2}{c^2}}}, \quad \sinh a = \frac{v_a}{c \sqrt{1 - \frac{v_a^2}{c^2}}}, \quad \tanh a = \frac{v_a}{c}. \quad (16.3)$$

Substituting (16.2) and (16.3) into (16.1) we obtain

$$\cosh a = \cosh b \cdot \cosh c - \sinh b \cdot \sinh c \cdot \cos A, \quad (16.4)$$

$A$  is the angle between the velocities  $\vec{v}_b$  and  $\vec{v}_c$ . This is actually nothing, but **the law of cosines for a triangle in Lobachevsky's geometry**. It expresses the length of a side of a triangle in terms of the lengths of the two other sides and the angle between them. Finding, hence,  $\cos A$  and, then,  $\sin A$  etc., one thus establishes **the law of sines of the Lobachevsky geometry**

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}. \quad (16.5)$$

Below, following Lobachevsky, we shall obtain the **law of cosines for a triangle** in the form

$$\cos A = -\cos B \cos C + \sin B \sin C \cosh a. \quad (16.6)$$

We write (10.4) in the form

$$\tanh b \tanh c \cos A = 1 - \frac{\cosh a}{\cosh b \cosh c}. \quad (16.7)$$

From the law of sines (16.5) we have

$$\frac{1}{\cosh c} = \frac{\sin A}{\sin C} \cdot \frac{\tanh c}{\sinh a}. \quad (16.8)$$

Substituting this expression into (16.7) we find

$$\tanh b \tanh c \cos A = 1 - \frac{\sin A}{\sin C} \cdot \frac{\tanh c}{\cosh b \tanh a}. \quad (16.9)$$

Hence we find  $\tanh c$

$$\tanh c = \frac{\tanh a \sin C}{\cos A \sin C \tanh a \tanh b + \frac{1}{\cosh b} \sin A}. \quad (16.10)$$

With the aid of the law of cosines, Lobachevsky further established the identity

$$(1 - \tanh b \tanh c \cos A)(1 - \tanh a \tanh b \cos C) = \frac{1}{\cosh^2 b}. \quad (16.11)$$

Applying (16.10), we find

$$1 - \tanh b \tanh c \cos A = \frac{\frac{1}{\cosh b} \sin A}{\cos A \sin C \tanh a \tanh b + \frac{1}{\cosh b} \sin A}. \quad (16.12)$$

Substitution of this expression into identity (16.11) yields

$$\frac{1}{\cosh b} = \frac{\sin A - \sin A \cos C \tanh a \tanh b}{\cos A \sin C \tanh a \tanh b + \frac{1}{\cosh b} \sin A}. \quad (16.13)$$

With account for

$$1 - \frac{1}{\cosh^2 b} = \tanh^2 b, \quad (16.14)$$

Eq. (16.13) assumes the form

$$\frac{\tanh b}{\tanh a} - \cos C = \cot A \frac{\sin C}{\cosh b}. \quad (16.15)$$

In a similar manner one obtains the relation

$$\frac{\tanh a}{\tanh b} - \cos C = \cot B \frac{\sin C}{\cosh a}. \quad (16.16)$$

From the **law of sines** we have

$$\frac{1}{\cosh b} = \frac{\sin A}{\sin B} \cdot \frac{\tanh b}{\sinh a}. \quad (16.17)$$

Substituting this expression into (16.15), we obtain

$$1 - \frac{\tanh a}{\tanh b} \cos C = \frac{\cos A \sin C}{\cosh a \sin B}. \quad (16.18)$$

Applying expressions (16.16) in (16.18), we find

$$\cos A = -\cos B \cos C + \sin B \sin C \cosh a. \quad (16.19)$$

In a similar manner one obtains the relations:

$$\cos B = -\cos A \cos C + \sin A \sin C \cosh b, \quad (16.20)$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cosh c.$$

