

ON FIBONACCI POWERS

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Fibonacci numbers have engaged the attention of mathematicians for several centuries, and whilst many of their properties are easy to establish by very simple methods, there are several unsolved problems connected to them. In this paper we review the history of the conjecture that the only perfect powers in Fibonacci sequence are 1, 8, and 144. Afterwards we consider more stronger conjecture and give the new characterization of closely related Wall-Sun-Sun primes.

1. INTRODUCTION

The Fibonacci numbers have engaged the attention of mathematicians for several centuries, and whilst many of their properties are easy to establish by very simple methods, there are several unsolved problems connected to them. Fibonacci numbers are defined with recurrence $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$, and initial terms $F_0 = 0$, and $F_1 = 1$. In this paper we consider Fibonacci power problem, i.e. the equation $F_n = C^q$, where C and q are integers > 1 .

Conjecture 1 *The only Fibonacci numbers F_n ($n > 0$) which are perfect powers are 1, 8, and 144.*

We now review some known results about special cases of Conjecture 1. The first question related to Conjecture 1 appeared in 1962 in the book by Ogilvy [16], the problem about square Fibonacci numbers (case $q = 2$). In 1963, both Moser and Carlitz [15] and Rollett [24] proposed that problem. By ingenious sieving computational method Wunderlich [31] showed, that except for the known cases (F_1 and F_{12}), F_n cannot be a square for $n < 10^6$. In 1964 square conjecture was proved analytically by Cohn [5, 6] and independently by Wyler [32]. Problem of Fibonacci cubes (case $q = 3$) was solved by Finkelstein in his doctoral dissertation in 1968 [8, 27]. Since then several other proofs were published: London and Finkelstein in 1969 [12], Steiner in 1978 [26], Lagarias and Weisser in 1981 [11], Pethö in 1983 [17].

In the beginning of 1980's, thanks to Baker's theory of lower bounds for linear forms in logarithms of algebraic numbers, it was proved by Shorey and Stewart [25] and independently by Pethö [18] that there were only finitely many effectively computable Fibonacci powers. The method of Shorey and Stewart implies that $q < 5.1 \cdot 10^{17}$ [19, 20], but this upper bound is huge for solutions by brutal

Research partially supported by the Ministry of Science and Environmental Protection, project 144032

Date: May 5, 2005; revised January 15, 2006.

2000 Mathematics Subject Classification: 11B39

Keywords and Phrases: Fibonacci number, perfect power, Wall-Sun-Sun prime

enumeration. From the other side, the conjecture was proved for small q by combining that method with some algebraic tools and computer calculations: Pethö for $q = 5$ [21] and McLaughlin for $q = 5, 7, 11, 13, 17$ [14]. Thanks to important progress in computational number theory and theory of linear forms in logarithms, Bugeaud, Mignotte, and Siksek [3] recently proved the conjecture using some of the ideas of the proof of Fermat's Last Theorem.

2. PRELIMINARIES

We give now some definitions and known facts about Fibonacci numbers without proof.

Definition 1

1. $Z(m) = \min\{t : m|F_t\}$
2. $P(m) = \min\{t : F_t \equiv 0, F_{t+1} \equiv 1 \pmod{m}\}$
3. $Y(m) = \max\{t : Z(m^t) = Z(m)\}$

$Z(m)$ is so called Fibonacci entry point (or rank of apparition, or restricted period [9]). This is the index of the smallest Fibonacci number divisible by m . $P(m)$ is the length of the Fibonacci sequence modulo m period. $Y(m)$ is the greatest integer Y such that $F_{Z(m)}$ is divisible by m^Y . It can be proved that $Z(n) \leq 2n$, and $P(n) \leq 6n$.

Now we give a few very old lemmas [9]:

Lemma 1 $m|F_n \Leftrightarrow Z(m)|n$.

Lemma 2 $p|F_{p-(\frac{p}{5})}$ for all primes p . (where $(\frac{p}{5})$ is the Legendre symbol)

Lemma 3 $Z(p^k) = p^{k-Y(p)}Z(p)$ for odd primes p and $k \geq Y(p)$.

Consider the equation $F_n = C^q$. Without loss of generality, we may require that q be prime. We can also suppose that n is prime, see Robbins [22, 23] and Pethö [17].

