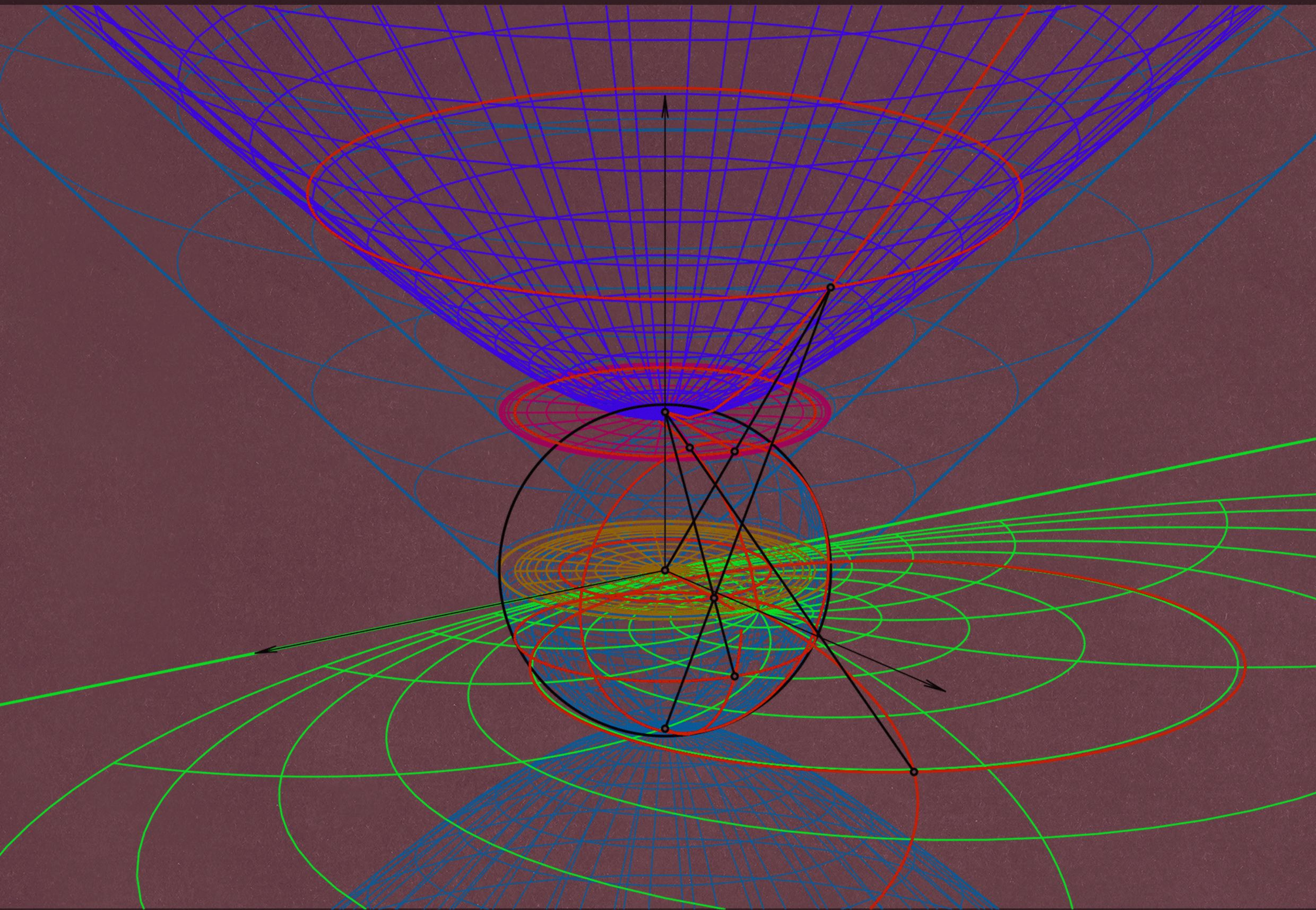

GEOMETRY

SMOOTH MANIFOLDS
PSEUDO-RIEMANNIAN GEOMETRY
OSSERMAN MANIFOLDS

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Smooth Manifolds
Pseudo-Riemannian Geometry
Osserman Manifolds

BELGRADE, 02.07.2025

The History of every major Galactic Civilization tends to pass through three distinct and recognizable phases, those of Survival, Inquiry and Sophistication, otherwise known as the How, Why, and Where phases. For instance, the first phase is characterized by the question "How can we eat?" the second by the question "Why do we eat?" and the third by the question "Where shall we have lunch?"

Douglas Adams
The Hitchhikers Guide to the Galaxy

PREFACE

The curvature is the most natural and most important invariant of pseudo-Riemannian geometry. According to Osserman[1] [97], the notion of curvature is one of the central concepts of differential geometry, distinguishing the geometric core of the subject from those aspects that are analytic, algebraic, or topological. The curvature information is contained in the curvature tensor, which is difficult to work with, despite the many symmetries it possesses. Extracting the geometrical information that is encoded therein is often quite a challenging task. That is why Gromov[2] [64] described the curvature tensor as a little monster of (multi)linear algebra whose full geometric meaning remains obscure. Therefore, instead of working with the curvature tensor itself, we often use Jacobi operators or sectional curvature that are easier to handle and have a better geometric interpretation, while they contain the complete curvature information.

Our general goal is to find some kind of bridge between the curvature of a pseudo-Riemannian manifold and its geometric properties. This aim can be approached in two ways. The first approach is direct and begins by endowing a manifold with a metric, and then its curvature is completely determined. We can compute it locally by calculating Christoffel symbols of the Levi-Civita connection according to the formula (5.11), and then use the formula (6.3) to get the components of the curvature operator, which consequently gives the Jacobi operators.

If the initial metric has some nice features, like a large group of isometries, then it is natural to expect that it implies the curvature that behaves nicely. We usually start from highly symmetric manifolds which consequently have a relatively simple curvature tensor, and then we study their geometric properties, which are often generalisations of properties common in Euclidean geometry. These simple starting manifolds, such as two-point homogeneous spaces, are called model spaces, and are used for more simple and concrete expression of the essential characteristics of Riemannian (or pseudo-Riemannain) geometry.

The second approach goes in the opposite direction. We look at more general manifolds to discover which of their properties are representative to a particular model space. If we cannot recognize a model space based on these certain properties, then we want to establish a complete classification of pseudo-Riemannian manifolds with these features. Without an explicit description of the curvature tensor, we try to conclude to what extant the special properties of Jacobi operators can determine the metric.

The simplest case in Riemannian geometry is when all the reduced Jacobi operators have the single eigenvalue, which implies a space of constant sectional curvature. The characteristic polynomial of the Jacobi operator which is independent on the unit tangent bundle is the subject of the next simplest case. The key question is whether this condition determines the curvature and metric tensor of such Riemannian manifolds, which are known as Osserman manifolds.

This book is meant to be a monograph on Osserman manifolds and brings together a large number of results published by the author in this topic. In order for as many read-

[1] Robert Osserman (1926–2011), American mathematician
[2] Mikhael Leonidovich Gromov (1943), Russian mathematician

ers as possible to follow these results, the book also contains a detailed introduction to smooth manifolds, as well as the basics of Riemannian and pseudo-Riemannian geometry, and can be used as a textbook for the course in differential geometry. The book can be divided into three imaginary parts: **Part I** Smooth manifolds, consisting of chapters 1–3, **Part II** Pseudo-Riemannian geometry, consisting of chapters 4–7, and **Part III** Osserman manifolds, consisting of chapters 8–10.

Part I (Chapters 1–3) is an introduction to smooth manifolds. We introduce the basic mathematical objects, notation and terminology which will be used throughout this book. The material of this part borrows from many sources, including the standard texts on manifolds and differential geometry: O'Neill[3] [96], Jeffrey Lee[4] [76], John Lee[5] [78], Gross[6] and Meinrenken[7] [65], Gallier[8] and Quaintance[9] [59], Morita[10] [86, Chapter 1], Tu[11] [114], Antić[12] [2].

Part II (Chapters 4–7) is an introduction to a pseudo-Riemannian geometry. An excellent reference for the classical treatment of pseudo-Riemannian geometry is the book by O'Neill [96]. We also recommend Jeffrey Lee [76]. Some standard texts for Riemannian geometry are given by John Lee [79] and do Carmo[13] [31]. We insist on the indefinite case, so we find valuable texts by Meinrenken [83] and Clark[14] [38]. For the submanifold theory we recommend Dajczer[15] and Tojeiro[16] [41].

Part III (Chapters 8–10) deals with the study of Osserman manifolds and related tensors. The basic reference on Osserman manifolds is certainly the monograph written by García-Río[17], Kupeli[18], and Vázquez-Lorenzo[19] [62]. We can also recommend the books written by Gilkey[20] [54, 55].

Belgrade, September 2024
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[3]Barrett O'Neill (1924–2011), American mathematician
[4]Jeffrey M Lee (1956), American mathematician
[5]John M Lee (1950), American mathematician
[6]Gal Gross, Canadian mathematician
[7]Eckhard Meinrenken, German-Canadian mathematician
[8]Jean Henri Gallier (1949), French-American mathematician
[9]Jocelyn Quaintance, америчка математичарка
[10]Shigeyuki Morita (1946), Japanese mathematician
[11]Loring Wuliang Tu (1952), Taiwanese-American mathematician
[12]Miroslava Antić (1978), Serbian mathematician
[13]Manfredo Perdigão do Carmo (1928–2018), Brazilian mathematician
[14]Pete Louis Clark (1976), American mathematician
[15]Marcos Dajczer (1948), Argentinian-Brazilian mathematician
[16]Ruy Tojeiro, Brazilian mathematician
[17]Eduardo García-Río, Spanish mathematician
[18]Demir Nuri Kupeli (1957), Turkish mathematician
[19]Ramón Vázquez-Lorenzo, Spanish mathematician
[20]Peter Belden Gilkey (1946), American mathematician

CONTENTS

Chapter 1

SMOOTH MANIFOLDS AND MAPS

One of the most significant events in the history of geometry was on June 10, 1854, when Bernhard Riemann[1] held a public lecture entitled "On the hypotheses which lie at the foundations of geometry" (*Über die Hypothesen, welche der Geometrie zu Grunde liegen*) at the Philosophical Faculty at Göttingen.

In 1851, Riemann completed his doctoral dissertation (it was on the foundations of complex analysis) under Gauss's[2] supervision. The next step in his academic career was to qualify as a privatdocent, a lecturer who received no salary, but was merely forwarded fees paid by these students who elected to attend his lectures.

This position was obtained through the process of habilitation, where candidates were asked to submit an inaugural paper (*habilitationsschrift*) as well as to hold a public lecture (*habilitationsvortrag*). An inaugural lecture topic was chosen by the faculty from a list of three proposals made by the candidate. The first two topics which Riemann submitted were ones on which he had already worked. However, contrary to tradition, Gauss passed over the first two and picked the third topic, about the foundations of geometry.

Riemann worked hard to make the lecture understandable to non-mathematicians in the audience (there were only a few geometricians), so he had a great presentation in which the ideas were clearly defined without the help of analytic techniques. Although Gauss was very impressed since the lecture surpassed all his expectations, the material was presented orally and without technical details, so it is not surprising that the lecture did not achieve the immediate impact on the mathematical world. However, the first publication of this lecture (by Dedekind[3] in 1868 [106], two years after Riemann's death) caused significant reactions of mathematicians and became a milestone in the history of geometry.

At that time was the investigation concerning Euclid's[4] fifth postulate, looking for axioms to solidly define the basic space of geometry. There was also the quest of the structure of the space we are living in, and some mathematicians were starting to think that other structures might exist. This led us to the non-Euclidean geometries dealt with by Bolyai[5] and Lobachevsky[6] (before them pioneering works had Saccheri[7] and Omar Khayyam[8]), which suddenly become only special cases of more general theory.

Riemann was the first to give a comprehensive contribution to the generalisation of the idea of surface to higher dimensions. His vision set up a new concept of space for which he used the word *mannigfaltigkeit*, which was first translated by Clifford[9] as *manifoldness*

[1]Georg Friedrich Bernhard Riemann (1826–1866), German mathematician
[2]Johann Carl Friedrich Gauß (1777–1855), German mathematician
[3]Richard Dedekind (1831–1916), German mathematician
[4]Euclid of Alexandria (fl. 300 BCE), Greek mathematician
[5]János Bolyai (1802–1860), Hungarian mathematician
[6]Nikolai Ivanovich Lobachevsky (1792–1856), Russian mathematician
[7]Giovanni Girolamo Saccheri (1667–1733), Italian Jesuit priest and mathematician
[8]Omar Khayyam (1048–1131), Persian mathematician, philosopher, and poet
[9]William Kingdon Clifford (1845–1879), English mathematician and philosopher

[107], while the word manifold was later established. The famous Riemann's lecture and the story about it can be found at Spivak[10] [109, Chapter 4].

Manifolds are the fundamental objects of Differential Geometry. However, Riemann did not have a precise definition of the concept. At that time there were technical difficulties to give a formal definition, because the general notion of topological space was not defined before Fréchet[11] (1906 [48]) and Hausdorff[12] (1914 [66]). Poincaré[13] in 1895 [100] introduced the idea of a manifold atlas and gave a definition of a manifold which served as a precursor to the modern concept of a manifold. However, the first rigorous axiomatic definition of manifolds was given by Veblen[14] and Whitehead[15] in 1931 [115] and 1932 [116].

Manifolds are all around us in many guises and these can be seen as higher dimensional generalisations of smooth curves and smooth surfaces. Generally speaking, these are geometrical objects that locally look like some Euclidean space $\mathbb{R}^n$, and on which one can do differential and integral calculus. Of course, the basic examples of manifolds are Euclidean spaces themselves, smooth plane curves (such as ellipses, hyperbolas, and parabolas), and smooth surfaces (such as ellipsoids, hyperboloids, paraboloids, tori). Higher-dimensional examples include the unit n-sphere in $\mathbb{R}^{n+1}$ and graphs of smooth maps between Euclidean spaces.

In contrast to mentioned examples, a manifold does not always appear in a well known space. In general it is rather difficult to think of it as a geometric figure, since it lives in a very abstract environment. However, when we find local coordinates on such a set and study the relationship among them, it often happens that a hidden geometric structure gradually comes to light. Since we want to involve as many objects as possible, it is unavoidable for the definition of manifolds to be abstract. Though, once an object is recognized to be a manifold based on this abstract definition, it appears in a known space through the coordinates, and turns out to be a very practical object.

The study of manifolds involves topology, so we assume that the reader is familiar with the definition and basic properties of topological spaces. We can consider a topological space with certain properties that tell us what exactly means that it locally looks like an Euclidean space. However, this resemblance to an Euclidean space should be sharp enough to allow partial differentiation and consequently all important features of differential and integral calculus on a manifold. Once we have established the calculus on a manifold, many concrete applications are possible. We can compute a volume (by integration) or a curvature (by formulas involving second derivatives), we can work in classical mechanics (where we solving systems of ordinary differential equations) or general relativity (where we solving systems of partial differential equations).

1.1 Locally Euclidean spaces

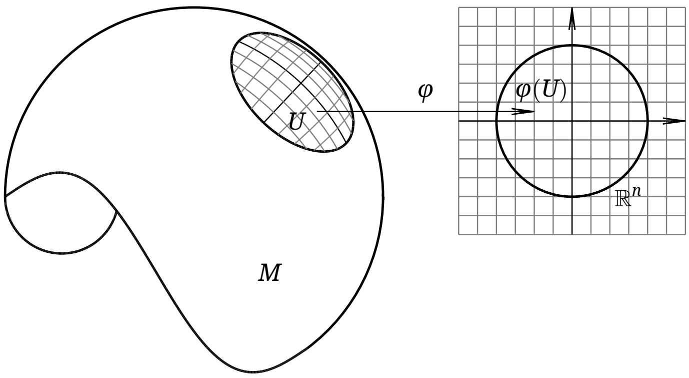
One can imagine a manifold as a dark room with a lamp available. At any given time, the lamp will permit us to look at a certain region of the room only, giving us a local idea about how the room looks like. The lamp here is an Euclidean space which comes equipped with a nice coordinate system.

In studying the geography of Earth as a whole, it is convenient to use geographical maps (charts), which provide a scaled-down representation of some portions of Earth on a flat

[10] Michael David Spivak (1940), American mathematician
[11] Maurice René Fréchet (1878–1973), French mathematician
[12] Felix Hausdorff (1868–1942), German mathematician
[13] Jules Henri Poincaré (1854–1912), French mathematician
[14] Oswald Veblen (1880–1960), American mathematician
[15] John Henry Constantine Whitehead (1904–1960), British mathematician

sheet of paper. Maps are often preferred over globes because they are more practical for everyday use. A collection of geographical maps such that every point of Earth is shown in at least one map is called an atlas and it gives an excellent description of the entire planet, without its actual visualization in three-dimensional space.

Let us formalize this idea of local coordinates for an arbitrary nonempty set M. A ***chart*** or a ***local coordinate system*** of ***dimension*** $n \in \mathbb{N}_0$ on the set M is an injective map $\varphi\colon U \to \mathbb{R}^n$, where $U \subseteq M$ and $\varphi(U) \subseteq \mathbb{R}^n$ is a nonempty open set. The domain U of a chart φ is called a ***coordinate neighbourhood***, and because of the frequent use, that chart φ is traditionally recorded as the ordered pair (U, φ). We say that (U, φ) is a ***chart at*** $p \in M$ if $p \in U$, and if $\varphi(p) = 0 \in \mathbb{R}^n$ additionally holds we say that the chart is ***centred*** at p. The inverse $\varphi^{-1}\colon \varphi(U) \to U$ of a chart φ is called a ***local parametrization***.



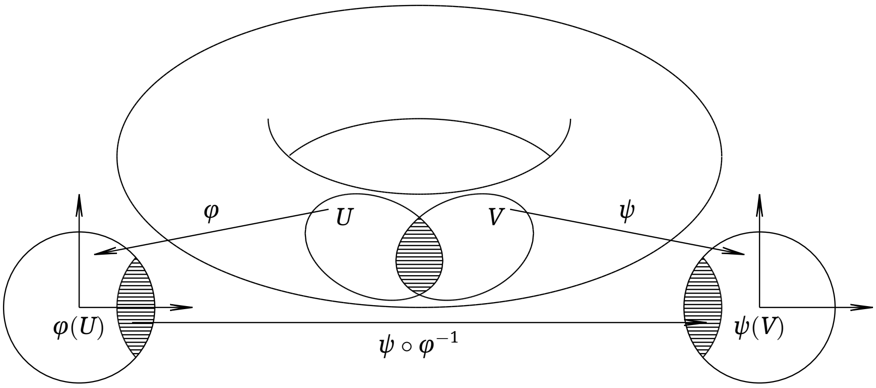
Throughout this book, $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ denotes the natural projection defined by $\pi_i(u_1, \ldots, u_n) = u_i$ for every $1 \leq i \leq n$. The ***coordinate functions*** of an n-dimensional chart (U, φ) on M are the functions $x_i\colon U \to \mathbb{R}$ defined by $x_i = \pi_i \circ \varphi$ for $1 \leq i \leq n$. Their values at some point $p \in U$ are the ***coordinates***, while we have

$$\varphi(p) = (x_1(p), \ldots, x_n(p)).$$

If $U \subseteq \mathbb{R}^m$ is an open subset of a Euclidean space, we say that a function $f\colon U \to \mathbb{R}^n$ is of class C^r on U for $r \in \mathbb{N}$, if each of its component functions $\pi_i \circ f$ has continuous partial derivatives of order r. If f is of class C^r for every $r \in \mathbb{N}$, then it is of class C^∞, and then we say that f is ***smooth*** or ***differentiable***. Functions of class C^0 are continuous, and those of class C^ω are analytic.

A function between open subsets of Euclidean spaces is a C^r-diffeomorphism if it is bijective and both it and its inverse are of class C^r. We allow a choice $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, but most often we use $r = \infty$, where a C^∞-diffeomorphism we simply call a ***diffeomorphism***, while a C^0-diffeomorphism is actually a ***homeomorphism***.

The link between two charts (U, φ) and (V, ψ) on a set M is established by the ***transition function***, which is the bijective map $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$. We say that two charts are C^r-***compatible*** if their transition function is a C^r-diffeomorphism (we assume that then both domain and codomain are open sets). In the special case when the subsets U and V are disjoint, we assume that the charts are trivially C^r-compatible.



A collection of charts on the set M that are mutually C^r-compatible and such that the union of all coordinate neighbourhoods is equal to M (every point of M is contained in the domain of some chart) is called a ***C^r-atlas*** for M. If each chart in a C^r-atlas is of dimension $n \in \mathbb{N}_0$, then n is the ***dimension*** of that atlas.

Since all functions of class C^r for $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ are certainly continuous (of class C^0), every atlas we consider is at least C^0-atlas, that is, a ***topological atlas***. We say that a set M is a ***locally Euclidean space*** of dimension n if it has a topological atlas of dimension n.

It turns out that a locally Euclidean space M becomes a topological space in a natural way. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \Lambda\}$ be a topological atlas of dimension n for a set M. The ***atlas topology*** of $\mathcal{A}$ is the topology on M where we have

$$V \subseteq M \text{ is open } \iff (\forall \alpha \in \Lambda)\, \varphi_\alpha(U_\alpha \cap V) \subseteq \mathbb{R}^n \text{ is open.} \tag{1.1}$$

Since $\varphi_\alpha(\emptyset) = \emptyset$ and $\varphi_\alpha(U_\alpha \cap M) = \varphi_\alpha(U_\alpha)$ are open, $\emptyset$ and M are open in M. The intersection is distributive over the union, while according to Lemma A.1 the image of the union is equal to the union of images, so we have

$$\varphi_\alpha\Big(U_\alpha \cap \bigcup_\beta V_\beta\Big) = \varphi_\alpha\Big(\bigcup_\beta (U_\alpha \cap V_\beta)\Big) = \bigcup_\beta \varphi_\alpha(U_\alpha \cap V_\beta),$$

from which it can be seen that the union $\bigcup_\beta V_\beta$ of open sets V_β is an open set. Since φ_α is injective, by Lemma A.1 the image of the intersection is equal to the intersection of images, so we have

$$\varphi_\alpha\Big(U_\alpha \cap \bigcap_\beta V_\beta\Big) = \varphi_\alpha\Big(\bigcap_\beta (U_\alpha \cap V_\beta)\Big) = \bigcap_\beta \varphi_\alpha(U_\alpha \cap V_\beta),$$

from which it can be seen that the finite intersection $\bigcap_\beta V_\beta$ of open sets V_β is an open set, which proves that the formula (1.1) really defines a topology on M.

Lemma 1.1. *If $\mathcal{A}$ is a topological atlas of fixed dimension for a set M, then the atlas topology of $\mathcal{A}$ is a unique topology by which M becomes a topological space such that all charts of the atlas $\mathcal{A}$ are homeomorphisms onto their image.*

Proof. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \Lambda\}$ be a topological space of dimension n for a set M that becomes a topological space. As we require that $\varphi_\alpha \colon U_\alpha \to \varphi_\alpha(U_\alpha)$ are homeomorphisms, the family $\{U_\alpha\}_{\alpha \in \Lambda}$ becomes an open cover of M, so $V \subseteq M$ is open if and only if $U_\alpha \cap V \subseteq M$ is open for every $\alpha \in \Lambda$. Therefore, a required topology on M can only be the atlas topology given by (1.1).

If $W \subseteq \varphi_\alpha(U_\alpha)$ is open in $\mathbb{R}^n$, then $\varphi_\beta(U_\beta \cap \varphi_\alpha^{-1}(W)) = \varphi_\beta \circ \varphi_\alpha^{-1}(\varphi_\alpha(U_\alpha \cap U_\beta) \cap W)$ is open in $\mathbb{R}^n$, so $\varphi_\alpha^{-1}(W)$ is open in M, and thus φ_α is continuous. If $V \subseteq U_\alpha$ is open in M, then $\varphi_\alpha(V) = \varphi_\alpha(U_\alpha \cap V)$ is open in $\mathbb{R}^n$, so φ_α^{-1} is continuous and therefore φ_α is a homeomorphism onto its image. $\square$

Some authors allow that the dimension of a chart from a C^r-atlas is not fixed in advance. However, according to the dimension invariance theorem (Theorem A.5), a nonempty subset of $\mathbb{R}^m$ is not homeomorphic to an open subset of $\mathbb{R}^n$ except for $m = n$. Therefore, n is constant on the connected component of M, but it is possible for each component to have a different dimension. We do not want to use the word manifold for spaces of mixed dimension, such as the disjoint union of different Euclidean spaces, so we require that a C^r-atlas has a fixed dimension.

Lemma 1.1 shows that the basic manifold concept implies a locally Euclidean topological space M. However, this approach has a minor technical problem as there are many choices of a topological atlas that give the same topological space.

For a chart (U, φ) on a set M we say that is C^r-***compatible*** with a C^r-atlas $\mathcal{A}$ for M if it is C^r-compatible with each chart from $\mathcal{A}$, which happens if and only if the union $\mathcal{A} \cup \{(U, \varphi)\}$ is also a C^r-atlas.

Consider a C^r-atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \Lambda\}$ of dimension n for a set M. For $V, W \subseteq M$ and an arbitrary map $\psi \colon V \to \mathbb{R}^n$, since the intersection is distributive over the union, and the formulas (A.1) and (A.2) from Lemma A.1 hold for the injective φ_α, we obtain

$$\begin{aligned} \psi(V \cap W) &= \bigcup_\alpha \psi(U_\alpha \cap V \cap W) = \bigcup_\alpha (\psi \circ \varphi_\alpha^{-1})(\varphi_\alpha(U_\alpha \cap V \cap W)) \\ &= \bigcup_\alpha (\psi \circ \varphi_\alpha^{-1})(\varphi_\alpha(U_\alpha \cap V) \cap \varphi_\alpha(U_\alpha \cap W)). \end{aligned} \tag{1.2}$$

Let (V, ψ) and (W, θ) be charts on the set M that are C^r-compatible with $\mathcal{A}$ Since the transition function $\psi \circ \varphi_\alpha^{-1}$ is a homeomorphism (as a C^r-diffeomorphism) for each $\alpha \in \Lambda$, the formula (1.2) shows that $\psi(V \cap W)$ is open in $\mathbb{R}^n$, as well as $\theta(V \cap W)$. For the transition function $\theta \circ \psi^{-1} \colon \psi(V \cap W) \to \theta(V \cap W)$ we have

$$\theta \circ \psi^{-1}\restriction_{\psi(U_\alpha \cap V \cap W)} = (\theta \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \psi^{-1}),$$

so $\theta \circ \psi^{-1}$ is of class C^r on $\psi(U_\alpha \cap V \cap W)$ for every $\alpha \in \Lambda$, and thus of class C^r on $\psi(V \cap W)$. Similarly, $\psi \circ \theta^{-1}$ is of class C^r on $\theta(V \cap W)$, so the charts (V, ψ) are (W, θ) C^r-compatible, which proves the following lemma.

Lemma 1.2. *if two charts are C^r-compatible with a C^r-atlas, then they are mutually C^r-compatible.*

A ***complete*** C^r-atlas is a maximal C^r-atlas in the sense that it is not properly contained in any larger C^r-atlas. In other words, every chart that is compatible with a complete C^r-atlas is already in the atlas.

Lemma 1.3. *Every C^r-atlas for a set M generates a unique complete C^r-atlas for M that contains it.*

Proof. Let $\mathcal{A}$ be a C^r-atlas for M, and let $\overline{\mathcal{A}}$ be the family of all charts on M that are C^r-compatible with $\mathcal{A}$. Obviously $\mathcal{A} \subseteq \overline{\mathcal{A}}$, while by Lemma 1.2 all charts from $\overline{\mathcal{A}}$ are mutually C^r-compatible. Every chart that is C^r-compatible with $\overline{\mathcal{A}}$ is also C^r-compatible with $\mathcal{A}$, so it is already in $\overline{\mathcal{A}}$, which proves that $\overline{\mathcal{A}}$ is a complete C^r-atlas. □

The complete smooth atlas allows great flexibility in the choice of a chart, which is discussed in the following lemma.

Lemma 1.4. *Let (U, φ) ba a chart from a complete C^r-atlas $\overline{\mathcal{A}}$ of dimension n on a set M. If a nonempty subset $V \subset U$ is such that $\varphi(V) \subset \mathbb{R}^n$ is open, then $(V, \varphi\restriction_V) \in \overline{\mathcal{A}}$. If a map $f \colon \varphi(U) \to f(\varphi(U)) \subseteq \mathbb{R}^n$ is a C^r-diffeomorphism, then $(U, f \circ \varphi) \in \overline{\mathcal{A}}$.*

Proof. Since $\varphi\restriction_V(V) = \varphi(V)$ is open, and the transition function $\varphi\restriction_V \circ \psi^{-1} = (\varphi \circ \psi^{-1})\restriction_{\psi(V\cap W)}$ is a C^r-diffeomorphism for each $(W, \psi) \in \overline{\mathcal{A}}$, we have $(V, \varphi\restriction_V) \in \overline{\mathcal{A}}$. Since $f \circ \varphi(U)$ is open, and $(f \circ \varphi) \circ \psi^{-1} = f \circ (\varphi \circ \psi^{-1})$ is a C^r-diffeomorphism for each $(W, \psi) \in \overline{\mathcal{A}}$, we have $(U, f \circ \varphi) \in \overline{\mathcal{A}}$. □

A frequent use of Lemma 1.4 refers to the case where a C^r-diffeomorphism f is the translation given by the formula $f(x) = x - \varphi(p)$, which maps $\varphi(p)$ to $0 \in \mathbb{R}^n$. Consequently, for every point $p \in M$ there exists a chart on M that is centred at p.

Of course, the topologies on M generated by Lemma 1.1 using an arbitrary topological atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ for M and using its complete topological atlas $\overline{\mathcal{A}} \supseteq \mathcal{A}$, are equal, since if $\varphi_\alpha(U_\alpha \cap W) \subseteq \mathbb{R}^n$ are open for each α, then for every $(V, \psi) \in \overline{\mathcal{A}}$, the formula (1.2) implies that $\psi(V \cap W) \subseteq \mathbb{R}^n$ is open.

In this sense, a locally Euclidean space of dimension n is an ordered pair $(M, \overline{\mathcal{A}})$, where M is a set, and $\overline{\mathcal{A}}$ is a complete topological atlas of dimension n for M, and therefore it is a topological space whose topology is determined by an arbitrary topological atlas $\mathcal{A} \subseteq \overline{\mathcal{A}}$, while all charts of that atlas are homeomorphisms onto their image. However, a topological space does not have to be locally Euclidean, as we see in the next example.

Example 1.1. Consider the crossing lines $M = \{(x, y) \in \mathbb{R}^2 : x^2 = y^2\} \subset \mathbb{R}^2$ with the subspace topology. If $\varphi\colon V \to \varphi(V) \subseteq \mathbb{R}^n$ is a homeomorphism for a connected neighbourhood $V \subseteq M$ of the point $0 = (0, 0)$, then $V \setminus \{0\}$ and $\varphi(V) \setminus \{\varphi(0)\}$ are homeomorphic. However, $V \setminus \{0\}$ has four connected components, whereas $\varphi(V) \setminus \{\varphi(0)\}$ has two components when $n = 1$ or only one component when $n > 1$, and therefore M is not locally Euclidean. △

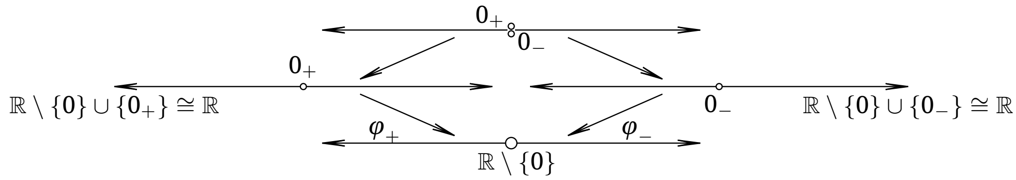
Usually, additional technical assumptions on the topological space are made to avoid some pathological examples that do not arise in practice. We naturally assume that M is a Hausdorff space, which means that for any two distinct points $p, q \in M$ there exist disjoint open subsets $P, Q \subset M$ such that $p \in P$ and $q \in Q$. In that case, instead of charts $(U, \varphi), (V, \psi) \in \overline{\mathcal{A}}$ with $p \in U$ and $q \in V$, by Lemma 1.4 we can take their restrictions $(U \cap P, \varphi\restriction_{U\cap P}), (V \cap Q, \psi\restriction_{V\cap Q}) \in \overline{\mathcal{A}}$ which have disjoint coordinate neighbourhoods. The ***Hausdorff condition*** is satisfied if for every two distinct points $p, q \in M$ there are charts $(U, \varphi), (V, \psi) \in \overline{\mathcal{A}}$ with $p \in U$ and $q \in V$ such that $U \cap V = \emptyset$, which is equivalent to the fact that the induced topology is Hausdorff.

It is useful to know that the Hausdorff condition is automatically satisfied for points from the same coordinate neighbourhood, which we see in the following lemma.

Lemma 1.5. *Let $(M, \overline{\mathcal{A}})$ be a locally Euclidean space and $(U, \varphi) \in \overline{\mathcal{A}}$. For distinct points $p, q \in U$ there exist $(V_p, \psi_p), (V_q, \psi_q) \in \overline{\mathcal{A}}$ such that $p \in V_p$, $q \in V_q$, and $V_p \cap V_q = \emptyset$.*

Proof. Distinct points $\varphi(p), \varphi(q) \in \varphi(U) \subseteq \mathbb{R}^n$ can be separated by open sets $P \ni \varphi(p)$ and $Q \ni \varphi(q)$, where $P \cap Q = \emptyset$, so we obtain disjoint sets $V_p = \varphi^{-1}(P) \ni p$ and $V_q = \varphi^{-1}(Q) \ni q$, while the required charts are the corresponding restrictions of the chart φ given by Lemma 1.4. □

Example 1.2. The ***line with two origins*** $M = \{(x, y) : x \in \mathbb{R}, y \in \{-1, 1\}\}/_\sim$ is the quotient space, where the equivalence relation $\sim$ is given by $(x, -1) \sim (x, 1)$, if $x \neq 0$. We can see it as $M = (\mathbb{R} \setminus \{0\}) \cup \{0_+, 0_-\}$, so let $\mathcal{A} = \{(U_+, \varphi_+), (U_-, \varphi_-)\}$, where $\varphi_\pm\colon M \setminus \{0_\mp\} \to \mathbb{R}$ are given by $\varphi_\pm(x) = x$ for $x \in \mathbb{R} \setminus \{0\}$ and $\varphi_\pm(0_\pm) = 0$. The transition function $\varphi_- \circ \varphi_+^{-1} = \mathbb{1}_{\mathbb{R}\setminus\{0\}}$ is a homeomorphism, so $\mathcal{A}$ is a topological atlas for M. Additionally, the charts φ_+ and φ_- are homeomorphisms with respect to the quotient topology of the space M, which is precisely the topology that, according to Lemma 1.1 will be generated by the atlas $\mathcal{A}$. Every neighbourhood of the point $0_\pm$ contains $\varphi_\pm^{-1}(-\varepsilon, \varepsilon) = (-\varepsilon, 0) \cup \{0_\pm\} \cup (0, \varepsilon)$ for some small $\varepsilon > 0$, so the points 0_+ and 0_- cannot be separated by open sets. The line with two origins is a locally Euclidean space of dimension 1 which does not satisfy the Hausdorff condition. △



An additional requirement is that M is second countable (satisfying the second axiom of countability), which means that its topology has a countable basis. In chart terms, this means that the complete atlas $\overline{\mathcal{A}}$ has a countable subatlas $\mathcal{A} \subseteq \overline{\mathcal{A}}$. Namely, for every $p \in M$ there is a chart at p which by Lemma 1.4 can be replaced by restriction to some basis set, so if M has a countable basis, then it also has a countable subatlas. Conversely, since open subsets of $\mathbb{R}^n$ have a countable basis, if there is a countable subatlas $\mathcal{A}$ then M is a countable union of open sets with a countable basis, ant it itself has a countable basis.

Example 1.3. An ordinary line $\mathbb{R}$ can be constructed in an unusual way as $L = I \times [0, 1)$ by taking the half-open intervals indexed by $I = \mathbb{Z}$ equipped with the topology induced by the lexicographical order on L. Let ω_1 be the first uncountable ordinal. Since every set can be well-ordered (according to Zermelo[16], where the axiom of choice is used), we can naturally introduce the well-ordered set $I = -\omega_1 \cup \{0\} \cup \omega_1$ on the model of $\mathbb{Z} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}$. The ***long line*** (or ***Alexandroff line***[17]) is $L = I \times [0, 1)$ with the lexicographical order topology. The set $\{x \in \omega_1 : x < \alpha\}$ is countable for every $\alpha \in \omega_1$, and so is the set $\{x \in I : \alpha < x < \beta\}$ for $\alpha, \beta \in I$, which implies that every $\alpha \in I$ has its successor $S(\alpha) \in I$. For each $\alpha \in I$ we introduce homeomorphisms $\varphi_\alpha \colon U_\alpha \to (0, 2)$, where $U_\alpha = (\{\alpha\} \times (0, 1)) \cup (\{S(\alpha)\} \times [0, 1))$, given by $\varphi_\alpha(\alpha, t) = t$ and $\varphi_\alpha(S(\alpha), t) = t + 1$, so L is a locally Euclidean space of dimension 1 with the topological atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$. The long line has no countable basis because $\{(\alpha, t) \in L : t \in (1/4, 3/4)\}_{\alpha \in I}$ is an uncountable family of disjoint nonempty open subsets of L. △

The second countable atlas condition, among other things, can be used in proofs by mathematical induction. Most of our examples have an atlas consisting of finitely many charts, and it is not hard to notice that if M is compact, instead of a complete atlas we can take a finite subatlas.

Finally, we introduce the definition of ***C^r-manifold*** of dimension $n \in \mathbb{N}_0$, which is the set M together with a complete C^r-atlas of dimension n that has a countable subatlas and satisfies the Hausdorff condition. In general, it is not convenient to explicitly define a complete because it contains too many charts. Therefore, along with the set M we usually specify some countable C^r-atlas $\mathcal{A}$ (which solves the countable basis problem), and we say that $(M, \mathcal{A})$ is a C^r-manifold, assuming that it is the C^r-manifold $(M, \overline{\mathcal{A}})$, where $\overline{\mathcal{A}} \supseteq \mathcal{A}$ is the complete C^r-atlas generated by Lemma 1.3.

Example 1.4. Of course, the basic example of a C^r-manifold of dimension n is the Euclidean space $\mathbb{R}^n$. It is locally Euclidean of dimension n, which is justified by the C^r-atlas with the single chart $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})$, and the Hausdorff condition is satisfied because $\mathbb{R}^n$ is Hausdorff as a metric space. △

1.2 Topological properties of manifolds

A ***topological manifold*** of dimension n or ***topological n-manifold*** is a C^0-manifold of dimension n. First of all, it is a topological space in accordance with Lemma 1.1. A ***chart*** (U, φ) of dimension n on a topological space $M \supseteq U$ is every homeomorphism

[16]Ernst Friedrich Ferdinand Zermelo (1871–1953), German logician and mathematician
[17]Pavel Sergeyevich Alexandrov (1896–1982), Soviet mathematician

$\varphi\colon U \to \varphi(U) \subseteq \mathbb{R}^n$. Any two charts of the same dimensions on a topological space are C^0-compatible because their transition function is a homeomorphism as a composition of homeomorphisms. Therefore, it is sufficient to find charts of dimension n on M whose domains form an open cover of M to obtain a topological atlas and showed that the topological space M is locally Euclidean of dimension n.

If a topological space M as a set has a topological atlas $\mathcal{A}$ by which M becomes a topological manifold whose topology induced by the atlas $\mathcal{A}$ agrees with the topology of M, then a complete topological atlas $\overline{\mathcal{A}} \supseteq \mathcal{A}$ is uniquely determined, as well as a topological manifold. This allows us to describe a topological manifold without any specific atlas, which is actually the standard definition of almost all authors.

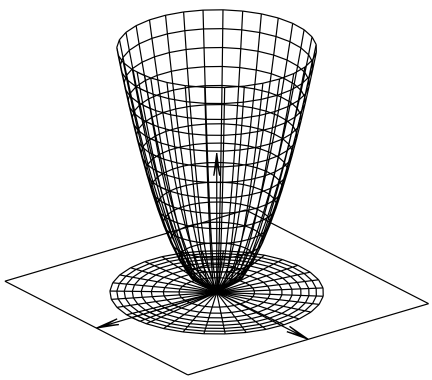
Lemma 1.6. *A topological n-manifold is a second countable Hausdorff topological space that is locally Euclidean of dimension n.*

Although there are some important locally Euclidean spaces that do not satisfy the additional conditions (see Example 1.2 and Example 1.3), it can be said that almost all ordinary geometrical figures satisfy them, and it is extremely rare to meet a space in nature without all of these conditions.

The Hausdorff condition has two essential properties. First, that the limit of a convergent sequence is unique. Second, that compact subsets are closed sets (especially, any finite set is closed) and thus have complements that are open. Hausdorffness together with the second countability implies the existence of partitions of unity, a very important tool in a manifold theory, and would suffice to justify these conditions. It follows that these conditions ensure that every manifold embeds in some finite-dimensional Euclidean space, and it can be endowed with a Riemannian metric.

The simplest examples of manifolds have an atlas consisting of only one chart. Then we automatically have a countable basis, while the Hausdorff condition follows directly from Lemma 1.5.

Example 1.5. Let $M = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\} \subset \mathbb{R}^3$ be the circular paraboloid with the subspace topology inherited from $\mathbb{R}^3$. It is enough to observe the single chart $\varphi\colon M \to \mathbb{R}^2$ given by $\varphi(x, y, z) = (x, y)$ that is the normal projection onto the plane $z = 0$, which is a homeomorphism between the paraboloid and the plane. Therefore, the paraboloid M is a topological 2-manifold. △



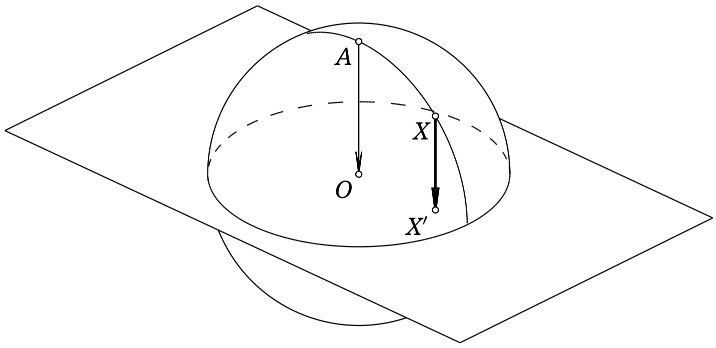
Example 1.6. For any open set $U \subseteq \mathbb{R}^n$, the ***graph*** of a continuous function $f\colon U \to \mathbb{R}^m$ is the subset of $\mathbb{R}^n \times \mathbb{R}^m$ defined by $\Gamma(f) = \{(x, y) : x \in U, y = f(x)\}$. The natural projection $\varphi\colon \Gamma(f) \to U$ given by $\varphi(x, y) = x$ is continuous, invertible, and its inverse $\varphi^{-1}(x) = (x, f(x))$ is continuous. Since φ is a homeomorphism, $\Gamma(f)$ with the atlas $\{(\Gamma(f), \varphi)\}$ is a locally Euclidean space, so $\Gamma(f)$ with the subspace topology inherited from $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ is a topological n-manifold. We can notice that the paraboloid from Example 1.5 is just one special case of graph. △

Of course, we cannot expect that a topological space M always has an atlas with a single chart (M, φ). If M is compact, then so is its image $\varphi(M) \subseteq \mathbb{R}^n$ under homeomorphism φ, but a nonempty open subset of Euclidean space is never compact, so we need at least two charts. In case when an atlas has more charts, it is necessary to check the Hausdorff condition, but in practice this is often straightforward. For example, if M is a topological subspace of Euclidean space, then the Hausdorff condition (as well as the second countability) is automatically inherited.