**Thus, the space of velocities in pseudo-Euclidean geometry is the Lobachevsky space.**

For a rectangular triangle  $C = \frac{\pi}{2}$ , according to (16.4) we have

$$\cosh c = \cosh a \cosh b. \quad (16.21)$$

From the theorems of sines, (16.5), and of cosines, (16.4) we obtain

$$\sin A = \frac{\sinh a}{\sinh c}, \quad \cos A = \frac{\tanh b}{\tanh c}. \quad (16.22)$$

In line with the obvious equality

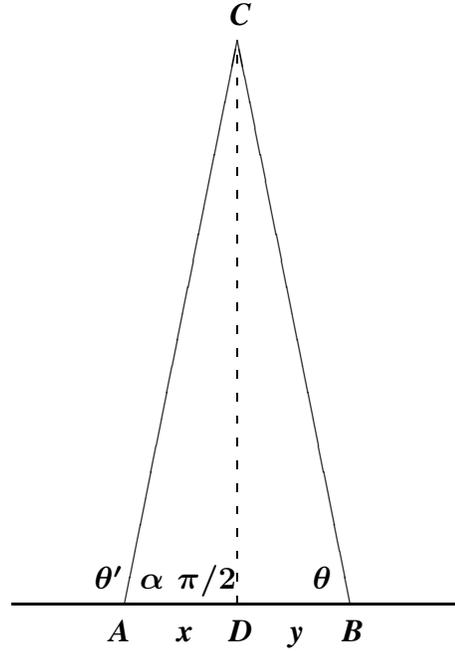
$$\sin^2 A + \cos^2 A = 1 \quad (16.23)$$

one can, making use of expressions (16.22) and (16.21), obtain the relation

$$\sin^2 A \cosh^2 b + \cos^2 A \frac{1}{\cosh^2 a} = 1. \quad (16.24)$$

Consider, as an example [16], the phenomenon of **light aberration**, i. e. the change in direction of a beam of light, when transition occurs from one inertial reference system to another. So, in two reference systems, moving with respect to each other, the directions toward one and the same source  $C$  will differ. Let  $\theta$  and  $\theta'$  be the angles at which the light from the source at point  $C$  is seen from two inertial reference systems  $A$  and  $B$ , moving with respect to each other with a velocity  $v$ . In the Lobachevsky velocity space we shall construct the triangle  $ACD$  (see Fig. 1), with angle  $C$  equal to zero, since light has the limit velocity.

Now, we join points  $A$  and  $B$  by a line, and we drop a perpendicular to this line from point  $C$ . It will intersect the line at point  $D$ . We denote the distance from point  $A$  to point  $D$  by  $x$  and the distance from point  $D$  to  $B$  by  $y$ .



**Fig. 1**

Applying for given triangle  $ACD$  the **law of cosines** (16.20), we obtain

$$\cosh x = \frac{1}{\sin \alpha}, \quad \sinh x = \frac{\cos \alpha}{\sin \alpha}, \quad (16.25)$$

hence

$$\tanh x = \cos \alpha = \cos(\pi - \theta') = -\cos \theta', \quad (16.26)$$

similarly

$$\tanh y = \cos \theta. \quad (16.27)$$

In accordance with formula (16.3),  $\tanh(x + y)$  is the velocity of one reference system with respect to the other in units of the velocity of light

$$\frac{v}{c} = \tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} = \frac{\cos \theta - \cos \theta'}{1 - \cos \theta \cdot \cos \theta'}. \quad (16.28)$$

Hence follow the known formulae for **aberration**

$$\cos \theta' = \frac{\cos \theta - \frac{v}{c}}{1 - \frac{v}{c} \cos \theta}, \quad (16.29)$$

$$\sin \theta' = \sqrt{1 - \frac{v^2}{c^2}} \cdot \frac{\sin \theta}{\left(1 - \frac{v}{c} \cos \theta\right)}. \quad (16.30)$$

Applying formulae (16.29) and (16.30) we obtain

$$\cos(\theta - \theta') = \frac{\left(\cos \theta - \frac{v}{c}\right) \cos \theta + \sqrt{1 - \frac{v^2}{c^2}} \sin^2 \theta}{1 - \frac{v}{c} \cos \theta}. \quad (16.31)$$

Let us determine the square distance between infinitesimally close points in the Lobachevsky space. From (16.1) we find

$$\vec{v}'^2 = \frac{(\vec{u} - \vec{v})^2 - \frac{1}{c^2} [\vec{u}, \vec{v}]^2}{\left(1 - \frac{\vec{u}\vec{v}}{c^2}\right)^2}, \quad (16.32)$$

$v'$  is the relative velocity.