Lemma 4 If $n > 12$ and F_n is a perfect q -th power, then there exists a prime $p > 5$, such that F_p is a perfect q -th power.

3. FIBONACCI CONJECTURE

Concerning Lemma 3, one can ask whether $Z(p^2) = Z(p)$. If $Z(p^2) \neq Z(p)$ for all primes p , then we can determine whole Z function from $Z(p)$ for primes p , because of nice $Z([m, n]) = [Z(m), Z(n)]$, where $[-, -]$ is the least common multiple of two numbers. Thus we have natural question.

Conjecture 2 $Y(p) = 1$ for all primes p .

Of course, if Conjecture 2 holds, then we have beautiful $Z(p^k) = p^{k-1}Z(p)$. As far as I know, Conjecture 2 appeared in the work of Wall in 1960 [29] in the equivalent form $P(p^2) \neq P(p)$. The period and the entry point functions are closely related. One can prove that $P(p^k) = p^{k-Y(p)}P(p)$ holds for all odd primes p , and that $P(p^2) = P(p) \Leftrightarrow Z(p^2) = Z(p)$.

Theorem 1 *Conjecture 2 implies Conjecture 1.*

Proof. If there exist perfect powers different from 1, 8 and 144, then by Lemma 4 there exist primes $p > 5$ and $q \geq 2$ with $F_p = C^q$. If r is any prime factor of C , then we have $r^2 | F_p$. Thus $Z(r^2) | p$ by Lemma 1, hence $Z(r^2) = p$, which means that $Z(r^2) = Z(r)$. If Conjecture 2 holds this is impossible and we have a contradiction.

REMARK 1. A proof that 1, 8, and 144 are the only Fibonacci powers was given by Buchanan in 1964 [1], but unfortunately, Buchanan's proof was incomplete and it was retracted later by himself [2]. He made an error assuming that Conjecture 2 holds, so he actually proved Theorem 1.

REMARK 2. We say that p is a primitive factor of F_n if $p | F_n$, and $p \nmid F_m$ for any $m < n$. Carmichael's theorem states that if $n > 12$, then F_n has a primitive prime factor [9]. It is clear that for an arbitrary primitive prime factor p of C^q we have $Z(p) = Z(p^2) = \dots = Z(p^q)$. In proof we used the fact that all prime factors of F_q for prime q are primitive.

There are many equivalent statements of Conjecture 2. If $Y(p) > 1$ is true for some prime p , then we have $Z(p^2) = Z(p)$. From Lemmas 1 and 2 we know that $Z(p) | p - (\frac{p}{5})$, and therefore $p^2 | F_{p - (\frac{p}{5})}$. It is clear that $p \nmid F_{p + (\frac{p}{5})}$ (if $p | F_{p + (\frac{p}{5})}$, then $Z(p) | p + (\frac{p}{5})$ and $Z(p) | p - (\frac{p}{5})$, i.e. $Z(p) | 2$, which is impossible). Therefore, $p^2 | F_{p-1}F_{p+1}$.

Lemma 5 *The following four statements are equivalent:*

1. $Y(p) > 1$;
2. $Z(p^2) = Z(p)$;
3. $p^2 | F_{p - (\frac{p}{5})}$;
4. $p^2 | F_{p-1}F_{p+1}$.

Crandall, Dilcher, and Pomerance [7] called prime p satisfying Lemma 5 statements the Wall-Sun-Sun prime. There is no known way to resolve congruence like $F_{p-\left(\frac{p}{5}\right)} \equiv 0 \pmod{p^2}$, other than through explicit powering computations. On the basis of a search conducted by McIntosh we have learned that there are no Wall-Sun-Sun primes less than 10^{14} , and no Wall-Sun-Sun primes are known [13]. Statistical considerations show that in an interval $[x, y]$ it is expected to be $\sum_{x \leq p \leq y} \frac{1}{p} \approx \ln(\ln y / \ln x)$ Wall-Sun-Sun primes [7].