Example 1.7. The unit circle $\mathbf{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ with the subspace topology is a topological 1-manifold, since the map $\psi\colon \mathbb{R} \to \mathbf{S}^1$ given by $\psi(t) = (\cos t, \sin t)$ has restrictions to small open sets which are homeomorphisms. More generally, the n-dimensional sphere of unit radius centred at the origin,

$$\mathbf{S}^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1},$$

is called the ***n*-sphere**. Hausdorffness and second countability for $\mathbf{S}^n$ with the subspace topology are inherited from $\mathbb{R}^{n+1}$. An arbitrary point $(a_1, \dots, a_{n+1}) \in \mathbf{S}^n$ can be assigned to the open hemisphere $U = \{(x_1, \dots, x_{n+1}) \in \mathbf{S}^n : a_1 x_1 + \dots + a_{n+1} x_{n+1} > 0\}$. The orthogonal projection of U onto the hyperplane given by $a_1 x_1 + \dots + a_{n+1} x_{n+1} = 0$ has formulas $(x_1, \dots, x_{n+1}) \mapsto (x_1', \dots, x_{n+1}')$ where $x_i' = x_i - a_i(a_1 x_1 + \dots + a_{n+1} x_{n+1})$ for $1 \le i \le n+1$. This projection determines a homeomorphism between the hemisphere U and the disc described by the set $\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_{n+1}^2 < 1,\ a_1 x_1 + \dots + a_{n+1} x_{n+1} = 0\}$, so the n-sphere is a topological n-manifold by Lemma 1.6. △



Manifolds, as topological spaces, share many important properties with Euclidean spaces. Since a topological manifold is locally Euclidean, local properties of Euclidean spaces (such as local compactness or local path-connectedness) are inherited. Usually, the open balls of radius $r > 0$ and centre $p \in \mathbb{R}^n$ form a topology basis of the Euclidean space $\mathbb{R}^n$, and we use the notation

$$B_r(p) = \{x \in \mathbb{R}^n : \|x - p\| < r\} \subset \mathbb{R}^n.$$

Lemma 1.7. *Every topological manifold is locally compact and locally path-connected.*

Proof. Let M be a topological n-manifold, and $U \subseteq M$ be a neighbourhood of a point $p \in M$. Since M is locally Euclidean, there exists a chart at $p \in M$, and this chart can be modified according to Lemma 1.4 so there is a chart (V, φ) centred at $p \in M$ such that $V \subseteq U$. The set $\varphi(V) \ni 0$ is open, so there is a sufficiently small $\varepsilon > 0$ such that $B_{2\varepsilon}(0) \subset \varphi(V)$. The ball $B_\varepsilon(0) \subset \mathbb{R}^n$ is path-connected and relatively compact, and so is its homeomorphic preimage $\varphi^{-1}(B_\varepsilon(0)) \subset \varphi^{-1}(\overline{B_\varepsilon(0)}) \subset V \subseteq U$. Thus M is locally path-connected and locally compact. □

If M is locally compact and Hausdorff, then all compact sets in M are closed. Hence, every point $p \in M$ lies in some open set whose closure is compact.

Lemma 1.8. *A topological manifold is connected if and only if it is path-connected.*

Proof. Since every path-connected topological space is connected, and every topological manifold is locally path-connected (Lemma 1.7), it suffices to prove that a connected and locally path-connected topological space is path-connected. Since any path-connected component of a locally path-connected space is open, the space itself is a union of disjoint open sets which are path-connected components. Thus, connected components and path-connected components are the same, with the lemma being a special case of this. □

A ***refinement*** of a cover $\mathcal{U}$ of M is a new cover $\mathcal{V}$ of M, such that every $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$. A simple example of refinement is a subcover, but refinements are much more flexible. For example, the covering of a metric space by open balls of radius 2 is refined by the covering by open balls of radius 1. A collection of subsets of M is said to be ***locally finite***, if each point of M has a neighbourhood that intersects only finitely many of the sets in the collection. We say that M is ***paracompact*** if every open cover of M has a locally finite open refinement.

The notion of paracompactness is introduced by Dieudonne[18] [42] as a generalisation of the notion of compactness. Every compact space is, certainly, paracompact, while the most famous counterexample is the long line from Example 1.3, which is not paracompact. Hausdorffness of paracompact spaces extend their properties.

Lemma 1.9. *Every paracompact Hausdorff space is regular.*

Proof. Let M be a paracompact Hausdorff space, $p \in M$, and C is a closed set not containing p. For any $q \in C$, by Hausdorffness, we have disjoint open sets $U_q \ni p$ and $V_q \ni q$. Since $C \subseteq \bigcup_{q \in C} V_q$, the sets V_q and $M \setminus C$ form an open cover of M. By paracompactness of M, there is a locally finite open refinement. Throwing out from this any open subset not intersecting C, we still get a locally finite collection $\mathcal{A}$ of open subsets, each contained in some V_q, that cover C. By local finiteness of $\mathcal{A}$, there exists an open set $W \ni p$ such that there are only finitely many members $A_1, \dots, A_k$ of $\mathcal{A}$ that intersects W. Let $q_1, \dots, q_k$ be points in C such that $A_i \subseteq V_{q_i} \ni q_i$ holds for $1 \le i \le k$. Let us define $U = W \cap U_{q_1} \cap \dots \cap U_{q_k}$ and $V = \bigcup \mathcal{A}$. It is easy to see that U and V are disjoint open sets. Thus $p \in U$ and $C \subseteq V$ can be separated by neighbourhoods and therefore M is regular. □

Moreover, the original theorem of Dieudonne states that every paracompact Hausdorff space is normal, but we do not need that fact. Every locally compact second countable Hausdorff space is paracompact, so this holds in particular for a topological manifold.

Lemma 1.10. *Every topological manifold is paracompact.*

Proof. Let M be a locally compact second countable Hausdorff space. Then there exists a countable basis $\{W_i\}_{i \in \mathbb{N}}$ consisting of relatively compact sets. Put $K_1 = \overline{W_1}$ and assume inductively that compact K_j has been defined for $1 \le j \le n$. Since $\{W_i\}_{i \in \mathbb{N}}$ is an open cover of compact K_n, there is the smallest $m \in \mathbb{N}$ such that $K_n \subset \bigcup_{i=1}^m W_i$, so we define $K_{n+1} = \bigcup_{i=1}^m \overline{W_i}$ which is compact as a finite union of compact sets. Thus we have an ***exhaustion*** of M as a family $\{K_n\}_{n \in \mathbb{N}}$ of compact subsets of M such that $K_n \subset \operatorname{Int}(K_{n+1})$ and $M = \bigcup_{n \in \mathbb{N}} K_n$.

Let $\mathcal{U}$ be an open cover of M for which we seek a locally finite refinement. For $p \in M$ there is $U_p \ni p$ from $\mathcal{U}$ and a minimal $n \in \mathbb{N}$ such that $p \in \operatorname{Int}(K_n)$ and $p \notin \operatorname{Int}(K_{n-1})$. The set $(K_n \setminus \operatorname{Int}(K_{n-1}))$ is a compact subset of open $(\operatorname{Int}(K_{n+1}) \setminus K_{n-2})$, and therefore there exists a finite subfamily that covers it, consists of sets of shape $V_p = (\operatorname{Int}(K_{n+1}) \setminus K_{n-2}) \cap U_p$. The cases $n = 1$ and $n = 2$ need a slight modification, for example $V_p = \operatorname{Int}(K_3) \cap U_p$. Anyway, such V_p can intersects only sets of such shape whose n differs from the original n for no

[18] Jean Dieudonné (1906–1992), French mathematician

more than one, which is a finite number. Hence, the sets of shape V_p make a new basis, which is a locally finite refinement of $\mathcal{U}$. □

Let us remark that some authors define a topological manifold as a locally Euclidean paracompact Hausdorff space. In this case we should know that each connected component of a locally Euclidean paracompact Hausdorff space has a countable basis. Thus, if a locally Euclidean paracompact Hausdorff space has a countable number of components, then it is second countable.

1.3 Smooth manifolds

The definition of topological manifolds is sufficient for studying topological properties, such as compactness and connectedness. However, smoothness is not invariant under homeomorphisms (a circle and a square are homeomorphic, but the square is not smooth), and therefore the topological structure is insufficient to provide the calculus. Therefore, a topological manifold requires an additional structure.

A ***smooth manifold*** or ***differential manifold*** of dimension n is a C^∞-manifold of dimension n. According to the definition, it is a set M furnished with a countable C^∞-atlas $\mathcal{A}$ of dimension n which satisfies the Hausdorff condition. We say that $\mathcal{A}$ is a ***smooth atlas*** for M, and every two charts from $\mathcal{A}$ are ***smoothly compatible*** in the sense that their transition function is a diffeomorphism.

Since our theory predominantly assumes the smoothness of fundamental objects, we often omit the explicit use of the adjective "smooth". Throughout this book, unless explicitly stated otherwise, ***manifold*** will refer to a smooth manifold, ***atlas*** will refer to a smooth atlas, and ***compatible charts*** will imply smooth compatibility. The dimension of a manifold M is denoted by $\dim M$, and if we want to emphasize that $n = \dim M$, we say that M is an ***n-manifold***, or we indicate the dimension through a superscript and write that M^n is a manifold.

According to Lemma 1.1, a smooth atlas $\mathcal{A}$ uniquely determines the topology on M, but in addition to the topological one, we also get an additional structure, called the ***smooth structure***, which allows changing of local coordinates during the transition from one chart to another always be smooth. A smooth structure on M is essentially defined by a complete smooth atlas $\overline{\mathcal{A}}$, but thanks to Lemma 1.3 it is sufficient to have a smooth subatlas $\mathcal{A} \subseteq \overline{\mathcal{A}}$.

By a common abuse of language, we usually speak of a manifold M, where a certain (complete smooth) atlas $\mathcal{A}$ is implicitly understood. Therefore, a chart on M will always mean a chart in the atlas $\mathcal{A}$. Let us survey some common examples of manifolds to which we naturally assign a smooth atlas to define smooth structure that we call the ***standard smooth structure***, which will, unless otherwise explicitly specified, be our default.

Example 1.8. The only neighbourhood of a point p in a 0-manifold M that is homeomorphic to an open subset of $\mathbb{R}^0$ is $\{p\}$ itself, and therefore there is exactly one chart $\varphi\colon \{p\} \to \mathbb{R}^0$. All charts on M are trivially compatible, and every 0-manifold is just a countable discrete space. We can notice that the countable subatlas (second countable) condition from the manifold definition does not allow to consider $\mathbb{R}$ as a 0-manifold. △

Example 1.9. If a topological manifold M has a topological atlas with a single chart, the compatibility condition holds trivially, this atlas is automatically smooth and determines a smooth structure on M. Euclidean space $\mathbb{R}^n$ with the single chart $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})$ from Example 1.4 is a topological n-manifold, and that identity map determines the standard smooth structure on $\mathbb{R}^n$. Also, the graph of any continuous function on an open subset of $\mathbb{R}^n$ is a topological n-manifold with a single chart from Example 1.6, and therefore it is an n-manifold. This shows that many of the familiar surfaces are manifolds, for example an elliptic paraboloid or a hyperbolic paraboloid. △

Example 1.10. Example 1.9 shows the 1-manifold $\mathbb{R}$ with the standard smooth structure is determined by the single chart $(\mathbb{R}, \mathbb{1}_{\mathbb{R}})$. However, we can use another single chart $(\mathbb{R}, \varphi)$, where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is given by $\varphi(x) = x^3$, to determine a new smooth structure on $\mathbb{R}$. Since the transition map $\mathbb{1}_{\mathbb{R}} \circ \varphi^{-1}(y) = y^{1/3}$ is not smooth at the origin, two charts are not compatible and therefore they determine different smooth structures. Thus, we have a new 1-manifold $\widetilde{\mathbb{R}}$ which convinces us that the same topological manifold can have many various smooth structures. △

Every open subset $U \subset \mathbb{R}^n$ can be seen as a manifold using the atlas with the single chart $(U, \mathbb{1}_U)$. Moreover, this kind of construction is possible for any open subset of an arbitrary manifold, as we see in the following examples.

Example 1.11. Let M be an n-manifold with a complete smooth atlas $\overline{\mathcal{A}}$. The natural atlas on an open subset $U \subset M$ consists of the chart $(V, \varphi) \in \overline{\mathcal{A}}$ such that $V \subseteq U$. Each point $p \in U$ is contained in the domain of some chart (V, φ) on M, while $(V \cap U, \varphi{\restriction}_{V\cap U})$ is a chart at $p \in U$ by Lemma 1.4. The second countability and Hausdorff condition are inherited by subspaces, so $U \subset M$ becomes an n-manifold in a natural way, and we call it an ***open submanifold*** of M. △

Example 1.12. For $m, n \in \mathbb{N}$, we denote the vector space of all real $m \times n$ matrices by $\mathbb{R}^{m\times n}$. Since $\mathbb{R}^{m\times n} \cong \mathbb{R}^{mn}$, according to Example 1.9, $\mathbb{R}^{m\times n}$ is a manifold of dimension mn. The ***general linear group*** of degree $n \in \mathbb{N}$ is a group $\mathrm{GL}(n, \mathbb{R})$ of invertible real square matrices of order n, that is,

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\} = \det^{-1}(\mathbb{R} \setminus \{0\}).$$

Since the determinant function $\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous, $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$, and therefore $\mathrm{GL}(n, \mathbb{R})$ is a manifold of dimension n^2 by Example 1.11 as an open submanifold of $\mathbb{R}^{n\times n}$. Similarly, the complex general linear group $\mathrm{GL}(n, \mathbb{C})$, which is a group of invertible $n \times n$ complex matrices of order n, is an open subset of $\mathbb{C}^{n\times n} \cong \mathbb{R}^{2n^2}$ and it is a manifold of dimension $2n^2$. △

Example 1.13. We can generalise Example 1.12 to rectangular matrices of full rank. Let $\mathbb{R}^{m\times n}_m$ be the subset of $\mathbb{R}^{m\times n}$ consists of matrices of rank $m < n$. For any $A \in \mathbb{R}^{m\times n}_m$ there is a submatrix $f_A(A) \in \mathrm{GL}(m, \mathbb{R})$, where $f_A\colon \mathbb{R}^{m\times n}_m \to \mathbb{R}^{m\times m}$ is a concrete projection. By continuity of the determinant function there exists a neighbourhood $U \subset \mathbb{R}^{m\times n}_m$ of matrix A such that $\det(f_A(X)) \neq 0$ holds for all $X \in U$. Hence $\mathbb{R}^{m\times n}_m$ is an open subset of $\mathbb{R}^{m\times n}$, and thus a manifold of dimension mn as an open submanifold of $\mathbb{R}^{m\times n}$. △

In Example 1.7 we showed that the sphere $\mathbf{S}^n$ is a topological n-manifold even though we did not specify the charts of topological atlas. In order to show that $\mathbf{S}^n$ is a smooth n-manifold, it is necessary to define a concrete smooth atlas, preferably with as few charts as possible, as we see in the following examples.

Example 1.14. Consider the topological atlas for the sphere $\mathbf{S}^n$ from Example 1.7. If we use only those points of the sphere whose coordinates are all zero except one, the corresponding hemispheres will cover the whole $\mathbf{S}^n$. The number of charts in the atlas is reduced to $2n+2$, where all of them have very simple formulas. Concretely, for each $1 \leq i \leq n+1$ we have projections on the hemispheres $U_{\pm i} = \{(x_1, \ldots, x_{n+1}) \in \mathbf{S}^n : \pm x_i > 0\}$ given by $\varphi_{\pm i}(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$, while $\varphi_{\pm i}(U_{\pm i}) = B_1(0) = \mathbf{B}^n \subset \mathbb{R}^n$ is the open unit ball. The projections $\varphi_{\pm i}\colon U_{\pm i} \to \mathbf{B}^n$ are bijections since we easily calculate their inverses,

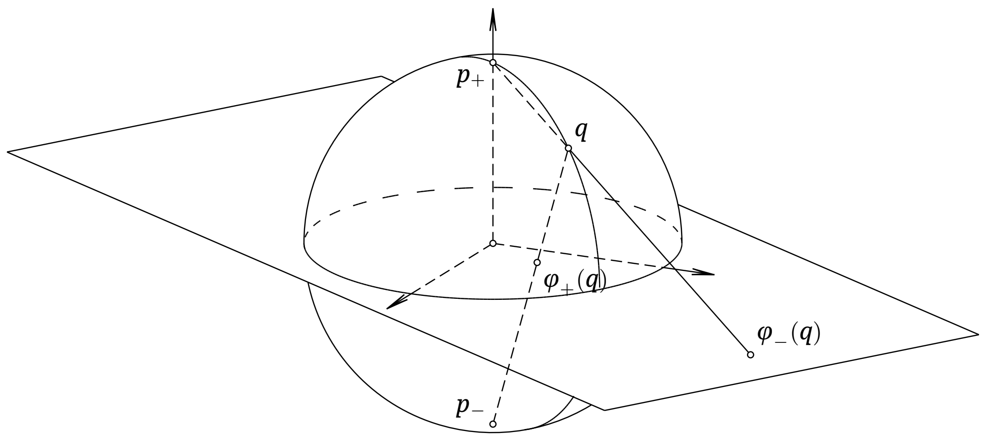
$$\varphi_{\pm i}^{-1}(y_1,\dots,y_n) = (y_1,\dots,y_{i-1},\pm\sqrt{1-y_1^2-\cdots-y_n^2},y_i,\dots,y_n),$$

which means that $(U_{\pm i},\varphi_{\pm i})$ are charts on the sphere $\mathbf{S}^n$. This charts are mutually compatible because all transition functions are smooth. For example, for $j<i$ we have

$$\begin{aligned}\varphi_{+j}\circ\varphi_{\pm i}^{-1}(y_1,\dots,y_n) &= \varphi_{-j}\circ\varphi_{\pm i}^{-1}(y_1,\dots,y_n)\\ &= (y_1,\dots,y_{j-1},y_{j+1},\dots,y_{i-1},\pm\sqrt{1-y_1^2-\cdots-y_n^2},y_i,\dots,y_n),\end{aligned}$$

which is smooth, so $\varphi_{+j}\circ\varphi_{\pm i}^{-1}$ is a diffeomorphism between V_{+j} and $V_{\pm(i-1)}$, and $\varphi_{-j}\circ\varphi_{\pm i}^{-1}$ is a diffeomorphism between V_{-j} and $V_{\pm(i-1)}$, where $V_{\pm k} = \{(y_1,\dots,y_n)\in\mathbf{B}^n : \pm y_k>0\}$ are open sets. The case $j>i$ can be solved similarly, while the case $j=i$ essentially does not exist since $U_{+i}\cap U_{-i}=\emptyset$, and therefore the charts $(U_{\pm i},\varphi_{\pm i})$ form a smooth atlas for $\mathbf{S}^n$. $\triangle$

Example 1.15. In Example 1.7 we use $2n+2$ charts to show that the n-sphere $\mathbf{S}^n$ is a smooth n-manifold. The number of charts in the atlas can be reduced to 2, which is optimal because $\mathbf{S}^n$ is compact. Let us consider two points $p_\pm = (0,\dots,0,\pm 1)\in\mathbf{S}^n$ and two sets $U_\pm = \mathbf{S}^n\setminus\{p_\mp\}$ which cover the whole $\mathbf{S}^n$.



The atlas will consists of two ***stereographic projections*** $\varphi_\pm\colon U_\pm\to\mathbb{R}^n$ defined by

$$\varphi_\pm(x_1,\dots,x_n,x_{n+1}) = \frac{1}{1\pm x_{n+1}}(x_1,\dots,x_n).$$

Such $\varphi_\pm$ are continuous, invertible, and the inverse

$$\varphi_\pm^{-1}(y_1,\dots,y_n) = \frac{1}{1+y_1^2+\cdots+y_n^2}(2y_1,\dots,2y_n,\mp(-1+y_1^2+\cdots+y_n^2)) \tag{1.3}$$

is also continuous. The transition map

$$\varphi_-\circ\varphi_+^{-1}(y_1,\dots,y_n) = \frac{1}{y_1^2+\cdots+y_n^2}(y_1,\dots,y_n) \tag{1.4}$$

is an obvious diffeomorphism from $\mathbb{R}^n\setminus\{0\}$ onto itself (geometrically, it is the inversion with respect to the sphere $\mathbf{S}^{n-1}\subset\mathbb{R}^n\setminus\{0\}$, which is an involution), and therefore these two charts are compatible. In this way, the stereographic projections on the n-sphere form a smooth atlas, and determine the smooth structure of the n-manifold $\mathbf{S}^n$.

Moreover, the stereographic projections are compatible with the projections from Example 1.14 since the transition functions are smooth,

$$\varphi_s \circ \varphi_{\pm i}^{-1}(y_1,\dots,y_n) = \frac{1}{1+sy_n}\left(y_1,\dots,y_{i-1},\pm\sqrt{1-y_1^2-\cdots-y_n^2},y_i,\dots,y_{n-1}\right)$$
$$\varphi_{\pm i} \circ \varphi_s^{-1}(y_1,\dots,y_n) = \frac{1}{1+y_1^2+\cdots+y_n^2}\left(2y_1,\dots,2y_{i-1},2y_{i+1},\dots,2y_n,-s(-1+y_1^2+\cdots+y_n^2)\right)$$

for each $1 \le i \le n$, as well as

$$\varphi_s \circ \varphi_{\pm(n+1)}^{-1}(y_1,\dots,y_n) = \frac{1}{1 \pm s\sqrt{1-y_1^2-\cdots-y_n^2}}(y_1,\dots,y_n)$$
$$\varphi_{\pm(n+1)} \circ \varphi_s^{-1}(y_1,\dots,y_n) = \frac{2}{1+y_1^2+\cdots+y_n^2}(y_1,\dots,y_n),$$

where $s \in \{+,-\}$, so the atlas with ordinary projections and the atlas with stereographic projections determine the same smooth structure, which is the standard smooth structure of the n-sphere. △

It is common to build new manifolds out of old. For example, if we have two manifolds of the same dimensions, we can take their disjoint union with the union of their atlases and get a new manifold, although it is no more interesting than considering the manifolds separately.

Example 1.16. Consider the product $M \times N$ for an m-manifold M and an n-manifold N. For any point $(p,q) \in M \times N$ we can choose a chart (U,φ) at $p \in M$ and a chart (V,ψ) at $q \in N$, which allows us to construct product charts in an obvious way $\varphi\times\psi\colon U\times V \to \varphi(U)\times\psi(V) \subseteq \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ by $(\varphi \times \psi)(u,v) = (\varphi(u),\psi(v))$. Any two charts $(\varphi_1 \times \psi_1)$ and $(\varphi_2 \times \psi_2)$ are compatible because $(\varphi_2 \times \psi_2) \circ (\varphi_1 \times \psi_1)^{-1} = (\varphi_2 \circ \varphi_1^{-1}) \times (\psi_2 \circ \psi_1^{-1})$. The product atlas is smooth, while the Hausdorffness and second countability are hereditary properties for products, so the ***product manifold*** $M \times N$ is a manifold of dimension $m+n$. Therefore, the torus $\mathbf{S}^1 \times \mathbf{S}^1$ and the infinite cylinder $\mathbf{S}^1 \times \mathbb{R}$ are manifolds. This construction can be easily extended to more manifolds, so the ***n-torus***, $\mathbf{T}^n = \mathbf{S}^1 \times\cdots\times \mathbf{S}^1$ (n factors) is a smooth n-manifold. △

Gluing is a good method to construct new spaces from known ones. For example, gluing together the upper and lower edges of a square gives a cylinder and gluing together the boundaries of the cylinder gives a torus. The quotient construction is a process starting with an equivalence relation $\sim$ on a set M, where we identify each equivalence class to a point. The ***quotient set*** $M/_\sim$ is the set of equivalence classes, and there is the natural projection map $\pi\colon M \to M/_\sim$ that sends $p \in M$ to its equivalence class $\pi(p) = [p]$.

If M is a topological space, it is natural that the quotient set $M/_\sim$ becomes the ***quotient space*** by obtaining the quotient topology in which $U \subseteq M/_\sim$ is open if and only if $\pi^{-1}(U) \subseteq M$ is open. Then the projection π is continuous, and we have additional topology compatibility if π is open. This motivates us to make all charts (U,φ) on $M/_\sim$ such that $\varphi \circ \pi$ is both continuous and open, because then the topology of $M/_\sim$ induced by Theorem 1.1 agrees with the quotient topology induced by the projection π. However, even if the original space is a manifold, a quotient space is often not a manifold, since the Hausdorff condition must be checked.

Example 1.17. The ***real projective space*** of dimension n is the set of all one-dimensional linear subspaces of the vector space $\mathbb{R}^{n+1}$. It is the quotient set $\mathbb{R}\mathbf{P}^n$ determined by the projection $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}\mathbf{P}^n$ sending each point $x \in \mathbb{R}^{n+1} \setminus \{0\}$ to the equivalence class $\pi(x) = [x] = \operatorname{Span}\{x\}$, which becomes a point of the space $\mathbb{R}\mathbf{P}^n = (\mathbb{R}^{n+1} \setminus \{0\})/_\sim$. Usually, we write the equivalence class of a vector $(x_1, \ldots, x_{n+1}) \neq 0$ in homogeneous coordinates, $(x_1 : \ldots : x_{n+1}) = [(x_1, \ldots, x_{n+1})]$.

Let us define the sets $\widetilde{U}_i = \{(x_1, \ldots, x_{n+1}) : x_i \neq 0\} \subset \mathbb{R}^{n+1} \setminus \{0\}$ and $U_i = \pi(\widetilde{U}_i) \subseteq \mathbb{R}\mathbf{P}^n$ for $1 \leq i \leq n+1$. The function $\varphi_i \colon U_i \to \mathbb{R}^n$ given by

$$\varphi_i(x_1 : \ldots : x_{n+1}) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right)$$

is well defined, since the value $\varphi_i([x])$ is invariant on multiplying $x = (x_1, \ldots, x_{n+1})$ by a nonzero scalar, and it is obviously bijective with the inverse

$$\varphi_i^{-1}(y_1, \ldots, y_n) = (y_1 : \ldots : y_{i-1} : 1 : y_i : y_{i+1} : \ldots : y_n).$$

In this way we make the standard charts (U_i, φ_i) on $\mathbb{R}\mathbf{P}^n$ for $1 \leq i \leq n+1$. They are mutually compatible since for $i > j$ the transition function gives

$$\varphi_j \circ \varphi_i^{-1}(y_1, \ldots, y_n) = \left(\frac{y_1}{y_j}, \ldots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \ldots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_i}{y_j}, \ldots, \frac{y_n}{y_j}\right),$$

which is a diffeomorphism between the open sets $\varphi_i(U_i \cap U_j) = \mathbb{R}^{j-1} \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-j}$ and $\varphi_j(U_i \cap U_j) = \mathbb{R}^{i-2} \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n+1-i}$, and it can be shown similarly in the case $i < j$. Since every nonzero vector has some nonzero coordinate, we have $\mathbb{R}\mathbf{P}^n = \bigcup_{i=1}^{n+1} U_i$, so our charts form a smooth atlas.

It remains to check the Hausdorff condition for distinct points $p = (p_1 : \ldots : p_{n+1})$ and $q = (q_1 : \ldots : q_{n+1})$ from $\mathbb{R}\mathbf{P}^n$. If $p, q \in U_i$ holds for some $1 \leq i \leq n+1$, the matter is solved by Lemma 1.5. Otherwise, there exist $1 \leq i \neq j \leq n+1$ such that $p_i \neq 0, p_j = 0$ and $q_i = 0, q_j \neq 0$. Consider the disjoint sets $P = \{(x_1 : \ldots : x_{n+1}) : x_i^2 > x_j^2\}$ and $Q = \{(x_1 : \ldots : x_{n+1}) : x_i^2 < x_j^2\}$, where $p \in P \subset U_i$ and $q \in Q \subset U_j$. Since for each $1 \leq k \leq n$ the sets $\{(y_1, \ldots, y_n) : |y_k| < 1\}$ are open in $\mathbb{R}^n$, then $\varphi_i(P)$ and $\varphi_j(Q)$ are open in $\mathbb{R}^n$, so by Lemma 1.4 $(P, \varphi_i\restriction_P)$ and $(Q, \varphi_j\restriction_Q)$ are charts of the complete smooth atlas (as $(1 : \ldots : 1) \in U_i$ for each i we could not use the original charts only), which proves the Hausdorff condition. Finally, $\mathbb{R}\mathbf{P}^n$ with the standard atlas introduced is an n-manifold.

It is important to emphasize that the topology of the manifold $\mathbb{R}\mathbf{P}^n$ induced by Theorem 1.1 agrees with the quotient topology induced by the projection π. We consider the map $\varphi_i \circ \pi \colon \widetilde{U}_i \to \mathbb{R}^n$ given by $\varphi_i \circ \pi(x_1, \ldots, x_{n+1}) = (x_1/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_{n+1}/x_i)$, for each $1 \leq i \leq n+1$. It is obviously continuous, but also open because for an open $W \subseteq \widetilde{U}_i$, as multiplication by $\lambda \neq 0$ is a homeomorphism of $\widetilde{U}_i \subset \mathbb{R}^{n+1} \setminus \{0\}$, we have open $\lambda W \subseteq \widetilde{U}_i$, from where $(\varphi_i \circ \pi)^{-1}((\varphi_i \circ \pi)(W)) = \bigcup_{\lambda \neq 0} \lambda W$ is open, and therefore $(\varphi_i \circ \pi)(W)$ is open. Since all the maps $\varphi_i \circ \pi$ are continuous and open, the topologies agree. △

Example 1.18. ***Complex projective space*** of dimension n is the quotient set $\mathbb{C}\mathbf{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/_\sim$ obtained in complete analogy with $\mathbb{R}\mathbf{P}^n$, where $\sim$ is the proportionality relation (scalars are nonzero complex numbers) that induces the natural projection $\pi \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}\mathbf{P}^n$ and homogeneous coordinates $\pi(z_1, \ldots, z_{n+1}) = (z_1 : \ldots : z_{n+1})$. The standard atlas charts are $\varphi_i \colon U_i \to \mathbb{C}^n \cong \mathbb{R}^{2n}$, where $U_i = \{(z_1 : \ldots : z_{n+1}) : z_i \neq 0\}$ for $1 \leq i \leq n+1$, given by

$$\varphi_i(z_1 : \ldots : z_{n+1}) = \left(\frac{z_1}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_{n+1}}{z_i}\right).$$

They are mutually compatible since for $i > j$ the transition function gives

$$\varphi_j \circ \varphi_i^{-1}(z_1, \ldots, z_n) = \left(\frac{z_1}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_{i-1}}{z_j}, \frac{1}{z_j}, \frac{z_i}{z_j}, \ldots, \frac{z_n}{z_j}\right),$$

which is a diffeomorphosm between open sets $\varphi_i(U_i \cap U_j) = \mathbb{C}^{j-1} \times (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{n-j}$ and $\varphi_j(U_i \cap U_j) = \mathbb{C}^{i-2} \times (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{n+1-i}$, and similarly in the case of $i < j$. Of course, we should always keep in mind that

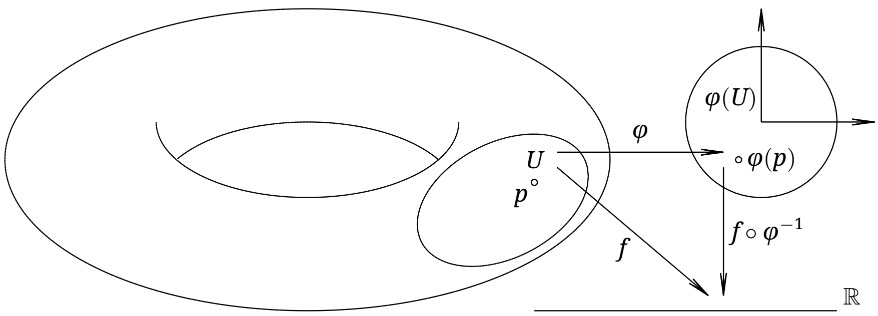
$$\frac{z_k}{z_j} = \frac{x_k + iy_k}{x_j + iy_j} = \frac{(x_k + iy_k)(x_j - iy_j)}{x_j^2 + y_j^2} = \left(\frac{x_k x_j + y_k y_j}{x_j^2 + y_j^2}, \frac{y_k x_j - x_k y_j}{x_j^2 + y_j^2} \right).$$

Analogous to the proof for $\mathbb{R}\mathbf{P}^n$, the Hausdorff condition holds for $\mathbb{C}\mathbf{P}^n$ because the points $p = (p_1 : \ldots : p_{n+1})$ and $q = (q_1 : \ldots : q_{n+1})$ from $\mathbb{C}\mathbf{P}^n$ with $p_i \neq 0, p_j = 0$ and $q_i = 0, q_j \neq 0$ can be separated by $P = \{(z_1 : \ldots : z_{n+1}) : \|z_i\| > \|z_j\|\}$ and $Q = \{(x_1 : \ldots : x_{n+1}) : \|z_i\| < \|z_j\|\}$. Finally, $\mathbb{C}\mathbf{P}^n$ with the introduces standard atlas is a $2n$-manifold whose topology agrees with the quotient topology because as in Example 1.17 it can be shown that all maps $\varphi_i \circ \pi$ are continuous and open. △

1.4 Smooth maps

The smooth structure on a manifold M with a smooth atlas $\mathcal{A}$ allows us to unambiguously define which real-valued functions on M are smooth.

Let $f\colon M \to \mathbb{R}$ be a real-valued function on M. If $(U, \varphi) \in \mathcal{A}$ is a chart at $p \in M$, we can naturally identify f with the composition function $f \circ \varphi^{-1}\colon \varphi(U) \to \mathbb{R}$, where $\varphi(U) \subseteq \mathbb{R}^n$, thus enabling the notion of smoothness of a function on a manifold. We say that a function $f\colon M \to \mathbb{R}$ is ***smooth at*** $p \in M$ if there exists a chart $(U, \varphi) \in \mathcal{A}$ at $p \in M$ such that $f \circ \varphi^{-1}$ is smooth at $\varphi(p)$ in the sense of ordinary calculus. A function $f\colon M \to \mathbb{R}$ is ***smooth*** if it is smooth at all points of M.



If $(V, \psi) \in \mathcal{A}$ is another chart at $p \in M$, the transition function $\varphi \circ \psi^{-1}$ is smooth, and because of $(f \circ \psi^{-1})\restriction_{\psi(U \cap V)} = (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1})$, we have that $f \circ \psi^{-1}$ is smooth at $\psi(p)$ if and only if $f \circ \varphi^{-1}$ is smooth at $\varphi(p)$. In this way, the smooth atlas allowed us that our definition does not depend on the choice of chart, while $f\colon M \to \mathbb{R}$ is smooth if and only if $f \circ \varphi^{-1}$ is smooth for every chart $(U, \varphi) \in \mathcal{A}$.

Let $\mathfrak{F}(M)$ denotes the set of all smooth real-valued functions $f\colon M \to \mathbb{R}$ on a manifold M. For two functions $f, h \in \mathfrak{F}(M)$ we naturally define the functions $f + h$ (sum), fh (product), and af (multiplication with a scalar $a \in \mathbb{R}$) by the equalities

$$(f + h)(p) = f(p) + h(p), \quad (fh)(p) = f(p)h(p), \quad (af)(p) = af(p),$$

that hold for all $p \in M$. It is easy to see that our new functions are also in $\mathfrak{F}(M)$. Equipped with these operations, $\mathfrak{F}(M)$ is a commutative ring (with unity), and also an algebra over $\mathbb{R}$.

Example 1.19. The coordinate functions $x_i = \pi_i \circ \varphi$ of an arbitrary chart (U, φ) on a manifold M are smooth as functions on the open submanifold U since their coordinate representation $x_i \circ \varphi^{-1}$ is a restriction of the projection π_i to $\varphi(U)$, and therefore smooth. △

Example 1.20. The height function $f\colon \mathbf{S}^n \to \mathbb{R}$ on the n-sphere with its standard smooth structure is given by $f(x_1,\dots,x_{n+1}) = x_{n+1}$. Using the formula (1.15) for the stereographic projections from Example 1.15 we can calculate

$$(f \circ \varphi_\pm^{-1})(y_1,\dots,y_n) = \mp\frac{-1+y_1^2+\cdots+y_n^2}{1+y_1^2+\cdots+y_n^2},$$

that is smooth on $\mathbb{R}^n$, and therefore f is smooth. Alternatively, we could use the atlas with projections on the hemispheres from Example 1.14 (since it also gives the standard smooth structure of the n-sphere) to show that the functions $f \circ \varphi_{\pm i}^{-1}$ are smooth.

More generally, let us consider a smooth function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ and its restriction $f\restriction_{\mathbf{S}^n} = f \circ \imath$, where $\imath\colon \mathbf{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is the inclusion map. The coordinate representation of inclusion is $\imath \circ \varphi_\pm^{-1}\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by

$$\imath \circ \varphi_\pm^{-1}(y_1,\dots,y_n) = \frac{1}{1+y_1^2+\cdots+y_n^2}(2y_1,\dots,2y_n,\mp(-1+y_1^2+\cdots+y_n^2)),$$

which is smooth on $\mathbb{R}^n$, hence $f\restriction_{\mathbf{S}^n} \circ \varphi_\pm^{-1} = f \circ \imath \circ \varphi_\pm^{-1}$ is also smooth, and therefore $f\restriction_{\mathbf{S}^n}$ is smooth. △

The ***support*** of a function $f\colon M \to \mathbb{R}$ on a manifold M is the closure of the set of points where f is nonzero,

$$\operatorname{supp}(f) = \overline{\{p \in M : f(p) \neq 0\}} = \overline{f^{-1}(\mathbb{R}\setminus\{0\})}.$$

Therefore, the support of a function f is the smallest closed subset of M outside of which f is zero. In other words, the complement of $\operatorname{supp}(f)$ is the largest open subset on which f is identically zero. If $\operatorname{supp}(f)$ is contained in some set $U \subseteq M$, we say that f is ***supported*** in U. If $\operatorname{supp}(f)$ is a compact set we say that f is ***compactly supported***. Of course, every function on a compact space is compactly supported.

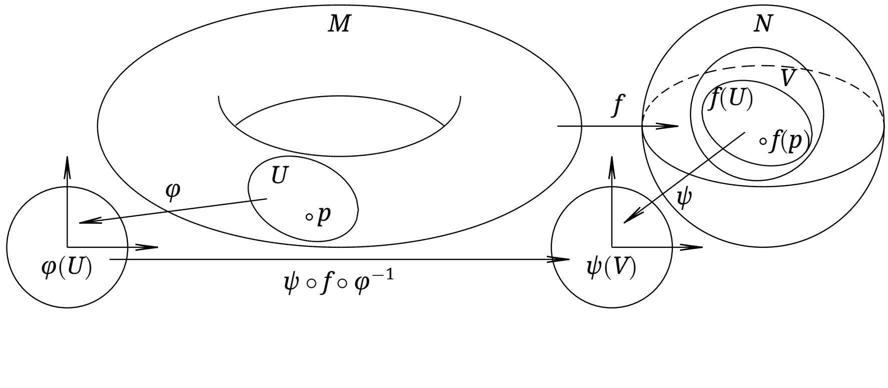
If a smooth function on an open submanifold of M is supported in a closed subset of M, then it can be extended by zero to the whole manifold.

Lemma 1.11. *If U is an open subset of a manifold M, and $f \in \mathfrak{F}(U)$ is supported in a closed subset of M, then the function $h\colon M \to \mathbb{R}$ defined by $h\restriction_U = f$ and $h\restriction_{M\setminus U} = 0$ is smooth.*

Proof. By assumption, $\operatorname{supp}(f) \subseteq A \subseteq U$ holds for some closed subset $A \subseteq M$, and therefore $h = 0$ (and therefore smooth) on the open subset $M \setminus A$. Of course, since $h\restriction_U = f$ is smooth on U, h is smooth on $(M\setminus A) \cup U \supseteq (M \setminus U) \cup U = M$. □

A smooth structure allows us to extend the notion of smoothness to maps between two manifolds M and N. The natural idea is to identify a map $f\colon M \to N$ with its ***coordinate representation*** $\psi \circ f \circ \varphi^{-1}$, where (U,φ) is a chart on M and (V,ψ) is a chart on N. The domain of the representation is $(f\circ\varphi^{-1})^{-1}(V) = \varphi(U \cap f^{-1}(V))$, but it is not necessarily an open set, which can be a problem. That problem can be removed by requiring that $f(U) \subseteq V$ because then $\psi \circ f \circ \varphi^{-1}$ is a map between open sets $\varphi(U \cap f^{-1}(V)) = \varphi(U)$ and $\psi(V)$.

A map $f\colon M \to N$ between manifolds M and N is said to be ***smooth at*** $p \in M$ if there exist charts (U,φ) at $p \in M$ and (V,ψ) at $f(p) \in N$, with $f(U) \subseteq V$ such that $\psi \circ f \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ is Euclidean smooth. A map $f\colon M \to N$ is ***smooth*** if it is smooth at all points of M.



An open domain of a coordinate representation can naturally be ensured by the requirement that the map f is continuous, and as expected, smoothness implies continuity, which we see in the following lemma.

Lemma 1.12. *A smooth map between manifolds is continuous.*

Proof. Let $f\colon M \to N$ be a smooth map between manifolds. For $p \in M$ there exist a chart (U, φ) at $p \in M$ and a chart (V, ψ) at $f(p) \in N$ with $f(U) \subseteq V$, such that $\psi \circ f \circ \varphi^{-1}$ is Euclidean smooth, and therefore continuous. The restriction $f{\restriction_U} = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi$ is continuous as a composition of continuous maps, and since f is continuous in a neighbourhood of each point $p \in M$ it is continuous on the whole M. $\square$

In order to check by definition whether a map $f\colon M \to N$ is smooth, it is enough to take some smooth atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in \Lambda\}$ for M with the property that $f(U_\alpha) \subseteq V_\alpha$ holds for some chart (V_α, ψ_α) on N and then check whether all maps $\psi_\alpha \circ f \circ \varphi_\alpha^{-1}$ are smooth. Of course, thanks to the smooth structure, the smoothness of a map does not depend on the choice of charts, which gives an alternative way to set a definition of smooth maps.

Lemma 1.13. *A map $f\colon M \to N$ between manifolds is smooth if and only if f is continuous and the function $\psi \circ f \circ \varphi^{-1}$ is Euclidean smooth for every chart (U, φ) on M and every chart (V, ψ) on N.*

Proof. Let (U, φ) be a chart on M and (V, ψ) be a chart on N. By Lemma 1.12, a smooth f is continuous which implies that $f^{-1}(V)$ is open, and therefore the function $\psi \circ f \circ \varphi^{-1}$ has the open domain $D = \varphi(U \cap f^{-1}(V))$. For an arbitrary point $d \in D \neq \emptyset$, denote $p = \varphi^{-1}(d)$, so there exist charts (U_1, φ_1) at $p \in M$ and (V_1, ψ_1) at $f(p) \in N$ with $f(U_1) \subseteq V_1$ and smooth $\psi_1 \circ f \circ \varphi_1^{-1}$. Since

$$(\psi \circ f \circ \varphi^{-1}){\restriction_{\varphi(U \cap U_1)}} = (\psi \circ \psi_1^{-1}) \circ (\psi_1 \circ f \circ \varphi_1^{-1}) \circ (\varphi_1 \circ \varphi^{-1}),$$

while the transition maps $\psi \circ \psi_1^{-1}$ and $\varphi_1 \circ \varphi^{-1}$ are smooth on appropriate neighbourhoods, $\psi \circ f \circ \varphi^{-1}$ is smooth on $\varphi(U \cap U_1) \ni d$ as a composition of smooth maps. Thus, smooth f implies smooth $\psi \circ f \circ \varphi^{-1}$.

The converse is obvious because for $p \in M$ and a chart (V, ψ) y $f(p) \in N$, an arbitrary chart (U, φ) at $p \in M$ can be restricted by Lemma 1.4 to the open set $W = U \cap f^{-1}(V)$ and obtain a new chart $\varphi{\restriction_W}$ with the condition $f(W) \subseteq V$. $\square$

Lemma 1.13 covers the most common definition of smoothness in the literature. A map $f\colon M \to N$ between manifolds is smooth if the function $\psi \circ f \circ \varphi^{-1}$ is Euclidean smooth for every chart (U, φ) on M and every chart (V, ψ) on N, with the stipulation that $\varphi(U \cap f^{-1}(V))$ is an open set. For example, $\varphi(U \cap f^{-1}(V))$ is open if f happens to be continuous.

There is a corresponding notion of a smooth function whose domain is an arbitrary subset of Euclidean space, which can be generalised to manifolds. A map $f\colon A \to N$, whose domain $A \subseteq M$ is an arbitrary subset of manifold and the codomain N is a manifold, is said to be ***smooth at*** $p \in A$, if there is a smooth map $f_p\colon U_p \to N$ where $U_p \subseteq M$ is a neighbourhood of p, such that $f = f_p$ on $U_p \cap A$. Of course, $f\colon A \to N$ is ***smooth*** if it is smooth at every point of A. If we allow the domain of the coordinate representation not to be an open set, a problem from the following example may occur.

Example 1.21. Consider the characteristic (indicator) function $f\colon \mathbb{R} \to \{0, 1\} \subset \mathbb{R}$ of the set $\{x \in \mathbb{R} : x \geq 0\}$. Let us take the single chart $(\mathbb{R}, \mathbb{1})$ on $M = \mathbb{R}$, and two charts on $N = \mathbb{R}$, for example $((-\infty, 1), \mathbb{1})$ and $((0, +\infty), \mathbb{1})$. The coordinate representation of f is just its restriction to the domain $D = f^{-1}(V)$, where V is the domain of the corresponding chart on N. For $V = (-\infty, 1)$, the set $D = (-\infty, 0)$ is open and $f{\restriction_D} = 0$, so it is smooth. However, for $V = (0, +\infty)$, the set $D = [0, +\infty)$ is not open, but $f{\restriction_D} = 1$ holds and therefore it has a smooth extension to an open set. Our goal is to generalise the definition of a smooth function between Euclidean spaces, so we do not want that such non-continuous f, and therefore not ordinary smooth function, be smooth. △

The definition of smooth map between manifolds generalises the previous concepts. For example, a map between open subsets of Euclidean spaces has the coordinate representation (relative to the identity charts) equal to some of its restriction, and is smooth if and only if it is Euclidean smooth. Similarly, in the case of a real-valued function on a manifold, we have a coincidence of smooth maps between manifolds with smooth functions determined by the smooth structure.