Setting  $\vec{u} = \vec{v} + d\vec{v}$  and substituting into (16.32) we find

$$(dl_v)^2 = c^2 \frac{(c^2 - v^2)(d\vec{v})^2 + (\vec{v}d\vec{v})^2}{(c^2 - v^2)^2}. \quad (16.33)$$

Passing to spherical coordinates in velocity space

$$v_x = v \sin \theta \cos \phi, \quad v_y = v \sin \theta \sin \phi, \quad v_z = v \cos \theta, \quad (16.34)$$

we obtain

$$(d\ell_v)^2 = c^2 \left[ \frac{c^2 (dv)^2}{(c^2 - v^2)^2} + \frac{v^2}{(c^2 - v^2)} (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (16.35)$$

Hence it is evident that the ratio between the length of the circle and the radius is

$$\frac{\ell}{v} = \frac{2\pi}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (16.36)$$

and is always greater than  $2\pi$ .

We now introduce the new variable

$$r = \frac{cv}{\sqrt{c^2 - v^2}}, \quad (16.37)$$

the range of which extends from zero to infinity. In the new variables we have

$$d\ell_v^2 = \frac{dr^2}{1 + \frac{r^2}{c^2}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2); \quad (16.38)$$

if we introduce the variable

$$r = c \sinh Z, \quad (16.39)$$

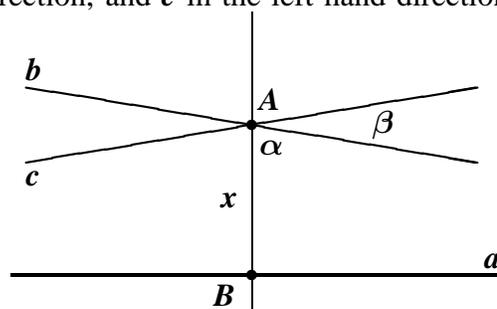
we obtain

$$d\ell_v^2 = c^2 dZ^2 + c^2 \sinh^2 Z (d\theta^2 + \sin^2 \theta d\phi^2). \quad (16.40)$$

Usually the space metric in cosmology is written this form, when dealing with the open Universe.

Further we shall dwell, in a descriptive manner, on certain theorems of Lobachevsky's geometry, following the book by N. V. Efimov ("Higher geometry" M.: Nauka, 1978 (in Russian)) and the lectures of N. A. Chernikov delivered at the Novosibirsk State University and published as a preprint in 1965.

In the Lobachevsky geometry, through point  $A$ , not lying on the straight line  $a$ , there pass an infinite number of straight lines, that do not intersect line  $a$ , but not all these straight lines are considered to be parallel to line  $a$ . Let  $a$  be a straight line in the plane, and let  $A$  be a point outside it (see Fig. 2),  $b$  and  $c$  are boundary straight lines that do not intersect straight line  $a$ . Any straight line passing through point  $A$  inside angle  $\beta$  will also not intersect straight line  $a$ , while any straight line passing through point  $A$  inside the angle containing point  $B$  will necessarily intersect straight line  $a$ . The straight line  $b$  is called the right boundary straight line, and  $c$  the left boundary straight line. It turns out to be that this property is conserved for any point lying on straight line  $b$ . Precisely such a boundary straight line  $b$  is parallel to  $a$  in the right-hand direction, and  $c$  in the left-hand direction. Thus, two



**Fig. 2**

straight lines parallel to  $a$  can be drawn through any one point: one going to the right and the other to the left. In the Lobachevsky geometry, the reciprocity theorem is proven: if one of two straight

lines is parallel to the other in a certain direction, then the second straight line is parallel to the first in the same direction. In a similar manner, it is established, that two straight lines parallel to a third in a certain direction are parallel to each other, also, in the same direction. Two straight lines, perpendicular to a third straight line, diverge. Two divergent straight lines always have one common perpendicular, to both sides of which they diverge indefinitely from each another.