However, Conjecture 2 is more stronger than Theorem 1. There are few results connecting Fermat's last theorem and Wall-Sun-Sun primes. In 1974 Bruckner showed, using the theory of cyclotomic fields, that if there exists a Wall-Sun-Sun number, then Fermat's last theorem has a solution in $Q(\sqrt{5})$ satisfying certain side conditions [27]. From the other side, in 1992 Zhi-Hong Sun and Zhi-Wei Sun [28] showed that if p is a failing exponent in the first case of Fermat's last theorem ($x^p + y^p = z^p$ with $p \nmid xyz$), then p must be Wall-Sun-Sun prime. Unfortunately, it never does, but if one could prove the converse of this result, then Wiles' work would guarantee that there are no Wall-Sun-Sun primes.

4. NEW CHARACTERIZATION OF WALL-SUN-SUN PRIMES

We shall denote n -th Lucas number with L_n , and use known formula for Lucas' numbers. For every integer p we have

$$L_p = \left(\frac{1 + \sqrt{5}}{2} \right)^p + \left(\frac{1 - \sqrt{5}}{2} \right)^p,$$

and therefore for odd p holds

$$2^{p-1} L_p = \sum_{k=0}^{\frac{p-1}{2}} \binom{p}{2k} 5^k = 1 + \sum_{k=1}^{\frac{p-1}{2}} \frac{p}{2k} \binom{p-1}{2k-1} 5^k. \quad (1)$$

From now till the end, we shall suppose that p is odd prime. Taking equation (1) modulo p , and using little Fermat theorem we obtain

Lemma 6 $L_p \equiv 1 \pmod{p}$.

Now from $\binom{p}{t} = \binom{p-1}{t} + \binom{p-1}{t-1}$ and $p \mid \binom{p}{t}$ for $1 \leq t \leq p-1$ we can conclude $\binom{p-1}{t} \equiv (-1)^t \pmod{p}$ and therefore

$$\binom{p-1}{2k-1} \equiv -1 \pmod{p}.$$

Substituting in (1) modulo p^2 we get

$$2^{p-1}L_p \equiv 1 + p \sum_{k=1}^{\frac{p-1}{2}} \frac{-1}{2k} 5^k \pmod{p^2},$$

and

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{5^k}{k} \equiv \frac{2 - 2^p L_p}{p} \pmod{p}. \quad (2)$$

Lemma 7 *Prime p is a Wall-Sun-Sun prime iff $L_p \equiv 1 \pmod{p^2}$.*

Proof. It is easy to show that $5F_{p-1}F_{p+1} = L_p^2 - 1$ holds for all odd p . If p is a Wall-Sun-Sun odd prime, then by Lemma 5 we have $L_p^2 \equiv 1 \pmod{p^2}$. But $L_p \equiv -1$ is impossible by Lemma 6, and therefore $L_p \equiv 1 \pmod{p^2}$.

Theorem 2 *Prime p is a Wall-Sun-Sun prime iff*

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{5^k - 1}{k} \equiv 0 \pmod{p}$$

Proof. From Lemma 7 and (2) it follows that p is a Wall-Sun-Sun prime iff

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{5^k}{k} \equiv \frac{2 - 2^p}{p} \pmod{p}.$$

Next we can use known fact

$$\sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p}$$

(actually Wolstenholme's theorem states that the congruence above is also satisfied modulo p^2). Transforming the expansion $\sum_{k=0}^p \binom{p}{k} = 2^p$, we obtain

$$2 + \sum_{k=1}^{p-1} \frac{p}{k} \binom{p-1}{k-1} = 2^p.$$

Consequently,

$$\frac{2 - 2^p}{p} \equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k} \equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k} + \sum_{k=1}^{p-1} \frac{1}{k} \equiv 2 \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{2k} \equiv \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k} \pmod{p},$$

implying the claim of Theorem.

REMARK 3. A similar identity for Wall-Sun-Sun primes p is given by Ward and published posthumously [30]:

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{\left(\frac{5}{9}\right)^k}{k} \equiv 2 \frac{\left(\frac{3}{2}\right)^{p-1} - 1}{p} \pmod{p}.$$

For the proof reader can consult [4] and [10].

ACKNOWLEDGMENT: I am grateful to Daniel Weisser, Ray Stainer, Attila Pethö, and Richard McIntosh for our communication. I am also grateful to Miodrag Živković who read carefully the article, and made many valuable suggestions.

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