Since the smoothness of map between manifolds is by definition a local property, we immediately have the ***Gluing lemma for smooth maps***.

Lemma 1.14. *Let M and N be manifolds and $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of M. If there are smooth maps $f_\alpha\colon U_\alpha \to N$ for $\alpha \in \Lambda$ such that $f_\alpha = f_\beta$ holds on $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in \Lambda$, then there exists a unique smooth map $f\colon M \to N$ such that $f\restriction_{U_\alpha} = f_\alpha$ for each $\alpha \in \Lambda$.*

Example 1.22. The obvious example of smooth map is the identity map, or more generally the inclusion map $\imath\colon U \hookrightarrow M$ for an open submanifold $U \subseteq M$. Every constant map $c\colon M \to N$ is also smooth. △

Lemma 1.15. *A composition of smooth maps between manifolds is smooth.*

Proof. Let $f\colon M \to N$ and $h\colon N \to P$ be smooth maps. Since h is smooth, for each $p \in M$ there exist charts (V, ψ) at $f(p) \in N$ and (W, θ) at $h(f(p)) \in P$, with $h(V) \subseteq W$ such that $\theta \circ h \circ \psi^{-1}$ is smooth. The map f is continuous by Lemma 1.12, so $f^{-1}(V) \ni p$ is open and there exists a chart (U, φ) at $p \in M$ with $U \subseteq f^{-1}(V)$. Since f is smooth, $\psi \circ f \circ \varphi^{-1}$ is smooth by Lemma 1.13. Hence $(h \circ f)(U) \subseteq h(V) \subseteq W$ and $\theta \circ (h \circ f) \circ \varphi^{-1} = (\theta \circ h \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1})$ is smooth as a composition of smooth functions. □

Although the definition is not simple, it is often straightforward to prove that a particular map is smooth. The most common way is to write the map in local coordinates and recognize its component functions as compositions of smooth elementary functions or maps that are known to be smooth.

Example 1.23. Any map from a 0-dimensional manifold into a smooth manifold is smooth, since each coordinate representation is constant. △

Example 1.24. Let M and N be manifolds and let $\pi_M\colon M \times N \to M$, $\pi_M(p, q) = p$ be the projection to the first component. Let (U, φ) be a chart at $p \in M$ and (V, ψ) be a chart at $q \in N$, then $(U \times V, \varphi \times \psi)$ is a chart at $(p, q) \in M \times N$. The coordinate representation of π_M is $\varphi \circ \pi_M \circ (\varphi \times \psi)^{-1}\colon (\varphi \times \psi)(U \times V) \to \varphi(U)$ given by $(a_1, \dots, a_m, b_1, \dots, b_n) \mapsto (a_1, \dots, a_m)$, which is smooth, and therefore π_M is smooth. △

Example 1.25. The quotient map $\pi\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}\mathbf{P}^n$ from Example 1.17 is smooth. Its coordinate representation related to introduced coordinates for $\mathbb{R}\mathbf{P}^n$ and standard coordinates for $\mathbb{R}^{n+1} \setminus \{0\}$ is

$$\varphi_i \circ \pi \circ \mathbb{1}^{-1}(x_1, \dots, x_{n+1}) = \varphi_i(x_1 : \dots : x_{n+1}) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right),$$

which is obviously smooth for $(x_1, \dots, x_{n+1}) \in \widetilde{U}_i$, that is, for $x_i \neq 0$. △

A ***Lie group***[19] is a manifold G that is also a group in the algebraic sense with the property that the group operations are compatible with the smooth structure. Concretely, the multiplication map $\mu\colon G \times G \to G$, $\mu(a, b) = ab$ and the inversion map $\imath\colon G \to G$, $\imath(a) = a^{-1}$ are both smooth.

[19] Marius Sophus Lie (1842–1899), Norwegian mathematician

Example 1.26. Let us mention basic examples of Lie groups. The Euclidean space $\mathbb{R}^n$ is a Lie group under addition. The set of nonzero complex numbers $\mathbb{C} \setminus \{0\}$ is a Lie group under multiplication. The unit circle $\mathbf{S}^1$ in $\mathbb{C} \setminus \{0\}$ is a Lie group under multiplication. The product $G_1 \times G_2$ of two Lie groups (G_1, μ_1) and (G_2, μ_2) is a Lie group under coordinatewise multiplication $\mu_1 \times \mu_2$. △

Example 1.27. The general linear group $\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\}$ from Example 1.12 is a manifold as an open subset of $\mathbb{R}^{n\times n}$. Since each component of the product AB of two matrices $A, B \in \mathrm{GL}(n, \mathbb{R})$ is a (quadratic) polynomial in the entries of A and B, the matrix multiplication is clearly smooth. By Cramer's[20] rule, the inverse matrix A^{-1} is the quotient of the adjugate matrix of A and the determinant of A. The adjugate of A is the transpose of the cofactor matrix of A, so the entries of the inverse matrix are (degree $n-1$) polynomials in the entries of A, and therefore the inversion map is smooth. △

1.5 Diffeomorphisms

A ***diffeomorphism*** between manifolds M and N is a smooth bijective map $f\colon M \to N$ that has a smooth inverse. If such diffeomorphism exists we say that M and N are ***diffeomorphic***. A diffeomorphism of manifolds is a bijection of the underlying sets that identifies their complete smooth atlases. Similarly, a ***homeomorphism*** between manifolds M and N is a continuous bijective map $f\colon M \to N$ that has a continuous inverse, and if it exists we say that manifolds are ***homeomorphic***. Since every smooth map is continuous, every diffeomorphism is a homeomorphism.

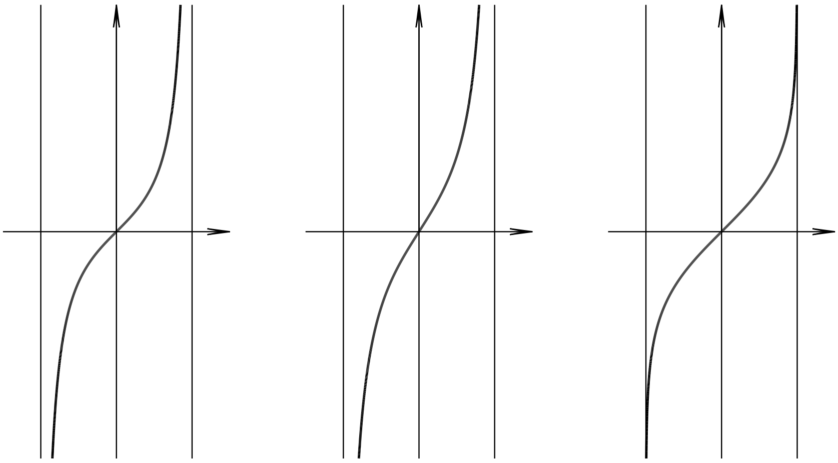
The manifold theory investigates properties of manifolds that are preserved by diffeomorphisms, so we consider diffeomorphic manifolds to be the same. Manifolds that are homeomorphic are considered the same topologically.

Example 1.28. Let M be an n-manifold. Every chart (U, φ) on M gives a diffeomorphism $\varphi\colon U \to \varphi(U)$ from the open submanifold $U \subseteq M$ onto an open subset of $\mathbb{R}^n$. The maps φ and φ^{-1} are smooth because such are their representations related to charts, (U, φ) on U and $(\varphi(U), \mathbb{1}_{\varphi(U)})$ on $\varphi(U)$, $\mathbb{1}_{\varphi(U)} \circ \varphi \circ \varphi^{-1} = \mathbb{1}_{\varphi(U)} = \varphi \circ \varphi^{-1} \circ \mathbb{1}_{\varphi(U)}^{-1}$. Conversely, every diffeomorphism $f\colon U \to f(U)$ from an open subset $U \subseteq M$ onto an open subset of $\mathbb{R}^n$ is a chart in the complete atlas. Namely, if (V, ψ) is a chart of the atlas for M, we know that ψ and ψ^{-1} are smooth, so the transition maps $f \circ \psi^{-1}$ and $\psi \circ f^{-1}$ are smooth as compositions by Lemma 1.15, which implies that the chart (U, f) is compatible with the atlas. △

Example 1.29. If f is a bijection from a set P onto a manifold M, then there is a unique way to make P a manifold such that f is a diffeomorphism. In that case, an atlas for P consists of charts $(f^{-1}(U), \varphi \circ f)$ where (U, φ) is a chart on M. △

Example 1.30. The open unit ball $\mathbf{B}^n = B_1(0)$ in $\mathbb{R}^n$ is diffeomorphic with $\mathbb{R}^n$ itself. In the case of dimension $n = 1$ there are many ways to set a smooth increasing bijection $f\colon (-1, 1) \to \mathbb{R}$ such as

[20] Gabriel Cramer (1704–1752), Swiss mathematician



$$f(x) = \frac{x}{1-x^2}, \quad f(x) = \tan\left(\frac{\pi}{2}x\right), \quad f(x) = \operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

In general, we can set a diffeomorphism $f\colon \mathbf{B}^n \to \mathbb{R}^n$ by

$$f(x) = \frac{x}{\sqrt{1-\|x\|^2}}, \quad f^{-1}(y) = \frac{y}{\sqrt{1+\|y\|^2}},$$

where the argument of the roots never vanishes, $1-\|x\|^2 \neq 0$ and $1+\|y\|^2 \neq 0$, so f and f^{-1} are smooth. The map

$$f(x) = \frac{x}{1-\|x\|^2}$$

is also a diffeomorphism, but it is more difficult to prove that f^{-1} is smooth without an explicit formula. However,

$$f(x) = \frac{x}{1-\|x\|}$$

is not a diffeomorphism, since the norm $\|\cdot\|\colon \mathbb{R}^n \to [0,\infty)$, unlike its square $\|\cdot\|^2$, is not smooth in general. Concretely, for $n=1$ we have

$$f(x) = \frac{x}{1-|x|}, \quad f'(x) = \frac{1}{(1-|x|)^2}, \quad f''(x) = \frac{2x}{(1-|x|)^3|x|},$$

and therefore $f''(0)$ do not exists and f is not smooth. △

Example 1.31. The map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (xe^y + y, xe^y - y)$ is a diffeomorphism. From $f(x,y) = (xe^y + y, xe^y - y) = (u,v)$ we have $u - v = 2y$ and $u + v = 2xe^y$, and therefore

$$f^{-1}(u,v) = \left(\frac{u+v}{2e^{\frac{u-v}{2}}}, \frac{u-v}{2}\right),$$

so it is easy to see that f is bijective, and both f and f^{-1} are smooth. △

Example 1.32. The projective line $\mathbb{R}\mathbf{P}^1$ is diffeomorphic to the circle $\mathbf{S}^1$. To obtain a diffeomorphism explicitly we can construct a bijection that identifies the standard atlas $\{(U_1,\varphi_1),(U_2,\varphi_2)\}$ for $\mathbb{R}\mathbf{P}^1$ with the standard stereographic atlas $\{(U_+,\varphi_+),(U_-,\varphi_-)\}$ for $\mathbf{S}^1$, where

$$\begin{aligned} U_1 &= \{(1:x) \in \mathbb{R}\mathbf{P}^1 : x \in \mathbb{R}\}, \quad \varphi_1(1:x) = x,\\ U_2 &= \{(x:1) \in \mathbb{R}\mathbf{P}^1 : x \in \mathbb{R}\}, \quad \varphi_2(x:1) = x,\\ U_+ &= \{(x,y) \in \mathbf{S}^1 : y \neq -1\}, \quad \varphi_+(x,y) = x/(1+y),\\ U_- &= \{(x,y) \in \mathbf{S}^1 : y \neq 1\}, \quad \varphi_-(x,y) = x/(1-y). \end{aligned}$$

For $f\colon \mathbb{R}\mathbf{P}^1 \to \mathbf{S}^1$, the identification φ_1 with φ_+ means that $\varphi_+ \circ f \circ \varphi_1^{-1}\colon \mathbb{R} \to \mathbb{R}$ is the identity, so we have $f{\restriction_{U_1}} = \varphi_+^{-1} \circ \varphi_1$. The calculations yield

$$f(x:y) = \left(\frac{2xy}{x^2+y^2}, \frac{x^2-y^2}{x^2+y^2}\right)$$

for $x \neq 0$, so we can naturally redefine f by the same formula for $x = 0$ and get the required diffeomorphism whose inverse is

$$f^{-1}(x,y) = \begin{cases} (1+y:x) \text{ за } y \neq -1 \\ (x:1-y) \text{ за } y \neq 1 \end{cases} .$$

The calculations show $\varphi_- \circ f \circ \varphi_2^{-1} = \mathbb{1}_{\mathbb{R}}$, so f is a diffeomorphism. Note that $x = \cos t$, $y = \sin t$ leads to the formula $f(\cos t : \sin t) = (\sin 2t, \cos 2t)$, so a required diffeomorphism can also be $[e^{it}] \mapsto e^{2it}$. △

Example 1.33. The classical example of a homeomorphism between manifolds that is not a diffeomorphism is the map $\varphi\colon \mathbb{R} \to \mathbb{R}$ given by $\varphi(x) = x^3$. It is smooth and invertible, but the inverse map is not smooth. In Example 1.10 we created the manifold $\widetilde{\mathbb{R}}$ from $\mathbb{R}$ endowed with the smooth structure determined by the chart $(\mathbb{R}, \varphi)$. This smooth structure is different than the standard one, but manifolds $\mathbb{R}$ and $\widetilde{\mathbb{R}}$ are diffeomorphic. The map $f\colon \widetilde{\mathbb{R}} \to \mathbb{R}$ given by $f(x) = x^3$ is a diffeomorphism, because its representation $\mathbb{1}_{\mathbb{R}} \circ f \circ \varphi^{-1} = \mathbb{1}_{\mathbb{R}}$ is obviously smooth, like its inverse. △

Example 1.34. Let M be a manifold with a complete atlas $\mathcal{A}$. Any homeomorphism $f\colon M \to M$ determines a new atlas $\mathcal{A}'$ on M with charts $(f(U), \varphi \circ f^{-1}) \in \mathcal{A}'$ obtained from charts $(U, \varphi) \in \mathcal{A}$. Then f is a diffeomorphism if and only if $\mathcal{A} = \mathcal{A}'$. Therefore, if f is a homeomorphism which is not a diffeomorphism, then f defines a new atlas $\mathcal{A}' \neq \mathcal{A}$. However, the new smooth structure on M is not essentially different from the old one. Although $f\colon M \to M$ is not a diffeomorphism it defines a diffeomorphism between M and the manifold with the atlas $\mathcal{A}'$. Hence, even though $\mathcal{A}$ and $\mathcal{A}'$ are different atlases, the resulting smooth structures are still diffeomorphic. △

Example 1.35. For every $s > 0$ we can construct the map $F_s\colon \mathbf{B}^n \to \mathbf{B}^n$ defined by

$$F_s(x) = \|x\|^s \frac{x}{\|x\|}$$

for $x \neq 0$ and $F_s(0) = 0$. Since $x \in \mathbf{B}^n$ means $\|x\| < 1$, it follows $\|F_s(x)\| < 1$ and we obtain $F_s(\mathbf{B}^n) \subseteq \mathbf{B}^n$. It is easy to check that $F_s \circ F_t = F_{st}$, so F_s is a bijection where $(F_s)^{-1} = F_{1/s}$, while $F_{1/s}(\mathbf{B}^n) \subseteq \mathbf{B}^n$ implies $F_s(\mathbf{B}^n) = \mathbf{B}^n$. Obviously, the map F_s is smooth on $\mathbf{B}^n \setminus \{0\}$, so there remains to check the smoothness at the point $x = 0$. Since $F_s(x) \to 0$ as $x \to 0$, we know that F_s is a homeomorphism. Since F_s is not smooth at 0 for $s < 1$, while $(F_s)^{-1} = F_{1/s}$ is not smooth at 0 for $s > 1$, we know that F_s is a diffeomorphism only for $s = 1$. In this way we construct an uncountable family of homeomorphisms on $\mathbf{B}^n$ that are not diffeomorphisms, but their restriction to $\mathbf{B}^n \setminus \{0\}$ are diffeomorphisms.

Let $\mathcal{A} = \{(V_\alpha, \psi_\alpha) : \alpha \in \Lambda\}$ be an atlas for M. For $p \in M$ there exists a chart (U, φ) of the complete atlas centred at p such that $\varphi(U) = \mathbf{B}^n$. For every $s > 0$ we construct the new atlas $\mathcal{A}_s = \{(V_\alpha \setminus \{p\}, \psi_\alpha{\restriction_{V_\alpha \setminus \{p\}}}) : \alpha \in \Lambda\} \cup \{(U, F_s \circ \varphi)\}$. Since $(F_s \circ \varphi) \circ (F_t \circ \varphi)^{-1} = F_s \circ F_t^{-1} = F_{s/t}$ is not a diffeomorphism for $s \neq t$, our complete smooth atlases determined by $\mathcal{A}_s$ are different and there are uncountable many of them. We have shown that if a topological manifold of dimension $n \geq 1$ has a smooth structure, then it has uncountable many different smooth structures. △

Unlike homeomorphisms that are not diffeomorphisms, it is difficult to find two homeomorphic manifolds that are not diffeomorphic. It turns out that for $n \leq 3$ every topological n-manifold has a smooth structure that is unique up to diffeomorphism (see Moise[21] [85]). However, the first example of exotic manifold structures was discovered by Milnor[22] in 1956 [84], who found that 7-sphere $\mathbf{S}^7$ admits smooth structures that are not diffeomorphic to the standard structure.

It is an interesting question whether a given topological manifold can carry smooth structures that are not diffeomorphic. This question is very complicated, even for Euclidean spaces. It appears that $\mathbb{R}^n$ for $n \neq 4$ has a unique smooth structure, up to diffeomorphism (see Stallings[23] [110]). However, a combination of results due to Donaldson[24] [43] and Freedman[25] [49] led to the discovery of non-standard smooth structures on $\mathbb{R}^4$. Moreover, $\mathbb{R}^4$ has uncountably many distinct smooth structures, no two of which are diffeomorphic to each other (see Taubes[26] [112]).

On the other hand, there exist topological manifolds that do not admit smooth structures at all. The first example of this deep result was a compact 10-dimensional manifold found by Kervaire[27] in 1960 [73]. In fact, examples of non-smoothable topological n-manifolds are known for each $n \geq 4$. The most famous example is so-called E_8 manifold, which is a topological 4-manifold discovered by Freedman in 1982 [49].

1.6 Partitions of unity

Since manifolds are constructed by gluing open sets in $\mathbb{R}^n$ by diffeomorphisms, working on the entire manifold can be inconvenient. The theory of partitions of unity is a rather technical but important tool that allows us work in local coordinates. The basic idea is to break the constant value function 1 into a bunch of smooth pieces that are easier to work with.

There is a huge number of smooth functions on a manifold, but the heart of partitions of unity are the bump functions. These are smooth equivalents of characteristic functions, and we use them to camouflage a function $f\colon M \to \mathbb{R}$. If we multiply f by a characteristic function of $U \subset M$, the result is zero outside of U, but this causes some discontinuities. A bump function b repairs the problem by smoothly decreasing to zero between an inner set V and an outer open set $U \supset \overline{V}$. Then, for $x \in V$ we have $b(x) = 1$ and therefore $(fb)(x) = f(x)$, while for $x \notin U$ we have $(fb)(x) = 0$. Additionally, the product fb will be at least as smooth as f was (except in the case of analytic functions).

The core of the theory is a smooth real function that vanishes for all negative values of the domain and is strictly positive for its positive values. The famous non-analytic smooth function $f\colon \mathbb{R} \to \mathbb{R}$ from Example A.7, defined by

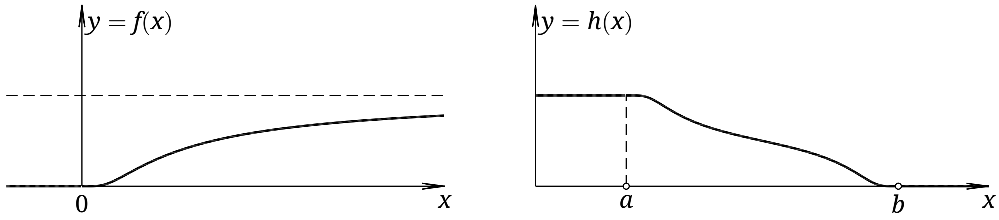
$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

has exactly such properties. For arbitrary $a, b \in \mathbb{R}$ such that $a < b$, we define a smooth function $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \frac{f(b-x)}{f(b-x) + f(x-a)}.$$

[21] Edwin Evariste Moise (1918–1998), American mathematician
[22] John Willard Milnor (1931), American mathematician
[23] John Robert Stallings Jr. (1935–2008), American mathematician
[24] Simon Kirwan Donaldson (1957), English mathematician
[25] Michael Hartley Freedman (1951), American mathematician
[26] Clifford Henry Taubes (1954), American mathematician
[27] Michel André Kervaire (1927–2007), French mathematician

Since at least one of the expressions $b - x$ and $x - a$ is positive, the denominator is positive for all x. Thus, a so-called ***cutoff function*** h is well defined with the properties $h(x) = 1$ for $x \leq a$, $0 < h(x) < 1$ for $a < x < b$, and $h(x) = 0$ for $x \geq b$.



A cutoff function can be further generalised in $\mathbb{R}^n$ where we use the open balls $B_r(p) = \{x \in \mathbb{R}^n : \|x - p\| < r\}$. For $0 < a < b$ we set a function $H\colon \mathbb{R}^n \to \mathbb{R}$ by $H(x) = h(\|x\|)$. The function H is smooth on $\mathbb{R}^n \setminus \{0\}$ as a composition of smooth functions, while it is identically equal to 1 on $B_a(0)$, so it is also smooth at $0 \in \mathbb{R}^n$. This construction proves the following lemma.

Lemma 1.16. *Given any $a, b \in \mathbb{R}$ such that $0 < a < b$, there is $H \in \mathfrak{F}(\mathbb{R}^n)$ such that $H(x) = 1$ for $x \in \overline{B_a(0)}$, $0 < H(x) < 1$ for $x \in B_b(0) \setminus \overline{B_a(0)}$, and $H(x) = 0$ for $x \in \mathbb{R}^n \setminus B_b(0)$.*

This concept is generalised from Euclidean space to manifolds, and the following lemma is fundamental in our theory.

Lemma 1.17. *Let M be a manifold and U be some neighbourhood of $p \in M$. Then there are a neighbourhood of $V \ni p$ and a compactly supported smooth function $b\colon M \to [0,1] \subset \mathbb{R}$ such that $b{\restriction}_V = 1$ and $\operatorname{supp}(b) \subset U$.*

Proof. According to Lemma 1.4 there is a chart (W, φ) centred at $p \in M$ such that $W \subseteq U$. There is a small enough $\varepsilon > 0$ such that $B_{3\varepsilon}(0) \subseteq \varphi(W) \subseteq \mathbb{R}^n$. Let $H\colon \mathbb{R}^n \to \mathbb{R}$ for $n = \dim M$ be a smooth function from Lemma 1.16 with $H(x) = 1$ for $\|x\| \leq \varepsilon$, $0 < H(x) < 1$ for $\varepsilon < \|x\| < 2\varepsilon$, and $H(x) = 0$ for $\|x\| \geq 2\varepsilon$. The new function $f = H \circ \varphi\colon W \to [0,1] \subset \mathbb{R}$ is smooth and supported in $\varphi^{-1}(\overline{B_{2\varepsilon}(0)}) \subset W$, so by Lemma 1.11 it extends by zero to $b \in \mathfrak{F}(M)$. For $V = \varphi^{-1}(B_\varepsilon(0)) \ni p$ we have $b{\restriction}_V = 1$, while $\operatorname{supp}(b) = \varphi^{-1}(\overline{B_{2\varepsilon}(0)}) \subset W \subseteq U$. □

The function $f \in \mathfrak{F}(M)$ from Lemma 1.17 is called a ***bump function*** at p and it easily extends a smooth function on the whole manifold M, which we see in the following lemma.

Lemma 1.18. *Let M be a manifold and U be some neighbourhood of $p \in M$. For $f \in \mathfrak{F}(U)$ there exist a neighbourhood $V \ni p$ and $F \in \mathfrak{F}(M)$ such that $\operatorname{supp}(F) \subseteq U$ and $F{\restriction}_V = f{\restriction}_V$.*

Proof. Applying Lemma 1.17 we find an open $V \ni p$ and a bump function $b \in \mathfrak{F}(M)$ supported in U with $b{\restriction}_V = 1$. The function $bf \in \mathfrak{F}(U)$ is supported in the closed subset $\operatorname{supp}(b) \subset U \subseteq M$, so by Lemma 1.11 it extends by zero to $F \in \mathfrak{F}(M)$, where we have $\operatorname{supp}(F) \subseteq U$ and $F{\restriction}_V = (bf){\restriction}_V = f{\restriction}_V$. □

When in Lemma 1.17, we replace a point from a neighbourhood by a compact subset, we get the important consequence.

Lemma 1.19. *If $K \subseteq U \subseteq M$ holds for a manifold M, a compact K, and an open U, then there is a smooth function $f\colon M \to [0,1] \subset \mathbb{R}$ supported in U with $f{\restriction}_K = 1$.*

Proof. According to Lemma 1.17, for each $p \in K \subseteq U$ there exist an open $V_p \ni p$ and a smooth function $f_p\colon M \to [0,1] \subset \mathbb{R}$ supported in U with $f_p{\restriction}_{V_p} = 1$. The family $\{V_p\}_{p \in K}$ is an open cover of the compact K and therefore it has a finite subcover $\{V_{p_1}, \ldots, V_{p_k}\}$. The function $f\colon M \to [0,1] \subset \mathbb{R}$ given by $f = 1 - (1 - f_{p_1})(1 - f_{p_2}) \cdots (1 - f_{p_k})$ is clearly smooth. Since $f(x) \neq 0$, if there exists $1 \leq i \leq k$ such that $f_{p_i}(x) \neq 0$, we have $\operatorname{supp}(f) \subseteq \bigcup_i \operatorname{supp}(f_{p_i}) \subseteq U$, which means that f is supported in U. Finally, $f{\restriction}_{V_p} = 1$ holds for $1 \leq i \leq k$, and therefore because of $K \subseteq \bigcup_{i=1}^k V_{p_i}$ we have $f{\restriction}_K = 1$. □

A partition of unity is a decomposition $\sum_{\alpha\in\Lambda} f_\alpha = 1$ of constant function 1 (the word unity stands for this function) into a sum of smooth functions f_α. We usually deal with infinite sums, so our functions are indexed by some infinite set Λ, so it is natural to require that for each $p \in M$ we have $f_\alpha(p) = 0$ for all but a finite number of $\alpha \in \Lambda$. The sum is then well defined as a function on M and we can retain the smoothness through the following definition.

A ***partition of unity*** on a manifold M is a family $\{f_\alpha\}_{\alpha\in\Lambda}$ of smooth functions $f_\alpha : M \to [0,1] \subset \mathbb{R}$, such that the family of supports $\{\operatorname{supp}(f_\alpha)\}_{\alpha\in\Lambda}$ is locally finite and $\sum_{\alpha\in\Lambda} f_\alpha(p) = 1$ holds for all $p \in M$. We say that a partition of unity is ***subordinate*** to some cover of M if the family of all supports is a refinement of this cover.

This definition implies that for every $p \in M$ there exists some $\alpha \in \Lambda$ such that $f_\alpha(p) > 0$, and thus $\{\operatorname{supp}(f_\alpha)\}_{\alpha\in\Lambda}$ is a cover of M. This cover is locally finite, so the (infinite) sum $\sum_{\alpha\in\Lambda} f_\alpha(p)$ is well defined. If $\{U_\alpha\}_{\alpha\in\Lambda}$ is a cover of M and $\operatorname{supp}(f_\alpha) \subseteq U_\alpha$ holds for all $\alpha \in \Lambda$, then we say that $\{f_\alpha\}_{\alpha\in\Lambda}$ is subordinate to $\{U_\alpha\}_{\alpha\in\Lambda}$ with the same index set as the partition of unity. However, if $\{f_\alpha\}_{\alpha\in\Lambda}$ is a partition of unity subordinate to a cover $\{V_\beta\}_{\beta\in\Delta}$, then $\{\operatorname{supp}(f_\alpha)\}_{\alpha\in\Lambda}$ is a refinement of $\{V_\beta\}_{\beta\in\Delta}$ and we can use the refinement map $\varphi\colon \Lambda \to \Delta$ satisfying $\operatorname{supp}(f_\alpha) \subseteq V_{\varphi(\alpha)}$ for every $\alpha \in \Lambda$, and get the partition of unity $\{\sum_{\alpha\in\varphi^{-1}(\beta)} f_\alpha\}_{\beta\in\Delta}$ that is subordinate to $\{V_\beta\}_{\beta\in\Delta}$ with the same index set.

Theorem 1.20. *For any open cover of a manifold there exists a partition of unity subordinate to it.*

Proof. Let $\mathcal{U}$ be an open cover of M. Since by Lemma 1.7 M is locally compact, there is an open refinement $\mathcal{U}'$ of $\mathcal{U}$ consists of relatively compact sets. Then, since by Lemma 1.10 M is paracompact, we may find a new open refinement $\mathcal{V}$ of $\mathcal{U}'$ which is locally finite.

Consider the set $\mathcal{W} = \{U \subseteq M : U \text{ is open and } \overline{U} \subseteq V \text{ for some } V \in \mathcal{V}\}$. Each $p \in M$ has a neighbourhood $V \in \mathcal{V}$, and since by Lemma 1.9 M is regular, the point p and the closed set $M \setminus V \not\ni p$ are separated by some open disjoint sets $P \ni p$ and $Q \supseteq M \setminus V$. Because of $P \cap Q = \emptyset$ we have $P \subseteq M \setminus Q$, so $\overline{P} \subseteq M \setminus Q$, while $M \setminus V \subseteq Q$ implies $M \setminus Q \subseteq V$, and therefore $\overline{P} \subseteq M \setminus Q \subseteq V$. Hence $p \in P \in \mathcal{W}$, which means that $\mathcal{W}$ is a cover of M, and therefore it is an open refinement of $\mathcal{V}$.

Another use of paracompactness gives a locally finite open refinement $\mathcal{W}'$ of $\mathcal{W}$. The family $\mathcal{W}' = \{W_\beta\}_{\beta\in\Delta}$ is a refinement of $\mathcal{V} = \{V_\alpha\}_{\alpha\in\Lambda}$ and for each $\beta \in \Delta$ there is $\varphi(\beta) \in \Lambda$ such that $\overline{W_\beta} \subseteq V_{\varphi(\beta)}$, which defines $\varphi\colon \Delta \to \Lambda$ and we can set

$$X_\alpha = \bigcup_{\beta\in\phi^{-1}(\alpha)} W_\beta,$$

where by convention it is the empty set in a case of $\varphi^{-1}(\alpha) = \emptyset$. Since $\mathcal{W}'$ is a locally finite family, for every $p \in M \setminus \bigcup_{\beta\in\phi^{-1}(\alpha)} \overline{W_\beta}$ there is a neighbourhood $U \ni p$ which intersects just finite number of members, for example $W_1, \dots, W_k$. For other W_β we have $W_\beta \cap U = \emptyset$, it implies $W_\beta \subseteq M \setminus U$, so $\overline{W_\beta} \subseteq M \setminus U$ and therefore $\overline{W_\beta} \cap U = \emptyset$. Because of this the set $U \setminus \bigcup_{\beta\in\phi^{-1}(\alpha)} \overline{W_\beta} = U \setminus (\overline{W_1} \cup \dots \cup \overline{W_k})$ is a neighbourhood of p, which means that $M \setminus \bigcup_{\beta\in\phi^{-1}(\alpha)} \overline{W_\beta}$ is open, so $\bigcup_{\beta\in\phi^{-1}(\alpha)} \overline{W_\beta}$ is closed. Thus, we have

$$\overline{X_\alpha} = \overline{\bigcup_{\beta\in\phi^{-1}(\alpha)} W_\beta} = \bigcup_{\beta\in\phi^{-1}(\alpha)} \overline{W_\beta} \subseteq V_\alpha,$$

so for any $\alpha \in \Lambda$ there exists a compact $\overline{X_\alpha}$ contained in open V_α, and by Lemma 1.19 there is a smooth function $f_\alpha \in \mathfrak{F}(M)$ supported in V_α and equal to 1 on $\overline{X_\alpha}$. The sum $f = \sum_{\alpha\in\Lambda} f_\alpha$ is a smooth function which is always finite non-zero. Therefore, the sum of new functions f_α/f is 1 everywhere. ☐

Example 1.36. Urysohn's[28] lemma is known from topology and states that a topological space is normal if and only if any two nonempty disjoint closed subsets can be separated by a continuous function in the sense that a continuous function exists which takes value 0 on one of the two subsets and value 1 on the other. Let A and B be two disjoint closed subsets in a manifold M. According to Theorem 1.20 there exists a partition of unity $\{\psi_A, \psi_B\}$ subordinate to the open cover $\{M \setminus A, M \setminus B\}$. The function $\psi_A \colon M \to [0,1] \subset \mathbb{R}$ is smooth and it satisfies $\psi_A\restriction_A = 0$ as well as $\psi_A\restriction_B = 1 - \psi_B\restriction_B = 1$, which gives the smooth version of Urysohn's lemma. In particular, any manifold is normal. △

Example 1.37. Let $f\colon A \to \mathbb{R}$ be a smooth function on an arbitrary subset $A \subseteq M$. Then for every $a \in A$ there exists $f_a \in \mathfrak{F}(U_a)$ such that $U_a \subseteq M$ is a neighbourhood of a and $f = f_a$ on $U_a \cap A$. By Theorem 1.20 there exists a partition of unity $\{\psi_a\}_{a\in A}$ subordinate to the open cover $\{U_a : a \in A\}$ of $U = \bigcup_{a\in A} U_a \subseteq M$. Since $f_a \in \mathfrak{F}(U_a)$ and $\operatorname{supp}(\psi_a) \subseteq U_a$ we obtain $\psi_a f_a \in \mathfrak{F}(U)$, which allows us to define $F = \sum_{a\in A} \psi_a f_a \in \mathfrak{F}(U)$. For $x \in A$ we have $F(x) = \sum_{a\in A} \psi_a f_a(x) = \sum_{a\in A} f(x)\psi_a(x) = f(x)$. Hence, any smooth $f\colon A \to \mathbb{R}$ has a smooth extension $F\colon U \to \mathbb{R}$ to an open set $U \supseteq A$.

If we additionally assume that A is closed, then there exists a partition of unity $\{\psi_1, \psi_2\}$ subordinate to the open cover $\{U, M \setminus A\}$ of M. Since $F \in \mathfrak{F}(U)$ and $\operatorname{supp}(\psi_1) \subseteq U$ we obtain $\psi_1 F \in \mathfrak{F}(M)$, while $\operatorname{supp}(\psi_2) \subseteq M \setminus A$ implies $\psi_1\restriction_A = 1 - \psi_2\restriction_A = 1$, and therefore $\psi_1 F = f$ on A. Thus, any smooth function on a closed subset of M can be extended to a smooth function on M. △

1.7 Problems

Problem 1.1. Is the smooth compatibility of charts on M a relation of equivalence for every nonempty set M?

Problem 1.2. Given the set $M = \mathbb{R} \cup \{\infty\}$, as well as the functions $\varphi = \mathbb{1}_{\mathbb{R}} \colon M \setminus \{\infty\} = \mathbb{R} \to \mathbb{R}$ and $\psi\colon M \setminus \{0\} \to \mathbb{R}$, where $\psi(x) = x$ for $x \neq 0$ and $\psi(\infty) = 0$. Does M with the atlas $\{\varphi, \psi\}$ form a smooth manifold?

Problem 1.3. Given the topological surface $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2 + r\} \subset \mathbb{R}^3$ with the subspace topology. For which values $r \in \mathbb{R}$ there exist an atlas such that M becomes a smooth manifold?

Problem 1.4. Determine (the most natural) atlas so that the set of all affine lines in $\mathbb{R}^2$ forms a smooth manifold.

Problem 1.5. Examine whether the function $f\colon \mathbf{S}^2 \to \mathbb{R}$ given by $f(x, y, z) = \sqrt{1 - z^2}$ is smooth.

Problem 1.6. Prove that the map $f\colon \mathbb{R}\mathbf{P}^2 \to \mathbb{R}$ given by

$$(x : y : z) \mapsto \frac{yz + xz + xy}{x^2 + y^2 + z^2}$$

is well-defined and smooth.

Problem 1.7. Prove that the map $f\colon \mathbb{R}\mathbf{P}^1 \to \mathbb{R}\mathbf{P}^1$ given by $(t : 1) \mapsto (t + 1 : 1)$ for $t \in \mathbb{R}$ and $(1 : 0) \mapsto (1 : 0)$ is smooth.

Problem 1.8. Prove that for every $p \in \mathbf{B}^n = B_1(0) \subset \mathbb{R}^n$ there exists a diffeomorphism $f\colon \mathbf{B}^n \to \mathbf{B}^n$ such that $f(0) = p$.

[28] Pavel Samuilovich Urysohn (1898–1924), Soviet mathematician

Problem 1.9. Given the set $M = \mathbb{C} \cup \{\infty\}$, as well as the functions $\varphi = \mathbb{1}_{\mathbb{C}} \colon M \setminus \{\infty\} = \mathbb{C} \to \mathbb{C} \cong \mathbb{R}^2$ and $\psi \colon M \setminus \{0\} \to \mathbb{C} \cong \mathbb{R}^2$, where $\psi(z) = 1/z$ for $z \neq 0$ and $\psi(\infty) = 0$. Prove that M with the atlas $\{\varphi, \psi\}$ forms a smooth manifold that is diffeomorphic to $\mathbf{S}^2$.

Problem 1.10. Exemine whether the map $f \colon \mathbb{F}\mathbf{P}^1 \to \mathbb{F}\mathbf{P}^1$ given by $f(t : 1) = (e^{t^2} : 1)$ for $t \in \mathbb{F}$ and $f(1 : 0) = (1 : 0)$ is smooth, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

Problem 1.11. Let $f \colon M \to \mathbb{R}$ be a positive continuous function on a manifold M. Prove that there exists $h \in \mathfrak{F}(M)$ such that for every $p \in M$ we have $0 < h(p) < f(p)$.

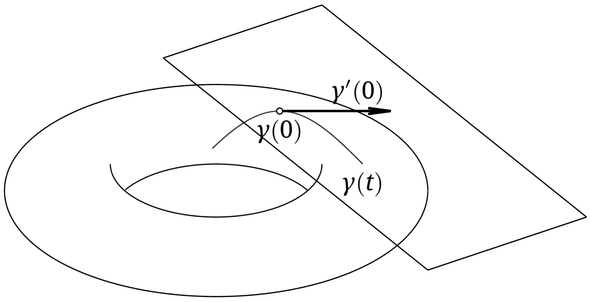
Problem 1.12. Let M be a manifold, $A \subset M$ a closed subset, and $f \colon A \to (0, \infty)$ smooth. Prove that there exists a smooth function $F \colon M \to (0, \infty)$ such that $F{\restriction_A} = f$.

CHAPTER 2

TANGENT SPACES AND MAPS

2.1 Tangent vectors

The most intuitive way to define a tangent vector is to use curves on a manifold. Let $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be a smooth curve in an n-manifold M which goes through a point $p \in M$, where $\varepsilon > 0$ is a small real number. In other words γ is a smooth map such that $\gamma(0) = p$. A tangent vector of manifold M at a point p can be seen as a derivative of a curve γ at zero. However, if M is not contained in some Euclidean space, the derivative $\gamma'(0)$ does not make sense. The common idea is to choose a chart (U, φ) at $p \in M$ and identify the curve γ on M with the curve $\varphi \circ \gamma$ on $\mathbb{R}^n$, so our tangent vector can be well defined $(\varphi \circ \gamma)'(0)$.



Since different smooth curves on M which go through p can give the same tangent vector, there is an obvious equivalence relation among these curves. Two such curves γ_1 and γ_2 are equivalent if there is a chart (U, φ) at p, such that $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$. Let (V, ψ) be a chart at p, then $p \in U \cap V \neq \emptyset$ and $\omega = \psi \circ \varphi^{-1}$ is smooth, so

$$\begin{aligned}(\psi \circ \gamma_1)'(0) &= (\omega \circ \varphi \circ \gamma_1)'(0) = \omega'(\varphi(p)) \cdot (\varphi \circ \gamma_1)'(0) \\ &= \omega'(\varphi(p)) \cdot (\varphi \circ \gamma_2)'(0) = (\omega \circ \varphi \circ \gamma_2)'(0) = (\psi \circ \gamma_2)'(0),\end{aligned}$$

and therefore the curve equivalence is chart independent.

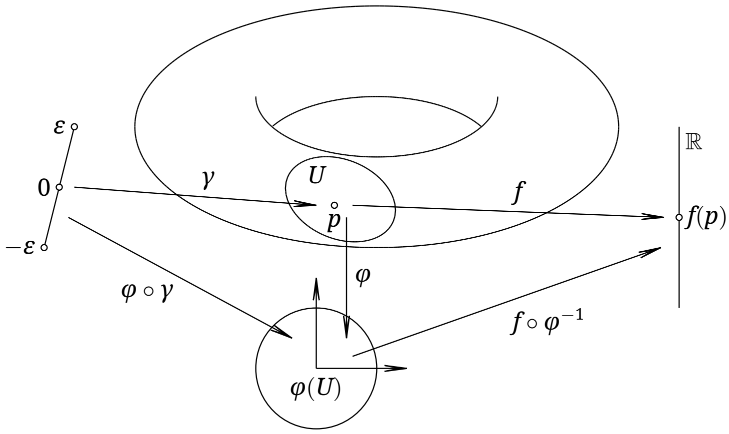
Alternatively, we can define $\gamma'(0)\colon \mathfrak{F}(M) \to \mathbb{R}$ by

$$(\gamma'(0))(f) = (f \circ \gamma)'(0) = \left.\frac{d(f \circ \gamma)}{dt}\right|_{t=0}$$

and get the relation between different definitions by $\Phi\colon (\varphi \circ \gamma)'(0) \mapsto \gamma'(0)$. Equivalent curves γ_1 and γ_2 imply

$$(f \circ \gamma_1)'(0) = (f \circ \varphi^{-1})'(\varphi(p)) \cdot (\varphi \circ \gamma_1)'(0) = (f \circ \varphi^{-1})'(\varphi(p)) \cdot (\varphi \circ \gamma_2)'(0) = (f \circ \gamma_2)'(0),$$

so Φ depends only on the equivalence class of γ.



The definition of a tangent vector over curves is very geometric, but it is not immediately obvious where the vector space structure comes from. However, since we have $(\alpha f + \beta h) \circ \gamma = \alpha(f \circ \gamma) + \beta(h \circ \gamma)$, an evident property of the function $\gamma'(0)$ is linearity. From the other side, since $(fh) \circ \gamma = (f \circ \gamma)(h \circ \gamma)$ holds, we have

$$\begin{aligned}(\gamma'(0))(fh) &= ((f \circ \gamma)(h \circ \gamma))'(0) = (f \circ \gamma)'(0) \cdot h(\gamma(0)) + (h \circ \gamma)'(0) \cdot f(\gamma(0)) \\ &= h(p)(\gamma'(0))(f) + f(p)(\gamma'(0))(h),\end{aligned}$$

which means that $\gamma'(0)$ satisfies the Leibniz[1] law (product rule). Hence, the function $\gamma'(0)\colon \mathfrak{F}(M) \to \mathbb{R}$ has properties characteristic for the derivative (an $\mathbb{R}$-linear function which is Leibnizian at p), so a real-valued function on $\mathfrak{F}(M)$ with this features we call the ***derivation at point*** p. This properties for a map $\gamma'(0)$ motivate us to introduce the definition of a tangent vector in the following way.

A ***tangent vector*** to a manifold M at a point $p \in M$ is any derivation at p. The set T_pM of all tangent vectors on a manifold M at a point $p \in M$ is called the ***tangent space*** to M at p. Accordingly, if $X\colon \mathfrak{F}(M) \to \mathbb{R}$ has $X \in T_pM$, then for all $\alpha, \beta \in \mathbb{R}$ and every $f, h \in \mathfrak{F}(M)$ hold linearity $X(\alpha f + \beta h) = \alpha X(f) + \beta X(h)$ and Leibnizian $X(fh) = f(p)X(h) + h(p)X(f)$.

The addition and the scalar multiplication we introduce naturally. If $X, Y \in T_pM$ then $(X+Y)(f) = X(f) + Y(f)$ and $(\alpha X)(f) = \alpha X(f)$ hold for each $f \in \mathfrak{F}(M)$ and all $\alpha \in \mathbb{R}$. Such defined $X + Y$ and αX are also tangent vectors and thus T_pM is a vector space over $\mathbb{R}$.

By their very nature manifolds are curved spaces, and they can be very complicated objects to study, while vector spaces are much simpler with many benefits. The key idea of calculus is linear approximation. A tangent space T_pM is a vector space, and it can be thought of as the best linear approximation to a manifold M at a point $p \in M$. This is the reason why we are minded to use a tangent space instead of an original manifold.