Parallel straight lines, indefinitely receding from each other in one direction, asymptotically approach each other in the other. The angle  $\alpha$  is called the parallelism angle at point  $A$  with respect to straight line  $a$ .

From the law of cosines (16.6) we find

$$1 = \sin \alpha \cosh x.$$

In obtaining this expression we took into account that straight line  $b$  asymptotically approaches straight line  $a$ , so, therefore, the angle between straight lines  $a$  and  $b$  is zero. Hence we obtain Lobachevsky's formula

$$\alpha(x) = 2 \arctan e^{-x},$$

here  $a$  is the distance from point  $A$  to straight line  $a$ . This function plays a fundamental part in the Lobachevsky geometry. This is not seen from our exposition, because we obtained the Lobachevsky geometry as the geometry of velocity space. proceeding from the pseudo-Euclidean geometry of space-time. Function  $\alpha(x)$  decreases monotonously. The area of the triangle is

$$S = d^2 \cdot (\pi - A - B - C), \quad (16.41)$$

here  $d$  is a constant value. Below we shall derive this formula. From the formula it is evident that in the Lobachevsky geometry similar triangles do not exist.

Following Lobachevsky, we express the function

$$\cos \Delta, \quad \text{where} \quad 2\Delta = A + B + C, \quad (16.42)$$

via the sides of the triangle. Applying the law of cosines (16.6) and, also, the formulae

$$\sin^2 \frac{A}{2} = \frac{1 - \cos A}{2}, \quad \cos^2 \frac{A}{2} = \frac{1 + \cos A}{2}, \quad (16.43)$$

we find

$$\sin^2 \frac{A}{2} = \frac{\sinh(p-b) \cdot \sinh(p-c)}{\sinh b \sinh c}, \quad (16.44)$$

$$\cos^2 \frac{A}{2} = \frac{\sinh p \cdot \sinh(p-a)}{\sinh b \sinh c}, \quad (16.45)$$

here  $p$  is the half-perimeter of the triangle

$$2p = a + b + c.$$

With the aid of formulae (10.44) and (10.45) we obtain

$$\sin \frac{A}{2} \cos \frac{B}{2} = \frac{\sinh(p-b)}{\sinh c} \cos \frac{C}{2}, \quad (16.46)$$

$$\sin \frac{B}{2} \cos \frac{A}{2} = \frac{\sinh(p-a)}{\sinh c} \cos \frac{C}{2}. \quad (16.47)$$

Hence we have

$$\sin \frac{A+B}{2} = \frac{\cosh\left(\frac{a-b}{2}\right)}{\cosh \frac{c}{2}} \cos \frac{C}{2}. \quad (16.48)$$

Applying the formulae

$$\cos \frac{A}{2} \cos \frac{B}{2} = \frac{\sinh p}{\sinh c} \sin \frac{C}{2}, \quad (16.49)$$

$$\sin \frac{A}{2} \sin \frac{B}{2} = \frac{\sinh(p-c)}{\sinh c} \sin \frac{C}{2}, \quad (16.50)$$

we find

$$\cos \frac{A+B}{2} = \frac{\cosh\left(\frac{a+b}{2}\right)}{\cosh \frac{c}{2}} \sin \frac{C}{2}. \quad (16.51)$$

From (16.48) and (16.51) we have

$$\cos \Delta = 2 \frac{\sinh \frac{a}{2} \sinh \frac{b}{2}}{\cosh \frac{c}{2}} \sin \frac{C}{2} \cos \frac{C}{2}. \quad (16.52)$$