Naturally, the derivative of a constant function vanishes. For $X \in T_pM$ and the unit function $1 \in \mathfrak{F}(M)$ we have $X(1) = X(1 \cdot 1) = 1(p)X(1) + 1(p)X(1) = 2X(1)$ and therefore $X(1) = 0$, while for an arbitrary $\alpha \in \mathbb{R}$, linearity gives $X(\alpha 1) = \alpha X(1) = 0$.

Lemma 2.1. *If $f \in \mathfrak{F}(M)$ is a constant function on a manifold $M \ni p$, then $Xf = 0$ holds for each $X \in T_pM$.*

The tangent space is defined in terms of smooth functions on the whole manifold, while charts are in general defined on some open subsets. However, tangent vectors act locally on $\mathfrak{F}(M)$.

Lemma 2.2. *If $f \in \mathfrak{F}(M)$ and $h \in \mathfrak{F}(M)$ agree on some neighbourhood of a point p of a manifold M, then $Xf = Xh$ holds for each $X \in T_pM$.*

[1] Gottfried Wilhelm von Leibniz (1646–1716)

Proof. Let $f = h$ on some neighbourhood $U \ni p$. By Lemma 1.17 there exists a function $b \in \mathfrak{F}(M)$ supported in U such that $b(p) = 1$. Since $(f - h)b = 0$ holds on all of M, we have $0 = X((f - h)b) = b(p)X(f - h) + (f - h)(p)X(b) = X(f - h)$, and hence $Xf = Xh$. □

The previous lemma is very important, because it allows the notion of germs. Two functions $f\colon U \to \mathbb{R}$ and $h\colon V \to \mathbb{R}$, locally defined on some neighbourhoods of the fixed point $p \in M$, are equivalent if there exists some open subset $W \subseteq U \cap V$, containing p, so that $f{\restriction_W} = h{\restriction_W}$. The equivalence class of such functions is called a ***germ*** at p.

Since we work with smooth functions (instead of C^k functions with $k < \infty$), any locally defined smooth function by Lemma 1.18 has an equivalent smooth function defined on the whole manifold M. Let us notice that if f and h are extensions of equivalent smooth functions, by Lemma 2.2 we have $Xf = Xh$ for any $X \in T_pM$. Thus, a tangent vector $X \in T_pM$ is essentially defined on the germ at p, so we also use the notation Xf for functions $f \in \mathfrak{F}(U)$ where $U \ni p$ is an open subset of M, which takes the value of X in an arbitrary (thus, each) function in $\mathfrak{F}(M)$ that is equivalent to f at p.

To define partial differentiation on a manifold, we should move the function $f \in \mathfrak{F}(M)$ back to the Euclidean space using some chart, and then take the usual partial derivatives. Let (U, φ) be a chart on an n-manifold M at a point $p \in M$, and $x_i = \pi_i \circ \varphi$ for $1 \leq i \leq n$ are appropriate coordinate functions. The ***partial derivative*** of a function $f \in \mathfrak{F}(M)$ for $1 \leq i \leq n$ we define by

$$(\partial_i)_p(f) = \left(\frac{\partial}{\partial x_i}\right)_p(f) = \frac{\partial f}{\partial x_i}(p) = \frac{\partial(f \circ \varphi^{-1})}{\partial \pi_i}(\varphi(p)).$$

Since our definition comes from a derivative we immediately have linearity and the Leibniz's law, so such defined function $(\partial_i)_p\colon \mathfrak{F}(M) \to \mathbb{R}$ is a tangent vector to M at p. Moreover, these partial derivatives form a basis of our tangent space T_pM.

We start from the Hadamard's[2] lemma (Lemma A.29) according to which if $U \ni a$ is an open convex neighbourhood in $\mathbb{R}^n$, then for $f \in \mathfrak{F}(U)$ there exist functions $l_i \in \mathfrak{F}(U)$ for $1 \leq i \leq n$ such that for each $x \in U$ we have

$$f(x) = f(a) + \sum_{i=1}^{n}(\pi_i(x) - \pi_i(a))l_i(x),$$

where $l_i(a) = (\partial f/\partial \pi_i)(a)$.

The Hadamard's lemma is essentially a first-order form of Taylor's[3] theorem. By iterating the previous expansion we arrive at a second-order form,

$$\begin{aligned} f(x) &= f(a) + \sum_{i=1}^{n}(\pi_i(x) - \pi_i(a))\left(l_i(a) + \sum_{j=1}^{n}(\pi_j(x) - \pi_j(a))l_{ij}(x)\right) \\ &= f(a) + \sum_{i=1}^{n}(\pi_i(x) - \pi_i(a))\frac{\partial f}{\partial \pi_i}(a) + \sum_{i,j=1}^{n}(\pi_i(x) - \pi_i(a))(\pi_j(x) - \pi_j(a))l_{ij}(x), \end{aligned}$$

where $l_{ij} \in \mathfrak{F}(U)$ for $1 \leq i, j \leq n$.

Let (U, φ) be a chart at $p \in M$ such that $\varphi(U)$ is convex (for instance, an open ball), and let $f \in \mathcal{F}(U)$. Applying the previous formula for the smooth function $f \circ \varphi^{-1} \in \mathfrak{F}(\varphi(U))$ and points $a = \varphi(p)$ and $x = \varphi(q)$, we get

$$f(q) = f(p) + \sum_{i=1}^{n}(x_i(q) - x_i(p))\frac{\partial(f \circ \varphi^{-1})}{\partial \pi_i}(\varphi(p)) + \sum_{i,j=1}^{n}(x_i(q) - x_i(p))(x_j(q) - x_j(p))l_{ij}(\varphi(q)),$$

[2] Jacques Salomon Hadamard (1865–1963), French mathematician
[3] Brook Taylor (1685–1731), English mathematician

that is,

$$f = f(p) + \sum_{i=1}^{n}(x_i - x_i(p))\frac{\partial f}{\partial x_i}(p) + \sum_{i,j=1}^{n}(x_i - x_i(p))(x_j - x_j(p))(l_{ij} \circ \varphi),$$

where $l_{ij} \circ \varphi$ is smooth for all $1 \leq i, j \leq n$.

This allows us to examine the value of Xf for an arbitrary tangent vector $X \in T_pM$. The first term is $f(p)$, which is constant, so it vanishes by Lemma 2.1. The Leibnizian property implies that X vanishes the last sum since each of three obtained terms has at least one factor of the form $(x_i - x_i(p))(p) = 0$. It remains the sum in the middle, where

$$Xf = X\left(\sum_{i=1}^{n}(x_i - x_i(p))\frac{\partial f}{\partial x_i}(p)\right) = \sum_{i=1}^{n} X(x_i)\left(\frac{\partial}{\partial x_i}\right)_p f,$$

and therefore we obtain

$$X = \sum_{i=1}^{n} X(x_i)\left(\frac{\partial}{\partial x_i}\right)_p. \tag{2.1}$$

The formula (2.1) implies that the partial derivatives $(\partial_i)_p$ generate the tangent space T_pM. On the other hand, applying these partial derivatives on the coordinate functions we get $(\partial_i)_p(x_j) = \delta_{ij}$. Then $\sum_{i=1}^{n} \lambda_i(\partial_i)_p = 0$ implies $\lambda_j = (\lambda_1(\partial_1)_p + \cdots + \lambda_n(\partial_n)_p)(x_j) = 0$ for all $1 \leq j \leq n$, so $(\partial_1)_p, \ldots, (\partial_n)_p$ are linearly independent, and therefore they form a basis of the tangent space. The following theorem summarizes the previous discussion and establishes the fundamental link between coordinates and tangent vectors.

Theorem 2.3. *Let M be an n-manifold and (U, φ) be a chart at $p \in M$ with $x_i = \pi_i \circ \varphi$. Then the partial derivatives $(\partial_1)_p, \ldots, (\partial_n)_p$ form a basis of T_pM and the formula* (2.1) *holds for all $X \in T_pM$.*

Let us go back to the beginning where we had a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$, a chart (U, φ) at $p = \gamma(0)$, and the relation $\Phi\colon (\varphi \circ \gamma)'(0) \mapsto \gamma'(0)$ which does not depend of the choice of an equivalent curve. We have already seen that $\gamma'(0)$ is a derivation at p, which means that $\gamma'(0) \in T_pM$.

The map Φ is injective since $(\varphi \circ \gamma_1)'(0) \neq (\varphi \circ \gamma_2)'(0)$ means that there exists some $1 \leq i \leq n$ such that $(\pi_i \circ \varphi \circ \gamma_1)'(0) \neq (\pi_i \circ \varphi \circ \gamma_2)'(0)$, respectively $(\gamma_1'(0))(x_i) \neq (\gamma_2'(0))(x_i)$, and therefore $\gamma_1'(0) \neq \gamma_2'(0)$. The map Φ is also surjective since any $X \in T_pM$ by (2.1) has the form $X = \sum_{i=1}^{n} \lambda_i(\partial_i)_p$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. This allows us to construct the curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ for small $\varepsilon > 0$ with $\gamma(t) = \varphi^{-1}(\varphi(p) + t(\lambda_1, \ldots, \lambda_n))$, for which we have

$$(\gamma'(0))(f) = (f \circ \gamma)'(0) = (f \circ \varphi^{-1})'(\varphi(p)) \cdot (\lambda_1, \ldots, \lambda_n) = X(f).$$

In this way we have shown that T_pM can be identified with the equivalence classes of smooth curves which go through the point $p \in M$,

2.2 Tangent maps

The basic idea of differential calculus is to approximate smooth objects by linear objects. The tangent space T_pM is a linear approximation of a manifold M at a point $p \in M$. The next idea is to approximate a smooth map between manifolds by a linear transformation of tangent spaces.

Let M and N be manifolds, and $f\colon M \to N$ is a smooth map. For $X \in T_pM$ and $h \in \mathfrak{F}(N)$ we can naturally set a linear map $T_pf(X)\colon \mathfrak{F}(N) \to \mathbb{R}$ by

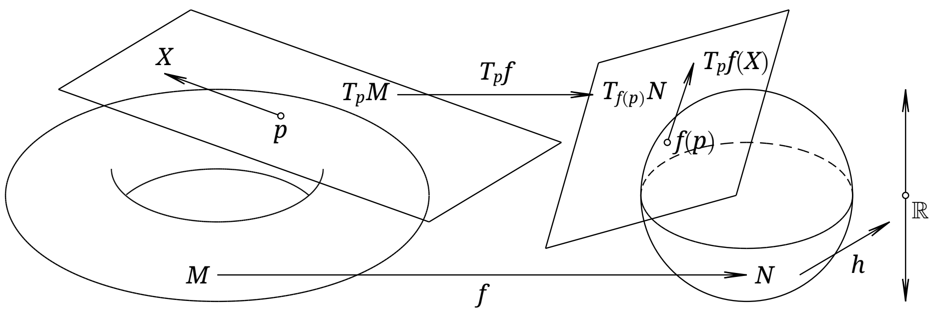
$$(T_pf(X))(h) = X(h \circ f).$$

Moreover, for $h_1, h_2 \in \mathfrak{F}(N)$ we have

$$\begin{aligned} T_pf(X)(h_1h_2) &= X((h_1h_2)\circ f) = X((h_1\circ f)(h_2\circ f)) = (h_1\circ f)(p)X(h_2\circ f) + (h_2\circ f)(p)X(h_1\circ f) \\ &= h_1(f(p))T_pf(X)(h_2) + h_2(f(p))T_pf(X)(h_1), \end{aligned}$$

so it is Leibnizian at $f(p)$, while the linearity is obvious. We can conclude that $T_pf(X)$ is a derivation at $f(p) \in N$, that is, $T_pf(X) \in T_{f(p)}N$ holds.

The previous result for any smooth $f\colon M \to N$ allows to define the ***tangent map*** of f at a point $p \in M$ by $T_pf\colon T_pM \to T_{f(p)}N$ and $T_pf(X)(h) = X(h\circ f)$, where $X \in T_pM$ and $h \in \mathfrak{F}(N)$. The tangent map T_pf is linear, and it is worth noting that some authors call it the ***differential*** of f at p, while the notations df_p, f_p, and D_pf are also used.



Example 2.1. The tangent map of the identity map $\mathbb{1}_M\colon M \to M$ at some point $p \in M$ is the identity map $\mathbb{1}_{T_pM}\colon T_pM \to T_pM$ since for $X \in T_pM$ and $f \in \mathfrak{F}(M)$ we have the equation $((T_p(\mathbb{1}_M))(X))(f) = X(f\circ \mathbb{1}_M) = X(f)$. △

The chain rule can easily be generalised to manifolds, which gives that the tangent map of composition is the composition of tangent maps.

Lemma 2.4. *Given any two smooth maps $f\colon M \to N$ and $h\colon N \to P$ between manifolds, for each point $p \in M$ we have $T_p(h\circ f) = T_{f(p)}h \circ T_pf$.*

Proof. The result follows directly from the straightforward calculation for $X \in T_pM$ and $l \in \mathfrak{F}(N)$, where we obtain $T_p(h\circ f)(X)(l) = X(l\circ h\circ f) = T_pf(X)(l\circ h) = (T_{f(p)}h(T_pf(X)))(l)$. □

As a consequence, a diffeomorphism between manifolds induces an isomorphism between the corresponding vector spaces.

Lemma 2.5. *Let $f\colon M \to N$ be a diffeomorphism between manifolds and $p \in M$, then $T_pf\colon T_pM \to T_{f(p)}N$ is an isomorphism between vector spaces and holds $(T_pf)^{-1} = T_{f(p)}(f^{-1})$.*

Proof. Since f and f^{-1} are smooth, $\mathbb{1}_{T_pM} = T_p(\mathbb{1}_M) = T_p(f^{-1}\circ f) = T_{f(p)}(f^{-1})\circ T_pf$ holds by Lemma 2.4, as well as $\mathbb{1}_{T_{f(p)}N} = T_{f(p)}(\mathbb{1}_N) = T_{f(p)}(f\circ f^{-1}) = T_pf\circ T_{f(p)}(f^{-1})$. □

A direct consequence of the previous lemma is the smooth version of the the dimension invariance theorem (Theorem A.5), which says that an m-manifold and an n-manifold cannot be diffeomorphic, unless $m = n$. The original theorem (when we replace the word diffeomorphic by homeomorphic) is not so easy to prove, but now we see that it is not necessary for our theory of smooth manifolds.

If U is an open submanifold of a manifold M with inclusion $\imath$, then both $\gamma\colon (-\varepsilon,\varepsilon) \to U$ and $\imath\circ\gamma\colon (-\varepsilon,\varepsilon) \to M$ essentially represent the same curve, so we expect that the tangent map of the inclusion is an isomorphism between the corresponding tangent spaces.

Lemma 2.6. *Let $U \subseteq M$ be an open subset with the inclusion $\imath\colon U \hookrightarrow M$. For each $p \in U$, the tangent map $T_p\imath\colon T_pU \to T_pM$ is an isomorphism.*

Proof. For $f \in \mathfrak{F}(U)$, by Lemma 1.18 there is an extension $F \in \mathfrak{F}(M)$ supported in U which agrees with f on some neighbourhood of p. If $T_p\imath(X) = 0 \in T_pM$ for some $X \in T_pU$, then by Lemma 2.2 $X(f) = X(F\restriction_U) = X(F \circ \imath) = ((T_p\imath)(X))(F) = 0$ holds for any $f \in \mathfrak{F}(U)$, which gives $X = 0$ and proves that $T_p\imath$ is injective. Since $\dim U = \dim M$ the map $T_p\imath$ is bijective, and therefore an isomorphism. $\square$

Example 2.2. Let M and N be manifolds and let $\pi_M \colon M \times N \to M$ and $\pi_N \colon M \times N \to N$ be canonical projections of the product $M \times N$. For each $(p,q) \in M \times N$, we can define the map $\pi \colon T_{(p,q)}(M \times N) \to T_pM \times T_qN$ by $\pi(X) = (T_{(p,q)}\pi_M(X), T_{(p,q)}\pi_N(X))$. In the opposite direction, we can define the map $\theta \colon T_pM \times T_qN \to T_{(p,q)}(M \times N)$ by $\theta(Y,Z) = T_p\mu(Y) + T_q\nu(Z)$ where $\mu \colon M \to M \times N$ is given by $\mu(x) = (x,q)$, and $\nu \colon N \to M \times N$ is given by $\nu(x) = (p,x)$. The calculation shows

$$\begin{aligned}\pi \circ \theta(Y,Z) &= (T_{(p,q)}\pi_M(T_p\mu(Y) + T_q\nu(Z)), T_{(p,q)}\pi_N(T_p\mu(Y) + T_q\nu(Z))) \\ &= (T_p(\pi_M \circ \mu)(Y) + T_q(\pi_M \circ \nu)(Z), T_p(\pi_N \circ \mu)(Y) + T_q(\pi_N \circ \nu)(Z)) = (Y,Z),\end{aligned}$$

from where we get that $\pi \circ \theta = \mathbb{1}_{T_pM \times T_qN}$ holds, so we can conlude that π is surjective, and since $\dim(T_{(p,q)}(M \times N)) = \dim(T_pM \times T_qN) = \dim M + \dim N$, we obtain that π is an isomorphism of vector spaces. $\triangle$

In the next two examples, we give some special cases in which the tangent map is equal to zero.

Example 2.3. If $f \colon M \to N$ is a constant map, then for $h \in \mathfrak{F}(N)$ the map $h \circ f$ is also constant, so by Lemma 2.1 for every $p \in M$ and $X \in T_pM$ we have $(T_pf(X))(h) = X(h \circ f) = 0$. Therefore for every $p \in M$ and constant map f we have $T_pf = 0$. $\triangle$

Example 2.4. Suppose that $f \in \mathfrak{F}(M)$ has a local extremum (minimum or maximum) at a point $p \in M$, then for a smooth curve $\gamma \colon (-\varepsilon, \varepsilon) \to M$ for which $\gamma(0) = p$ and every $h \in \mathfrak{F}(\mathbb{R})$ we have $(T_pf(\gamma'(0)))(h) = \gamma'(0)(h \circ f) = (h \circ f \circ \gamma)'(0) = h'(f(p))(f \circ \gamma)'(0) = 0$, and therefore $T_pf = 0$. $\triangle$

Although so far tangent spaces and tangent maps look rather abstract, in local coordinates things are becoming more practical. Let $f \colon M \to N$ be a smooth map between an m-manifold M and an n-manifold N. Let (U,φ) be a chart at $p \in M$ with $x_j = \pi_j \circ \varphi$ for $1 \le j \le m$, and (V,ψ) be a chart at $f(p) \in N$ with $y_i = \pi_i \circ \psi$ for $1 \le i \le n$. According to the formula (2.1), for each $1 \le j \le m$ we have

$$T_pf\left(\frac{\partial}{\partial x_j}\right)_p = \sum_{i=1}^{n} \left(T_pf\left(\frac{\partial}{\partial x_j}\right)_p\right)(y_i)\left(\frac{\partial}{\partial y_i}\right)_{f(p)} = \sum_{i=1}^{n} \frac{\partial(y_i \circ f)}{\partial x_j}(p)\left(\frac{\partial}{\partial y_i}\right)_{f(p)}. \tag{2.2}$$

The matrix of the tangent map T_pf with respect to these coordinate bases is

$$\begin{pmatrix} \frac{\partial(y_1 \circ f)}{\partial x_1}(p) & \cdots & \frac{\partial(y_1 \circ f)}{\partial x_m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial(y_n \circ f)}{\partial x_1}(p) & \cdots & \frac{\partial(y_n \circ f)}{\partial x_m}(p) \end{pmatrix} = \left(\frac{\partial(y_i \circ f)}{\partial x_j}(p)\right)_{1 \le i \le n, 1 \le j \le m}$$

called the ***Jacobian matrix***[4] of f at a point p relative to charts φ and ψ.

Since the entries of the Jacobian matrix satisfy

$$\frac{\partial(y_i \circ f)}{\partial x_j}(p) = \frac{\partial(\pi_i \circ (\psi \circ f \circ \varphi^{-1}))}{\partial \pi_j}(\varphi(p)),$$

[4] Carl Gustav Jacob Jacobi (1804–1851), German mathematician

we see that the tangent map is actually the derivative of the coordinate representation at the corresponding point, $T_pf = (\psi \circ f \circ \varphi^{-1})'(\varphi(p))$, as well as $T_pf = f'(p)$ in the case that f is a map between open subsets of Euclidean spaces.

Especially, if we set $f = \mathbb{1}_M : M \to M$, then we have the formula

$$\left(\frac{\partial}{\partial x_j}\right)_p = \sum_{i=1}^n \frac{\partial y_i}{\partial x_j}(p)\left(\frac{\partial}{\partial y_i}\right)_p , \tag{2.3}$$

which gives the rule for the change of coordinates, and the Jacobian matrix here is the Jacobian matrix of the transition map $\psi \circ \varphi^{-1}$ at $\varphi(p)$.

Example 2.5. The transition map between polar coordinates and standard coordinates in suitable open subsets of the plane is given by $(x, y) = (r\cos\theta, r\sin\theta)$. Let $p \in \mathbb{R}^2$ has polar coordinates $(r, \theta) = (3, \pi/2)$ and let $X \in T_p\mathbb{R}^2$ be the tangent vector whose polar coordinate representation is $X = 2(\partial/\partial r)_p - (\partial/\partial\theta)_p$. Concrete calculations by the formula (2.3) give

$$\left(\frac{\partial}{\partial r}\right)_p = \frac{\partial x}{\partial r}(p)\left(\frac{\partial}{\partial x}\right)_p + \frac{\partial y}{\partial r}(p)\left(\frac{\partial}{\partial y}\right)_p = \cos\frac{\pi}{2}\left(\frac{\partial}{\partial x}\right)_p + \sin\frac{\pi}{2}\left(\frac{\partial}{\partial y}\right)_p = \left(\frac{\partial}{\partial y}\right)_p$$
$$\left(\frac{\partial}{\partial \theta}\right)_p = \frac{\partial x}{\partial \theta}(p)\left(\frac{\partial}{\partial x}\right)_p + \frac{\partial y}{\partial \theta}(p)\left(\frac{\partial}{\partial y}\right)_p = -3\sin\frac{\pi}{2}\left(\frac{\partial}{\partial x}\right)_p + 3\cos\frac{\pi}{2}\left(\frac{\partial}{\partial y}\right)_p = -3\left(\frac{\partial}{\partial x}\right)_p ,$$

and therefore $X = 2(\partial/\partial r)_p - (\partial/\partial\theta)_p = 3(\partial/\partial x)_p + 2(\partial/\partial y)_p$. △

Example 2.6. It is important to note that a coordinate vector $(\partial_i)_p$ of the tangent space T_pM does not depend only on the specific coordinate function x_i, but on the entire coordinate system, because the derivative is obtained by differentiation with respect to x_i but while remaining coordinates are fixed. Let (x, y) denote the standard coordinates in $\mathbb{R}^2$, while the new coordinates $(\bar{x}, \bar{y})$ are given by $\bar{x} = x$ and $\bar{y} = y + x$. Then we have

$$\left(\frac{\partial}{\partial x}\right)_p = \left(\frac{\partial}{\partial \bar{x}}\right)_p + \left(\frac{\partial}{\partial \bar{y}}\right)_p \neq \left(\frac{\partial}{\partial \bar{x}}\right)_p ,$$

which differs at each point $p \in \mathbb{R}^2$. △

2.3 Submersions and immersions

Let $f: M \to N$ be a smooth map between manifolds. The tangent map $T_pf: T_pM \to T_{f(p)}N$ should represent the best linear approximation of f at $p \in M$, and investigating algebraic properties of the map T_pf we can conclude a lot about f itself. The most important property of the tangent map (as a linear map) is its rank, since it is independent of choices of bases. The ***rank*** of a map f at a point $p \in M$ is the rank of the tangent map,

$$\operatorname{rank}_p f = \operatorname{rank} T_pf = \dim \operatorname{Im}(T_pf).$$

If the rank is independent of the choice of point $p \in M$, we say that f has ***constant rank***, and the rank is denoted by $\operatorname{rank} f$.

The rank of a linear map has natural upper bounds, because it is never higher than the dimension of either its domain ($\operatorname{rank}_p f \leq \dim M$) or its codomain ($\operatorname{rank}_p f \leq \dim N$). If the rank $\operatorname{rank}_p f$ is equal to this upper bound we say that f has ***full rank at*** p, and if additionally f has constant rank we say that f has ***full rank***. The most important constant rank maps are those of full rank.

In the case of $\operatorname{rank} f = \dim M$, which means that T_pf is injective for any $p \in M$, we say that f is an ***immersion***, and then f locally looks like an injective map. In the case of $\operatorname{rank} f = \dim N$, which means that $T_p f$ is surjective for any $p \in M$, we say that f is a ***submersion***, and then f locally looks like a surjective map. Let us note that if a map is an immersion or submersion, according to our definition we always assume that it is smooth.

A prototype of immersion is the inclusion $\imath\colon \mathbb{R}^m \hookrightarrow \mathbb{R}^n$ of Euclidean space into Euclidean space of higher dimension ($m \leq n$) given by $\imath(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$. A prototype of submersion is the projection $\pi\colon \mathbb{R}^m \to \mathbb{R}^n$ of Euclidean space onto Euclidean space of lower dimension ($m \geq n$) given by $\pi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) = (x_1, \ldots, x_n)$. Moreover, it turns out (Theorem 2.10) that every immersion is locally an inclusion, while every submersion is locally a projection.

Example 2.7. Every regular curve γ on a manifold is an immersion since its derivative never vanishes, which gives $\operatorname{rank}\gamma = 1$. Especially, $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (\cos t, \sin t)$ is an immersion, but although $T_t\gamma$ is injective for all $t \in \mathbb{R}$, γ is not injective. △

Lemma 2.7. *Let $f\colon M \to N$ be a smooth map between manifolds and $p \in M$. If $T_p f$ is injective, then p has a neighbourhood U such that $f\restriction_U$ is an immersion. If $T_p f$ is surjective, then p has a neighbourhood U such that $f\restriction_U$ is a submersion.*

Proof. Let $\dim M = m$ and $\dim N = n$. If we choose a chart (U, φ) at $p \in M$ and a chart (V, ψ) at $f(p) \in N$, either hypothesis means that the Jacobian matrix of f relative to φ and ψ has full rank at p. Consider the continuous map $T\colon U \to \mathbb{R}^{n\times m}$ given by $T(p) = (\psi\circ f\circ\varphi^{-1})'(\varphi(p))$. Since matrices of full rank by Example 1.13 form an open subset of $\mathbb{R}^{n\times m}$, their preimage gives a neighbourhood of p where the Jacobian matrix of f has full rank. □

The inverse function theorem from real analysis (Theorem A.30) gives a sufficient condition for a smooth function to be invertible in a neighbourhood of a point in its domain. The ***inverse function theorem*** can be generalised in terms of smooth maps between manifolds.

Theorem 2.8. *Let $f\colon M \to N$ be a smooth map between manifolds. If $T_p f$ is invertible for some $p \in M$, then there exists a neighbourhood $U \subseteq M$ of p such that $f\restriction_U\colon U \to f(U)$ is a diffeomorphism.*

Proof. Since $T_p f$ is a bijection, we have $\dim M = \dim N = n$. Choose a chart (U, φ) centred at $p \in M$ and a chart (V, ψ) centred at $f(p) \in N$ such that $f(U) \subseteq V$, and $\psi \circ f \circ \varphi^{-1}$ is smooth. Since φ and ψ are diffeomorphisms, $(\psi\circ f\circ\varphi^{-1})'(0) = T_0(\psi\circ f\circ\varphi^{-1}) = T_{f(p)}\psi \circ T_p f \circ T_0\varphi^{-1}$ is invertible. Theorem A.30 guarantees a neighbourhood $W \subseteq \varphi(U) \subseteq \mathbb{R}^n$ of 0 such that the restriction $\psi\circ f\circ\varphi^{-1}\restriction_W$ is a diffeomorphism onto its image, so the restriction of f to $\varphi^{-1}(W)$ is a diffeomorphism onto its image as a composition of diffeomorphisms. □

A map $f\colon M \to N$ between manifolds is a ***local diffeomorphism*** if each point $p \in M$ has a neighbourhood U such that $f(U)$ is open in N and $f\restriction_U\colon U \to f(U)$ is a diffeomorphism. The key properties of local diffeomorphisms can be seen in the following theorem.

Theorem 2.9. *A smooth map $f\colon M \to N$ is a local diffeomorphism if and only if it is both immersion and submersion.*

Proof. if f is a local diffeomorphism then for each $p \in M$ there is a neighbourhood $U \ni p$ such that $f\restriction_U\colon U \to f(U)$ is a diffeomorphism, so $T_p(f\restriction_U)\colon T_pU \to T_{f(p)}f(U)$ is an isomorphism by Lemma 2.5, as well as $T_p f\colon T_pM \to T_{f(p)}N$ by Lemma 2.6, which proves that f is immersion and submersion. Conversely, if f is both immersion and submersion, then $T_p f$ is an isomorphism for every $p \in M$, so by Theorem 2.8, any p has a neighbourhood on which the restriction of f is a diffeomorphism onto its image. □

In the case that M and N have the same dimension, it is enough to know that f is an immersion or a submersion, since then the other condition is immediate and f is a local diffeomorphism by Theorem 2.9.

Example 2.8. The map $\psi\colon \mathbb{R} \to \mathbf{S}^1$ given by $\psi(t) = (\cos t, \sin t)$ has coordinate representations (related to the projections of hemispheres) given by $t \mapsto \cos t$ for points where $\sin t \neq 0$ and $t \mapsto \sin t$ for points where $\cos t \neq 0$, so ψ is a local diffeomorphism. For an arbitrary $a \in \mathbb{R}$ the restriction $\psi\restriction_{(a,a+2\pi)}\colon (a, a+2\pi) \to \mathbf{S}^1 \setminus \{\psi(a)\}$ is a homeomorphism, and therefore a diffeomorphism. An ***angle function*** on $U \subset \mathbf{S}^1$ is the restriction of the inverse of that diffeomorphism $\theta \in \mathfrak{F}(U)$, so (U, θ), according to Example 1.28, is a chart of the circle $\mathbf{S}^1$ with the standard smooth structure. If we identify $\mathbf{S}^1 \subset \mathbb{C} \cong \mathbb{R}^2$, then the angle function has $e^{i\theta(z)} = z$ for all $z \in U$. Now it is easy to see that the map $f\colon \mathbf{S}^1 \to \mathbf{S}^1$ given by $f(z) = z^2$ is smooth, since in new charts it has the representation given by $t \mapsto 2t + 2k\pi$ for some $k \in \mathbb{Z}$. △

Example 2.9. Let $\pi\colon \mathbf{S}^n \to \mathbb{R}\mathbf{P}^n$ be the canonical quotient map that sends x to its equivalence class $[x] = \{x, -x\}$, that is, $\pi(x_1, \dots, x_{n+1}) = (x_1 : \dots : x_{n+1})$. If we use the standard charts φ_i on $\mathbb{R}\mathbf{P}^n$ and their corresponding charts on $\mathbf{S}^n$ which are projections of the hemispheres to the planes $x_i = 0$, then we get coordinate representations that are up to sign equal to the diffeomorphism between $\mathbf{B}^n$ and $\mathbb{R}^n$ from Example 1.30, so π is a local diffeomorphism but not a diffeomorphism since it is not bijective. A local diffeomorphism is a continuous map, and since $\mathbf{S}^n$ is compact, so is $\mathbb{R}\mathbf{P}^n$. △

In the case that a smooth map has constant rank it can be locally written in a simple canonical form using the change of coordinates, which we see in the ***constant rank theorem***.

Theorem 2.10. *Let $f\colon M \to N$ be a smooth map of constant rank $r = \operatorname{rank} f$ between manifolds M and N. For every $p \in M$ there is a chart (U, φ) centred at $p \in M$ and a chart (V, ψ) centred at $f(p) \in N$ such that $f(U) \subseteq V$ where f has a coordinate representation of the form $\psi \circ f \circ \varphi^{-1}(x_1, \dots, x_r, x_{r+1}, \dots, x_m) = (x_1, \dots, x_r, 0, \dots, 0)$.*

Proof. For every $p \in M$ there is a chart (U_1, φ_1) centred at $p \in M$ and a chart (V_1, ψ_1) centred at $f(p) \in N$ such that $f(U_1) \subseteq V_1$ and $\psi_1 \circ f \circ \varphi_1^{-1}$ is smooth. The Jacobian matrix of f at a point p has entries in the form

$$\frac{\partial(y_i \circ f)}{\partial x_j}(p) = \frac{\partial(\pi_i \circ \psi_1 \circ f \circ \varphi_1^{-1})}{\partial \pi_j}(0)$$

for $1 \leq i \leq n = \dim N$, $1 \leq j \leq m = \dim M$. Since $\operatorname{rank}_p f = r$, we can permute the appropriate coordinates, that is, we use permutations of rows and columns of the Jacobian matrix in such a way that in the upper left corner we get an invertible square submatrix of order r. Formally, a permutation of coordinates is a diffeomorphism, which by Lemma 1.4 leads to new charts (U_1, φ_2) and (V_1, ψ_2) related to which f has the coordinate representation $\psi_2 \circ f \circ \varphi_2^{-1}$, such that the submatrix of the Jacobian matrix of f at p with entries

$$\frac{\partial(\pi_i \circ \psi_2 \circ f \circ \varphi_2^{-1})}{\partial \pi_j}(0)$$

for $1 \leq i, j \leq r$, is invertible. Consider the function $\theta_1\colon \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$\theta_1(a_1, \dots, a_m) = (\pi_1 \circ \psi_2 \circ f \circ \varphi_2^{-1}(a_1, \dots, a_m), \dots, \pi_r \circ \psi_2 \circ f \circ \varphi_2^{-1}(a_1, \dots, a_m), a_{r+1}, \dots, a_m),$$

which the corresponding Jacobian matrix at $0 \in \mathbb{R}^m$ is a block upper triangular matrix

$$\begin{pmatrix}
\frac{\partial \pi_1 \psi_2 f \varphi_2^{-1}}{\partial \pi_1}(0) & \cdots & \frac{\partial \pi_1 \psi_2 f \varphi_2^{-1}}{\partial \pi_r}(0) & \frac{\partial \pi_1 \psi_2 f \varphi_2^{-1}}{\partial \pi_{r+1}}(0) & \cdots & \frac{\partial \pi_1 \psi_2 f \varphi_2^{-1}}{\partial \pi_m}(0) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial \pi_r \psi_2 f \varphi_2^{-1}}{\partial \pi_1}(0) & \cdots & \frac{\partial \pi_r \psi_2 f \varphi_2^{-1}}{\partial \pi_r}(0) & \frac{\partial \pi_r \psi_2 f \varphi_2^{-1}}{\partial \pi_{r+1}}(0) & \cdots & \frac{\partial \pi_r \psi_2 f \varphi_2^{-1}}{\partial \pi_m}(0) \\
0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}.$$

As on the diagonal we have the previously mentioned square submatrix of rank r, as well as the unit submatrix of order $m-r$, we obtain $\operatorname{rank}_0 \theta_1 = m$, so by Theorem A.30 we have a diffeomorphism on some neighbourhood $W_1 \ni 0$. This is a legitimate change of coordinates that according to Lemma 1.4 brings us a new chart (U_3, φ_3) at $p \in M$, where $U_3 = U_1 \cap \varphi_2^{-1}(W_1)$ and $\varphi_3 = \theta_1 \circ \varphi_2\restriction_{U_3}$, which is centred at p because of $\psi_2 \circ f \circ \varphi_2^{-1}(0) = 0$. A new coordinate representation of f is of form

$$\psi_2 \circ f \circ \varphi_3^{-1}(b_1, \dots, b_r, a_{r+1}, \dots, a_m) = (b_1, \dots, b_r, b_{r+1}, \dots, b_n),$$

where $b_i = \pi_i \circ \psi_2 \circ f \circ \varphi_2^{-1}(a_1, \dots, a_m)$ for $1 \le i \le n$. Thus, the Jacobian matrix of f relative to new charts is a block lower triangular matrix

$$\begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \frac{\partial b_{r+1}}{\partial b_1} & \cdots & \frac{\partial b_{r+1}}{\partial b_r} & \frac{\partial b_{r+1}}{\partial a_{r+1}} & \cdots & \frac{\partial b_{r+1}}{\partial a_m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial b_n}{\partial b_1} & \cdots & \frac{\partial b_n}{\partial b_r} & \frac{\partial b_n}{\partial a_{r+1}} & \cdots & \frac{\partial b_n}{\partial a_m} \end{pmatrix}.$$

Since on the diagonal we first have the unit submatrix of order $r = \operatorname{rank} f$, the other component on the diagonal is the zero submatrix, which means that $b_{r+1}, \dots, b_n$ are independent of $a_{r+1}, \dots, a_m$, but depend only on $b_1, \dots, b_r$. This allows us to introduce a new change of coordinates $\theta_2 \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$\theta_2(b_1, \dots, b_r, c_{r+1}, \dots, c_n) = (b_1, \dots, b_r, c_{r+1} - b_{r+1}(b_1, \dots, b_r), \dots, c_n - b_n(b_1, \dots, b_r)),$$

on some neighbourhood $W_2 \ni 0$, because the corresponding Jacobian matrix is a block lower triangular matrix with the unit submatrices on the diagonal. Hence, we have a new chart (V, ψ) centred at $f(p) \in N$, where $V = V_1 \cap \psi_2^{-1}(W_2)$ and $\psi = \theta_2 \circ \psi_2\restriction_V$, so with additional $U = U_3 \cap f^{-1}(V)$ and $\varphi = \varphi_3\restriction_U$ we have the desired coordinate representation

$$\psi \circ f \circ \varphi^{-1}(b_1, \dots, b_r, b_{r+1}, \dots, b_m) = (b_1, \dots, b_r, 0, \dots, 0),$$

which finishes the proof. □

Example 2.10. The constant rank theorem often applies to immersions and submersions. For example, it shows that any submersion locally looks like a projection with respect to suitable charts. Since a projection is an open map, while charts are diffeomorphisms, we conclude that any submersion is an open map. △

Example 2.11. Is there a submersion $f \colon \mathbf{S}^m \to \mathbb{R}^n$ for some $m, n \in \mathbb{N}$? The domain $M = \mathbf{S}^m$ is compact, so for a smooth $f \colon M \to \mathbb{R}^n$ we consider $\pi_1 \circ f \colon M \to \mathbb{R}$, where π_1 is the projection to the first component, which attains a maximum on M at some point $p \in M$. According to Example 2.4 we have $0 = T_p(\pi_1 \circ f) = T_{f(p)}\pi_1 \circ T_p f$, but $T_{f(p)}\pi_1$ is surjective, so $T_p f$ is not surjective, and consequently f cannot be a submersion.

Moreover, every continuous map with a compact domain is closed (Lemma A.2), while according to Example 2.10, a submersion is an open map. If $f \colon \mathbf{S}^m \to \mathbb{R}^n$ is a submersion, then the image of $\mathbf{S}^m$ is a nonempty open and closed subset in $\mathbb{R}^n$, so $f(\mathbf{S}^m) = \mathbb{R}^n$, and since it is not compact, a submersion f does not exist. △

Example 2.12. Let $f \colon \mathbf{S}^1 \times \mathbb{R} \to \mathbb{R}^2$ be defined by $f((x,y), r) = (e^r x, e^r y)$. If we use an angle function θ from Example 2.8 as a chart of the circle $\mathbf{S}^1$ we get the coordinate representation $f \circ (\theta \times \mathbb{1}_{\mathbb{R}})^{-1}(t, r) = (e^r \cos t, e^r \sin t)$. Therefore f is smooth with the Jacobian matrix

$$T_{(t,r)}(f \circ (\theta \times \mathbb{1}_{\mathbb{R}})^{-1}) = \begin{pmatrix} -e^r \sin t & e^r \cos t \\ e^r \cos t & e^r \sin t \end{pmatrix},$$

whose determinant is $-e^{2r} \neq 0$, which gives $\operatorname{rank} f = 2$, so f is a submersion and a local diffeomorphism. △

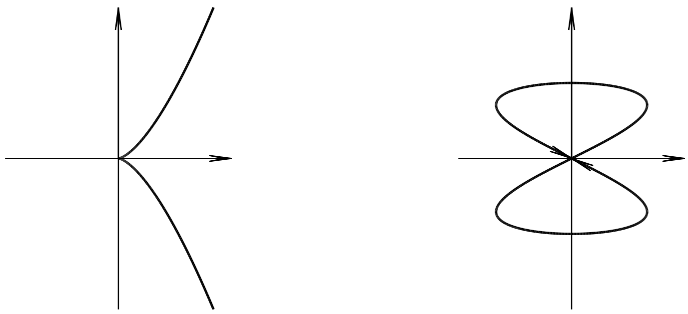
A ***topological embedding*** is a homeomorphism onto its image. More explicitly, it is a map $f\colon M \to N$ which yields a homeomorphism between topological spaces M and $f(M)$, where $f(M) \subseteq N$ carries the subspace topology inherited from N. As a consequence, any topological embedding is injective and continuous.

An ***embedding*** (or ***imbedding***) is an immersion that is also a topological embedding. In other words, an embedding is an injective (smooth) immersion which domain is diffeomorphic to its image. This special case of immersion is rather important, and it allows us to introduce the notion of submanifold.

Example 2.13. According to the constant rank theorem any immersion $f\colon M \to N$ has a coordinate representation in the form of inclusion $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)$. Thus, for each $p \in M$ there is an neighbourhood $U \subseteq M$ of p such that $f{\restriction}_U$ is an embedding, which means that any immersion is a local embedding. △

Example 2.14. Let $f\colon M \to N$ be an injective immersion. The answer to the question whether f is an embedding is reduced to checking whether $f^{-1}\colon f(M) \to M$ is continuous. With the additional condition that f is open or closed map, this is obviously satisfied. If M is compact, f is closed (Lemma A.2), so it is an embedding. If $\dim M = \dim N$, then f is a local diffeomorphism (Theorem 2.9), so f is open and therefore an embedding. △

Example 2.15. A smooth map $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (t^2, t^3)$ is a topological embedding. However, although it is injective, it is not an immersion since $\gamma'(0) = 0$, which gives $\operatorname{rank}_0 \gamma = 0$. △



Example 2.16. Consider a map $\gamma\colon (-\pi, \pi) \to \mathbb{R}^2$ given by $\gamma(t) = (\sin 2t, \sin t)$. Since for any $t \in (-\pi, \pi)$ we have $\gamma'(t) \neq 0$, the map γ is an injective immersion. However, its image $\{(x, y) \in \mathbb{R}^2 : x^2 = 4y^2(1 - y^2)\}$ is a compact set in $\mathbb{R}^2$, but since the domain is not compact then γ is not a topological embedding, and therefore γ is not an embedding. △

Example 2.17. Let $\gamma\colon \mathbb{R} \to \mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1 \subset \mathbb{C}^2 \cong \mathbb{R}^4$ be a winding of the torus given by

$$\gamma(t) = (e^{2\pi i t}, e^{2\pi i c t}) = (\cos 2\pi t, \sin 2\pi t, \cos 2\pi c t, \sin 2\pi c t),$$

for some irrational $c \in \mathbb{R} \setminus \mathbb{Q}$. Since $\gamma'(t)$ never vanishes, γ is an immersion. It is also injective since $\gamma(t_1) = \gamma(t_2)$ implies both $t_1 - t_2 \in \mathbb{Z}$ and $c(t_1 - t_2) \in \mathbb{Z}$, and consequently $t_1 = t_2$. According to the Dirichlet's[5] approximation theorem (Lemma A.33), for every $\varepsilon > 0$ there exist integers $m, n \in \mathbb{Z}$ such that $|cn - m| < \varepsilon$. Since the line segment is shorter than the circular arc we have

$$|\gamma(n) - \gamma(0)| = |(e^{2\pi i n}, e^{2\pi i c n}) - (1, 1)| = |e^{2\pi i c n} - e^{2\pi i m}| \leq 2\pi |cn - m| < 2\pi\varepsilon,$$

[5] Peter Gustav Lejeune Dirichlet (1805–1859), German mathematician

so $\gamma(0)$ is a limit point of $\gamma(\mathbb{Z})$. However, $\mathbb{Z}$ has no limit point in $\mathbb{R}$ which proves that γ is not a homeomorphism onto its image $\gamma(\mathbb{R})$, hence γ is not an embedding.

The second argument is that $\gamma(\mathbb{R})$ is not locally path-connected since in a ball we have a countably infinite number of parallel lines shading it, each of which forms a (local) path-component.

The third argument is that the image of an embedding cannot be dense, while it is not hard to show that $\gamma(\mathbb{R})$ is dense in $\mathbf{T}^2$ Let $f\colon M \to N$ be an embedding which is not a submersion. The constant rank theorem gives a chart (V, ψ) on N, where $P = V \cap f(M) \neq \emptyset$ such that $\psi(P) \subseteq \mathbb{R}^m \times \{0\}$. However, then the open set $\psi^{-1}(\psi(V) \cap (\mathbb{R}^m \times (\mathbb{R}^{n-m} \setminus \{0\})))$ lies in V and does not intersect P, so $f(M)$ is not dense in N. △

2.4 Submanifolds

Many familiar manifolds appear naturally as subsets of other manifolds. Roughly speaking, submanifolds of some manifold M is a subset $P \subseteq M$ that has a manifold structure in its own right. That structure comes from M through the appropriate inclusion that has some nice properties. A ***submanifold*** of a manifold M is a subset $P \subseteq M$ which is a manifold such that the appropriate inclusion $\imath\colon P \hookrightarrow M$ is an embedding. Thus, a submanifold has the subspace topology inherited from its containing manifold and a smooth structure with respect to which the inclusion map is embedding.

If P is a submanifold of M, then M is the ***ambient manifold*** for P, and the difference $\dim M - \dim P$ is the ***codimension*** of P in M. A submanifold of codimension 1 is called a ***hypersurface***. Let us notice that such defined submanifolds are also called ***embedded submanifolds*** or ***regular submanifolds*** by some authors. In a more relaxed variant where the corresponding inclusion $\imath\colon P \hookrightarrow M$ is an immersion (not necessarily an embedding), we say that P is an ***immersed submanifold***.

Example 2.18. In Example 1.20 we show that the inclusion $\imath\colon \mathbf{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth. This time it is more convenient to use ordinary projections from Example 1.14, where the coordinate representation is given by

$$\imath \circ \varphi_{\pm i}^{-1}(y_1, \dots, y_n) = (y_1, \dots, y_{i-1}, \pm\sqrt{1 - y_1^2 - \cdots - y_n^2}, y_i, \dots, y_n),$$

so the Jacobian matrix contains the $n \times n$ identity submatrix, from where $T_p\imath$ is injective for every $p \in \mathbf{S}^n$. Therefore, $\imath$ is an injective immersion, while $\mathbf{S}^n$ is compact (closed and bounded in $\mathbb{R}^{n+1}$), so $\imath$ is embedding by Example 2.14, which implies that $\mathbf{S}^n$ is a submanifold of $\mathbb{R}^{n+1}$. △

Example 2.19. Submanifolds of codimension 0 of a manifold M are exactly the open submanifolds. If $U \subseteq M$ is an open submanifold with the inclusion $\imath\colon U \hookrightarrow M$, then $\imath$ is obviously an embedding (U has the subspace topology, while the coordinate representation of $\imath$ is the identity) with the codimension 0, which means $\dim U = \dim M$. Conversely, if $\imath\colon U \hookrightarrow M$ is an embedding of codimension 0, then it is both an immersion and a submersion, so by Theorem 2.9 it is a local diffeomorphism, and therefore U is an open subset of M. △

We often find submanifolds in the codomain of some embedding, which is discussed in the following theorem.

Theorem 2.11. *If $f\colon M \to N$ is an embedding between manifolds, then $P = f(M)$ with the subspace topology is a topological manifold, and it has a unique smooth structure making it into a submanifold of N such that f is a diffeomorphism between M and P.*

Proof. Since f is an embedding, it is a homeomorphism between M and P. For any chart (U, φ) on M we take a corresponding chart $(f(U), \varphi \circ f^{-1})$ on P which determines a smooth structure such that f is the appropriate diffeomorphism (see Example 1.29). The inclusion $P \hookrightarrow N$ is a composition $f \circ f^{-1}$ of diffeomorphism $f^{-1}: P \to M$ and embedding $f: M \to N$, so it is an embedding, which means that P is a submanifold of N. □

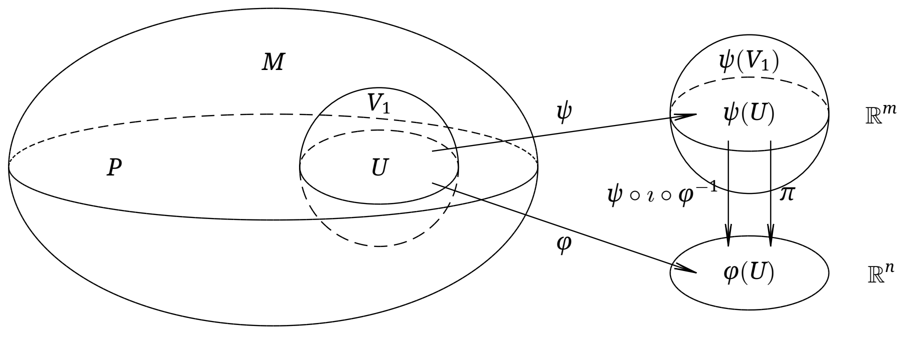
Example 2.20. For some fixed $p \in N$, the map $f: M \to M \times N$ given by $f(x) = (x, p)$ is an embedding, so a ***slice*** $M \times \{p\}$ is a submanifold of $M \times N$ diffeomorphic to M. △

Example 2.21. Let M and N be manifolds, and $f: U \to N$ is smooth for an open set $U \subseteq M$. Then the graph $\Gamma(f) = \{(x, y) \in M \times N : x \in U, y = f(x)\}$ is a submanifold of $M \times N$. The map $\gamma: U \to M \times N$ given by $\gamma(x) = (x, f(x))$ is smooth and $\gamma(U) = \Gamma(f)$. From $\pi_M \circ \gamma = \mathbb{1}_U$ follows $T_{\gamma(x)}\pi_M \circ T_x\gamma = \mathbb{1}_{T_xM}$ for all $x \in U$, so $T_x\gamma$ is injective and γ is an immersion. The inverse of $\gamma: U \to \Gamma(f)$ is the continuous map $\pi_M\restriction_{\Gamma(f)}$, so γ is a homeomorphism, which implies that $\Gamma(f)$ is a submanifold of $M \times N$ diffeomorphic to U. △

It turns out that all submanifolds are actually slices, which leads to the following theorem which represents an alternative (and actually the most common) definition of submanifolds.

Theorem 2.12. *If P is a submanifold of dimension n of a manifold M, then for each point $p \in P$ there is a chart (V, ψ) at $p \in M$ such that $\psi(V \cap P) = \psi(V) \cap (\mathbb{R}^n \times \{0\})$.*

Proof. If $P \subseteq M$ is a submanifold of dimension n then the inclusion $\imath: P \hookrightarrow M$ is an embedding. Since $\imath$ is an immersion, by the constant rank theorem (Theorem 2.10) for every $p \in P$ there are charts, (U, φ) centred at $p \in P$ and (V, ψ) centred at $p \in M$, such that $U = \imath(U) \subseteq V$ and $\psi \circ \imath \circ \varphi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$. Since $\imath$ is a topological embedding and $U \subseteq P$ is open, there exists an open $W \subseteq M$ such that $U = W \cap P$. Let $\pi: \mathbb{R}^m \to \mathbb{R}^n$ be the projection onto the first n coordinates, then we have $\varphi(U) = \pi \circ \psi(U)$, so $C = \psi^{-1}((\mathbb{R}^n \setminus \varphi(U)) \times \{0\}) \subseteq V$ is closed disjoint from U. If we restrict ψ to the open subset $V_1 = V \cap W \setminus C$, then we obtain $\psi(V_1 \cap P) = \psi(U \setminus C) = \psi(U) = \psi(V_1) \cap (\mathbb{R}^n \times \{0\})$, so $(V_1, \psi\restriction_{V_1})$ is a chart at $p \in M$ that satisfies the required condition. □



Charts (V, ψ) on M for which $\psi(V \cap P) = \psi(V) \cap (\mathbb{R}^n \times \{0\})$ holds are ***slice charts*** adapted to a subset $P \subseteq M$ and they naturally induce n-dimensional charts $(V \cap P, \psi_P)$ on P with $\psi_P = \pi \circ \psi$, where $\pi: \mathbb{R}^m \to \mathbb{R}^n$ is the projection to the first n coordinates. If (U, φ) is another slice chart on M and $\varphi_P = \pi \circ \varphi$, then we have $\varphi_P \circ \psi_P^{-1} = \pi \circ (\varphi \circ \psi^{-1}) \circ \jmath$, where $\jmath(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$, which is smooth as a composition of smooth maps. If there is a slice map at every point $p \in P$, then the induced charts form a smooth atlas for P, and it is clear that it assigns P the subspace topology from M. The coordinate representation of the inclusion $\imath: P \hookrightarrow M$ is $\psi \circ \imath \circ \psi_P^{-1} = \jmath$ which is smooth, while the Jacobian matrix contains the $n \times n$ identity submatrix, so $\imath$ is an immersion, and consequently it is an embedding.

Thus, the slice charts on P induces a natural atlas with whicn P becomes a submanifold of M. However, it turns out that the complete atlas of a submanifold is uniquely determined.

Theorem 2.13. *Let M be a manifold, $P \subseteq M$, and $n \in \mathbb{N}$. If for each $p \in P$ there exists a chart (V, ψ) at $p \in M$ such that $\psi(V \cap P) = \psi(V) \cap (\mathbb{R}^n \times \{0\})$, then there is a unique complete atlas for P with which it becomes a submanifold of M of dimension n.*

Proof. Let $\widetilde{P}$ be a submanifold of M consisting of the set P and some other atlas such that the inclusion $\widetilde{\imath}\colon \widetilde{P} \hookrightarrow M$ is an embedding. This inclusion when restricting the codomain to $P = \widetilde{\imath}(\widetilde{P})$ becomes $\widetilde{\imath_1}\colon \widetilde{P} \hookrightarrow P$, where $\imath \circ \widetilde{\imath_1} = \widetilde{\imath}$. Since $\widetilde{\imath_1}$ is continuous (a homeomorphism because $\widetilde{\imath}$ is an embedding), and $\psi_P \circ \widetilde{\imath_1} \circ \varphi^{-1} = \pi \circ (\psi \circ \widetilde{\imath} \circ \varphi^{-1})$ is smooth for a chart (U, φ) on $\widetilde{P}$ and a slice chart (V, ψ) on M that induces the chart $(V \cap P, \psi_P)$ on P, $\widetilde{\imath_1}$ is smooth according to Lemma 1.13. Since for $p \in P$ we have $T_p\widetilde{\imath} = T_p\imath \circ T_p\widetilde{\imath_1}$, and both $T_p\widetilde{\imath}$ and $T_p\imath$ are injective, then $T_p\widetilde{\imath_1}$ is injective, which means that $\widetilde{\imath_1}$ is an immersion. However, due to $\dim \widetilde{P} = \dim P$ it is a local diffeomorphism, and since it is a ijection, the it is also a diffeomorphism, which means that $\widetilde{P}$ and P have the same complete smooth atlas. $\square$

Theorem 2.11 finds submanifolds in the codomain of smooth maps, but we can notice them in the domain as well, because in practice, submanifolds are most often shown as a set of solutions of equations or a system of equations.

Theorem 2.14. *If $f\colon M \to N$ is a smooth map of constant rank r, then for every $q \in f(M)$, the level set $P = f^{-1}(q)$ is a submanifold in M of codimension r.*

Proof. By the constant rank theorem, for every $p \in P$ there is a chart (U, φ) centred at $p \in M$ and a chart (V, ψ) centred at $q = f(p) \in N$ such that f related to these charts has a coordinate representation in the form $\psi \circ f \circ \varphi^{-1}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0)$. Because of $\varphi(U \cap P) = \{(x_1, \ldots, x_m) \in \varphi(U) : x_1 = \cdots = x_r = 0\} = \varphi(U) \cap (\{0\} \times \mathbb{R}^{m-r})$, Theorem 2.13 finishes the proof. $\square$

Especially, if f from the previous theorem is a submersion, then the level set is a submanifold of codimension $\dim N$. We say that a point $p \in M$ is a ***regular point*** if T_pf is surjective; otherwise, we say that p is a ***critical point***. A point $q \in N$ is called a ***regular value*** if its preimage $f^{-1}(q)$ contains only regular points. A level set $f^{-1}(q)$ is called a ***regular level set*** if q is a regular value of f.

Theorem 2.15. *Every regular level set of a smooth map between manifolds is a submanifold whose codimension is equal to the dimension of codomain.*

Proof. Let $f\colon M \to N$ be a smooth map and let $q \in N$ be a regular value. By Lemma 2.7 the set U of points $p \in M$ such that T_pf has rank $\dim N$ is open in M and contains $f^{-1}(q)$. It follows that $f{\restriction_U}$ is a submersion, so $f^{-1}(q)$ is a submanifold of U by Theorem 2.14. The composition of embeddings $f^{-1}(q) \hookrightarrow U \hookrightarrow M$ is again embedding, which proves the statement. $\square$

Example 2.22. If a function $f\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is given by $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$, then f is a submersion, so $\mathbf{S}^n = f^{-1}(1)$ is a submanifold in $\mathbb{R}^{n+1}$ of dimension n. $\triangle$

Example 2.23. Consider the subspace $M = \{(x : y : z) \in \mathbb{R}\mathbf{P}^2 : xy = z^2\} \subset \mathbb{R}\mathbf{P}^2$ which is first of all well defined because $(tx)(ty) = (tz)^2$ for $t \neq 0$ holds if and only if $xy = z^2$. Also, it is easy to see that $f\colon \mathbb{R}\mathbf{P}^2 \to \mathbb{R}$ is well defined by

$$f(x : y : z) = \frac{xy - z^2}{x^2 + y^2 + z^2},$$

and that $M = f^{-1}(0)$. In the first chart we have the smooth $f \circ \varphi_1^{-1}(x, y) = (x - y^2)/(1 + x^2 + y^2)$ with the Jacobian matrix

$$(f \circ \varphi_1^{-1})'(x, y) = \begin{pmatrix} \frac{(1+x^2+y^2)-2x(x-y^2)}{(1+x^2+y^2)^2} & \frac{-2y(1+x^2+y^2)-2y(x-y^2)}{(1+x^2+y^2)^2} \end{pmatrix} = \begin{pmatrix} \frac{1-x^2+y^2+2xy^2}{(1+x^2+y^2)^2} & \frac{-2y(1+x+x^2)}{(1+x^2+y^2)^2} \end{pmatrix},$$

which is of rank 1 except for $y = 0$ and $x = \pm 1$, that is for the points $(1 : \pm 1 : 0) \notin M$. In the second chart we have $f \circ \varphi_2^{-1}(x,y) = (x - y^2)/(1 + x^2 + y^2)$, with a symmetric result. In the third chart we have the smooth $f \circ \varphi_3^{-1}(x,y) = (xy - 1)/(1 + x^2 + y^2)$ with the Jacobian matrix

$$(f \circ \varphi_3^{-1})'(x,y) = \begin{pmatrix} \frac{y(1+x^2+y^2)-2x(xy-1)}{(1+x^2+y^2)^2} & \frac{x(1+x^2+y^2)-2y(xy-1)}{(1+x^2+y^2)^2} \end{pmatrix} = \begin{pmatrix} \frac{y+2x-x^2y+y^3}{(1+x^2+y^2)^2} & \frac{x+2y-xy^2+x^3}{(1+x^2+y^2)^2} \end{pmatrix},$$

which is of rank 1 except for the solution of the system $y + 2x - x^2y + y^3 = 0$ and $x + 2y - xy^2 + x^3 = 0$. If we multiply the first equation by x, and the second by y and add them, we get $2(x^2 + xy + y^2) = 0$, which is possible only for $x = y = 0$ that corresponds to the point $(0 : 0 : 1) \notin M$. Therefore, f is a submersion, except at three concrete points that do not belong to M, and in any case M is a regular level set and therefore a submanifold of $\mathbb{R}\mathbf{P}^2$. △

It turns out that at each point of a regular level set we have a nice decomposition of the tangent space.

Lemma 2.16. *If P is a regular level set of smooth map $f\colon M \to N$, then $T_pP = \operatorname{Ker} T_pf$ for each $p \in P$.*

Proof. Let $\imath\colon P = f^{-1}(q) \hookrightarrow M$ be the natural inclusion. Since $f\restriction_P = f \circ \imath$ is constantly equal to $q \in N$, we have $T_pf \circ T_p\imath = T_pf\restriction_P = 0$, and therefore $T_p\imath(T_pP) \subseteq \operatorname{Ker} T_pf \subseteq T_pM$. Since the subspace T_pP has the same dimension as $\operatorname{Ker} T_pf$, we conclude that they are equal under the natural identification $T_p\imath(T_pP) \cong T_pP$. □

The concept od submanifolds allows us to restrict the domain or codomain of a smooth map between manifolds to their submanifolds, whereby the restriction will remain smooth. Of course, for the domain restriction it is obvious because we have a composition of a smooth map and an inclusion which is an embedding.

Theorem 2.17. *Let $f\colon M \to N$ be a smooth map and P be a submanifold of N, then the restriction $f\colon M \to P$ is also smooth.*

Proof. If $f_1\colon M \to P$ is the observed restriction, then $f = \imath \circ f_1$, where the inclusion $\imath\colon P \hookrightarrow N$ is an embedding. A coordinate representation $\psi_P \circ f_1 \circ \varphi^{-1} = \pi \circ \psi \circ f \circ \varphi^{-1}$ is smooth for a chart (U, φ) on M and a slice chart (V, ψ) on N that induces the chart $(V \cap P, \psi_P)$ on P, while f_1 is continuous since $\imath$ is a homeomorphism onto its image. Therefore f_1 is smooth by Lemma 1.13. □

2.5 Vector fields

Basically, a vector field on a manifold M is a map $p \mapsto X(p)$ that assigns to each point $p \in M$ a tangent vector $X(p) \in T_pM$. We would like such assignments to have some smoothness properties when p varies in M. To do this we need to fit together all the tangent spaces T_pM as p ranges over M into a single manifold. The ***tangent bundle*** of a manifold M is the disjoint union of tangent spaces to M,

$$TM = \bigsqcup_{p\in M} T_pM = \bigcup_{p\in M} (\{p\} \times T_pM) = \bigcup_{p\in M} \{(p,X) : X \in T_pM\}.$$

An element of the tangent bundle TM can be thought of as a pair (p,X), where p is a point in M and X is a tangent vector to M at p. The tangent bundle comes equipped with the natural projection map $\pi\colon TM \to M$ defined by $\pi(p,X) = p$, that maps each tangent vector from T_pM to the single base point p. We often conveniently identify the pair $(p,X) \in TM$ with the tangent vector $X \in T_pM$ and the fiber $\pi^{-1}(p)$ with T_pM.

To make a manifold from the set TM, it is necessary to give a smooth atlas in some natural way. Let (U,φ) be a chart on M, and let $x_i = \pi_i \circ \varphi$ be the coordinate functions. By Lemma 2.6, for $p \in U$ we have $T_pU = T_pM$, and therefore $TU = \bigsqcup_{p\in U} T_pU = \bigsqcup_{p\in U} T_pM \subseteq TM$. A tangent vector $X \in T_pM$ at $p \in U$ is the unique linear combination $X = \sum_{i=1}^n X(x_i)(\partial_i)_p$, which motivate us to define the map $\widetilde{\varphi}\colon \pi^{-1}(U) = TU \to \varphi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$ by

$$\widetilde{\varphi}(p,X) = (x_1(p),\ldots,x_n(p),X(x_1),\ldots,X(x_n)). \tag{2.4}$$

We can notice that $\widetilde{\varphi}$ is a bijection with the inverse

$$\widetilde{\varphi}^{-1}(\varphi(p),\lambda_1,\ldots,\lambda_n) = \left(p,\sum_{i=1}^n \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p\right).$$

Let (U,φ) and (V,ψ) be charts on M, then the corresponding charts on TM are $(TU,\widetilde{\varphi})$ and $(TV,\widetilde{\psi})$. The sets $\widetilde{\varphi}(TU\cap TV) = \varphi(U\cap V)\times\mathbb{R}^n$ and $\widetilde{\psi}(TU\cap TV) = \psi(U\cap V)\times\mathbb{R}^n$ are open in $\mathbb{R}^{2n}$, while for the map $\widetilde{\psi}\circ\widetilde{\varphi}^{-1}\colon \varphi(U\cap V)\times\mathbb{R}^n \to \psi(U\cap V)\times\mathbb{R}^n$ holds

$$\begin{aligned}
(\widetilde{\psi}\circ\widetilde{\varphi}^{-1})(\varphi(p),\lambda_1,\ldots,\lambda_n) &= \widetilde{\psi}\left(p,\sum_{i=1}^n \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p\right)\\
&= \left(\psi(p),\sum_{i=1}^n \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p (\pi_1\circ\psi),\ldots,\sum_{i=1}^n \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p (\pi_n\circ\psi)\right)\\
&= \left((\psi\circ\varphi^{-1})(\varphi(p)),\sum_{i=1}^n \lambda_i \frac{\partial(\pi_1\circ(\psi\circ\varphi^{-1}))}{\partial \pi_i}(\varphi(p)),\ldots,\sum_{i=1}^n \lambda_i \frac{\partial(\pi_n\circ(\psi\circ\varphi^{-1}))}{\partial \pi_i}(\varphi(p))\right),
\end{aligned}$$

and therefore it is smooth. In this way, the original atlas for M induces a smooth atlas for TM.

Since M has a countable subatlas, then TM also has one. For $X \in T_pM$ and $Y \in T_qM$ where $p \neq q$ (otherwise Lemma 1.5 works) there exist trivially compatible charts, φ at $p \in M$ and ψ at $q \in M$, and therefore $\widetilde{\varphi}$ and $\widetilde{\psi}$ have disjoint domains that separate points (p,X) and (q,Y), which proves the Hausdorff condition. Hence, the tangent bundle TM with the induced atlas is a smooth manifold of dimension $2n$.

A ***section*** of a tangent bundle TM is a map $X\colon M \to TM$ such that $\pi\circ X = \mathbb{1}_M$, where $\pi\colon TM \to M$ is the natural projection. A ***vector field*** on a manifold M is a smooth section of the tangent bundle TM, that is a smooth function $X\colon M \to TM$ that assigns a tangent vector $X_p = X(p) \in T_pM$ to each point $p \in M$. The set of all vector fields on a manifold M we denote by $\mathfrak{X}(M)$.

Let (U,φ) be a chart on M with coordinate functions $x_i = \pi_i\circ\varphi$. A section $X\colon M \to TM$ has a coordinate representation $\widetilde{\varphi}\circ X\circ\varphi^{-1}\colon \varphi(U) \to \varphi(U)\times\mathbb{R}^n$ which due to (2.4) can be expressed with $\varphi(p) \mapsto (\varphi(p),X_p(x_1),\ldots,X_p(x_n))$ for $p \in U$. Every section X can be seen as a map that assigns to each smooth function $f \in \mathfrak{F}(M)$ a function $Xf\colon M \to \mathbb{R}$ given by $(Xf)(p) = X_p(f)$. Then $X_p(x_i) = (Xx_i\circ\varphi^{-1})(\varphi(p))$ and for $v \in \varphi(U)$ we have

$$(\widetilde{\varphi}\circ X\circ\varphi^{-1})(v) = (v, Xx_1\circ\varphi^{-1}(v),\ldots,Xx_n\circ\varphi^{-1}(v)), \tag{2.5}$$

so the section X is smooth on U if and only if Xx_i are smooth functions for $1 \le i \le n$.

For $1 \le i \le n$ we define the i-th ***coordinate vector field*** $\partial_i\colon U \to TU$ by $\partial_i(p) = (\partial_i)_p$, but since $\partial_i(x_j)\colon p \mapsto (\partial_i)_p(x_j) = \delta_{ij}$ for all $1 \le j \le n$, thus $\partial_i(x_j) \in \mathfrak{F}(U)$, so we have $\partial_i \in \mathfrak{X}(U)$ as a basic example of a vector field.

For $f \in \mathfrak{F}(M)$, the coordinate representation of $\partial_i f$ is the smooth function

$$\partial_i f\circ\varphi^{-1}\colon \varphi(p) \mapsto \left(\frac{\partial}{\partial x_i}\right)_p f = \frac{\partial(f\circ\varphi^{-1})}{\partial \pi_i}(\varphi(p)),$$

and therefore $\partial_i f \in \mathfrak{F}(U)$, while at a point $p \in U$ for a section $X\colon M \to TM$ we have

$$(Xf)(p) = X_p(f) = \left(\sum_{i=1}^{n} X_p(x_i)\left(\frac{\partial}{\partial x_i}\right)_p\right) f = \sum_{i=1}^{n} X_p(x_i)\frac{\partial f}{\partial x_i}(p) = \left(\sum_{i=1}^{n}(Xx_i)\frac{\partial f}{\partial x_i}\right)(p),$$

so on the coordinate neighbourhood U we get

$$Xf = \sum_{i=1}^{n}(Xx_i)\partial_i f.$$

If $X \in \mathfrak{X}(M)$, the section is smooth, so Xx_i are smooth, and Xf is smooth on U, but since this holds for any chart, we obtain $Xf \in \mathfrak{F}(M)$. Any particular function $X_p\colon \mathfrak{F}(M) \to \mathbb{R}$ is a derivation at p, so linearity and Leibnizian properties can simply be extended to the smooth section $X\colon \mathfrak{F}(M) \to \mathfrak{F}(M)$. The Leibnizian property for points gives $X(fh) = f(Xh) + h(Xf)$ for every $f, h \in \mathfrak{F}(M)$, and if X is also linear then we say that X is a ***derivation***, so we have proved that any vector field is a derivation.

Conversely, every derivation $X\colon \mathfrak{F}(M) \to \mathfrak{F}(M)$ induces a derivation at a point $p \in M$, so $X_p(f) = (Xf)(p)$ for $f \in \mathfrak{F}(M)$ defines a tangent vector at p, and therefore the section $X\colon M \to TM$ is also defined. Since in any chart (U, φ) we have $x_i = \pi_i \circ \varphi \in \mathfrak{F}(U)$, then $Xx_i \in \mathfrak{F}(U)$, so using (2.5) we see that this section is smooth on U, and therefore smooth on the whole M. This completes the proof that vector fields have a dual nature, which we see in the following theorem.

Theorem 2.18. *Derivations on a manifold are exactly smooth sections of its tangent bundle.*

We can introduce basic operations on $\mathfrak{X}(M)$ in a natural way. The addition and the scalar multiplication $(\alpha X + \beta Y)(h) = \alpha(X(h)) + \beta(Y(h))$ (for $\alpha, \beta \in \mathbb{R}$, $h \in \mathfrak{F}(M)$) make $\mathfrak{X}(M)$ a vector space over $\mathbb{R}$. Additionally, a vector field can be multiplied by $f \in \mathfrak{F}(M)$ with $(fX)(h) = fX(h)$, which makes $\mathfrak{X}(M)$ a module over the ring $\mathfrak{F}(M)$. In accordance with the introduced operations, analogous to the formula (2.1), we obtain the decomposition of an arbitrary vector field via the coordinate vector fields in some chart,

$$X = \sum_{i=1}^{n} Xx_i\frac{\partial}{\partial x_i}.$$

Example 2.24. If $V \in T_pM$ is an arbitrary tangent vector of a manifold $M \ni p$, then there exists a vector field $X \in \mathfrak{X}(M)$ such that $X_p = V$. By choosing a chart (U, φ) at $p \in M$, we get coordinate functions and corresponding base tangent vectors. Since $V = \sum_i v_i(\partial_i)_p$ holds for some $v_i = V(x_i) \in \mathbb{R}$, we can define $Y \in \mathfrak{X}(U)$ by $Y = \sum_i v_i\partial_i$, to get $Y_p = V$. According to Lemma 1.18 there exists $b \in \mathfrak{F}(M)$ supported in U with $b(p) = 1$. If we define $X = bY$ on U and $X = 0$ on $M \setminus \operatorname{supp}(b)$, we obtain $X \in \mathfrak{X}(M)$ and $X_p = b(p)Y_p = V$ holds. △

The multiplication for $X, Y \in \mathfrak{X}(M)$ can be seen as $(XY)(h) = X(Y(h))$ for $h \in \mathfrak{F}(M)$, which is clearly $\mathbb{R}$-linear. However, using the Leibniz property for X and Y we have

$$XY(fh) = X(Y(fh)) = X(fYh + hYf) = fX(Yh) + (Xf)(Yh) + hX(Yf) + (Xh)(Yf).$$

Looking closely at this formula, we see two extra terms $(Xf)(Yh)$ and $(Xh)(Yf)$. Since in general there is no reason for these terms to cancel each other, XY is not Leibnizian, and thus it is not a vector field. However, our extra terms are symmetric in X and Y, so if we compute $YX(fh)$ as well and subtract it from $XY(fh)$, these terms will disappear. Hence, $XY - YX$ will be Leibnizian and therefore it is a vector fields called the ***commutator*** or the ***Lie bracket***, with the notation $[X, Y] = XY - YX \in \mathfrak{X}(M)$.

The commutator on $\mathfrak{X}(M)$ naturally has some nice properties. From the definition follows the anti-symmetry $[Y,X] = -[X,Y]$. The linearity of vector fields immediately gives the $\mathbb{R}$-bilinearity, $[\alpha X + \beta Y, Z] = \alpha[X,Z] + \beta[Y,Z]$ and $[X, \alpha Y + \beta Z] = \alpha[X,Y] + \beta[X,Z]$ for $\alpha, \beta \in \mathbb{R}$, and $X, Y, Z \in \mathfrak{X}(M)$.

However, although the commutator is $\mathbb{R}$-bilinear it is not $\mathfrak{F}(M)$-bilinear. Since for every $X, Y \in \mathfrak{X}(M)$ and $f, h \in \mathfrak{F}(M)$ we have $X(fY(h)) = fXY(h) + (Xf)(Yh)$, we obtain the useful formula

$$X(fY) = fXY + (Xf)Y,$$

from which we can easily calculate

$$[fX, hY] = fh[X,Y] + f(Xh)Y - h(Yf)X. \tag{2.6}$$

Another nice feature of vector fields is the ***Jacobi identity***,

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0, \tag{2.7}$$

that holds for all $X, Y, Z \in \mathfrak{X}(M)$, which can be easily checked, or shown by applying Lemma A.31 for the subalgebra $\mathfrak{X}(M)$ of the algebra of all linear endomorphisms $\mathrm{End}(\mathfrak{F}(M))$ of the vector space $\mathfrak{F}(M)$ over the field $\mathbb{R}$.

A ***Lie algebra*** (over $\mathbb{R}$) is a vector space endowed with an antisymmetric $\mathbb{R}$-bilinear operation $[-,-]$ that satisfies the Jacobi identity. Since vector fields on an arbitrary manifold M has all the above properties, we can say that $\mathfrak{X}(M)$ is a Lie algebra.

Example 2.25. Let us determine all vector fields $V \in \mathfrak{X}(\mathbb{R}^2)$ that satisfy

$$\left[\frac{\partial}{\partial x}, V\right] = V = \left[V, \frac{\partial}{\partial y}\right].$$

Their general form is $V = a\partial_x + b\partial_y$, where $a = a(x,y)$ and $b = b(x,y)$ are some smooth functions. From the initial conditions we get

$$\begin{aligned} \partial_x(a\partial_x + b\partial_y) - (a\partial_x + b\partial_y)\partial_x &= \frac{\partial a}{\partial x}\partial_x + \frac{\partial b}{\partial x}\partial_y = a\partial_x + b\partial_y, \\ (a\partial_x + b\partial_y)\partial_y - \partial_y(a\partial_x + b\partial_y) &= -\frac{\partial a}{\partial y}\partial_x - \frac{\partial b}{\partial y}\partial_y = a\partial_x + b\partial_y. \end{aligned}$$

From $\partial a/\partial x = a$ follows $a = f(y)e^x$, while $\partial b/\partial x = b$ implies $b = h(y)e^x$. From $-\partial a/\partial y = a$ we have $-f'(y) = f(y)$, so $f(y) = Ce^{-y}$, while from $-\partial b/\partial y = b$ follows $-h'(y) = h(y)$, and therefore $h(y) = De^{-y}$. Hence, we obtain $a = Ce^{x-y}$ and $b = De^{x-y}$ for some constants C and D. The wanted vector fields are of form $V = e^{x-y}(C\partial_x + D\partial_y)$, where C and D are arbitrary constants. △

Example 2.26. Let us express the coordinate vector fields $\partial/\partial x$ and $\partial/\partial y$ in polar coordinates, given by $x = r\cos\theta, y = r\sin\theta$ for $r > 0$, $-\pi < \theta < \pi$. The calculations give

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}.$$

The matrix of coefficients is the Jacobian matrix of the change of coordinates, with the inverse

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}^{-1} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

and therefore

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta}.$$

△

It is important to see that a commutator is always zero in two special cases. The first case is a consequence of anti-symmetry, $[X,X]=0$ holds for all $X\in\mathfrak{X}(M)$. The second case concerns coordinate vector fields from the same chart (U,φ). Partial derivatives are the same in either order on smooth functions,

$$\begin{aligned}\frac{\partial^2 f}{\partial x_i\partial x_j}&=\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}f=\frac{\partial}{\partial x_i}\left(\frac{\partial(f\circ\varphi^{-1})}{\partial\pi_j}\circ\varphi\right)=\frac{\partial}{\partial\pi_i}\left(\frac{\partial(f\circ\varphi^{-1})}{\partial\pi_j}\circ\varphi\circ\varphi^{-1}\right)\circ\varphi\\&=\frac{\partial^2(f\circ\varphi^{-1})}{\partial\pi_i\partial\pi_j}\circ\varphi=\frac{\partial^2(f\circ\varphi^{-1})}{\partial\pi_j\partial\pi_i}\circ\varphi=\frac{\partial^2 f}{\partial x_j\partial x_i},\end{aligned}$$

and therefore

$$\left[\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right]=0. \tag{2.8}$$

Example 2.27. Consider the following two vector fields on $\mathbb{R}^2$,

$$X=\frac{\partial}{\partial x}\quad\text{and}\quad Y=(1+x^2)\frac{\partial}{\partial y}.$$

Then

$$XY=(1+x^2)\frac{\partial^2}{\partial x\partial y}+2x\frac{\partial}{\partial y}\quad\text{and}\quad YX=(1+x^2)\frac{\partial^2}{\partial y\partial x},$$

so we obtain

$$[X,Y]=XY-YX=2x\frac{\partial}{\partial y}.$$

Is it possible to find new coordinates $(u,v)=f(x,y)$ such that X and Y are exactly vector fields $\partial/\partial u$ and $\partial/\partial v$? We can notice that the vector fields X and Y are linearly independent everywhere, but this is not good enough. By (2.8) for the coordinate vectors fields holds $[\partial/\partial u,\partial/\partial v]=0$, so the answer is negative. △

Example 2.28. Consider the previous example for vector fields on $\mathbb{R}^2$, on the open subset where $xy>0$,

$$X=\frac{x}{y}\frac{\partial}{\partial x}+\frac{\partial}{\partial y},\quad Y=2\sqrt{xy}\frac{\partial}{\partial x}.$$

A straightforward calculations show that XY and YX are both equal to $2\sqrt{x/y}(\partial/\partial x)$ plus some second order derivatives which we need not to calculate, because we know in advance that these are the same, and therefore $[X,Y]=0$. So, we can consider the change of coordinates $x=x(u,v)$, $y=y(u,v)$. We should set

$$\begin{aligned}\frac{x}{y}\frac{\partial}{\partial x}+\frac{\partial}{\partial y}&=X=\frac{\partial}{\partial u}=\frac{\partial x}{\partial u}\frac{\partial}{\partial x}+\frac{\partial y}{\partial u}\frac{\partial}{\partial y}\\2\sqrt{xy}\frac{\partial}{\partial x}&=Y=\frac{\partial}{\partial v}=\frac{\partial x}{\partial v}\frac{\partial}{\partial x}+\frac{\partial y}{\partial v}\frac{\partial}{\partial y},\end{aligned}$$

which implies

$$\frac{\partial x}{\partial u}=\frac{x}{y},\quad\frac{\partial y}{\partial u}=1,\quad\frac{\partial x}{\partial v}=2\sqrt{xy},\quad\frac{\partial y}{\partial v}=0.$$

First, we conclude $y=u+C$ for a constant C. Then, $\partial x/x=\partial u/(u+C)$ gives $x=(u+C)f(v)$. Finally, we have $(u+C)f'(v)=2\sqrt{(u+C)f(v)(u+C)}$, that is $f'(v)=\pm2\sqrt{f(v)}$ and therefore $\sqrt{f}=\pm v+\text{Const}$, which gives $f=(v+D)^2$ for a constant D. Hence, for arbitrary constants C and D, the change of coordinates $x=(u+C)(v+D)^2$, $y=u+C$ gives $X=\partial/\partial u$, $Y=\partial/\partial v$ and we have the affirmative answer. △

Example 2.29. Consider the same question as before, this time for vector fields

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

When we confirm $[X,Y] = 0$, we can introduce polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, which is well defined for $r > 0$, $-\pi < \theta < \pi$. Then we have

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} = \frac{1}{r}Y, \quad \frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = X.$$

We can introduce another change of coordinates with $\rho = f(r)$, where we keep the coordinate θ, in such a way to get the desired form $Y = \partial/\partial\rho$. Since

$$\frac{\partial}{\partial r} = \frac{\partial \rho}{\partial r}\frac{\partial}{\partial \rho} = f'(r)\frac{\partial}{\partial \rho},$$

we need $f'(r) = 1/r$, so $f(r) = \ln r + \text{Const}$, hence $r = Ce^{\rho}$ and we have $x = Ce^{\rho}\cos\theta$, $y = Ce^{\rho}\sin\theta$ for any constant $C > 0$. △

2.6 Global tangent maps

Let $f\colon M \to N$ be a smooth map between manifolds M and N. By fitting together the tangent maps of f at all points of M, we obtain a globally defined map between tangent bundles, $Tf\colon TM \to TN$ with $Tf(X_p) = T_pf(X_p)$ for all $X_p \in T_pM$. This map Tf is called ***the global tangent map*** and it is just the map whose restriction to each tangent space $T_pM \subseteq TM$ is T_pf.

The global tangent map Tf pushes a tangent vector $X \in T_pM$ forward from the domain manifold to the codomain manifold tangent vector $f_*(X) = T_pf(X)$ called the ***pushforward*** of X. The pushforward of X can be denoted by $T_pf(X)$, $Tf(X)$, or $f_*(X)$, depending what relations we want to point out and whether we want to emphasize the base point $p = \pi(X)$. Let us notice that if $\pi_M\colon TM \to M$ and $\pi_N\colon TN \to N$ are the canonical projections of tangent bundles, then we have an obvious relation $\pi_N \circ Tf = f \circ \pi_M$.

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\pi_M"'] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Theorem 2.19. *The global tangent map is a smooth map.*

Proof. For a smooth $f\colon M \to N$ and any $p \in M$ there is a chart (U,φ) at $p \in M$ with $x_j = \pi_j \circ \varphi$ $(1 \le j \le m = \dim M)$ and a chart (V,ψ) at $f(p) \in N$ with $y_i = \pi_i \circ \psi$ $(1 \le i \le n = \dim N)$, where $f(U) \subseteq V$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth. For associated charts $(TU,\widetilde{\varphi})$ on TM and $(TV,\widetilde{\psi})$ on TN, $Tf(TU) \subseteq TV$ obviously holds. The coordinate representation

$$\widetilde{\psi} \circ Tf \circ \widetilde{\varphi}^{-1}\colon \varphi(U) \times \mathbb{R}^m \to \psi(V) \times \mathbb{R}^n$$

of the global tangent map Tf can be calculated using the formula (2.2),

$$\left(\widetilde{\psi}\circ Tf\circ\widetilde{\varphi}^{-1}\right)(\varphi(p),\lambda_1,\dots,\lambda_m)=\left(\widetilde{\psi}\circ Tf\right)\left(p,\sum_{j=1}^{m}\lambda_j\left(\frac{\partial}{\partial x_j}\right)_p\right)$$
$$=\widetilde{\psi}\left(f(p),\sum_{j=1}^{m}\lambda_j T_pf\left(\frac{\partial}{\partial x_j}\right)_p\right)=\widetilde{\psi}\left(f(p),\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\frac{\partial(y_i\circ f)}{\partial x_j}(p)\left(\frac{\partial}{\partial y_i}\right)_{f(p)}\right)$$
$$=\left((\psi\circ f\circ\varphi^{-1})(\varphi(p)),\sum_{j=1}^{m}\lambda_j\frac{\partial(\pi_1\circ\psi\circ f\circ\varphi^{-1})}{\partial\pi_j}(\varphi(p)),\dots,\sum_{j=1}^{m}\lambda_j\frac{\partial(\pi_n\circ\psi\circ f\circ\varphi^{-1})}{\partial\pi_j}(\varphi(p))\right),$$

which is smooth since $\psi\circ f\circ\varphi^{-1}$ is smooth, and therefore Tf is smooth. □

If $f\colon M\to N$ is a constant map then Example 2.3 implies $T_pf=0$ for each $p\in M$, and therefore $Tf=0$. Moreover, the converse is also valid, but only on the connected components.

Lemma 2.20. *Let $f\colon M\to N$ be a smooth map between a connected manifold M and a manifold N. If the global tangent map $Tf\colon TM\to TN$ vanishes then f is constant.*

Proof. As before, for an arbitrary $p\in M$ we choose a chart (U,φ) at $p\in M$ and a chart (V,ψ) at $f(p)\in N$ with $f(U)\subseteq V$ such that $\psi\circ f\circ\varphi^{-1}$ is smooth. If we apply $Tf=0$ on the coordinate vectors from T_qM, then by (2.2) we have

$$\frac{\partial(y_i\circ f)}{\partial x_j}(q)=\frac{\partial(\pi_i\circ(\psi\circ f\circ\varphi^{-1}))}{\partial\pi_j}(\varphi(q))=0,$$

for all $q\in U$, $1\le j\le\dim M$, and $1\le i\le\dim N$. Thus $\psi\circ f\circ\varphi^{-1}$ is constant on some ball around $\varphi(p)$, so f is constant on some connected neighbourhood of p, for each $p\in M$. Consider a closed set $Q=f^{-1}(\{f(q)\})$ that contains some fixed point $q\in M$. Since each point from Q has a connected neighbourhood where f is constant, so Q is open. The only nonempty closed and open subset of a connected space M is M itself, hence f is constant on the whole M. □

Let $f\colon M\to N$ be a smooth map between manifolds M and N. If we apply the global tangent map on a vector field $X\in\mathfrak{X}(M)$, every point $p\in M$ is associated to the vector $T_pf(X_p)\in T_{f(p)}N$. However, this way in general does not establish a vector field on N. For example, if f is not surjective there is a problem to assign a vector to points from $N\setminus f(M)$. On the other hand, if f is not injective we have $f(p)=f(q)$ for some distinct points $p,q\in M$, and there is no reason why the vectors $T_pf(X_p)$ and $T_qf(X_q)$ from the same tangent space $T_{f(p)}N=T_{f(q)}N$ should be equal.

In the case that vector fields $X\in\mathfrak{X}(M)$ and $Y\in\mathfrak{X}(N)$ have the property that for each $p\in M$ holds $T_pf(X_p)=Y_{f(p)}$, we say that X and Y are ***f*-related** and write $X\sim_f Y$. Since we have $(X(h\circ f))(p)=X_p(h\circ f)=(T_pf(X_p))(h)$ and $(Yh\circ f)(p)=(Yh)(f(p))=Y_{f(p)}(h)$ for any $p\in M$ and $h\in\mathfrak{F}(N)$, the following statement holds.

Lemma 2.21. *Let $f\colon M\to N$ be a smooth map between manifolds, and let $X\in\mathfrak{X}(M)$ and $Y\in\mathfrak{X}(N)$ be vector fields. Then $X\sim_f Y$ holds if and only if $X(h\circ f)=(Yh)\circ f$ holds for all $h\in\mathfrak{F}(N)$.*

Example 2.30. Let $f\colon\mathbb{R}\to\mathbb{R}^2$ be the smooth map $f(t)=(\cos t,\sin t)$ and $X=d/dt\in\mathfrak{X}(\mathbb{R})$. There is a natural question whether there exists $Y\in\mathfrak{X}(\mathbb{R}^2)$ such that $X\sim_f Y$. For a potential solution $Y=\mu(x,y)(\partial/\partial x)+\nu(x,y)(\partial/\partial y)$, by the formula (2.2) we have

$$Y_{f(t)}=T_tf\left(\frac{d}{dt}\right)_t=\frac{\partial\cos t}{\partial t}\left(\frac{\partial}{\partial x}\right)_{f(t)}+\frac{\partial\sin t}{\partial t}\left(\frac{\partial}{\partial y}\right)_{f(t)}=-\sin t\left(\frac{\partial}{\partial x}\right)_{f(t)}+\cos t\left(\frac{\partial}{\partial y}\right)_{f(t)},$$

and therefore we need $\mu(\cos t, \sin t) = -\sin t$ and $\nu(\cos t, \sin t) = \cos t$. For example it works for $\mu(x,y) = -y$ and $\nu(x,y) = x$, which gives a solution $Y = -y(\partial/\partial x) + x(\partial/\partial y)$.