Replacing  $\sin \frac{C}{2} \cos \frac{C}{2}$  in (16.52) by the expressions from Eqs. (16.44) and (16.45) we find

$$\cos \Delta = \frac{\sqrt{\sinh p \cdot \sinh(p-a) \sinh(p-b) \sinh(p-c)}}{2 \cosh \frac{a}{2} \cosh \frac{b}{2} \cosh \frac{c}{2}}. \quad (16.53)$$

From (16.41) we have the equality

$$\sin \frac{S}{2d^2} = \cos \Delta. \quad (16.54)$$

Comparing (16.53) and (16.54) we obtain

$$\sin \frac{S}{2d^2} = \frac{\sqrt{\sinh p \cdot \sinh(p-a) \sinh(p-b) \sinh(p-c)}}{2 \cosh \frac{a}{2} \cosh \frac{b}{2} \cosh \frac{c}{2}}. \quad (16.55)$$

In our formulae the sides  $a, b, c$  are dimensionless quantities, in accordance with definition (16.3). Eq. (16.55) is the analog of the

Heron formula in Euclidean geometry. From (16.52) the expression for the area of the triangle can be written, also, in the form

$$\sin \frac{S}{2d^2} = \frac{\sinh \frac{a}{2} \sinh \frac{b}{2}}{\cosh \frac{c}{2}} \sin C. \quad (16.56)$$

The area  $S$  is expressed in dimensionless units, since the sides of the triangle are dimensionless. In our exposition, the constant  $d$  is unity, on the basis of the law of cosines (16.4).

From formula (16.41) it follows that in the Lobachevsky geometry the area of a triangle cannot be indefinitely large, it is restricted to the quantity  $d^2\pi$ . Thus, admitting the existence of a triangle of indefinitely large area is equivalent to Euclid's parallelism axiom. The areas of polygons can be indefinitely large in the Lobachevsky geometry.

The area of a spherical triangle in Euclidean geometry is

$$S_{\Delta} = R^2(A + B + C - \pi), \quad (16.57)$$

here  $R$  is the radius of the sphere. Comparing this expression with formula (16.41), we see that formula (16.41) can be derived from formula (16.57), if the radius of the sphere is chosen to be imaginary and equal to the value  $R = id$ . This circumstance was already noted by Lambert.

If one introduces the variables

$$x = \frac{v_x}{c}, \quad y = \frac{v_y}{c}, \quad z = \frac{v_z}{c}, \quad (16.58)$$

then formula (16.33), for the Lobachevsky geometry, in the  $x, y$  plane assumes the form

$$(d\ell_v)^2 = c^2 \frac{(1 - y^2) \cdot (dx)^2 + 2xy dx dy + (1 - x^2) \cdot (dy)^2}{(1 - x^2 - y^2)^2}, \quad (16.59)$$

the quantities  $x, y$  are called Beltrami coordinates in the Lobachevsky geometry.

Passing to new variables  $\xi, \eta$  with the aid of formulae

$$x = \tanh \xi, \quad y = \frac{\tanh \eta}{\cosh \xi}, \quad (16.60)$$

and calculating the differentials

$$dx = \frac{\xi}{\cosh^2 \xi}, \quad dy = \frac{1}{\cosh^2 \eta \cosh \xi} d\eta - \frac{\tanh \eta}{\cosh^2 \xi} \sinh \xi d\xi,$$

upon performing the required computations, we find

$$(dl_v)^2 = c^2(\cosh^2 \eta d\xi^2 + d\eta^2). \quad (16.61)$$

The net of coordinate lines

$$\xi = \text{const}, \quad \eta = \text{const}, \quad (16.62)$$

is orthogonal. The area of the triangle in these variables is

$$S = \iint_{(\Delta)} \cosh \eta d\xi d\eta. \quad (16.63)$$

For calculating the area of a triangle by formula (16.63) it is necessary to find the geodesic (extremal) line in the Lobachevsky geometry in coordinates  $\xi, \eta$ . To this end we shall take advantage of the principle of stationary action.

Length is

$$\begin{aligned} L &= \int ds = \int \sqrt{\cosh^2 \eta \cdot d\xi^2 + d\eta^2} = \\ &= \int_{\eta_1}^{\eta_2} d\eta \sqrt{\cosh^2 \eta \cdot \xi'^2 + 1}. \end{aligned} \quad (16.64)$$

Hence the extremal curve is found in accordance with the condition

$$\delta L = \int_{\eta_1}^{\eta_2} \frac{\xi' \cdot \cosh^2 \eta \cdot \delta(\xi')}{\sqrt{\cosh^2 \eta \cdot \xi'^2 + 1}} \cdot d\eta = 0, \quad \xi' = \frac{d\xi}{d\eta}. \quad (16.65)$$

The variation  $\delta$  commutes with differentiation, i. e.

$$\delta(\xi') = (\delta\xi)'; \quad (16.66)$$

taking this into account and integrating by parts in the integral (16.65) we obtain

$$\delta L = - \int_{\eta_1}^{\eta_2} d\eta \delta\xi \frac{d}{d\eta} \left( \frac{\cosh^2 \eta \cdot \xi'}{\sqrt{\cosh^2 \eta \cdot \xi'^2 + 1}} \right) = 0. \quad (16.67)$$

Here, it is taken into account that the variations  $\delta\xi$  at the limit points of integration are zero.

From equality (16.67), owing to the variation  $\delta\xi$  being arbitrary, it follows

$$\frac{d}{d\eta} \left( \frac{\cosh^2 \eta \cdot \xi'}{\sqrt{\cosh^2 \eta \cdot \xi'^2 + 1}} \right) = 0. \quad (16.68)$$

Hence we find the equation for the geodesic line

$$\frac{\cosh^2 \eta \cdot \xi'}{\sqrt{\cosh^2 \eta \cdot \xi'^2 + 1}} = c; \quad (16.69)$$

geodesic lines, as the shortest in the Lobachevsky geometry, are straight lines in it.

Resolving this equation, we obtain

$$\xi - \xi_0 = \pm c \int \frac{d\eta}{\cosh \eta \sqrt{\cosh^2 \eta - c^2}}. \quad (16.70)$$

Changing the variable of integration

$$u = \tanh \eta, \quad (16.71)$$

we find

$$\begin{aligned} \xi - \xi_0 &= \pm \int \frac{c du}{\sqrt{(1 - c^2) + c^2 u^2}} = \\ &= \pm \int \frac{dv}{\sqrt{1 + v^2}} = \pm \operatorname{arcsch} v. \end{aligned} \quad (16.72)$$

Here

$$v = \frac{cu}{\sqrt{1 - c^2}}. \quad (16.73)$$

It is suitable to take for variable  $c$  the following notation:

$$c = \sin \delta. \quad (16.74)$$

Thus, the equation of a geodesic line has the form

$$\sinh(\xi - \xi_0) = \pm \tan \delta \cdot \tanh \eta. \quad (16.75)$$

Let us, now, construct a triangle in the  $\xi, \eta$  plane (Fig. 3). The lines  $AB$  and  $AC$  are geodesic lines, that pass through point  $(\xi_0, 0)$ . The angles  $A_1$  and  $A_2$  are inferior to the parallelism angle  $\alpha$ :

$$A = A_1 + A_2.$$

From expression (16.75) we find the derivative of the geodesic line  $AC$  at point  $\xi_0$