Let us notice that although f is not surjective, there is a solution since the condition from the definition trivially holds for points that are not in the image $f(\mathbb{R})$. However, this is the reason why a required vector field Y is not uniquely determined. For example, the vector field $Y = -y(x^2+y^2)(\partial/\partial x) + x(x^2+y^2)(\partial/\partial y)$ also satisfies the problem conditions. △

Theorem 2.22. *If $f\colon M \to N$ is a diffeomorphism between manifolds M and N, then for any $X \in \mathfrak{X}(M)$ there is a unique $Y \in \mathfrak{X}(N)$ such that $X \sim_f Y$.*

Proof. Since f is a bijection, any point $q \in N$ is uniquely assigned with $Y_q = T_{f^{-1}(q)}f(X_{f^{-1}(q)})$, which defines a section $Y = Tf \circ X \circ f^{-1}\colon N \to TN$ that is smooth as a composition of smooth maps (Tf is smooth by Theorem 2.19), so $Y \in \mathfrak{X}(N)$. □

The unique vector field $Y \in \mathfrak{X}(N)$ from the previous theorem such that $X \sim_f Y$ is called the ***pushforward*** for f of $X \in \mathfrak{X}(M)$ and has the notation f_*X. So, $f_*X \in \mathfrak{X}(N)$ is defined only for diffeomorphisms $f\colon M \to N$ and can be calculated using the formula $(f_*X)_q = Tf(X_{f^{-1}(q)})$, while by Lemma 2.21 for any $h \in \mathfrak{F}(N)$ we have

$$((f_*X)h) \circ f = X(h \circ f). \tag{2.9}$$

The key thing about related vector fields is their cooperation respect to the commutator, as discussed in the next theorem.

Theorem 2.23. *Let $f\colon M \to N$ be a smooth map between manifolds M and N. If for vector fields $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ hold $X_1 \sim_f Y_1$ and $X_2 \sim_f Y_2$, then $[X_1, X_2] \sim_f [Y_1, Y_2]$ also holds.*

Proof. From Lemma 2.21 for $h \in \mathfrak{F}(N)$ we have $X_1X_2(h \circ f) = X_1((Y_2h) \circ f) = (Y_1Y_2(h)) \circ f$ and $X_2X_1(h \circ f) = (Y_2Y_1(h)) \circ f$, so $[X_1, X_2](h \circ f) = (Y_1Y_2(h)) \circ f - (Y_2Y_1(h)) \circ f = ([Y_1, Y_2]h) \circ f$. The proof ends with the reuse of Lemma 2.21. □

As a consequence, if we put a diffeomorphism $f\colon M \to N$ in the previous theorem, then for $X_1, X_2 \in \mathfrak{X}(M)$ we have a nice formula

$$f_*[X_1, X_2] = [f_*X_1, f_*X_2]. \tag{2.10}$$

2.7 Problems

Problem 2.1. In the half-plane $M = \{(x,y) \in \mathbb{R}^2 : y > 0\} \subset \mathbb{R}^2$, the point $p \in M$ is given by coordinates $(x,y) = (0,2)$. Express the tangent vector $T_pM \ni X = 4\big(\frac{\partial}{\partial x}\big)_p + \big(\frac{\partial}{\partial y}\big)_p$ in terms of $\big(\frac{\partial}{\partial r}\big)_p$ and $\big(\frac{\partial}{\partial \theta}\big)_p$ in polar coordinates.

Problem 2.2. Show that $f\colon \mathbb{R}\mathbf{P}^1 \to \mathbb{R}$ given by $f(x:y) = xy/(x^2+y^2)$ is well-defined and smooth. Determine the maximum of f on $\mathbb{R}\mathbf{P}^1$, if it exists.

Problem 2.3. Is the composition of two maps of constant rank always of constant rank?

Problem 2.4. Prove that any smooth map between manifolds can be decomposed into the composition of a submersion and an embedding.

Problem 2.5. For which $m, n \in \mathbb{N}$ does there exist a submersion $f\colon \mathbb{R}\mathbf{P}^m \to \mathbf{S}^n$ that is not surjective?

Problem 2.6. Let $f\colon \mathbb{R}\mathbf{P}^1 \to \mathbb{R}\mathbf{P}^n$ for $n \in \mathbb{N}$ be given by $f(x:y) = (x^n : x^{n-1}y : \ldots : xy^{n-1} : y^n)$. Prove that f is well-defined and examine whether it is an embedding.

Problem 2.7. Let $f\colon \mathbb{R}\mathbf{P}^1 \times \mathbb{R}\mathbf{P}^1 \to \mathbb{R}\mathbf{P}^3$ be given by $f((x:y),(z:w)) = (xz:xw:yz:yw)$. Prove that f is well-defined and determine whether f is an immersion, a submersion, or an embedding.

Problem 2.8. For which $r \in \mathbb{R}$ is the set $M = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1,\ x^2+z^2 = r^2\} \subset \mathbb{R}^3$ a one-dimensional submanifold of $\mathbb{R}^3$?

Problem 2.9. Prove that $\{(x:y:z) \in \mathbb{R}\mathbf{P}^2 : xy = z^2\}$ is a submanifold of $\mathbb{R}\mathbf{P}^2$.

Problem 2.10. Prove that $M = \{((x_1:\ldots:x_{n+1}),(y_1:\ldots:y_{n+1})) \in \mathbb{R}\mathbf{P}^n \times \mathbb{R}\mathbf{P}^n : \sum_{i=1}^{n+1} x_i y_i = 0\}$ is a submanifold of $\mathbb{R}\mathbf{P}^n \times \mathbb{R}\mathbf{P}^n$ and determine $\dim M$.

Problem 2.11. For the manifold $M = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\} \subset \mathbb{R}^2$, determine the vector field $V \in \mathfrak{X}(M)$ if

$$\left[\frac{\partial}{\partial y}, V\right] = 2\frac{x+y}{y^2}\frac{\partial}{\partial x} \quad \text{and} \quad \left[y\frac{\partial}{\partial x}, V\right] = 0.$$

Problem 2.12. Determine $V \in \mathfrak{X}(\mathbb{R}^2)$ for which $[\partial_x + x\partial_y, [\partial_x, V]] = 0 = [V, \partial_y]$ such that $V_{(0,0)} = (\partial_x)_{(0,0)} + (\partial_y)_{(0,0)}$ and $V_{(1,1)} = 3(\partial_x)_{(1,1)} + 3(\partial_y)_{(1,1)}$.

Problem 2.13. Determine all constants $c \in \mathbb{R}$ for which the vector fields $X, Y \in \mathfrak{X}(\mathbb{R}_+ \times \mathbb{R}_+)$ given by

$$X = (x+1)\frac{\partial}{\partial x} + (y\ln y + y)\frac{\partial}{\partial y} \quad \text{и} \quad Y = (2x+c)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$$

allow a coordinate change such that X and Y become coordinate vector fields.

Problem 2.14. Determine the real constants a and c for which the vector fields X and Y in $\mathbb{R}_+ \times \mathbb{R}_+$ given by

$$X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} \quad \text{и} \quad Y = a\frac{\partial}{\partial x} + x^c\frac{\partial}{\partial y}$$

allow a coordinate change such that X and Y become coordinate vector fields, and then find such a coordinate change.

Problem 2.15. If $f\colon D \to f(D)$ is a diffeomorphism given by $(x,y) = f(r,\theta) = (r\cos\theta, r\sin\theta)$, where $D, f(D) \subset \mathbb{R}^2$, compute $(f^{-1})_*(\partial_x)$.

Problem 2.16. Let $Q = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$ be an open submanifold of $\mathbb{R}^2$ and $f\colon Q \to Q$ be given by $(u,v) = f(x,y) = (xy, y/x)$. Show that f is a diffeomorphism and compute $f_*(x\partial_x + y\partial_y)$ and $f_*(y\partial_x)$.

Problem 2.17. Let $f\colon \mathbb{R}^2 \to \mathbb{R}\mathbf{P}^2$ be given by $f(x,y) = (x:y:1)$ and $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$. How many vector fields $Y \in \mathfrak{X}(\mathbb{R}\mathbf{P}^2)$ satisfy $X \sim_f Y$? Write down a solution (if one exists) for Y in all three standard charts.

Chapter 3

TENSOR BUNDLES AND FIELDS

3.1 Vector bundles

The tangent bundle of a manifold has a natural structure as a smooth manifold in its own right, which is a concept that can be generalised. Roughly speaking, a vector bundle is a family of vector spaces, one for each point of manifold, glued together in some smoothly varying manner.

Let an n-manifold M be the ***bundle base*** and let $r \in \mathbb{N}$ be the fixed ***bundle rank***. Consider a family of real vector spaces E_p of dimension r indexed by points $p \in M$, and let the ***bundle space*** be

$$E = \bigsqcup_{p \in M} E_p.$$

The ***bundle projection*** is the natural projection $\pi\colon E \to M$ defined by $\pi(v) = p$ for $v \in E_p$. A ***bundle chart*** on E is a map $\widetilde{\varphi}\colon \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^r$ for which there exists a map (U, φ) on M such that $\widetilde{\varphi}$ for each $p \in U$ restricts to a linear isomorphism

$$\widetilde{\varphi}\restriction_{E_p}\colon E_p = \pi^{-1}(p) \to \{\varphi(p)\} \times \mathbb{R}^r \cong \mathbb{R}^r.$$

A ***bundle atlas*** is a smooth atlas for E consisting of bundle charts on E. Every bundle atlas for E determines a ***complete bundle atlas*** consisting of all bundle charts that are compatible with all bundle charts from the given atlas.

A ***vector bundle*** (or ***smooth real vector bundle***) of rank r over M is a smooth manifold of dimension $n + r$ generated by a set E with a (complete) bundle atlas for E. The additional topological conditions from the definition of a manifold are automatically transferred from a base M. Since M has a countable subatlas, a bundle atlas also has a countable subatlas, and therefore E is second countable. If $x \in E_p$ and $y \in E_q$ hold for $p \neq q$ (the case $p = q$ is covered by Lemma 1.5), there exist open disjoint $V_p, V_q \subset M$ that separate $p \in V_p$ and $q \in V_q$, so $\pi^{-1}(V_p)$ and $\pi^{-1}(V_q)$ separate x and y, and therefore E is Hausdorff.

The bundle projection $\pi\colon E \to M$ is a smooth surjective map in which the essential information of a vector bundle is hidden, and they are often identified. Of course, when there is no danger of ambiguity, it is easier to identify a vector bundle by its total space E.

The key property of a bundle chart $\widetilde{\varphi}$ is that, at the points $p \in U$, it restricts to a linear isomorphism between a ***fiber*** $E_p = \pi^{-1}(p)$ over p which has the structure of a real vector space and $\mathbb{R}^r$. As a manifold M locally looks like an open subset of $\mathbb{R}^n$, a vector bundle E over M locally looks like the product of an open subset of $\mathbb{R}^n$ and a vector space $\mathbb{R}^r$.

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\pi"'] & \varphi(U) \times \mathbb{R}^r \arrow[d, "\pi_{\varphi(U)}"] \\
U \arrow[r, "\varphi"] & \varphi(U)
\end{tikzcd}$$

For a transition function $\widetilde{\psi} \circ \widetilde{\varphi}^{-1} \colon \varphi(U \cap V) \times \mathbb{R}^r \to \psi(U \cap V) \times \mathbb{R}^r$, where (U, φ) and (V, ψ) are the corresponding charts on M, we have $\pi_{\psi(U \cap V)} \circ \widetilde{\psi} \circ \widetilde{\varphi}^{-1} = \psi \circ \varphi^{-1} \circ \pi_{\varphi(U \cap V)}$, from which it can be seen that the transition function is of form

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1}(x, v) = ((\psi \circ \varphi^{-1})(x), \tau(x)v),$$

where $\tau \colon \varphi(U \cap V) \to \mathrm{GL}(r, \mathbb{R})$. It follows that to check the compatibility of charts it is sufficient to examine the smoothness of the map τ.

Example 3.1. The tangent bundle of an n-manifold M is a basic example of a vector bundle. For a chart (U, φ) on M we can consider the global tangent map $T\varphi \colon TU \to T\varphi(U) \cong \varphi(U) \times \mathbb{R}^n$, which at a point $p \in U$ is restricted to the linear isomorphism $T_p\varphi \colon T_pU = T_pM \to T_{\varphi(p)}\varphi(U) \cong \mathbb{R}^n$. If (V, ψ) is some other chart on M, then we have $T_p\psi \circ (T_p\varphi)^{-1} = T_{\varphi(p)}(\psi \circ \varphi^{-1})$, from which follows the compatibility of bundle charts $T\psi \circ (T\varphi)^{-1}(x, v) = (\psi \circ \varphi^{-1}(x), T_x(\psi \circ \varphi^{-1})(v))$ because the Jacobian matrix depend smoothly on x. Thus, an arbitrary smooth atlas for M generates a bundle atlas for TM, so TM is a vector bundle of rank n over M. Let us notice that from a simple calculation,

$$T_p\varphi \left(\frac{\partial}{\partial x_j} \right)_p = \sum_i \frac{\partial(\pi_i \circ \varphi)}{\partial x_j}(p) \left(\frac{\partial}{\partial \pi_i} \right)_{\varphi(p)} = \sum_i \frac{\partial \pi_i}{\partial \pi_j}(\varphi(p)) \left(\frac{\partial}{\partial \pi_i} \right)_{\varphi(p)} = \left(\frac{\partial}{\partial \pi_j} \right)_{\varphi(p)}$$

follows that the bundle charts $T\varphi$ correspond to the chart from (2.4) that we originally introduced when we defined the tangent bundle. △

Example 3.2. The simplest, but certainly important example of a vector bundle is a ***product bundle***, that exists for any n-manifold M and each $r \in \mathbb{N}$. This is a manifold $E = M \times \mathbb{R}^r$ of dimension $n + r$ with the natural projection $\pi_M \colon M \times \mathbb{R}^r \to M$, $\pi_M(p, v) = p$. The product bundle of rank r over M assigns the fiber $E_p = \pi_M^{-1}(p) = \{p\} \times \mathbb{R}^r$ to each point $p \in M$, while the bundle charts are $\varphi \times \mathbb{1}_{\mathbb{R}^r}$, where φ are charts on M. In particular, the cylinder $\mathbf{S}^1 \times \mathbb{R}$ is a product bundle over the circle $\mathbf{S}^1$ via the projection $\pi_{\mathbf{S}^1} \colon \mathbf{S}^1 \times \mathbb{R} \to \mathbf{S}^1$. △

Example 3.3. Let P be a submanifold of a manifold M, and let $\pi \colon E \to M$ be a vector bundle. Then its restriction $\pi_P \colon \pi^{-1}(P) \to P$ is also a vector bundle, we denote $E\restriction_P = \pi^{-1}(P)$ and call it the ***restriction*** of E to P. If (U, φ) is a chart at $p \in M$, and (U_P, φ_P) is a chart at $p \in P$ such that $U_P \subseteq U$, then from the induces chart $\widetilde{\varphi}$ on E we obtain the induced chart $\widetilde{\varphi}_P$ on $E\restriction_P$ by

$$\widetilde{\varphi}_P = ((\varphi_P \circ \varphi^{-1}) \times \mathbb{1}_{\mathbb{R}^r}) \circ \widetilde{\varphi}\restriction_{\pi^{-1}(U_P)} \colon \pi^{-1}(U_P) \to \varphi_P(U_P) \times \mathbb{R}^r,$$

while the transition function $\widetilde{\psi}_P \circ (\widetilde{\varphi}_P)^{-1}$ is smooth as a composition of smooth functions. △

Example 3.4. Let $\pi_E \colon E \to M$ and $\pi_F \colon F \to N$ be vector bundles of rank r_1 and r_2, then their Cartesian product $E \times F$ is a vector bundle of rank $r_1 + r_2$ over $M \times N$, whose fiber at $(p, q) \in M \times N$ is the vector space $E_p \times E_q$. △

Example 3.5. Let M be a manifold, and $\pi_E \colon E \to M$ and $\pi_F \colon F \to M$ be vector bundles of rank r_1 and r_2 over M. The ***Whitney sum***[1] of E and F is the vector bundle $E \oplus F$ over M which fiber at a point $p \in M$ is the direct sum $E_p \oplus F_p$, that is,

$$E \oplus F = \bigsqcup_{p \in M} (E_p \oplus F_p)$$

with the obvious projection $\pi = \pi_E \circ \pi_1 = \pi_F \circ \pi_2$. For each $p \in M$ there is a chart (U, φ) at $p \in M$ and bundle charts $\widetilde{\varphi} \colon \pi_E^{-1}(U) \to \varphi(U) \times \mathbb{R}^{r_1}$ and $\overline{\varphi} \colon \pi_F^{-1}(U) \to \varphi(U) \times \mathbb{R}^{r_2}$, where we

[1] Hassler Whitney (1907–1989), American mathematician

define $\Phi\colon \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^{r_1+r_2}$ by $\Phi = (\varphi \circ \pi, \pi_{\mathbb{R}^{r_1}} \circ \widetilde{\varphi} \circ \pi_1, \pi_{\mathbb{R}^{r_2}} \circ \overline{\varphi} \circ \pi_2)$. Let $\Psi\colon \pi^{-1}(V) \to \psi(V) \times \mathbb{R}^{r_1+r_2}$ be another bijective map, where (V, ψ) is a chart at $p \in M$. If the transition functions $\widetilde{\psi} \circ \widetilde{\varphi}^{-1}$ and $\overline{\psi} \circ \overline{\varphi}^{-1}$ correspond to smooth maps $\tau_1\colon \varphi(U \cap V) \to \mathrm{GL}(r_1, \mathbb{R})$ and $\tau_2\colon \varphi(U \cap V) \to \mathrm{GL}(r_2, \mathbb{R})$, then the transition function $\Psi \circ \Phi^{-1}$ corresponds to the map $\tau\colon \varphi(U \cap V) \to \mathrm{GL}(r_1 + r_2, \mathbb{R})$ where $\tau(p) = \tau_1(p) \oplus \tau_2(p)$ is the block diagonal matrix with $\tau_1(p)$ и $\tau_2(p)$ on the diagonal, which depends smoothly on p and proves that $E \oplus F$ is a vector bundle over M. △

Suppose $\pi_E\colon E \to M$ and $\pi_F\colon F \to N$ are vector bundles. A smooth map $\Phi\colon E \to F$ is called a (smooth) ***bundle homomorphism*** if there exists a map $f\colon M \to N$ satisfying $\pi_N \circ \Phi = f \circ \pi_M$ such that the restriction $\Phi\restriction_{E_p}\colon E_p \to F_{f(p)}$ is linear on fibers for all $p \in M$. In that case we say that f is the ***base map*** of Φ, as well as that F ***covers*** f.

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow[f]{} & N \end{array}$$

Example 3.6. If $f\colon M \to N$ is a smooth map between manifolds M and N, then its global tangent map $Tf\colon TM \to TN$ is a bundle homomorphism covering f. △

If a bundle homomorphism $\Phi\colon E \to F$ is additionally diffeomorphism (thus f is also a diffeomorphism), then we say that Φ is a ***bundle isomorphism***.

A bundle isomorphism $\Phi\colon E \to M \times \mathbb{R}^r$ with the product bundle from Example 3.2 is called a ***trivialization*** of E, and if it exists, then we say that a vector bundle E is ***trivial***. Each bundle chart $\widetilde{\varphi}$ defines a bundle isomorphism

$$(\varphi^{-1} \times \mathbb{1}_{\mathbb{R}^r}) \circ \widetilde{\varphi}\colon E\restriction_U = \pi^{-1}(U) \to U \times \mathbb{R}^r,$$

which we call a ***local trivialization*** of E over U. Of course, every vector bundle is locally trivial in the sense that a trivialization exists over some neighbourhood of every point from base. However, we cannot expect a local trivialization to exist for every subset $U \subseteq M$, but if it does, we say that the bundle is ***trivial over*** U.

An important special case occurs when both E and F are vector bundles over the same manifold M. In this case it is convenient to use slightly more restrictive definitions that include $f = \mathbb{1}_M$. A ***bundle homomorphism over*** M is a bundle homomorphism covering the identity map of M, and whose restriction to each fiber is linear. If there exists a bundle homomorphism $F\colon E \to F$ over M that is also a bundle isomorphism, then Φ is a ***bundle isomorphism over*** M and we say that E and F are ***isomorphic over*** M.

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ & {\scriptstyle \pi_E}\searrow \quad \swarrow{\scriptstyle \pi_F} & \\ & M & \end{array}$$

3.2 Local and global frames

A ***section*** of a vector bundle $\pi\colon E \to M$ over an open set $U \subseteq M$ is a map $\sigma\colon U \to E$ such that $\pi \circ \sigma = \mathbb{1}_U$. Hence, for each point $p \in U$, the section σ selects one element of the fiber E_p. The term section comes from the geometric interpretation of the image $\sigma(U) \subset E$ as a subset of the total space E. The basic example of section is a vector field on a manifold M, which is a smooth section of the tangent bundle TM over M.

If σ is a smooth section of a vector bundle $\pi\colon E \to M$, then $\pi \circ \sigma = \mathbb{1}_M$ implies the injectivity of σ and $T\sigma$, so from the continuity of π it follows that σ is an embedding, and $\sigma(M)$ is a submanifold of E. Especially, any vector bundle admits a canonical section called the ***zero section***, that maps each point $p \in M$ to the zero vector $0 \in E_p$, which gives a natural embedding $M \hookrightarrow E$.

Example 3.7. A section of a product bundle $\pi_M\colon M \times \mathbb{R}^r \to M$ is a map $\sigma\colon M \to M \times \mathbb{R}^r$ of the form $\sigma = (\mathbb{1}_M, f)$ for some map $f\colon M \to \mathbb{R}^r$, which is smooth if and only if f is smooth. Hence, a smooth section of a product bundle of rank r over M is essentially a smooth map $f\colon M \to \mathbb{R}^r$. △

The set of all smooth sections of E over $U \subseteq M$ is denoted by $\Gamma(U, E)$, while in the case that U is the manifold M we use the notation $\Gamma(E) = \Gamma(M, E)$. It is easy to notice that the set $\Gamma(U, E)$ is a vector space over $\mathbb{R}$ under addition and scalar multiplication defined pointwise by $(c_1\sigma_1 + c_2\sigma_2)(p) = c_1\sigma_1(p) + c_2\sigma_2(p)$ for $\sigma_1, \sigma_2 \in \Gamma(U, E)$, $c_1, c_2 \in \mathbb{R}$, and $p \in U$. Just like in the case of vector fields $\mathfrak{X}(U) = \Gamma(TU) = \Gamma(U, TM)$, smooth sections of E can be multiplied by smooth functions $f \in \mathfrak{F}(U)$ with $(f\sigma)(p) = f(p)\sigma(p)$, and $\Gamma(U, E)$ is a module over the ring $\mathfrak{F}(U)$.

Let M be a manifold of dimension n, and $U \subseteq M$ is some its open subset. Consider a vector bundle $\pi\colon E \to M$ of rank r. A ***local frame*** for E over U is an ordered r-tuple $(\sigma_1, \dots, \sigma_r)$ of smooth sections over U ($\sigma_i \in \Gamma(U, E)$ for $1 \leq i \leq r$) such that vectors $\sigma_1(p), \dots, \sigma_r(p)$ form a basis for the fiber E_p, for each $p \in U$. In particular, a local frame for E over M is called a ***global frame*** for E.

The most frequently observed vector bundle is the tangent bundle ($E = TM$), and then for frames for TM we say that they are ***frames*** for a manifold M. In other words, a local frame for a manifold M over U is an ordered m-tuple of vector fields on U whose values at each point $p \in U$ form a basis of the tangent space T_pM. A manifold that admits a global frame is called ***parallelisable***.

Example 3.8. If (U, φ) is an arbitrary chart on a manifold M with the corresponding coordinate functions $x_i = \pi_i \circ \varphi$, then the coordinate vector fields form a local frame

$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$$

for M over U called the ***coordinate frame***. Of course, any point of M has an neighbourhood over which there is such a local frame. Coordinate vector fields in some chart provide a convenient way of representing vector fields. △

Example 3.9. The standard basis $(e_1, \dots, e_r)$ for $\mathbb{R}^r$ by $E_i(p) = (p, e_i)$ for $1 \leq i \leq r$ and $p \in M$ gives a global frame $(E_1, \dots, E_r)$ for the product bundle $M \times \mathbb{R}^r$. △

Example 3.10. Let $\pi\colon E \to M$ be a vector bundle of rank r. A local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^r$ of E over $U \subseteq M$ generates a local frame based on Example 3.9, where we define $\sigma_i = \Phi^{-1} \circ E_i\colon U \to E$ for $1 \leq i \leq r$. Since $\pi = \pi_U \circ \Phi$ we have $\pi \circ \sigma_i = \pi \circ \Phi^{-1} \circ E_i = \pi_U \circ E_i = \mathbb{1}_U$, so σ_i is a section, while it is smooth because Φ is a diffeomorphism. Since Φ restricts to an isomorphism and $\Phi \circ \sigma_i = E_i$, then $(\sigma_1, \dots, \sigma_r)$ is a local frame for E over U and we say that this frame is ***associated with*** Φ.

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, bend left, "\pi"'] & & U \times \mathbb{R}^r \arrow[dl, bend right, "\pi_U"] \\
& U \arrow[ul, bend left, "\sigma_i"] \arrow[ur, bend right, "E_i"'] &
\end{tikzcd}$$

Conversely, if $(\sigma_1, \dots, \sigma_r)$ is a local frame for E over U, then $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^r$ given by $\Phi(\sum_{i=1}^{r} \lambda_i\sigma_i(p)) = (p, \lambda_1, \dots, \lambda_r)$ for each $p \in U$ is a local trivialization of E over U. △

From the previous example we see that the existence of a local frame for E over U is equivalent to the existence of a local trivialization over U. As a consequence, we have the following theorem.

Theorem 3.1. *A vector bundle E over a manifold M is trivial if and only if there exists a global frame for E over M. Especially, a tangent bundle TM is trivial if and only if a manifold M is parallelisable.*

Example 3.11. A manifold M of dimension n with a single chart atlas $\{(M,\varphi)\}$ induces a natural global trivialization $(\varphi^{-1} \times \mathbb{1}_{\mathbb{R}^n}) \circ \widetilde{\varphi}\colon TM \to M \times \mathbb{R}^n$, so the tangent bundle TM is trivial and the manifold M is parallelisable. △

Example 3.12. Let G be a Lie group with the neutral $e \in G$. Consider $f\colon G \times T_eG \to TG$ given by $f(p,X) = (T_eL_p)(X)$, where $L_p\colon G \to G$ is a diffeomorphism defined by $L_p(q) = pq$. Then the map f^{-1} is a global trivialization as the diffeomorphism between manifolds TG and $G \times T_eG \cong G \times \mathbb{R}^{\dim G}$, so any Lie group is parallelisable. △

Let $\pi\colon E \to M$ be a vector bundle. A ***subbundle*** of E is a subset $D \subseteq E$ such that the restriction $\pi\restriction_D\colon D \to M$ is a vector bundle, and the inclusion $\imath\colon D \hookrightarrow E$ is a bundle homomorphism over M. In particular, a ***distribution*** on M is a subbundle of the tangent bundle, and its dimension is the rank of this subbundle. The following theorem is the local frame criterion for checking that a union of subspaces is a subbundle.

Theorem 3.2. *Let E be a vector bundle over a manifold M and assume that for each $p \in M$ we have a k-dimensional linear subspace $D_p \subseteq E_p$. Then $D = \bigsqcup_{p\in M} D_p \subseteq E$ is a subbundle of E if and only if each point of M has a neighbourhood $U \subseteq M$ over which there exists a local frame for D.*

Proof. In this proof we use Example 3.10 and the relation between local trivializations and local frames. If D is a subbundle then a local trivialization easily gives a wanted local frame for D. On the other hand, let $(\sigma_1,\dots,\sigma_k)$ be a local frame for D over some open subset containing $p \in M$. We can complete it to a local frame $(\sigma_1,\dots,\sigma_r)$ for E over some neighbourhood U of p. It is associated with a local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^r$ defined by $\Phi(\sum_{i=1}^r \lambda_i\sigma_i(q)) = (q,\lambda_1,\dots,\lambda_r)$. However, Φ takes $D \cap \pi^{-1}(U)$ to the subset $\{(q,\lambda_1,\dots,\lambda_k,0,\dots,0) : q \in U; \lambda_1,\dots,\lambda_k \in \mathbb{R}\} \subseteq U \times \mathbb{R}^r$, which is a submanifold. Moreover, the map $\Psi\colon D \cap \pi^{-1}(U) \to U \times \mathbb{R}^k$ defined by $\Psi(\sum_{i=1}^k \lambda_i\sigma_i(q)) = (q,\lambda_1,\dots,\lambda_k)$ is a local trivialization of D, so D is a vector bundle. □

Suppose $\pi_E\colon E \to M$ and $\pi_F\colon F \to M$ are vector bundles and $\Phi\colon E \to F$ is a bundle homomorphism over M. The ***rank*** of Φ at a point $p \in M$ is the rank of the linear map $\Phi\restriction_{E_p}$, and if its rank is the same for all $p \in M$ we say that Φ has ***constant rank***. Let us define subsets $\operatorname{Ker}\Phi \subseteq E$ and $\operatorname{Im}\Phi \subseteq F$ by

$$\operatorname{Ker}\Phi = \bigsqcup_{p\in M} \operatorname{Ker}(\Phi\restriction_{E_p}) \quad \text{and} \quad \operatorname{Im}\Phi = \bigsqcup_{p\in M} \operatorname{Im}(\Phi\restriction_{E_p}).$$

Since the fibers of a bundle have the same dimension everywhere, in order for $\operatorname{Ker}\Phi$ and $\operatorname{Im}\Phi$ to be subbundles, it is necessary that F has constant rank. However, the converse is also true, as can be seen from the following theorem that allows us to obtain new subbundles (see Lee [78, Theorem 10.34]).

Theorem 3.3. *Let E and F be vector bundles over a manifold M and let $\Phi\colon E \to F$ be a bundle homomorphism over M. If Φ has constant rank, then* $\operatorname{Ker}\Phi$ *is a subbundle of E and* $\operatorname{Im}\Phi$ *is a subbundle of F.*

Proof. Let us suppose that Φ has constant rank ρ. For any $p \in M$ we can choose a local frame $(\sigma_1, \ldots, \sigma_r)$ for E over a neighbourhood $U \ni p$. The maps $\Phi \circ \sigma_i \colon U \to F$ for $1 \leq i \leq r$ are smooth sections of F over U. Since these sections span $(\operatorname{Im}\Phi)\restriction_U$, we can assume (after rearranging the indices if necessary) that the elements $\Phi \circ \sigma_1(p), \ldots, \Phi \circ \sigma_\rho(p)$ form a basis for $\operatorname{Im}(\Phi\restriction_{E_p})$. By continuity they remain linearly independent in some neighbourhood $V \ni p$, and because of constant rank, $(\Phi \circ \sigma_1, \ldots, \Phi \circ \sigma_\rho)$ is a local frame for $\operatorname{Im}\Phi$ over V, which proves that $\operatorname{Im}\Phi$ is a subbundle of F.

Consider a subbundle $E_0 \subseteq E\restriction_V$ spanned by $\sigma_1, \ldots, \sigma_\rho$. Since the bundle homomorphism $\Phi\restriction_{E_0} \colon E_0 \to (\operatorname{Im}\Phi)\restriction_V$ is bijective it is a bundle isomorphism. Define a bundle homomorphism $\Psi \colon E\restriction_V \to E\restriction_V$ by $\Psi = \mathbb{1} - (\Phi\restriction_{E_0})^{-1} \circ \Phi$ to see that $\Phi \circ \Psi = 0$ on both E_0 and $(\operatorname{Ker}\Phi)\restriction_V$. However, E_0 and $(\operatorname{Ker}\Phi)\restriction_V$ together span $E\restriction_V$, so Ψ takes its values in $(\operatorname{Ker}\Phi)\restriction_V$ and consequently $\operatorname{Im}\Psi = (\operatorname{Ker}\Phi)\restriction_V$ is a subbundle of $E\restriction_V$, which proves that $\operatorname{Ker}\Phi$ is a subbundle of E. □

3.3 Vector fields on a sphere

The question of how many linearly independent vector fields exist on a sphere is a classical problem of differential topology. Let $\operatorname{Span}(M) \in \mathbb{N}_0$ denotes the maximal number of linearly independent vector fields on a manifold M. We are interested in the values $\operatorname{Span}(\mathbf{S}^{n-1})$, so we consider vector fields on a sphere $\mathbf{S}^{n-1} \subset \mathbb{R}^n$.

The idea is to assign to each point $x \in \mathbf{S}^{n-1}$ a vector $V(x)$ tangent to x (orthogonal to the position vector of x) in a smooth way. The zero vector is orthogonal to everything, which is inconvenient (since it violates linear independence), so we look for nowhere-zero vector fields. Such a vector field can be normalized to get $\|V(x)\| = 1$ for all x, so we looking for smooth maps $V \colon \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ such that $V(x) \perp x$ holds for all $x \in \mathbf{S}^{n-1}$.

Example 3.13. Consider a standard topological argument, where a required vector field V would give a homotopy $h_t(x) = x\cos(\pi t) + V(x)\sin(\pi t)$ between the identity $h_0 = \mathbb{1}_{\mathbf{S}^{n-1}}$ and the antipodal map $h_1 = -\mathbb{1}_{\mathbf{S}^{n-1}}$. Then the degree of a continuous map from the n-sphere $\mathbf{S}^{n-1}$ to itself, is a homotopy invariant, and we have that the degree $(-1)^n$ of antipodal map (a composition of n reflections) is equal to the degree 1 of identity, and therefore n must be even. Hence, for an odd n there are no such vector fields, which gives $\operatorname{Span}(\mathbf{S}^{2k}) = 0$. This claim is well known as the ***hairy ball theorem*** (or the ***hedgehog theorem***) and states that there is no non-vanishing continuous vector field on even-dimensional spheres, or expressed colloquially, you cannot comb a hairy ball flat without creating a cowlick. △

Example 3.14. When n is even we can identify $\mathbb{R}^n \cong \mathbb{C}^{n/2}$, and put $V(x) = ix$ which is clearly a vector field on every sphere of odd dimension, which gives $\operatorname{Span}(\mathbf{S}^{2k+1}) > 0$. In the special case $n = 2$ we obtain a global frame for the circle $\mathbf{S}^1$ consists of one vector field given by $V(x) = ix$, that is a unit tangent vector field directed counter-clockwise. △

We can find a vector field on $\mathbf{S}^{n-1}$ of the form $V(x) = Ax$ for some matrix $A \in \mathbb{R}^{n\times n}$. As we require $Ax \perp x$, we have that

$$x^\intercal A x = 0 \tag{3.1}$$

holds for all $x \in \mathbf{S}^{n-1}$, and thus (by appropriate scaling) for all $x \in \mathbb{R}^n$. By polarizing (3.1), $0 = (x+y)^\intercal A(x+y) = x^\intercal Ay + y^\intercal Ax = x^\intercal Ay + x^\intercal A^\intercal y = x^\intercal(A+A^\intercal)y$ holds for all $x, y \in \mathbb{R}^n$, which yields $A + A^\intercal = 0$. Hence, as a consequence of the condition (3.1) we have a skew-symmetric matrix $A = -A^\intercal$. Conversely, a skew-symmetric matrix A implies $2x^\intercal Ax = x^\intercal(A + A^\intercal)x = 0$, so we conclude that a required vector field uniquely corresponds to some skew-symmetric matrix A.

Additionally, since V is normalized we have $\|V(x)\| = \|x\|$ that implies $(Ax)^\intercal(Ax) = x^\intercal x$, so $x^\intercal(A^\intercal A - \mathbb{1})x = 0$, which corresponds to the condition (3.1) for a matrix $A^\intercal A - \mathbb{1}$ that must be skew-symmetric. However, $A^\intercal A - \mathbb{1}$ is obviously symmetric, which yields $A^\intercal A = \mathbb{1}$, and therefore $A^2 = -\mathbb{1}$. A required vector field depends on a skew-symmetric orthogonal matrix A, and since then $(\det A)^2 = \det(A^2) = \det(-\mathbb{1}) = (-1)^n$ holds, we can additionally confirm that an odd n is out of the question.

If there exist vector fields $V_1, \dots, V_k \in \mathfrak{X}(\mathbf{S}^{n-1})$ such that $V_1(x), \dots, V_k(x) \in T_x\mathbf{S}^{n-1} \cong x^\perp$ are linearly independent for all $x \in \mathbf{S}^{n-1}$, then the Gram[2]–Schmidt[3] process makes vector fields with the condition that $V_1(x), \dots, V_k(x)$ are mutually orthogonal for all $x \in \mathbf{S}^{n-1}$. Thus, for appropriate matrices $A_1, \dots, A_{n-1} \in \mathbb{R}^{n\times n}$ we additionally suppose that $A_i x \perp A_j x$ holds for $1 \le i \ne j \le k$, which gives $x^\intercal(A_i^\intercal A_j)x = 0$ and again, like the condition (3.1), implies that $A_i^\intercal A_j$ is skew-symmetric. Thus, we obtain $-A_iA_j = A_i^\intercal A_j = -(A_i^\intercal A_j)^\intercal = -A_j^\intercal A_i = A_jA_i$. Finally, the required conditions are described with the Hurwitz[4] relation

$$A_iA_j + A_jA_i = -2\delta_{ij}\,\mathbb{1}, \tag{3.2}$$

which holds for all $1 \le i, j \le k$.

A classical problem is to determine parallelisable spheres $\mathbf{S}^{n-1} \subset \mathbb{R}^n$, which are characterized by $\operatorname{Span}(\mathbf{S}^{n-1}) = n - 1$. Using methods from algebraic topology it was proved in 1958 by Bott[5] and Milnor [25] and independently by Kervaire [72] that we have parallelisable spheres only for $n \in \{1, 2, 4, 8\}$. In fact, this question is closely related with existence of normed division algebra of the real numbers. There are only four such algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. Concretely, the only parallelisable spheres are $\mathbf{S}^0$, $\mathbf{S}^1$, $\mathbf{S}^3$, and $\mathbf{S}^7$, and they are assigned with the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ respectively.

Example 3.15. In the case $n = 4$ we can set three skew-symmetric orthogonal matrices that satisfy the Hurwitz relations (3.2) by

$$A_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In this way, we obtain three orthonormal vector fields on $\mathbf{S}^3$, so the sphere $\mathbf{S}^3$ is parallelisable. It is not hard to notice that the matrices A_1, A_2, A_3 correspond to multiplication by unit quaternions $i, j, k \in \mathbb{H}$. △

Skew-symmetric orthogonal matrices such that (3.2) holds were discussed independently by Hurwitz [70] and Radon[6] [101] (see also Eckmann[7] [44]), which is closely related to the ***Hurwitz–Radon function*** $\rho\colon \mathbb{N} \to \mathbb{N}$ given by

$$\rho(a \cdot 2^{4b+c}) = 8b + 2^c \quad \text{for} \quad 0 \le c \le 3 \quad \text{and} \quad 2 \nmid a. \tag{3.3}$$

Their algebraic construction gave the lower bound, $\operatorname{Span}(\mathbf{S}^{n-1}) \ge \rho(n) - 1$. However, Adams[8] used sophisticated techniques of homotopy theory to provide a definite solution in 1962 [1]. The answer is that no more independent vector fields can be found than those already constructed.

Theorem 3.4. *For any $n \in \mathbb{N}$ holds* $\operatorname{Span}(\mathbf{S}^{n-1}) = \rho(n) - 1$.

[2] Jørgen Pedersen Gram (1850–1916), Danish mathematician
[3] Erhard Schmidt (1876–1959), German mathematician
[4] Adolf Hurwitz (1859–1919), German mathematician
[5] Raoul Bott (1923–2005), Hungarian-American mathematician
[6] Johann Karl August Radon (1887–1956), Austrian mathematician
[7] Beno Eckmann (1917–2008), Swiss mathematician
[8] John Frank Adams (1930–1989), British mathematician

Consequently, the only parallelisable spheres $\mathbf{S}^{n-1}$ are those with $\rho(n) = n$, which happens only for $n \in \{1, 2, 4, 8\}$.

Example 3.16. Although the sphere $\mathbf{S}^2 \subset \mathbb{R}^3$ does not admit a nowhere-zero vector field, there exists a vector field on it that is zero at exactly one point. From the stereographic projection given by $\varphi_-(x,y,z) = (x/(1-z), y/(1-z)) = (u, v)$ we have coordinate vector fields

$$\frac{\partial}{\partial u} \in \mathfrak{X}(\mathbb{R}^2) \quad \text{and} \quad \frac{\partial}{\partial x_1} = (\varphi_-^{-1})_* \left(\frac{\partial}{\partial u}\right) \circ \varphi_- \in \mathfrak{X}(\mathbf{S}^2 \setminus \{p_+\}),$$

where $x_1 = u \circ \varphi_-$. The second chart, $\varphi_+(x,y,z) = (x/(1+z), y/(1+z)) = (\bar{u}, \bar{v})$ smoothly overlaps φ_- on the intersection $\mathbf{S}^2 \setminus \{p_+, p_-\}$, and the relation from (1.4) is

$$(\bar{u}, \bar{v}) = \varphi_+ \circ \varphi_-^{-1}(u, v) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right).$$

For $y_1 = \bar{u} \circ \varphi_+$ and $y_2 = \bar{v} \circ \varphi_+$ we can calculate

$$\frac{\partial y_1}{\partial x_1} = \frac{\partial(\bar{u} \circ \varphi_+ \circ \varphi_-^{-1})}{\partial u} \circ \varphi_- = \frac{\partial \frac{u}{u^2+v^2}}{\partial u} \circ \varphi_- = \frac{v^2 - u^2}{(u^2+v^2)^2} \circ \varphi_- = (\bar{v}^2 - \bar{u}^2) \circ \varphi_+ = \frac{y^2 - x^2}{(1+z)^2},$$
$$\frac{\partial y_2}{\partial x_1} = \frac{\partial(\bar{v} \circ \varphi_+ \circ \varphi_-^{-1})}{\partial u} \circ \varphi_- = \frac{\partial \frac{v}{u^2+v^2}}{\partial u} \circ \varphi_- = \frac{-2uv}{(u^2+v^2)^2} \circ \varphi_- = (-2\bar{u}\bar{v}) \circ \varphi_+ = \frac{-2xy}{(1+z)^2},$$

and obtain

$$\frac{\partial}{\partial x_1} = \frac{\partial y_1}{\partial x_1}\frac{\partial}{\partial y_1} + \frac{\partial y_2}{\partial x_1}\frac{\partial}{\partial y_2} = \frac{y^2 - x^2}{(1+z)^2}\frac{\partial}{\partial y_1} + \frac{-2xy}{(1+z)^2}\frac{\partial}{\partial y_2},$$

so the vector field $\partial/\partial x_1$ can be extended to the point p_+. The concrete formula

$$V_q = \begin{cases} \left(\dfrac{\partial}{\partial x_1}\right)_q & \text{for } q \in \mathbf{S}^2 \setminus \{p_+\} \\ \left(\dfrac{y^2 - x^2}{(1+z)^2}\dfrac{\partial}{\partial y_1} + \dfrac{-2xy}{(1+z)^2}\dfrac{\partial}{\partial y_2}\right)_q & \text{for } q \in \mathbf{S}^2 \setminus \{p_-\} \end{cases}$$

defines V at each point of the sphere $\mathbf{S}^2$, and since it is smooth in local coordinates, we have $V \in \mathfrak{X}(\mathbf{S}^2)$. Of course, $V_{p_+} = 0$, while $V_q = (\partial/\partial x_1)_q \neq 0$ holds on $\mathbf{S}^2 \setminus \{p_+\}$. △

Of course, any k linearly independent vector fields on a sphere span a k-dimensional distribution. However, although any distribution is not determined in this way by some vector fields, we know that a sphere $\mathbf{S}^m$ admits a k-dimensional continuous distribution for $2k \leq m$ if and only if it admits k linearly independent continuous vector fields (see Steenrod[9] [111, Theorem 27.16]). Hence we have the following consequence of Theorem 3.4.

Theorem 3.5. *The sphere $\mathbf{S}^{n-1}$ for $\rho(n) \leq k \leq n - 1 - \rho(n)$ does not admit a k-dimensional distribution.*

3.4 Covector fields

The ***cotangent space*** of a manifold M at a point $p \in M$ is the dual space of the tangent space,

$$T_p^*M = (T_pM)^* = \mathrm{Hom}(T_pM, \mathbb{R}).$$

[9]Norman Earl Steenrod (1910–1971), American mathematician

Elements of the cotangent space T_p^*M are ***tangent covectors*** at p, which are linear functions $\omega_p\colon T_pM \to \mathbb{R}$. Following the theory of vector fields, we want to investigate a map ω that smoothly assigns to each point $p \in M$ a covector ω_p at p.

For example, for every function $f \in \mathfrak{F}(M)$ we define its ***differential*** df, that assigns to each point $p \in M$ a covector $df_p \in T_p^*M$ given by $df_p(X_p) = X_p(f)$ for $X_p \in T_pM$. A tangent map in this important special case $f \in \mathfrak{F}(M)$ can be calculated in some chart (U, φ) at $p \in M$ by formula (2.2). We have $T_pf(\partial/\partial x_j)_p = (\partial f/\partial x_j)(p)(d/dt)_{f(p)}$, and therefore by linearity of tangent maps

$$T_pf(X_p) = X_p(f)\left(\frac{d}{dt}\right)_{f(p)}$$

holds for $X_p \in T_pM$. Thus, if we take advantage of the identification $T_{f(p)}\mathbb{R} \cong \mathbb{R}$, then we can naturally identify the differential df_p with the tangent map T_pf.