$$\xi'_0 = -\tan \delta_2. \quad (16.76)$$

Hence and from  $\triangle ALP$  we have

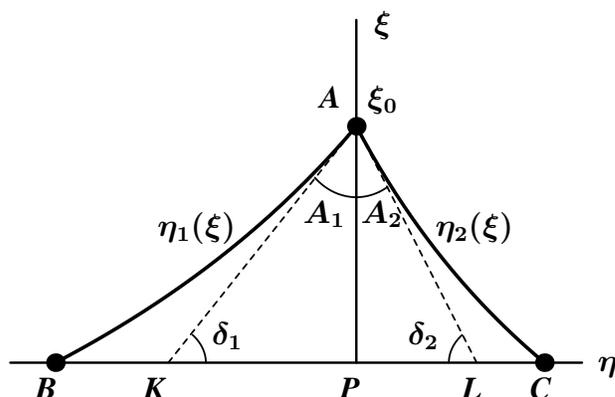


Fig. 3

$$\delta_2 = \frac{\pi}{2} - A_2, \quad (16.77)$$

similarly, from  $\triangle AKP$ , we also find for the geodesic line  $AB$

$$\delta_1 = \frac{\pi}{2} - A_1. \quad (16.78)$$

Thus, the constant  $c$  for each geodesic is expressed via the angles  $A_1, A_2$ . The geodesic lines  $AB$  and  $AC$  intersect the  $\eta$  axis at points  $\eta_1^0, \eta_2^0$ .

In accordance with (16.63) the area of the triangle  $ABC$  is

$$S_{\Delta} = \int_0^{\xi_0} d\xi \int_{\eta_1(\xi)}^{\eta_2(\xi)} \cosh \eta \cdot d\eta = \int_0^{\xi_0} \{\sinh \eta_2(\xi) - \sinh \eta_1(\xi)\} d\xi. \quad (16.79)$$

Taking advantage of expression (16.75), we find

$$\sinh \eta = \pm \frac{\sinh(\xi - \xi_0)}{\sqrt{\cos^{-2} \delta - \cosh^2(\xi - \xi_0)}}. \quad (16.80)$$

Hence we find

$$\sinh \eta_2(\xi) = -\frac{\sinh(\xi - \xi_0)}{\sqrt{\sin^{-2} A_2 - \cosh^2(\xi - \xi_0)}}, \quad (16.81)$$

$$\sinh \eta_1(\xi) = \frac{\sinh(\xi - \xi_0)}{\sqrt{\sin^{-2} A_1 - \cosh^2(\xi - \xi_0)}}. \quad (16.82)$$

Then the intersection points of the geodesic lines with the straight line  $\eta(\xi = 0)$  are

$$\sinh \eta_2^0 = \frac{\sinh \xi_0}{\sqrt{\sin^{-2} A_2 - \cosh^2 \xi_0}}, \quad (16.83)$$

$$\sinh \eta_1^0 = -\frac{\sinh \xi_0}{\sqrt{\sin^{-2} A_1 - \cosh^2 \xi_0}}. \quad (16.84)$$

From the law of sines (16.5) we have

$$\sin B = \sinh \xi_0 \cdot \frac{\sin A_1}{\sinh |\eta_1^0|}. \quad (16.85)$$

Substituting into this expression the value of  $\eta_1^0$  (16.84) we obtain

$$\sin B = \sqrt{1 - \sin^2 A_1 \cosh^2 \xi_0}, \quad \cos B = \sin A_1 \cosh \xi_0. \quad (16.86)$$

Similarly

$$\cos C = \sin A_2 \cosh \xi_0. \quad (16.87)$$

Introducing the variable

$$u = \cosh(\xi - \xi_0) \quad (16.88)$$

in the integral (16.79), we obtain

$$S_{\Delta} = \int_1^{\cosh \xi_0} \left\{ \frac{1}{\sqrt{\sin^{-2} A_1 - u^2}} + \frac{1}{\sqrt{\sin^{-2} A_2 - u^2}} \right\} du. \quad (16.89)$$

Hence follows

$$S_{\Delta} = \arcsin(\sin A_1 \cosh \xi_0) + \arcsin(\sin A_2 \cosh \xi_0) - (A_1 + A_2). \quad (16.90)$$

Taking into account (16.86) and (16.87), we obtain

$$S_{\Delta} = \arcsin(\cos B) + \arcsin(\cos C) - A. \quad (16.91)$$

Ultimately, we have

$$S_{\Delta} = \pi - A - B - C. \quad (16.92)$$

We have obtained the expression for the area of a triangle  $S_{\Delta}$  in the Lobachevsky geometry, that we earlier (16.41) made use of in finding formula (16.55).

From the above we saw that the Lobachevsky geometry, created by him as an “imaginary geometry”, has become a composite part of the physics of relativistic motions, as the geometry of velocity space.