Let M be an n-manifold, and let (U, φ) be a chart at arbitrary point $p \in M$ with coordinate functions $x_i = \pi_i \circ \varphi$. By Theorem 2.3, $(\partial/\partial x_1)_p, \ldots, (\partial/\partial x_n)_p$ form a basis of the tangent space T_pM. Since $(dx_i)_p(\partial/\partial x_j)_p = (\partial/\partial x_j)_p(x_i) = \delta_{ij}$ for $1 \leq i,j \leq n$, covectors $(dx_1)_p, \ldots, (dx_n)_p$ form a basis of the cotangent space T_p^*M that is dual to given basis for T_pM. Additionally, for $\omega_p \in T_p^*M$ and $X_p \in T_pM$, from (2.1) follows

$$\omega_p(X_p) = \omega_p\left(\sum_{i=1}^n X_p(x_i)\left(\frac{\partial}{\partial x_i}\right)_p\right) = \sum_{i=1}^n X_p(x_i)\,\omega_p\left(\frac{\partial}{\partial x_i}\right)_p = \sum_{i=1}^n \omega_p\left(\frac{\partial}{\partial x_i}\right)_p (dx_i)_p(X_p),$$

and we obtain

$$\omega_p = \sum_{i=1}^n \omega_p\left(\frac{\partial}{\partial x_i}\right)_p (dx_i)_p. \tag{3.4}$$

Theorem 3.6. *Let M be an n-manifold, and (U, φ) is a chart at $p \in M$ with $x_i = \pi_i \circ \varphi$. Then covectors $(dx_1)_p, \ldots, (dx_n)_p$ form a basis for T_p^*M and* (3.4) *holds for all $\omega_p \in T_p^*M$.*

Since we want a map $\omega\colon p \mapsto \omega_p$ to be smooth, it is necessary that its codomain is a manifold. Following the tangent bundle construction, we make the ***cotangent bundle***

$$T^*M = \bigsqcup_{p\in M} T_p^*M$$

of a manifold M as a union of all cotangent spaces. As in the case of the tangent bundle, the cotangent bundle goes together with the natural projection $\pi\colon T^*M \to M$ which is given by $\pi(\omega_p) = p$ for $\omega_p \in T_p^*M$. Similar to the construction of the tangent bundle, we use charts (U, φ) with coordinate functions $x_i = \pi_i \circ \varphi$ to make bundle charts $\widetilde{\varphi}\colon \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n$ by

$$\widetilde{\varphi}\left(\sum_{i=1}^n \lambda_i (dx_i)_p\right) = (\varphi(p), \lambda_1, \ldots, \lambda_n).$$

If $\widetilde{\psi}\colon \pi^{-1}(V) \to \psi(V) \times \mathbb{R}^n$ is another such map, this time induced by a chart (V, ψ) with coordinate functions $y_i = \pi_i \circ \psi$, then from

$$(dx_i)_p = \sum_{j=1}^n \left((dx_i)_p\left(\frac{\partial}{\partial y_j}\right)_p\right)(dy_j)_p = \sum_{j=1}^n \frac{\partial x_i}{\partial y_j}(p)(dy_j)_p$$

we have

$$\begin{aligned}(\widetilde{\psi} \circ \widetilde{\varphi}^{-1})(\varphi(p), \lambda_1, \ldots, \lambda_n) &= \widetilde{\psi}\left(\sum_{i=1}^n \lambda_i (dx_i)_p\right) = \widetilde{\psi}\left(\sum_{j=1}^n \left(\sum_{i=1}^n \lambda_i \frac{\partial x_i}{\partial y_j}(p)\right)(dy_j)_p\right) \\ &= \left(\psi(p), \sum_{i=1}^n \lambda_i \frac{\partial x_i}{\partial y_1}(p), \ldots, \sum_{i=1}^n \lambda_i \frac{\partial x_i}{\partial y_n}(p)\right).\end{aligned}$$

The corresponding map $\tau\colon \varphi(U\cap V)\to \mathrm{GL}(n,\mathbb{R})$ is of the form

$$\tau(\varphi(p)) = \begin{pmatrix} \frac{\partial x_1}{\partial y_1}(p) & \cdots & \frac{\partial x_n}{\partial y_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial y_n}(p) & \cdots & \frac{\partial x_n}{\partial y_n}(p) \end{pmatrix}$$

that is smooth, and it is the map that $z\in\varphi(U\cap V)$ maps to the transpose of the Jacobian matrix $T_{\psi\circ\varphi^{-1}(z)}(\varphi\circ\psi^{-1})$. Hence T^*M is a vector bundle with a bundle atlas consisting of bundle charts $\widetilde{\varphi}$, and therefore T^*M is also a smooth manifold of dimension $2n$.

Consider sections of the cotangent bundle, which are maps $\omega\colon M\to T^*M$ such that $\pi\circ\omega = \mathbb{1}_M$. The ***covector field*** on a manifold M is a smooth section of the cotangent bundle T^*M, and the set of all covector fields is denoted by $\mathfrak{X}^*(M)=\Gamma(T^*M)$.

If (U,φ) is a chart on M with coordinate functions $x_i=\pi_i\circ\varphi$, then every section ω of the cotangent bundle T^*U can be written as a linear combination $\omega=\sum_{i=1}^n a_i\,dx_i$ for some functions $a_i\colon U\to\mathbb{R}$. Smoothness of these coefficient functions depends of smoothness of sections, since for a bundle chart $\widetilde{\varphi}$ of T^*U we have $\widetilde{\varphi}\circ\omega=(\varphi,a_1,\dots,a_n)$, so ω is smooth if and only if $a_i\in\mathfrak{F}(U)$ for all $1\le i\le n$.

Lemma 3.7. *If (U,φ) is a chart on an n-manifold M with coordinate functions x_i, then a section $\omega=\sum_{i=1}^n a_i\,dx_i$ of the cotangent bundle T^*U is smooth if and only if all coefficient functions $a_i\colon U\to\mathbb{R}$ are smooth.*

For $f\in\mathfrak{F}(M)$, we can consider the differential df that in every chart (U,φ) has a form $df=\sum_{i=1}^n a_i\,dx_i$. Acting on coordinate vectors we get the coefficient functions

$$\frac{\partial f}{\partial x_j}=\left(\frac{\partial}{\partial x_j}\right)f=(df)\left(\frac{\partial}{\partial x_j}\right)=\sum_{i=1}^n a_i\,dx_i\left(\frac{\partial}{\partial x_j}\right)=\sum_{i=1}^n a_i\delta_{ij}=a_j,$$

that are obviously smooth, so we obtain a covector field $df\in\mathfrak{X}^*(M)$, and a local expression,

$$df=\sum_{i=1}^n\frac{\partial f}{\partial x_i}\,dx_i. \tag{3.5}$$

Example 3.17. The differential of a function $f\colon\mathbb{R}^2\to\mathbb{R}$ given by $f(x,y)=xy^2\sin x$ is calculated by (3.5),

$$df=\frac{\partial(xy^2\sin x)}{\partial x}\,dx+\frac{\partial(xy^2\sin x)}{\partial y}\,dy=y^2(\sin x+x\cos x)\,dx+2xy\sin x\,dy.$$

△

Example 3.18. Although the differential of a smooth function is a covector field, there are covector fields that are not the differential of some smooth function. For example, if there exists $f\in\mathfrak{F}(\mathbb{R}^2)$ such that $dx+x\,dy=df$, then we have $\partial f/\partial x=1$ and $\partial f/\partial y=x$, and therefore we obtain $f(x,y)=x+\gamma(y)$, and then $\gamma'(y)=x$, which is impossible. △

From the formula (3.5), directly follows that the differential $d\colon\mathfrak{F}(M)\to\mathfrak{X}^*(M)$ has the following nice properties.

Lemma 3.8. *For all $f,h\in\mathfrak{F}(M)$, $l\in\mathfrak{F}(\mathbb{R})$, $\alpha,\beta\in\mathbb{R}$, the formula* (3.5) *holds as well as*

$$\begin{aligned} d(\alpha f+\beta h)&=\alpha df+\beta dh,\\ d(fh)&=fdh+hdf,\\ d(l\circ f)&=(l'\circ f)df. \end{aligned}$$

Of course, for constant $f \in \mathfrak{F}(M)$ we have $df = 0$, while $df = 0$ for a connected M implies constant f, which is a consequence of Lemma 2.20.

For an arbitrary section ω of the cotangent bundle T^*M and a vector field $X \in \mathfrak{X}(M)$, we naturally define a function $\omega(X)\colon M \to \mathbb{R}$ by $\omega(X)_p = \omega_p(X_p)$ for $p \in M$. An arbitrary function $f\colon M \to \mathbb{R}$ commutes with ω because of $\omega(fX)_p = \omega_p(f(p)X_p) = f(p)\omega_p(X_p) = (f\omega(X))_p$, which implies $\omega(fX) = f\omega(X)$. In some chart (U, φ) on M, from the formulas (3.4) and (2.1) we can calculate

$$\omega(X) = \sum_{i=1}^{n} \omega\left(\frac{\partial}{\partial x_i}\right) dx_i \left(\sum_{j=1}^{n} Xx_j \frac{\partial}{\partial x_j}\right) = \sum_{i,j=1}^{n} \omega\left(\frac{\partial}{\partial x_i}\right) Xx_j\, dx_i\left(\frac{\partial}{\partial x_j}\right) = \sum_{i=1}^{n} \omega\left(\frac{\partial}{\partial x_i}\right) Xx_i.$$

If ω is a smooth section, Lemma 3.7 gives $\omega(\partial_i) \in \mathfrak{F}(U)$, and therefore $\omega(X)$ is smooth on an arbitrary coordinate neighbourhood, so $\omega(X) \in \mathfrak{F}(M)$. Conversely, if $\omega(X) \in \mathfrak{F}(M)$ holds for all $X \in \mathfrak{X}(M)$, it especially holds for ∂_i, so $\omega = \sum_{i=1}^{n} \omega(\partial_i)\, dx_i$ is smooth by Lemma 3.7. Thus, a covector field $\omega \in \mathfrak{X}^*(M)$ can be seen as a map $\omega\colon \mathfrak{X}(M) \to \mathfrak{F}(M)$.

Lemma 3.9. *A section ω of the cotangent bundle T^*M is smooth if and only if for every $X \in \mathfrak{X}(M)$ holds $\omega(X) \in \mathfrak{F}(M)$.*

Let $f\colon M \to N$ be a smooth map between manifolds M and N. The global tangent map $f_* = Tf\colon TM \to TN$ pushes tangent vector forward from M to N. Dualizing this leads to a map on covectors going in the opposite direction, from N to M. The ***pullback*** is a map $f^*\colon T^*N \to T^*M$ defined by

$$(f^*(\omega_{f(p)}))(X_p) = \omega_{f(p)}(f_*(X_p))$$

for a covector $\omega_{f(p)} \in T^*_{f(p)}N$ and a vector $X_p \in T_pM$ at a point $p \in M$.

Unlike vector fields which in general can not be pushed by smooth map, covector fields always pull back to covector fields. For a covector field $\omega \in \mathfrak{X}^*(N)$ we define the ***pullback*** $f^*\omega$ as a section of the cotangent bundle T^*M such that $(f^*\omega)_p = f^*(\omega_{f(p)})$ holds for $p \in M$, witch means $(f^*\omega)_p(X_p) = \omega_{f(p)}(f_*(X_p))$ for all $X_p \in T_pM$. There is also the pullback of a function $h \in \mathfrak{F}(N)$ defined by $f^*h = h \circ f \in \mathfrak{F}(M)$.

Lemma 3.10. *For a smooth map $f\colon M \to N$ and $\omega, \tau \in \mathfrak{X}^*(N)$, $h \in \mathfrak{F}(N)$ we have*

$$\begin{aligned} f^*(dh) &= d(f^*h), \\ f^*(\omega + \tau) &= f^*\omega + f^*\tau, \\ f^*(h\omega) &= (f^*h)(f^*\omega). \end{aligned}$$

Proof. The pullback and differential commutes, since for $p \in M$ and $X_p \in T_pM$ holds

$$(f^*dh)_p(X_p) = (dh)_{f(p)}(f_*(X_p)) = (f_*(X_p))h = X_p(h \circ f) = X_p(f^*h) = (d(f^*h))_p(X_p),$$

while the other two equalities are easy to show. □

Theorem 3.11. *If $f\colon M \to N$ is smooth and $\omega \in \mathfrak{X}^*(N)$ then $f^*\omega \in \mathfrak{X}^*(M)$.*

Proof. For $p \in M$ we can choose a chart (U, φ) at $p \in M$ with $x_j = \pi_j \circ \varphi$ for $1 \le j \le m = \dim M$ and a chart (V, ψ) at $f(p) \in N$ with $y_i = \pi_i \circ \psi$ for $1 \le i \le n = \dim N$, such that $f(U) \subseteq V$ and $\psi \circ f \circ \varphi^{-1}$ is smooth. Since $\omega\restriction_V = \sum_{i=1}^{n} a_i\, dy_i$ holds for some $a_i \in \mathfrak{F}(V)$, from (3.5) we have

$$(f^*\omega)\restriction_U = \sum_{i=1}^{n} (f^*a_i) f^*(dy_i) = \sum_{i=1}^{n} (f^*a_i)\, d(f^*y_i) = \sum_{i=1}^{n} (a_i \circ f)\, d(y_i \circ f) = \sum_{i=1}^{n}\sum_{j=1}^{m} (a_i \circ f) \frac{\partial(y_i \circ f)}{\partial x_j}\, dx_j.$$

Since the coefficient functions $\sum_{i=1}^{n} (a_i \circ f)(\partial(y_i \circ f)/\partial x_j)$ are smooth, $f^*\omega$ is smooth on U by Lemma 3.7. Thus, $f^*\omega$ is smooth at each point $p \in M$, so $f^*\omega \in \mathfrak{X}^*(M)$. □

Example 3.19. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ is given by $(u, v) = f(x, y, z) = (x^2y, y\sin z)$ and $\omega \in \mathfrak{X}^*(\mathbb{R}^2)$ is given by $\omega = u\,dv + v\,du$. By direct calculations we have

$$\begin{aligned} f^*\omega &= (f^*u)(f^*dv) + (f^*v)(f^*du) = (u\circ f)\,d(v\circ f) + (v\circ f)\,d(u\circ f) \\ &= (x^2y)\,d(y\sin z) + (y\sin z)\,d(x^2y) = x^2y(\sin z\,dy + y\cos z\,dz) + y\sin z(2xy\,dx + x^2\,dy), \end{aligned}$$

and we obtain $f^*\omega = 2xy^2\sin z\,dx + 2x^2y\sin z\,dy + x^2y^2\cos z\,dz$. △

3.5 Tensor fields

The notion of tensor field on a manifold M generalises the notions of smooth real-valued functions $\mathfrak{F}(M)$, vector fields $\mathfrak{X}(M)$, and covector fields $\mathfrak{X}^*(M)$, and therefore provides more complicated objects on a manifold. Tensors occur in many different guises, but their common characteristic is multilinearity.

In general, we consider a module $\mathcal{V}$ over ring K, and its dual module $\mathcal{V}^* = \operatorname{Hom}(\mathcal{V}, K)$. A ***tensor*** of type $(r, s) \in \mathbb{N}_0 \times \mathbb{N}_0$ on $\mathcal{V}$ is a K-multilinear map $A\colon (\mathcal{V}^*)^r \times (\mathcal{V})^s \to K$. The set $\mathfrak{T}^r_s(\mathcal{V})$ of all tensors of type (r, s) on $\mathcal{V}$ is a module over K with usual definitions of addition and scalar multiplication.

A ***tensor field*** A on a manifold M is a tensor on the $\mathfrak{F}(M)$-module $\mathfrak{X}(M)$. This actually means that an $\mathfrak{F}(M)$-multilinear map

$$A\colon \underbrace{\mathfrak{X}^*(M) \times \cdots \times \mathfrak{X}^*(M)}_{r} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s} \to \mathfrak{F}(M),$$

is a tensor field of type $(r, s) \in \mathbb{N}_0 \times \mathbb{N}_0$. Thus, the set $\mathfrak{T}^r_s(M)$ of all tensor fields on M of type (r, s) is a module over $\mathfrak{F}(M)$.

Example 3.20. Tensor fields generalise the previously introduced notions. In the special case $(r, s) = (0, 0)$, tensor fields are smooth functions on M, $\mathfrak{T}^0_0(M) = \mathfrak{F}(M)$. By Lemma 3.9, $\omega \in \mathfrak{X}^*(M)$ is an $\mathfrak{F}(M)$-linear map $\omega\colon \mathfrak{X}(M) \to \mathfrak{F}(M)$, so $\mathfrak{T}^0_1(M) = \mathfrak{X}^*(M)$. Finally, each $X \in \mathfrak{X}(M)$ determines a tensor field $\overline{X}\colon \mathfrak{X}^*(M) \to \mathfrak{F}(M)$ by $\overline{X}(\omega) = \omega(X)$, while the converse is $Xf = df(X) = \overline{X}(df)$ for $f \in \mathfrak{F}(M)$. This yields the identification $\mathfrak{T}^1_0(M) = \mathfrak{X}(M)$, so we use the same notation $\overline{X} = X \in \mathfrak{X}(M)$, that is, $X(\omega) = \omega(X)$ holds for every $\omega \in \mathfrak{X}^*(M)$. △

To prove that a given map $A\colon \mathfrak{X}^*(M)^r \times \mathfrak{X}(M)^s \to \mathfrak{F}(M)$ is a tensor field, we have to show that it is $\mathfrak{F}(M)$-linear in each slot (in each variable separately). Additivity in each slot is often obvious, so the main question is whether functions from $\mathfrak{F}(M)$ can be factored out A of each slot,

$$A(\omega_1, \ldots, \omega_r, X_1, \ldots, fX_i, \ldots, X_s) = fA(\omega_1, \ldots, \omega_r, X_1, \ldots, X_i, \ldots, X_s).$$

Example 3.21. Consider a function $E\colon \mathfrak{X}^*(M) \times \mathfrak{X}(M) \to \mathfrak{F}(M)$ given by $E(\omega, X) = \omega(X)$. It is obviously $\mathfrak{F}(M)$-linear in both slots, so $E \in \mathfrak{T}^1_1(M)$. △

Example 3.22. For a fixed $0 \neq \omega \in \mathfrak{X}^*(M)$ we can define $F\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{F}(M)$ by $F(X, Y) = X(\omega(Y))$. The map F is $\mathfrak{F}(M)$-linear in the first slot, but in the second slot it is only additive, because of $F(X, fY) = X(\omega(fY)) = X(f\omega(Y)) = (Xf)\omega(Y) + fF(X, Y)$, so F is not a tensor field. △

Example 3.23. Every $\mathfrak{F}(M)$-multilinear map $A\colon \mathfrak{X}(M)^s \to \mathfrak{X}(M)$ is naturally identified with $\overline{A}\colon \mathfrak{X}^*(M) \times \mathfrak{X}(M)^s \to \mathfrak{F}(M)$ given by $\overline{A}(\omega, X_1, \ldots, X_s) = \omega(A(X_1, \ldots, X_s))$, which induces the tensor field $\overline{A} \in \mathfrak{T}^1_s(M)$. Therefore, we can use the same notation, $\overline{A} = A$, and assume that $A \in \mathfrak{T}^1_s(M)$ holds. △

Tensors of type $(0, s)$ are said to be ***covariant*** of order s, while tensors of type $(r, 0)$ are said to be ***contravariant*** of order $r \geq 1$. For example, real-valued smooth functions and covector fields are covariant tensors, while vector fields are contravariant tensors.

While we can add only tensors of the same type, any two tensors can be multiplied. For $A_1 \in \mathfrak{T}^{r_1}_{s_1}(M)$ and $A_2 \in \mathfrak{T}^{r_2}_{s_2}(M)$ we define $A_1 \otimes A_2 \colon \mathfrak{X}^*(M)^{r_1+r_2} \times \mathfrak{X}(M)^{s_1+s_2} \to \mathfrak{F}(M)$ by

$$\begin{aligned}(A_1 \otimes A_2)&(\omega_1, \dots, \omega_{r_1+r_2}, X_1, \dots, X_{s_1+s_2}) \\ &= A_1(\omega_1, \dots, \omega_{r_1}, X_1, \dots, X_{s_1})\, A_2(\omega_{r_1+1}, \dots, \omega_{r_1+r_2}, X_{s_1+1}, \dots, X_{s_1+s_2}).\end{aligned}$$

Then $A_1 \otimes A_2$ is a tensor of type $(r_1 + r_2, s_1 + s_2)$ called the ***tensor product*** of A_1 and A_2. Especially, for $A \in \mathfrak{T}^r_s(M)$ and $f \in \mathfrak{F}(M)$ we have $A \otimes f = f \otimes A = fA$, while additional $A \in \mathfrak{F}(M)$ gives the ordinary multiplication of functions in $\mathfrak{F}(M)$.

The tensor product is $\mathfrak{F}(M)$-linear, since $(f_1 A_1 + f_2 A_2) \otimes B = f_1 A_1 \otimes B + f_2 A_2 \otimes B$ holds for $f_1, f_2 \in \mathfrak{F}(M)$, $A_1, A_2 \in \mathfrak{T}^{r_1}_{s_1}(M)$, $B \in \mathfrak{T}^{r_2}_{s_2}(M)$, as well as the similar identity by the other component. Also, directly from the definition we have that the tensor product is associative, so for tensors A, B, C of any types we often write without brackets $A \otimes B \otimes C$. We can notice that for a covariant tensor A and a contravariant tensor B by definition holds $A \otimes B = B \otimes A$. However, although functions commute with anything, $f(A \otimes B) = (fA) \otimes B = A \otimes (fB)$, the tensor product in general is not commutative.

Example 3.24. On a coordinate neighbourhood, $(dx_1 \otimes dx_2)(\partial_1, \partial_2) = dx_1(\partial_1)\, dx_2(\partial_2) = 1$ and $(dx_2 \otimes dx_1)(\partial_1, \partial_2) = dx_2(\partial_1)\, dx_1(\partial_2) = 0$, so $dx_1 \otimes dx_2 \neq dx_2 \otimes dx_1$. △

Just as for a vector field or a covector field, any tensor field A on a manifold M can be viewed as a field, assigning a value A_p at each point $p \in M$. It turns out that the value A_p does not depend on whole vector and covector fields, not even from their values in some neighbourhood of p, but solely from their individual values at the point p.

Lemma 3.12. *If anyone of covector fields $\omega_1, \dots, \omega_r$ or vector fields $X_1, \dots, X_s$ vanishes at $p \in M$, then $A(\omega_1, \dots, \omega_r, X_1, \dots, X_s)(p) = 0$ holds for $A \in \mathfrak{T}^r_s(M)$.*

Proof. Let (U, φ) be a chart at $p \in M$ with $x_i = \pi_i \circ \varphi \in \mathfrak{F}(U)$. By Lemma 1.17 there is a bump function $b \in \mathfrak{F}(M)$ supported in U with $p(p) = 1$, which allows extensions $bX_j(x_i) \in \mathfrak{F}(M)$ and $b\partial_i \in \mathfrak{X}(M)$ as well as $b\omega_j(\partial_i) \in \mathfrak{F}(M)$ and $b\, dx_i \in \mathfrak{X}^*(M)$. From the multilinearity of tensor we have equations

$$b^2 A(\dots, X_j, \dots) = A(\dots, \sum_{i=1}^n bX_j(x_i)\, b\partial_i, \dots) = \sum_{i=1}^n bX_j(x_i)\, A(\dots, b\partial_i, \dots),$$
$$b^2 A(\dots, \omega_j, \dots) = A(\dots, \sum_{i=1}^n b\omega_j(\partial_i)\, b\, dx_i, \dots) = \sum_{i=1}^n b\omega_j(\partial_i)\, A(\dots, b\, dx_i, \dots),$$

which we calculate at p. If $(X_j)_p = 0$ for some $1 \leq j \leq s$, then we have $(bX_j(x_i))(p) = 0$ which gives $A(\dots, X_j, \dots)(p) = 0$. Similarly, $(\omega_j)_p = 0$ for some $1 \leq j \leq r$ implies $(b\omega_j(\partial_i))(p) = 0$ and therefore $A(\dots, \omega_j, \dots)(p) = 0$. □

Theorem 3.13. *The value $A(\omega_1, \dots, \omega_r, X_1, \dots, X_s)(p)$ of a tensor $A \in \mathfrak{T}^r_s(M)$ at a point $p \in M$ depends only of the values $(\omega_1)_p, \dots, (\omega_r)_p, (X_1)_p, \dots, (X_s)_p$.*

Proof. Let $(\omega_i)_p = (\overline{\omega}_i)_p$ for $\omega_i, \overline{\omega}_i \in \mathfrak{X}^*(M)$, $1 \leq i \leq r$ and $(X_j)_p = (\overline{X}_j)_p$ for $X_j, \overline{X}_j \in \mathfrak{X}(M)$, $1 \leq j \leq s$. The value of difference

$$\begin{aligned}A(\omega_1, \dots, \omega_r, X_1, \dots, X_s) &- A(\overline{\omega}_1, \dots, \overline{\omega}_r, \overline{X}_1, \dots, \overline{X}_s) \\ &= A(\omega_1 - \overline{\omega}_1, \dots, \omega_r, X_1, \dots, X_s) + A(\overline{\omega}_1, \omega_2 - \overline{\omega}_2 \dots, \omega_r, X_1, \dots, X_s) \\ &\qquad + \dots + A(\overline{\omega}_1, \dots, \overline{\omega}_r, \overline{X}_1, \dots, X_s - \overline{X}_s)\end{aligned}$$

at the point $p \in M$ is equal to zero, since by Lemma 3.12 all $r + s$ terms vanish at p. □

Directly from Theorem 3.13 follows that a tensor field $A \in \mathfrak{T}^r_s(M)$ has a value A_p at each point $p \in M$, which is a funciton $A_p\colon (T^*_pM)^r \times (T_pM)^s \to \mathbb{R}$ given by

$$A_p((\omega_1)_p, \dots, (\omega_r)_p, (X_1)_p, \dots, (X_s)_p) = A(\omega_1, \dots, \omega_r, X_1, \dots, X_s)(p).$$

It is easy to check $\mathbb{R}$-multilinearity of the function A_p, so by restriction of tensor field $A \in \mathfrak{T}^r_s(M)$ at $p \in M$ we get the tensor $A_p \in \mathfrak{T}^r_s(T_pM)$. Just as a vector field is a smooth section of the tangent bundle, a tensor field $A \in \mathfrak{T}^r_s(M)$ can be considered as a smooth section of the appropriate tensor bundle that assigns to each point $p \in M$ the tensor A_p. Especially, this interpretation allows that the restriction $A\restriction_U$ of a tensor field $A \in \mathfrak{T}^r_s(M)$ to an open subest $U \subset M$ can be seen as a well defined tensor field on U, that is, $A\restriction_U \in \mathfrak{T}^r_s(U)$.

Let (U, φ) be a chart on an n-manifold M, and $x_i = \pi_i \circ \varphi$ are its coordinate functions. The coordinate formulas, (2.1) for vector fields and (3.4) for covector fields, can be extended to tensor fields of arbitrary type. ***Components*** of a tensor field $A \in \mathfrak{T}^r_s(M)$ relative to the given chart are functions

$$A^{i_1 \dots i_r}_{j_1 \dots j_s} = A\left(dx_{i_1}, \dots, dx_{i_r}, \frac{\partial}{\partial x_{j_1}}, \dots, \frac{\partial}{\partial x_{j_s}}\right) \in \mathfrak{F}(U),$$

which are defined for all indices $1 \le i_1, \dots, i_r, j_1, \dots, j_s \le n$.

The components of a covector field $\omega \in \mathfrak{T}^0_1(M) = \mathfrak{X}^*(M)$ are $\omega_j = \omega(\partial_j)$, and these can be seen in the formula (3.4), $\omega = \sum_j \omega_j \, dx_j$. Similarly, the components of a vector field $X \in \mathfrak{T}^1_0(M) = \mathfrak{X}(M)$ are $X^i = X(dx_i) = dx_i(X) = X(x_i)$, and we can see them in the formula (2.1), $X = \sum_i X^i \partial_i$.

Example 3.25. If $A \in \mathfrak{T}^1_s(M)$ is represented by $A\colon \mathfrak{X}(M)^s \to \mathfrak{X}(M)$ as in Example 3.23, then its components are determined by

$$A\left(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_s}}\right) = \sum_{j=1}^n A^j_{i_1 \dots i_s} \frac{\partial}{\partial x_j},$$

because of $A(dx_j, \partial_{i_1}, \dots, \partial_{i_s}) = dx_j(A(\partial_{i_1}, \dots, \partial_{i_s})) = \sum_k A^k_{i_1 \dots i_s} \, dx_j(\partial_k) = A^j_{i_1 \dots i_s}$. △

An arbitrary tensor field $A \in \mathfrak{T}^r_s(M)$ can be expressed by its components with

$$A(\omega_1, \dots, \omega_r, X_1, \dots, X_s) = \sum_{1 \le i_1, \dots, i_r, j_1, \dots, j_s \le n} (\omega_1)_{i_1} \cdots (\omega_r)_{i_r} (X_1)^{j_1} \cdots (X_s)^{j_s} A^{i_1 \dots i_r}_{j_1 \dots j_s}, \tag{3.6}$$

so if we use $(\omega_k)_i = \omega_k(\partial_i) = \partial_i(\omega_k)$ and $(X_k)^j = X_k(dx_j) = dx_j(X_k)$, we obtain

$$A = \sum_{1 \le i_1, \dots, i_r, j_1, \dots, j_s \le n} A^{i_1 \dots i_r}_{j_1 \dots j_s} \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes \cdots \otimes dx_{j_s}. \tag{3.7}$$

We can notice that in a fixed chart, components of a sum of tensors are just the sum of components, while components of a tensor product are given by

$$(A \otimes B)^{i_1 \dots i_{r+r'}}_{j_1 \dots j_{s+s'}} = A^{i_1 \dots i_r}_{j_1 \dots j_s} \cdot B^{i_{r+1} \dots i_{r+r'}}_{j_{s+1} \dots j_{s+s'}}.$$

There is an interesting operation on the set of tensors that tensors of type (r, s) shrink to tensors of type $(r-1, s-1)$, and its general definition is based on the following special case.

Lemma 3.14. *There is a unique $\mathfrak{F}(M)$-linear map $C\colon \mathfrak{T}^1_1(M) \to \mathfrak{F}(M)$ such that $C(X \otimes \omega) = \omega(X)$ for all $X \in \mathfrak{X}(M)$ and $\omega \in \mathfrak{X}^*(M)$.*

Proof. In a coordinate neighbourhood U, any $A \in \mathfrak{T}^1_1(M)$ by the formula (3.7) has a form $A = \sum_{i,j=1}^n A^i_j \partial_i \otimes dx_j$. Since it must be $C(\partial_i \otimes dx_j) = dx_j(\partial_i) = \delta_{ij}$, we are forced to define $C(A) = \sum_{i=1}^n A^i_i = \sum_{i=1}^n A(dx_i, \partial_i)$. Such C has the required properties on U. To extend the map on the whole manifold it suffices to show that this definition is independent of the choice of coordinate system, where

$$\begin{aligned}\sum_k A\left(dy_k, \frac{\partial}{\partial y_k}\right) &= \sum_k A\left(\sum_i \frac{\partial y_k}{\partial x_i} dx_i, \sum_j \frac{\partial x_j}{\partial y_k}\frac{\partial}{\partial x_j}\right) = \sum_{i,j,k} \frac{\partial y_k}{\partial x_i}\frac{\partial x_j}{\partial y_k} A\left(dx_i, \frac{\partial}{\partial x_j}\right)\\ &= \sum_{i,j} \delta_{ij} A\left(dx_i, \frac{\partial}{\partial x_j}\right) = \sum_i A\left(dx_i, \frac{\partial}{\partial x_i}\right)\end{aligned}$$

completes the proof. □

The map C from the previous lemma is called $(1,1)$ ***contraction***. In general, if we choose one contravariant slot $1 \le i \le r$ and one covariant slot $1 \le j \le s$, and apply C to the map

$$(\omega_i, X_j) \mapsto A(\omega_1, \dots, \omega_r, X_1, \dots, X_s),$$

we obtain a $(1,1)$ tensor for fixed $\omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_r, X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_s$. Therefore we have $(C^i_j A)(\omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \dots, \omega_r, X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_s) \in \mathfrak{F}(M)$, and thus $C^i_j A \in \mathfrak{T}^{r-1}_{s-1}(M)$. We say that $C^i_j A$ is the ***contraction*** of A over i, j. The components of the contraction are

$$(C^i_j A)^{p_1 \dots p_{r-1}}_{q_1 \dots q_{s-1}} = \sum_{k=1}^n A^{p_1 \dots p_{i-1}\, k\, p_i \dots p_{r-1}}_{q_1 \dots q_{j-1}\, k\, q_j \dots q_{s-1}}.$$

Example 3.26. Tensor contraction can be seen as a generalisation of the trace. A linear operator $L \in \mathrm{End}(\mathcal{V})$ on a finite-dimensional vector space $\mathcal{V}$ has a representation $L \in \mathfrak{T}^1_1(\mathcal{V})$ (see Example 3.23). Related to some basis $(e_1, \dots, e_n)$ in $\mathcal{V}$, the tensor components correspond to the matrix entries, because of $L(e_i) = \sum_j L^j_i e_j$ (see Example 3.25). Hence, $CL = \sum_j L^j_j = \mathrm{Tr}\, L$, which means that the contraction of L is the trace of L. △

The pullback of a covector field can be generalised to the pullback of a covariant tensor. Let $f\colon M \to N$ be a smooth map between manifolds. The ***pullback of a covariant tensor*** $A \in \mathfrak{T}^0_s(N)$ is defined by

$$(f^* A)(X_1, \dots, X_s) = A(f_*(X_1), \dots, f_*(X_s))$$

for all $p \in M$, $X_i \in T_pM$. At any point $p \in M$, the pullback $f^*(A)$ gives an $\mathbb{R}$-multilinear function from $(T_pM)^s$ to $\mathbb{R}$, which is a $(0,s)$ tensor on T_pM. Coordinate computations show that $f^*(A)$ is a covariant tensor field on M.

Thus, for a smooth $f\colon M \to N$ and each $s \ge 0$ we get the $\mathbb{R}$-linear $f^*\colon \mathfrak{T}^0_s(N) \to \mathfrak{T}^0_s(M)$. Additionally, for $A \in \mathfrak{T}^0_s(N)$ and $B \in \mathfrak{T}^0_t(N)$ we have $f^*(A \otimes B) = f^*(A) \otimes f^*(B)$. Also, for a smooth $h\colon N \to P$ holds $(h \circ f)^* = f^* \circ h^*\colon \mathfrak{T}^0_s(P) \to \mathfrak{T}^0_s(M)$.

In general, rearranging the arguments of a tensor need not have any expected result on its value. However, some tensors do not change their values when their arguments are rearranged. For a covariant or contravariant tensor (of order at least 2) we say that is ***symmetric*** if its value is unchanged by transposing any pair of its arguments. If each such transposition produces a sign change then we say that this tensor is ***skew-symmetric***.

The ***symmetric part*** of a covariant tensor A of order s is the new covariant tensor $\mathrm{Sym}\, A$ of order s given by

$$(\mathrm{Sym}\, A)(X_1, \dots, X_s) = \frac{1}{s!} \sum_{\sigma \in S_s} A(X_{\sigma(1)}, \dots, X_{\sigma(s)}),$$

where S_s is the symmetric group on s elements, that is, the group of permutations of the set $\{1, \dots, s\}$. It is easy to see that $\operatorname{Sym} A$ is symmetric and A is symmetric if and only if $\operatorname{Sym} A = A$.

If A and B are symmetric covariant tensors, then $A \otimes B$ in general is not symmetric. However, their ***symmetric product*** defined by $AB = \operatorname{Sym}(A \otimes B)$ is a symmetric tensor, and it is easy to check that it is commutative ($AB = BA$) and $\mathbb{R}$-bilinear ($(\alpha A + \beta B)C = \alpha AC + \beta BC$). Especially, if ω and τ are covector fields, then

$$\omega\tau = \frac{1}{2}(\omega \otimes \tau + \tau \otimes \omega),$$

while we often use ω^2 as a short notation for the symmetric product $\omega\omega$.

3.6 Tensor fields derivations

For now, we considered the algebraic side of the tensor fields, so it is time to establish a tensor calculus. A ***tensor field derivation*** on a manifold M is a collection of $\mathbb{R}$-linear maps

$$\nabla = \nabla^r_s \colon \mathfrak{T}^r_s(M) \to \mathfrak{T}^r_s(M), \quad (r, s) \in \mathbb{N}_0 \times \mathbb{N}_0,$$

such that for any tensor fields A, B and any contraction C we have

$$\nabla(A \otimes B) = \nabla A \otimes B + A \otimes \nabla B, \tag{3.8}$$

$$\nabla(CA) = C(\nabla A). \tag{3.9}$$

According to the definition, the derivation ∇ is $\mathbb{R}$-linear, preserves tensor type, obeys Leibniz rule, and commutes with all contractions.

For $f \in \mathfrak{F}(M)$ holds $fA = f \otimes A$, so we have $\nabla(fA) = (\nabla f)A + f\nabla A$. Especially, in the special case $(r, s) = (0, 0)$, ∇^0_0 is a derivation on $\mathfrak{T}^0_0(M) = \mathfrak{F}(M)$, so there is a unique vector field $X \in \mathfrak{X}(M)$ such that $\nabla f = Xf$ holds for all $f \in \mathfrak{F}(M)$.

Since a tensor field derivation is not $\mathfrak{F}(M)$-linear, in general the value of ∇A at a point $p \in M$ cannot usually be determined from A_p only. However, it can be found from the values of A on any arbitrarily small neighbourhood of p, and this local character of tensor field derivation can be expressed as follows.

Theorem 3.15. *if ∇ is a tensor field derivation on a manifold M and $U \subset M$ is an open subset, then there exists a unique derivation ∇_U on U such that $\nabla_U(A{\restriction_U}) = (\nabla A){\restriction_U}$ holds for each tensor field A on M.*

Proof. For each $p \in U$ by Lemma 1.17 there exists a bump function $b_p \in \mathfrak{F}(M)$ supported in U and identically equal to 1 on some neighbourhood $V_p \ni p$. For an arbitrary tensor field $T \in \mathfrak{T}^r_s(U)$ we have $b_p T \in \mathfrak{T}^r_s(M)$, which allows to define $(\nabla_U T)_p = (\nabla(b_p T))_p$. First, we can show that this definition does not depend on the choice of bump function, and $\nabla_U T \in \mathfrak{T}^r_s(U)$ holds. After that, it remains to show that ∇_U is a tensor field derivation on U, that has the stated restriction properties, and that it is unique. □

The Leibnizian formula for tensors (3.8) can be reformed and generalised. Comparing functions $A(\omega_1, \dots, X_s) \in \mathfrak{F}(M)$ from (3.6) with a tensor $A \otimes \omega_1 \otimes \dots \otimes X_s \in \mathfrak{T}^{r+s}_{r+s}(M)$ whose components are

$$(A \otimes \omega_1 \otimes \dots \otimes \omega_r \otimes X_1 \otimes \dots \otimes X_s)^{i_1 \dots i_r l_1 \dots l_s}_{j_1 \dots j_s k_1 \dots k_r} = A^{i_1 \dots i_r}_{j_1 \dots j_s} (\omega_1)_{k_1} \dots (\omega_r)_{k_r} (X_1)^{l_1} \dots (X_s)^{l_s},$$

we see that after $r + s$ contractions, in which we pair i_t with k_t, and j_t with l_t, we obtain

$$A(\omega_1, \dots, \omega_r, X_1, \dots, X_s) = C^1_1 C^2_2 \dots C^r_r C^{r+1}_1 \dots C^{r+s}_s (A \otimes \omega_1 \otimes \dots \otimes \omega_r \otimes X_1 \otimes \dots \otimes X_s),$$

and therefore the derivation gives

$$\begin{aligned}\nabla\big(A(\omega_1,\dots,\omega_r,X_1,\dots,X_s)\big) =& (\nabla A)(\omega_1,\dots,\omega_r,X_1,\dots,X_s)\\ &+\sum_{i=1}^{r} A(\omega_1,\dots,\nabla\omega_i,\dots,\omega_r,X_1,\dots,X_s)\\ &+\sum_{j=1}^{s} A(\omega_1,\dots,\omega_r,X_1,\dots,\nabla X_j,\dots,X_s).\end{aligned} \tag{3.10}$$

From the formula (3.10) follows that the term involving ∇A can be expressed in terms of ∇ applied solely to functions, vector fields, and covector fields. Moreover, since

$$(\nabla\omega)(X) = \nabla(\omega X) - \omega(\nabla X) \tag{3.11}$$

holds for $\omega \in \mathfrak{X}^*(M)$, it is enough to know only derivations of functions and vector fields. It turns out that the converse is also true, which means that from the suitable data on $\mathfrak{F}(M)$ and $\mathfrak{X}(M)$ we can construct a tensor field derivation.

Theorem 3.16. *For a vector field $\nabla_0^0\colon \mathfrak{F}(M) \to \mathfrak{F}(M)$ and an $\mathbb{R}$-linear map $\nabla_0^1\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that $\nabla_0^1(fX) = (\nabla_0^0 f)X + f\nabla_0^1 X$ holds for all $f \in \mathfrak{F}(M)$ and $X \in \mathfrak{X}(M)$, there is a unique tensor field derivation ∇ on M whose ∇_0^0 and ∇_0^1 are appropriate parts.*

Proof. Uniqueness follows from (3.10) which we also use to complete ∇. For $\omega \in \mathfrak{X}^*(M)$ we have to define $(\nabla_1^0\omega)(X) = \nabla_0^0(\omega(X)) - \omega(\nabla_0^1 X)$ because of (3.11), so

$$\begin{aligned}(\nabla_1^0\omega)(fX) &= \nabla_0^0(\omega(fX)) - \omega(\nabla_0^1 fX) = \nabla_0^0(f\omega(X)) - \omega((\nabla_0^0 f)X + f\nabla_0^1 X)\\ &= (\nabla_0^0 f)\omega(X) + f\nabla_0^0(\omega(X)) - (\nabla_0^0 f)\omega(X) - f\omega(\nabla_0^1 X) = f(\nabla_1^0\omega)(X),\end{aligned}$$

which means that $\nabla_1^0\omega$ is $\mathfrak{F}(M)$-linear and therefore $\nabla_1^0\omega \in \mathfrak{X}^*(M)$ holds. It is also easy to see that $\nabla_1^0\colon \mathfrak{X}^*(M) \to \mathfrak{X}^*(M)$ is $\mathbb{R}$-linear. By the formula (3.10), $\nabla_s^r A$ for $r+s \geq 2$ and $A \in \mathfrak{T}_s^r(M)$ must be expressed by

$$\begin{aligned}(\nabla_s^r A)(\omega_1,\dots,\omega_r,X_1,\dots,X_s) &= \nabla_0^0\big(A(\omega_1,\dots,\omega_r,X_1,\dots,X_s)\big)\\ &-\sum_{i=1}^{r} A(\omega_1,\dots,\nabla_1^0\omega_i,\dots,\omega_r,X_1,\dots,X_s) - \sum_{j=1}^{s} A(\omega_1,\dots,\omega_r,X_1,\dots,\nabla_0^1 X_j,\dots,X_s),\end{aligned}$$

so we define it in this way. As before, we should check that $\nabla_s^r A$ is $\mathfrak{F}(M)$-multilinear, i.e. $\nabla_s^r A \in \mathfrak{T}_s^r(M)$, and also that $\nabla_s^r\colon \mathfrak{T}_s^r(M) \to \mathfrak{T}_s^r(M)$ is $\mathbb{R}$-linear. The next step is a direct computation which shows that $\nabla(A\otimes B) = \nabla A \otimes B + A \otimes \nabla B$ holds.