The discovery of Lobachevsky had a great impact on the development of various parts of mathematics. Thus, for example, the French mathematician G. Hadamard, in the book “Non-Euclidean geometry” in Section devoted to the theory of automorphic functions noted:

*“We hope we have succeeded in showing, how Lobachevsky’s discovery permeates throughout Poincaré’s entire remarkable creation, for which it served, by the idea of Poincaré himself, as the foundation. We are sure that Lobachevsky’s discovery will play a great part, also, at the further stages of development of the theory we have considered”.*

Beltrami raised the question: “*Is it possible to realize Lobachevsky planimetry in the form of an internal geometry of a certain surface in Euclidean space?*” Hilbert has shown, that in Euclidean space no surface exists, that is isometric to the **entire** Lobachevsky plane. However, part of the plane of the Lobachevsky geometry can be realized in Euclidean space.

## Problems and exercises

### Section 2

- 2.1. An electric charge is in a falling elevator. Will it emit electromagnetic waves?
- 2.2. A charge is in a state of weightlessness in a space ship. Will it radiate?

### Section 3

- 3.1. Let the metric tensor of Minkowski space in a non-inertial coordinate system have the form  $\gamma_{\mu\nu}(x)$ . Show that there exists a coordinate system  $x'$ , in which the metric tensor has the same form  $\gamma_{\mu\nu}(x')$ , and that nonlinear transformations relating these systems constitute a group.

### Section 4

- 4.1. Is the following statement correct: "In a moving reference system (with a constant velocity  $v$ ) time flows slower, than in a reference system at rest"?
- 4.2. Is the Lorentz contraction of a rod (4.13) real or apparent?
- 4.3. Is it possible, by making use of the Lorentz effect of contraction, to achieve a high density of substance by accelerating a rod?

### Section 8

- 8.1. The electric charge of a body is independent on the choice of reference system. On the basis of this assertion find the transformation law of charge density, when transition occurs from one inertial reference system to another.

8.2. With the aid of Lorentz transformations find the field of a charge undergoing uniformly accelerated motion.

### Section 9

9.1. Three small space rockets  $A$ ,  $B$  and  $C$  are drifting freely in a region of space distant from other matter, without rotation and without relative motion, and  $B$  and  $C$  are equidistant from  $A$ . When a signal is received from  $A$ , the engines of  $B$  and  $C$  are switched on, and they start to smoothly accelerate. Let the rockets  $B$  and  $C$  be identical and have identical programs of acceleration. Suppose  $B$  and  $C$  have been connected from the very beginning by a thin thread. What will happen to the thread? Will it break or not?

(Problem by J. Bell)

9.2. Let some device emits electromagnetic energy with power 6000 Watt in a definite direction. What force is required due to the recoil to hold the device at rest?

### Section 10

10.1. Applying the principle of stationary action obtain the following formula for the Lorentz force:

$$\vec{f} = \rho \vec{E} + \frac{\rho}{c} [\vec{v}, \vec{H}],$$

where  $\rho$  is the electric charge density.

### Section 11

11.1. Does a charge, moving along a geodesic line in a uniformly accelerating reference system, radiate?

11.2. Does a charge, moving along a geodesic line in an arbitrary non-inertial reference system, radiate?

- 11.3. Does a charge, that is at rest in a non-inertial reference system, radiate?
- 11.4. Does an elevator, the rope of which has been torn, represent an inertial reference system?

### Section 12

- 12.1. Find the space geometry on a disk, rotating with a constant angular velocity  $\omega$ .
- 12.2. Consider an astronaut in a space ship moving with constant acceleration  $a$  away from the Earth. Will he be able to receive information from the Control Center during his trip?

### Section 16

- 16.1. Find a surface in the Lobachevsky geometry, on which the Euclidean planimetry is realized.
- 16.2. Explain the Thomas precession with the aid of the Lobachevsky geometry.
- 16.3. Does a triangle exist in the Lobachevsky geometry, all angles of which equal zero?
- 16.4. Find the area of a triangle on a sphere of radius  $R$  in Euclidean geometry.

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