It remains to prove that a derivation commutes with contractions. For $C\colon \mathfrak{T}_1^1(M) \to \mathfrak{F}(M)$ we have $\nabla C(X\otimes\omega) = \nabla(\omega(X)) = (\nabla\omega)(X) + \omega(\nabla X) = C(X\otimes\nabla\omega + \nabla X\otimes\omega) = C\nabla(X\otimes\omega)$, so C and ∇ commute on tensors of the form $X\otimes\omega$. Locality of ∇ allows calculations on coordinate neighbourhoods, where we know that all tensor fields of type $(1,1)$ are sums of tensor products. At the end, we should extend this process to contractions of tensor fields of arbitrary type. □

Example 3.27. For any $V \in \mathfrak{X}(M)$, by the previous theorem, we can set a tensor field derivation L_V on M by $L_V(f) = Vf$ for all $f \in \mathfrak{F}(M)$ and by $L_V(X) = [V,X]$ for all $X \in \mathfrak{X}(M)$. The conditions of the theorem are satisfied since $(L_V)_0^0 = V$ is a vector field, $(L_V)_0^1 = [V,\cdot]$ is $\mathbb{R}$-linear, and we also have the equality $L_V(fX) = [V,fX] = (Vf)X + f[V,X] = (L_V f)X + fL_V X$. This tensor derivative L_V we call the ***Lie derivative*** relative to V. △

Lemma 3.17. *If $X,Y \in \mathfrak{X}(M)$, then for the Lie derivative we have $[L_X,L_Y] = L_{[X,Y]}$.*

Proof. First, we check the formula for $f \in \mathfrak{F}(M) = \mathfrak{T}^0_0(M)$, where

$$[L_X, L_Y]f = L_X L_Y f - L_Y L_X f = XYf - YXf = [X,Y]f = L_{[X,Y]}f.$$

After that, we prove the formula for $V \in \mathfrak{X}(M) = \mathfrak{T}^1_0(M)$, where we use the Jacobi identity (2.7) for

$$L_X L_Y V - L_Y L_X V = [X,[Y,V]] - [Y,[X,V]] = [X,[Y,V]] + [X,[V,Y]] + [V,[Y,X]] = [[X,Y],V] = L_{[X,Y]}V.$$

Since $[L_X, L_Y]$ and $L_{[X,Y]}$ agree on $\mathfrak{T}^0_0(M)$ and $\mathfrak{T}^1_0(M)$, according to Theorem 3.16 it is enough to prove that $[L_X, L_Y]$ is a tensor field derivation. The initial conditions are obvious, so it remains to check the properties (3.8) and (3.9). The contractions commute with L_X and L_Y, so $[L_X, L_Y]C = L_X L_Y C - L_Y L_X C = C L_X L_Y - C L_Y L_X = C[L_X, L_Y]$, while from

$$L_X L_Y (A \otimes B) = L_X L_Y A \otimes B + L_X A \otimes L_Y B + L_Y A \otimes L_X B + A \otimes L_X L_Y B,$$

we confirm the Leibniz rule $[L_X, L_Y](A \otimes B) = [L_X, L_Y]A \otimes B + A \otimes [L_X, L_Y]B$, which completes the proof. □

Example 3.28. The calculations from Lemma 3.17 can be directly completed because for $\omega \in \mathfrak{X}^*(M) = \mathfrak{T}^0_1(M)$ from

$$\begin{aligned}
L_X L_Y \omega(V) &= L_X(L_Y\omega(V)) - L_Y\omega(L_X V)\\
&= L_X(L_Y(\omega(V)) - \omega(L_Y V)) - L_Y(\omega(L_X V)) + \omega(L_Y L_X V)\\
&= XY(\omega(V)) - X(\omega([Y,V])) - Y(\omega([X,V]) + \omega(L_Y L_X V)
\end{aligned}$$

we get

$$\begin{aligned}
(L_X L_Y - L_Y L_X)\omega(V) &= (XY - YX)(\omega(V)) - \omega((L_X L_Y - L_Y L_X)V)\\
&= [X,Y](\omega(V)) - \omega(L_{[X,Y]}V) = L_{[X,Y]}\omega(V),
\end{aligned}$$

which further generalises the formula for an arbitrary tensor field $A \in \mathfrak{T}^r_s(M)$, where

$$\begin{aligned}
L_X L_Y A(\omega_1, \dots, V_s) =& L_X(L_Y A(\omega_1, \dots, V_s)) - \sum_i L_Y A(\dots, L_X\omega_i, \dots) - \sum_j L_Y A(\dots, L_X V_j, \dots)\\
=& L_X L_Y(A(\omega_1, \dots, V_s)) - L_X \sum_k A(\dots, L_Y\omega_k, \dots) - L_X \sum_l A(\dots, L_Y V_l, \dots)\\
&- \sum_i L_Y(A(\dots, L_X\omega_i, \dots)) + \sum_i \sum_l A(\dots, L_X\omega_i, \dots, L_Y V_l, \dots)\\
&+ \sum_i \sum_{k \neq i} A(\dots, L_Y\omega_k, \dots, L_X\omega_i, \dots) + \sum_i A(\dots, L_Y L_X\omega_i, \dots)\\
&- \sum_j L_Y(A(\dots, L_X V_j, \dots)) + \sum_j \sum_k A(\dots, L_Y\omega_k, \dots, L_X V_j, \dots)\\
&+ \sum_j \sum_{l \neq j} A(\dots, L_Y V_l, \dots, L_X V_j, \dots) + \sum_j A(\dots, L_Y L_X V_j, \dots)
\end{aligned}$$

implies

$$\begin{aligned}
(L_X L_Y - L_Y L_X)A(\omega_1, \dots, V_s) =& (XY - YX)(A(\omega_1, \dots, V_s)) - \sum_i A(\dots, (L_X L_Y - L_Y L_X)\omega_i, \dots)\\
&- \sum_j A(\dots, (L_X L_Y - L_Y L_X)V_j, \dots)\\
=& [X,Y](A(\omega_1, \dots, V_s)) - \sum_i A(\dots, L_{[X,Y]}\omega_i, \dots) - \sum_j A(\dots, L_{[X,Y]}V_j, \dots)\\
=& L_{[X,Y]}A(\omega_1, \dots, V_s),
\end{aligned}$$

but it is certainly better and faster to avoid this unnecessary calculations. △

3.7 Problems

Problem 3.1. Prove that there exista a covector field on the sphere $\mathbf{S}^2 \subset \mathbb{R}^3$ that is zero at exaclty one point.

Problem 3.2. On the manifold $M = \{(x,y) \in \mathbb{R}^2 : x > 0\}$, consider the function $f \in \mathfrak{F}(M)$ given by $f(x,y) = x/(x^2+y^2)$. Compute the differential df both in standard and polar coordinates. Determine the set of all points $p \in M$ where $df_p = 0$.

Problem 3.3. For $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) = f(s,t) = (st, e^t)$ and $\omega = x\,dy - y\,dx$ compute $f^*\omega$.

Problem 3.4. For $f\colon \mathbb{R}^2 \to \mathbb{R}^3$, $(x,y,z) = f(\theta,\varphi) = ((\cos\varphi + 2)\cos\theta, (\cos\varphi+2)\sin\theta, \sin\varphi)$ and $\omega = z^2\,dx$ compute $f^*\omega$.

Problem 3.5. Let M be a manifold and $A \in \mathfrak{T}^0_s(M)$. Show that $\operatorname{Sym} A \in \mathfrak{T}^0_s(M)$ is the unique symmetric tensor satisfying $(\operatorname{Sym} A)(X, X, \dots, X) = A(X, X, \dots, X)$ for every $X \in \mathfrak{X}(M)$.

Problem 3.6. Show that the symmetric product ia associative for symmetric covariant tensors.

Problem 3.7. Let M be a parallelisable two-dimensional manifold with a global frame (∂_1, ∂_2). The map $\det\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{F}(M)$ is given in terms of the components of vector fields with respect to (∂_1, ∂_2) by $\det(X, Y) = X^1Y^2 - X^2Y^1$. Determine whether $\det \in \mathfrak{T}^0_2(M)$. Do there exist $\omega, \tau \in \mathfrak{X}^*(M)$ such that $\det = \omega \otimes \tau$?

Problem 3.8. Let ∇ be a tensor field derivative on the manifold $\mathbb{R}^n$ such that for every $f \in \mathfrak{F}(\mathbb{R}^n)$ and $1 \leq i \leq n$, we have $\nabla f = \partial_1 f$ and $\nabla(dx_i) = dx_1 + dx_i$. Find the general form of ∇X, where $X \in \mathfrak{X}(\mathbb{R}^n)$. In particular, compute $\nabla\partial_1$ and $\nabla\partial_2$.

CHAPTER 4

PSEUDO-RIEMANNIAN METRIC

4.1 Scalar products

Let $\mathcal{V}$ be a finite-dimensional vector space over $\mathbb{R}$. A ***bilinear form*** on $\mathcal{V}$ is an $\mathbb{R}$-bilinear function $g\colon \mathcal{V}\times\mathcal{V}\to\mathbb{R}$. A bilinear form g on $\mathcal{V}$ is called ***symmetric*** if $g(X,Y)=g(Y,X)$ holds for all $X,Y\in\mathcal{V}$. In other words, a symmetric bilinear form g on $\mathcal{V}$ is a symmetric covariant tensor of order two on $\mathcal{V}$. It is uniquely determined by the associated quadratic form $\varepsilon\colon \mathcal{V}\to\mathbb{R}$ given by

$$\varepsilon_X = g(X,X), \tag{4.1}$$

because we can recover it using the polarization identity

$$g(X,Y)=\frac{1}{4}(\varepsilon_{X+Y}-\varepsilon_{X-Y}), \tag{4.2}$$

or equivalently $g(X,Y)=(\varepsilon_{X+Y}-\varepsilon_X-\varepsilon_Y)/2$.

We say that a symmetric bilinear form g on $\mathcal{V}$ is ***nondegenerate*** if $g(X,Y)=0$ for all $Y\in\mathcal{V}$ implies $X=0$. Additionally, if for all $X\neq 0$ holds $g(X,X)>0$ then g is ***positive definite***. Similarly, if for all $X\neq 0$ holds $g(X,X)<0$ then g is ***negative definite***. If g is positive definite or negative definite we say that g is ***definite***, otherwise it is ***indefinite***. Evidently, if g is definite then it is nondegenerate, but the converse is not true.

A ***scalar product*** g on $\mathcal{V}$ is a nondegenerate symmetric bilinear form on $\mathcal{V}$. A finite-dimensional real vector space $\mathcal{V}$ together with a scalar product g on $\mathcal{V}$ will be referred to as a ***scalar product space*** $(\mathcal{V},g)$. Since the data of g and the associated quadratic form ε are interchangeable by (4.1) and (4.2) we sometimes say that $(\mathcal{V},g)$ is a ***quadratic vector space***. Especially, an ***inner product*** is a positive definite scalar product in which case $(\mathcal{V},g)$ is called an ***inner product space***.

In the presence of some basis $(E_1,\dots,E_n)$ in $\mathcal{V}$, each bilinear form g on $\mathcal{V}$ has the associated ***Gram matrix*** G with the entries $g_{ij}=g(E_i,E_j)$ for $1\le i,j\le n$. The Gram matrix G recovers g by the bilinearity $g(\sum_{i=1}^n \alpha_i E_i,\sum_{j=1}^n \beta_j E_j)=\sum_{i,j=1}^n \alpha_i\beta_j g_{ij}$, which can be written in the matrix notation as $g(X,Y)=X^\intercal GY$, where $X,Y\in\mathcal{V}$ are column matrices. The symmetry of g is obviously equivalent that its Gram matrix is symmetric, $G^\intercal=G$. Finally, in the following lemma we see that the condition of nondegeneracy implies $\det G\neq 0$, which means that the Gram matrix is invertible.

Lemma 4.1. *A symmetric bilinear form on $\mathcal{V}$ is nondegenerate if and only if its Gram matrix with respect to any basis in $\mathcal{V}$ is invertible.*

Proof. Let G be the Gram matrix of g with respect to a basis $(E_1,\dots,E_n)$. If $\det G=0$, then there exists a vector $Y\neq 0$ such that $GY=0$, which implies $g(X,Y)=X^\intercal GY=0$ for all $X\in\mathcal{V}$, and consequently g is not nondegenerate. Conversely, if $\det G\neq 0$, then $GY\neq 0$ holds for any $Y\neq 0$, there exists $1\le i\le n$ such that $g(E_i,Y)=E_i^\intercal GY\neq 0$, so g is nondegenerate. □

Example 4.1. The basic example of the inner product is the standard scalar product of the Euclidean space $\mathbb{R}^n$, given by $g(X,Y) = X^\intercal Y = \sum_i x_i y_i$, and its Gram matrix with respect to the canonical basis is the identity. The simplest example of an indefinite scalar product space is given by $g(X,Y) = x_1y_1 - x_2y_2$ on $\mathbb{R}^2$. Obviously, such g is bilinear, its Gram matrix G with respect to the canonical basis is diagonal with 1 and -1 on the diagonal, so G is symmetric and invertible. △

Let $(\mathcal{V}, g)$ be a scalar product space. The ***norm*** (or ***length***) of any vector $X \in \mathcal{V}$ is the nonnegative number $\|X\| = \sqrt{|g(X,X)|}$, while its ***quadratic norm*** is $\varepsilon_X = g(X,X)$, so we have $|\varepsilon_X| = \|X\|^2$. The sign of ε_X distinguishes all nonzero vectors $X \in \mathcal{V}$ into three different types that indicate the ***causal character*** of a vector. A nonzero vector $X \in \mathcal{V}$ is ***spacelike*** if $\varepsilon_X > 0$; ***timelike*** if $\varepsilon_X < 0$; ***null*** (***isotropic***, ***lightlike***) if $\varepsilon_X = 0$. Especially, a vector $X \in \mathcal{V}$ is ***nonnull*** (***non-isotropic***, ***anisotropic***, ***definite***) if $\varepsilon_X \neq 0$, and it is ***unit*** if $\varepsilon_X \in \{-1, 1\}$ ($\|X\| = 1$).

Let us remark that we left the causal character of the zero vector undetermined. Naturally, the zero vector could be considered null, but when we say that a vector is null we assume that it is not zero. Some authors, especially those who often work in Lorentzian geometry (for example O'Neill [96]), consider the zero vector to be spacelike. The last possibility is that the zero vector should have all three causal characters (see Lee [76]).

We say that two vectors $X, Y \in \mathcal{V}$ are ***mutually orthogonal*** if $g(X,Y) = 0$, and this is written as $X \perp Y$. Similarly, for subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{V}$, which are usually subspaces, we say that they are ***orthogonal*** and we write $\mathcal{A} \perp \mathcal{B}$ if $g(X,Y) = 0$ holds for all $X \in \mathcal{A}$ and all $Y \in \mathcal{B}$, while we can also naturally write $X \perp \mathcal{A}$ if $\operatorname{Span}\{X\} \perp \mathcal{A}$. The ***orthogonal subspace*** (or ***perpendicular subspace***) of a subspace $\mathcal{W} \leq \mathcal{V}$ is defined as $\mathcal{W}^\perp = \{X \in \mathcal{V} : X \perp \mathcal{W}\} \leq \mathcal{V}$, which is the maximal subspace of $\mathcal{V}$ orthogonal to $\mathcal{W}$.

In the definite case, the orthogonal subspace $\mathcal{W}^\perp$ is known as the ***orthogonal complement*** of $\mathcal{W}$, because of $\mathcal{V} = \mathcal{W} + \mathcal{W}^\perp$. However, when g is indefinite, $\mathcal{W} + \mathcal{W}^\perp$ is generally not all of $\mathcal{V}$, so sometimes the orthogonal $\mathcal{W}^\perp$ is not the complement of $\mathcal{W}$. Anyway, the perpendicular operation does have some common properties, which we emphasize in the following two lemmas.

Lemma 4.2. *A subspace $\mathcal{W}$ of a scalar product space $\mathcal{V}$ satisfies* $\dim \mathcal{W} + \dim \mathcal{W}^\perp = \dim \mathcal{V}$.

Proof. Let us extend a basis $(E_1, \dots, E_k)$ in $\mathcal{W}$ to a basis $(E_1, \dots, E_n)$ in $\mathcal{V}$. Then a vector $X = \sum_{i=1}^n \alpha_i E_i$ belongs to $\mathcal{W}^\perp$ if and only if $0 = g(X, E_j) = \sum_{i=1}^n \alpha_i g_{ij}$ for all $1 \leq j \leq k$. We obtain a homogeneous system of $k = \dim \mathcal{W}$ linear equations in $n = \dim \mathcal{V}$ unknowns, but by Lemma 4.1 the rows of the coefficient matrix are linearly independent, so this matrix has rank k, which implies that the space of solutions has dimension $n - k$. However, the system solutions $(\alpha_1, \dots, \alpha_n)$ by construction give exactly all vectors $X \in \mathcal{W}^\perp$ and therefore $\dim \mathcal{W}^\perp = n - k$. □

Lemma 4.3. *For a subspace $\mathcal{W}$ of a scalar product space $\mathcal{V}$ we have $(\mathcal{W}^\perp)^\perp = \mathcal{W}$.*

Proof. Since $\mathcal{W} \perp \mathcal{W}^\perp$ we have $\mathcal{W} \leq (\mathcal{W}^\perp)^\perp$. On the other hand, according to Lemma 4.2 we have $\dim \mathcal{W} + \dim \mathcal{W}^\perp = \dim \mathcal{V} = \dim \mathcal{W}^\perp + \dim(\mathcal{W}^\perp)^\perp$, so $\dim(\mathcal{W}^\perp)^\perp = \dim \mathcal{W}$, and therefore $(\mathcal{W}^\perp)^\perp = \mathcal{W}$. □

For any subspace $\mathcal{W}$ of a scalar product space $(\mathcal{V}, g)$, the restriction of the scalar product $g\restriction_{\mathcal{W}} = g\restriction_{\mathcal{W}\times\mathcal{W}}$ is a symmetric bilinear form on $\mathcal{W}$. Moreover, if g is positive definite then $g\restriction_{\mathcal{W}}$ is a positive definite scalar product on $\mathcal{W}$. However, if g is indefinite then $g\restriction_{\mathcal{W}}$ does not need be nondegenerate.

Example 4.2. If $S \in \mathcal{V}$ is spacelike, and $T \in \mathcal{V}$ is timelike, then we easily construct a null vector by

$$N = S + \frac{-g(S,T) \pm \sqrt{(g(S,T))^2 - \varepsilon_S \varepsilon_T}}{\varepsilon_S} T,$$

since the coefficient along T is the solution of the quadratic equation $\varepsilon_{S+xT} = \varepsilon_S + 2xg(S,T) + x^2\varepsilon_T = 0$. Ig g is indefinite then there exist both spacelike and timelike vectors, and therefore there is a null N, so for $\mathcal{W} = \mathrm{Span}\{N\} \le \mathcal{V}$ the restriction $g\restriction_{\mathcal{W}}$ is not nondegenerate. △

We transfer properties of the scalar product of scalar product space $(\mathcal{V}, g)$ to subspace $\mathcal{W}$, so we say that it is ***nondegenerate***, ***positive definite***, or ***negative definite***, if such is its restriction $g\restriction_{\mathcal{W}}$. The ***radical*** of $\mathcal{W}$ is the subspace $\mathrm{rad}(\mathcal{W}) = \mathcal{W} \cap \mathcal{W}^\perp$ which can help us to characterize nondegenerate subspaces.

Lemma 4.4. *Let $\mathcal{W}$ be a subspace of a scalar product space $(\mathcal{V}, g)$. Then the equivalent statements are:* $\mathrm{rad}(\mathcal{W})$ *is trivial, $\mathcal{W}$ is nondegenerate, $\mathcal{W}^\perp$ is nondegenerate, $\mathcal{V} = \mathcal{W} + \mathcal{W}^\perp$.*

Proof. By definition, $\mathcal{W}$ is nondegenerate if 0 is the only vector in $\mathcal{W}$ orthogonal to $\mathcal{W}$, which is equivalent to $\mathrm{rad}(\mathcal{W}) = \mathcal{W} \cap \mathcal{W}^\perp = \{0\}$. From Lemma 4.3, $\mathrm{rad}(\mathcal{W}) = \mathrm{rad}(\mathcal{W}^\perp)$, so nondegeneracy of $\mathcal{W}$ and $\mathcal{W}^\perp$ are equivalent. According to the Grassmann[1] formula $\dim(\mathcal{W} + \mathcal{W}^\perp) + \dim(\mathcal{W} \cap \mathcal{W}^\perp)$ is equal to $\dim \mathcal{W} + \dim \mathcal{W}^\perp$, which is $\dim \mathcal{V}$ by Lemma 4.2. Thus, $\mathcal{V} = \mathcal{W} + \mathcal{W}^\perp$ is equivalent to $\mathrm{rad}(\mathcal{W}) = \mathcal{W} \cap \mathcal{W}^\perp = \{0\}$. □

Of course, since a scalar product space is nondegenerate, we have $\mathrm{rad}(\mathcal{V}) = \mathcal{V}^\perp = \{0\}$. The orthogonal direct sum is denoted by the symbol $\oplus$, so if $\mathcal{W} \le \mathcal{V}$ is nondegenerate then we can write $\mathcal{V} = \mathcal{W} \oplus \mathcal{W}^\perp$.

A scalar product g on $\mathcal{V}$ can be diagonalized by applying Lemma 4.4. As usual, a set of mutually orthogonal unit vectors is said to be ***orthonormal***. Any set of $\dim \mathcal{V}$ orthonormal vectors in $\mathcal{V}$ is necessarily a basis in $\mathcal{V}$, and such basis always exists.

Lemma 4.5. *Any scalar product space has an orthonormal basis.*

Proof. The proof is by induction on $n = \dim \mathcal{V}$. The case $\dim \mathcal{V} = 1$ is obvious. If $n > 1$, choose a nonnull vector $X \in \mathcal{V}$ (otherwise $\varepsilon_X = 0$ holds for all $X \in \mathcal{V}$, so $g = 0$ is degenerate), and by rescaling it we obtain the unit $E_1 = (1/\sqrt{|\varepsilon_X|})X$. The one-dimensional subspace $\mathrm{Span}\{E_1\}$ is nondegenerate, so by Lemma 4.4 $\mathrm{Span}\{E_1\}^\perp$ is also nondegenerate. By induction hypothesis, there is an orthonormal basis $(E_2, \ldots, E_n)$ of $\mathrm{Span}\{E_1\}^\perp$. Since $\mathcal{V} = \mathrm{Span}\{E_1\} + \mathrm{Span}\{E_1\}^\perp$, we obtain a wanted orthonormal basis $(E_1, E_2, \ldots, E_n)$. □

An indefinite scalar product often has appearance of $\varepsilon_{E_i} \in \{-1, 1\}$ for vectors from an orthonormal basis $(E_1, \ldots, E_n)$ in formulas that would be familiar in the positive definite case. If $X = \sum_{i=1}^n \alpha_i E_i$ then $g(X, E_j) = \alpha_j g(E_j, E_j)$, so $\alpha_j = g(X, E_j)/g(E_j, E_j) = \varepsilon_{E_j} g(X, E_j)$, and therefore we obtain

$$X = \sum_{i=1}^{n} \varepsilon_{E_i} g(X, E_i) E_i. \tag{4.3}$$

Lemma 4.6. *If $(E_1, \ldots, E_n)$ is an orthonormal basis in a scalar product space $(\mathcal{V}, g)$, then each $X \in \mathcal{V}$ is uniquely expressed by the formula* (4.3).

The Gram matrix of a scalar product g with respect to an orthonormal basis $(E_1, \ldots, E_n)$ in $\mathcal{V}$ is diagonal, because of $g_{ij} = g(E_i, E_j) = \delta_{ij}\varepsilon_{E_i}$. Whenever it is convenient we order the vectors in an orthonormal basis so that negative signs come first, $\varepsilon_{E_i} = -1$ for $1 \le i \le \nu$, and then positive signs, $\varepsilon_{E_i} = 1$ for $\nu < i \le n$. The orthonormal expansion is still available taking these signs into account.

The ***index*** of a scalar product g on $\mathcal{V}$ is the largest integer $\mathrm{Ind}(g) \in \mathbb{N}_0$ which is the dimension of a negative definite subspace of $\mathcal{V}$. The number of negative ε_{E_i} is equal to the index of g which establishes the following theorem known as the ***Sylvester's law of inertia***[2].

[1] Hermann Günther Graßmann (1809–1877), German physicist, mathematician and linguist

[2] James Joseph Sylvester (1814–1897), English mathematician

Theorem 4.7. *The number of negative ε_{E_i} does not depend on the orthonormal basis $(E_1, \dots, E_n)$ in a scalar product space $(\mathcal{V}, g)$, and it is equal to the index of g.*

Proof. If $\mathcal{T} \leq \mathcal{V}$ is negative definite and $\mathcal{S} \leq \mathcal{V}$ is positive definite, we have $\mathcal{T} \cap \mathcal{S} = \{0\}$, whence Grassmann's formula gives $\dim \mathcal{T} + \dim \mathcal{S} = \dim(\mathcal{T} + \mathcal{S}) \leq \dim \mathcal{V} = n$. The subspace $\mathcal{S} = \operatorname{Span}\{E_{\nu+1}, \dots, E_n\}$ is positive definite, which gives $\dim \mathcal{T} \leq \nu$, while $\dim \mathcal{T} = \nu$ for the negative definite $\mathcal{T} = \operatorname{Span}\{E_1, \dots, E_\nu\}$, so we get $\operatorname{Ind}(g) = \nu$. □

The ***index*** of a scalar product space $(\mathcal{V}, g)$ of dimension n is the index of its scalar product $\nu = \operatorname{Ind}(\mathcal{V}) = \operatorname{Ind}(g)$, while the ***signature*** is an ordered pair (p, q) that represents the number of negative ε_{E_i} and the number of positive ε_{E_i}. According to Theorem 4.7, the numbers $0 \leq p, q \leq n$ are independent on a particular choice of orthonormal basis, and we have $p = \nu$, $q = n - \nu$.

It is worth noting that $(\mathcal{V}, -g)$ is also a scalar product space with $\operatorname{Ind}(-g) = n - \operatorname{Ind}(g)$. Scalar product spaces $(\mathcal{V}, g)$ and $(\mathcal{V}, -g)$ exchange numbers in the signature, but the difference between them is not essential, so it is usual to normalize the index with $\nu \leq n/2$, which would mean that there are no more temporal coordinates than spatial ones.

Let $(\mathcal{V}, g^{\mathcal{V}})$ and $(\mathcal{W}, g^{\mathcal{W}})$ be scalar product spaces. We say that a linear map $L \colon \mathcal{V} \to \mathcal{W}$ preserves the scalar product if $g^{\mathcal{W}}(LX, LY) = g^{\mathcal{V}}(X, Y)$ holds for all $X, Y \in \mathcal{V}$. Such map L is necessarily injective since $LX = 0$ implies $g^{\mathcal{V}}(X, Y) = 0$ for all $Y \in \mathcal{V}$ and therefore $X = 0$. Of course, if L preserves the scalar product then it preserves the associated quadratic forms, while the converse holds by polarization. A ***linear isometry*** is a linear bijective map that preserves the scalar product.

Lemma 4.8. *There exists a linear isometry $L \colon \mathcal{V} \to \mathcal{W}$ if and only if scalar product spaces $\mathcal{V}$ and $\mathcal{W}$ have the same dimension and index.*

Proof. If L is a linear isometry then $\dim \mathcal{V} = \dim \mathcal{W}$ and L maps an orthonormal basis in $\mathcal{V}$ to an orthonormal basis in $\mathcal{W}$, and therefore Theorem 4.7 gives $\operatorname{Ind} \mathcal{V} = \operatorname{Ind} \mathcal{W}$. Conversely, we can reorder orthonormal bases $(E_1, \dots, E_n)$ in $\mathcal{V}$ and $(F_1, \dots, F_n)$ in $\mathcal{W}$ such that $g^{\mathcal{V}}(E_i, E_i) = g^{\mathcal{W}}(F_i, F_i)$, and define the linear isometry by $L(E_i) = F_i$. □

4.2 Null vectors

In a scalar product space $(\mathcal{V}, g)$, a nonzero vector $X \in \mathcal{V}$ that is self-orthogonal ($X \perp X$) is referred to as a null vector. If g is definite, then the zero vector is the only vector with zero norm, so null vectors exist only in the case of indefinite g. The set of all null vectors $\{X \in \mathcal{V} : X \neq 0, \varepsilon_X = 0\}$ is called the ***nullcone*** or ***lightcone***. How our awareness readily accepts only positive definite metric, null vectors are quite inconvenient and often counter-intuitive, but they are also crucial in understanding pseudo-Riemannian manifolds.

Lemma 4.9. *Any scalar product space $\mathcal{V}$ of signature (p, q) can be decomposed as an orthogonal sum $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$, where $\mathcal{V}^+$ is a maximal positive definite subspace of dimension q, and $\mathcal{V}^-$ is a maximal negative definite subspace of dimension p.*

Proof. Let $(T_1, \dots, T_p, S_1, \dots, S_q)$ be an orthonormal basis of $\mathcal{V}$, with $\varepsilon_{T_i} = -1$ for $1 \leq i \leq p$ and $\varepsilon_{S_j} = 1$ for $1 \leq j \leq q$. Subspaces $\mathcal{V}^- = \operatorname{Span}\{T_1, \dots, T_p\}$ and $\mathcal{V}^+ = \operatorname{Span}\{S_1, \dots, S_q\}$ are mutually orthogonal and have trivial intersection, where $\mathcal{V} = \mathcal{V}^+ + \mathcal{V}^-$. □

Lemma 4.9 allows a decomposition $N = S + T$ of any null vector $N \in \mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$, such that $S \in \mathcal{V}^+$ and $T \in \mathcal{V}^-$. Since $\mathcal{V}^+ \perp \mathcal{V}^-$, we have $g(S, T) = 0$, and therefore we obtain $0 = \varepsilon_N = g(S + T, S + T) = \varepsilon_S + \varepsilon_T$. The condition $N \neq 0$ implies $S \neq 0$, so $\varepsilon_S \neq 0$ and finally $\varepsilon_S = -\varepsilon_T > 0$. Hence, any null $N \in \mathcal{V}$ can be decomposed as a sum $N = S + T$ of mutually orthogonal S and T, with $\varepsilon_S = -\varepsilon_T > 0$. However, the previous decomposition is not unique, even in the same plane $\mathrm{Span}\{S, T\}$.

Lemma 4.10. *Any null $N \neq 0$ from a scalar product space $\mathcal{V}$ can be decomposed as $N = S + T$, such that $S, T \in \mathcal{V}$ and $\varepsilon_S = -\varepsilon_T = 1$.*

Proof. We already have a decomposition $N = S + T$ with $g(S, T) = 0$ and $\varepsilon_S = -\varepsilon_T > 0$. Consider $S_1 = \theta S + (1-\theta)T$ and $T_1 = (1-\theta)S + \theta T$ for some $\theta > 1/2$. Then $S_1, T_1 \in \mathcal{V}$ with $S_1 + T_1 = S + T = N$ and

$$g(S_1, T_1) = g(\theta S + (1-\theta)T, (1-\theta)S + \theta T) = \theta(1-\theta)(\varepsilon_S + \varepsilon_T) = 0.$$

We have $\varepsilon_{S_1} = \theta^2\varepsilon_S + (1-\theta)^2\varepsilon_T = (\theta^2 - (1-\theta)^2)\varepsilon_S = (2\theta - 1)\varepsilon_S > 0$, while $g(S_1, T_1) = 0$ implies $\varepsilon_{S_1} + \varepsilon_{T_1} = \varepsilon_{S_1+T_1} = \varepsilon_N = 0$, which yields $\varepsilon_{S_1} = -\varepsilon_{T_1} > 0$. This construction, for any $\theta > 1/2$ gives a new decomposition, so for $\theta = (1 + \varepsilon_S)/(2\varepsilon_S) > 1/2$ we obtain $\varepsilon_{S_1} = -\varepsilon_{T_1} = 1$, and a new decomposition

$$N = \left(\frac{\varepsilon_S + 1}{2\varepsilon_S}S + \frac{\varepsilon_S - 1}{2\varepsilon_S}T\right) + \left(\frac{\varepsilon_S - 1}{2\varepsilon_S}S + \frac{\varepsilon_S + 1}{2\varepsilon_S}T\right)$$

with desired properties. $\square$

Example 4.3. New constructions are possible outside of the plane $\mathrm{Span}\{S, T\}$. If $\dim \mathcal{V} > 2$, there exists a nonnull $W \in \mathcal{V}$, such that $W \perp \mathrm{Span}\{S, T\}$. We want $\alpha, \beta, \gamma \in \mathbb{R}$ such that $S_1 = \alpha S + \beta T + \gamma W$ and $T_1 = (1-\alpha)S + (1-\beta)T - \gamma W$, which assures $S_1 + T_1 = S + T = N$. From $S_1 \perp T_1$ we have

$$0 = g(S_1, T_1) = \alpha(1-\alpha)\varepsilon_S + \beta(1-\beta)\varepsilon_T - \gamma^2\varepsilon_W = ((\alpha - \alpha^2) - (\beta - \beta^2))\varepsilon_S - \gamma^2\varepsilon_W,$$

and therefore

$$\gamma^2\varepsilon_W = (\alpha - \beta)(1 - \alpha - \beta)\varepsilon_S. \tag{4.4}$$

Since

$$\varepsilon_{S_1} = \alpha^2\varepsilon_S + \beta^2\varepsilon_T + \gamma^2\varepsilon_W = (\alpha^2 - \beta^2)\varepsilon_S + (\alpha - \beta)(1 - \alpha - \beta)\varepsilon_S = (\alpha - \beta)\varepsilon_S,$$

the last condition $\varepsilon_{S_1} > 0$ is true for $\alpha > \beta$. We choose $\alpha + \beta < 1$ for a spacelike W or $\alpha + \beta > 1$ for a timelike W. This, together with $\alpha > \beta$, according to (4.4) determines the final conditions for α and β. Finally, every α and β limited with the last two inequalities create a new decomposition, where $\gamma = \pm\sqrt{(\alpha - \beta)(1 - \alpha - \beta)\varepsilon_S/\varepsilon_W}$. $\triangle$

We say that a subspace of a scalar product space is ***totally isotropic*** if it consists just of vectors whose norm is zero. Using the identity (4.2) we see that any two vectors in a totally isotropic subspace are mutually orthogonal. Thus, for a totally isotropic subspace $\mathcal{W} \leq \mathcal{V}$ we have $g\restriction_{\mathcal{W}} = 0$. Therefore $\mathcal{W} \leq \mathcal{W}^\perp$ holds, and consequently $\mathrm{rad}(\mathcal{W}) = \mathcal{W}$.

It is worth noting that the radical of an arbitrary subspace $\mathcal{U} \leq \mathcal{V}$ is totally isotropic because of $\mathrm{rad}(\mathrm{rad}(\mathcal{U})) = \mathrm{rad}(\mathcal{U})$. In general, any complementary subspace $\mathcal{W}$ to $\mathrm{rad}(\mathcal{U})$ in the sense of usual linear algebra will give rise to a radical splitting $\mathcal{U} = \mathrm{rad}(\mathcal{U}) \oplus \mathcal{W}$. It is easy to see that $\mathcal{W}$ is far from being unique, but it is surely nondegenerate.

Example 4.4. Let $\mathcal{U}$ be an arbitrary subspace of a scalar product space $(\mathcal{V}, g)$. Assume that $\mathcal{U} = \mathcal{W} \oplus \mathcal{N}$ is the orthogonal sum of a nondegenerate $\mathcal{W} \leq \mathcal{V}$ and a totally isotropic $\mathcal{N} \leq \mathcal{V}$. First, we want to show that $\mathcal{N}$ is uniquely determined. On the one hand, we have $\mathcal{N} \leq \mathcal{U}$, while $\mathcal{N} \perp \mathcal{W}, \mathcal{N}$ gives $\mathcal{N} \perp \mathcal{U}$, so $\mathcal{N} \leq \mathcal{U} \cap \mathcal{U}^\perp = \mathrm{rad}(\mathcal{U})$. On the other hand, since nondegenerate and totally isotropic subspaces have a trivial intersection, we have the inequality $\dim \mathrm{rad}(\mathcal{U}) = \dim(\mathcal{W} + \mathrm{rad}(\mathcal{U})) - \dim \mathcal{W} \leq \dim \mathcal{U} - \dim \mathcal{W} = \dim \mathcal{N}$. Since

$\mathcal{N} \le \operatorname{rad}(\mathcal{U})$ and $\dim \operatorname{rad}(\mathcal{U}) \le \dim \mathcal{N}$, we obtain the uniqueness, $\mathcal{N} = \operatorname{rad}(\mathcal{U})$, which brings us to a radical splitting.

Suppose that there is another such decomposition, $\mathcal{U} = \mathcal{W} \oplus \operatorname{rad}(\mathcal{U}) = \mathcal{W}' \oplus \operatorname{rad}(\mathcal{U})$ for some nondegenerate $\mathcal{W}' \le \mathcal{V}$. If $(E_1, \dots, E_k)$ is an orthonormal basis for $\mathcal{W}$, then from $\mathcal{W} \le \mathcal{W}' \oplus \operatorname{rad}(\mathcal{U})$ we can decompose $E_i = E'_i + N_i$, where $E'_i \in \mathcal{W}'$ and $N_i \in \operatorname{rad}(\mathcal{U})$. However, since $g(E_i, E_j) = g(E'_i + N_i, E'_j + N_j) = g(E'_i, E'_j)$, we see that $(E'_1, \dots, E'_k)$ is an orthonormal basis for $\mathcal{W}'$, where $\mathcal{W}$ and $\mathcal{W}'$ have the same signature. Thus, we invariantly introduce the ***signature*** of a subspace $\mathcal{U}$ of a scalar product space $\mathcal{V}$ as an ordered triple (p, q, r), where (p, q) is the signature of the scalar product space $(\mathcal{W}, g\restriction_{\mathcal{W}})$, and $r = \dim \operatorname{rad}(\mathcal{U})$. △

For a totally isotropic subspace $\mathcal{W} \le \mathcal{V}$, by Lemma 4.2 holds $\dim \mathcal{W} + \dim \mathcal{W}^\perp = \dim \mathcal{V}$, so from $\mathcal{W} \le \mathcal{W}^\perp$ follows $\dim \mathcal{W} \le (\dim \mathcal{V})/2$. Moreover, the dimension of a totally isotropic subspace is not greater than the index.

Theorem 4.11. *For a totally isotropic subspace $\mathcal{W}$ of a scalar product space $(\mathcal{V}, g)$ we have $\dim \mathcal{W} \le \min(\operatorname{Ind}(g), \dim \mathcal{V} - \operatorname{Ind}(g))$, and especially $\dim \mathcal{W} \le (\dim \mathcal{V})/2$.*

Proof. Let $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ be a decomposition from Lemma 4.9. According to the Grassmann formula we have $\dim \mathcal{W} = \dim(\mathcal{W} + \mathcal{V}^+) + \dim(\mathcal{W} \cap \mathcal{V}^+) - \dim \mathcal{V}^+$. Using $\mathcal{W} + \mathcal{V}^+ \le \mathcal{V}$ and $\mathcal{W} \cap \mathcal{V}^+ = \{0\}$, it yields $\dim \mathcal{W} \le \dim \mathcal{V} - \dim \mathcal{V}^+ = \dim \mathcal{V}^- = \operatorname{Ind}(g)$. Consequently, $\dim \mathcal{W} \le \operatorname{Ind}(-g) = \dim \mathcal{V} - \operatorname{Ind}(g)$. □

Example 4.5. If $(\mathcal{V}, g)$ is a scalar product space of signature (p, q), then it has an orthonormal basis $(T_1, \dots, T_p, S_1, \dots, S_q)$, where T_i are timelike and S_i are spacelike. The bound from Theorem 4.11 is easily reached by constructing a totally isotropic subspace $\operatorname{Span}\{S_1 + T_1, S_2 + T_2, \dots, S_\nu + T_\nu\}$, where $\nu = \min(p, q)$. Many other constructions are also possible, because the elements in the basis can be permuted, while individual elements can be changed by its opposite vector. The dimension of the maximal totally isotropic subspace of $(\mathcal{V}, g)$ is called the ***Witt index***[3]. We can notice that the Witt index ν is in fact the minimum of the numbers $\operatorname{Ind}(g)$ and $\operatorname{Ind}(-g)$. △

Any totally isotropic subspace of a scalar product space has its isotropic supplement which we see in the following theorem (for example, see Clark [38, Theorem 6.2]).

Theorem 4.12. *Let $\mathcal{W}$ be a totally isotropic subspace of a scalar product space $(\mathcal{V}, g)$. For any basis $(N_1, \dots, N_k)$ of $\mathcal{W}$ there is a corresponding basis $(M_1, \dots, M_k)$ of some totally isotropic subspace of $\mathcal{V}$ which has a trivial intersection with $\mathcal{W}$ such that $g(N_i, M_j) = \delta_{ij}$ holds for $1 \le i, j \le k$.*

Proof. The proof is by induction on k, where the case $k = 0$ is trivial. Let us introduce $\mathcal{P} = \operatorname{Span}\{N_1, \dots, N_{k-1}\}$. Since $\operatorname{Span}\{N_k\}$ is not a subspace of $\mathcal{P}$, $\mathcal{P}^\perp$ is not a subspace of $\operatorname{Span}\{N_k\}^\perp$, and there exists $X \in \mathcal{P}^\perp$ such that $g(X, N_k) \ne 0$. Then

$$M_k = \frac{-\varepsilon_X}{2(g(X, N_k))^2} N_k + \frac{1}{g(X, N_k)} X$$

is null with $g(N_k, M_k) = 1$. The subspace $\operatorname{Span}\{N_k, M_k\} = \operatorname{Span}\{N_k, X\} \le \mathcal{P}^\perp$ is nondegenerate since it has an orthonormal basis $((N_k + M_k)/\sqrt{2}, (N_k - M_k)/\sqrt{2})$. Thus, $\mathcal{P}$ is a subspace of the nondegenerate $\operatorname{Span}\{N_k, M_k\}^\perp$ which we consider as a scalar product space, and we apply the induction hypothesis to get $(M_1, \dots, M_{k-1})$ which with M_k completes a basis of a totally isotropic subspace with desired properties. □

[3] Ernst Witt (1911–1991), German mathematician

4.3 Pseudo-Riemannian manifolds

A ***metric tensor*** or a ***metric*** on a manifold M is a symmetric covariant tensor field of order two on M which is nondegenerate at any point and has constant index. In other words, a metric is a map $g\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{F}(M)$ that smoothly assigns a scalar product $g_p\colon T_pM \times T_pM \to \mathbb{R}$ on the tangent space T_pM for each point $p \in M$, such that the index $\operatorname{Ind}(g_p)$ does not depend on p. A ***pseudo-Riemannian manifold*** is a manifold M endowed with a metric g.

Strictly speaking a pseudo-Riemannian manifold (also called ***semi-Riemannian manifold***) is an ordered pair (M, g). Two different metrics on the same manifold establish different pseudo-Riemannian manifolds. It often happens that a pseudo-Riemannian manifold has a default concrete metric, so we can denote it by the name of its smooth manifold.

The common index $0 \leq \nu = \operatorname{Ind}(g_p) = \operatorname{Ind}(g) = \operatorname{Ind}(M) \leq \dim M = n$ of all scalar products g_p is called the ***index*** of a pseudo-Riemannian manifold (M, g). Since the distinction between g and $-g$ in a pseudo-Riemannian manifold is not essential, without loss of generality we may assume that the index satisfies $\nu \leq n/2$.

It is interesting to look at some special cases, depending on the index. The most common case is $\nu = 0$, where we say that M is a ***Riemannian manifold***. In this case, g is called a ***Riemannian metric***, and it is characterized by the fact that g_p is a positive definite scalar product on T_pM for each $p \in M$. If $\nu = 1 \neq n$ then M is called a ***Lorentzian manifold***[4] and the corresponding metric is ***Lorentzian***. A pseudo-Riemannian manifold of signature (ν, ν) is a ***Kleinian manifold***[5], it is necessarily of even dimension, while its metric is said to be ***Kleinian*** or ***neutral***.

Let (U, φ) be a chart on a pseudo-Riemannian n-manifold (M, g) with coordinate functions $x_i = \pi_i \circ \varphi$. The components of a metric tensor g on U are $g_{ij} = g(\partial_i, \partial_j)$ for $1 \leq i, j \leq n$, so for vector fields $X, Y \in \mathfrak{X}(U)$ we have

$$g(X,Y) = g\left(\sum_{i=1}^{n} X(x_i)\partial_i, \sum_{j=1}^{n} Y(x_j)\partial_j\right) = \sum_{i,j=1}^{n} g(\partial_i, \partial_j) X(x_i) Y(x_j) = \sum_{i,j=1}^{n} g_{ij}\, dx_i(X)\, dx_j(Y),$$

and therefore the metric tensor can be expressed as

$$g = \sum_{i,j=1}^{n} g_{ij}\, dx_i \otimes dx_j = \sum_{i,j=1}^{n} g_{ij}\, dx_i\, dx_j,$$

where $dx_i\, dx_j = (dx_i \otimes dx_j + dx_j \otimes dx_i)/2$ is the symmetric product of covariant tensors.

Example 4.6. The simplest and most important example of Riemannian manifold is of course $\mathbb{R}^n$ with the ***Euclidean metric*** $\overline{g}$ which is the classic inner product on each tangent space under the natural identification $T_p\mathbb{R}^n \cong \mathbb{R}^n$. In standard coordinates this can be written by

$$\overline{g} = \sum_{i,j=1}^{n} \delta_{ij}\, dx_i\, dx_j = (dx_1)^2 + \cdots + (dx_n)^2, \tag{4.5}$$

that is, $\overline{g}_{ij} = \delta_{ij}$, which gives the identity Gram matrix $G = \mathbb{1}_{\mathbb{R}^n}$. Henceforward in any geometric context, the ***Euclidean space*** $\mathbb{R}^n$ will denote the Riemannian manifold $(\mathbb{R}^n, \overline{g})$. $\triangle$

Example 4.7. In the standard metric $\overline{g}$ from (4.5) for $\mathbb{R}^n$, for any $1 \leq \nu \leq n$ we can change the first ν signs from plus to minus, which brings us to the metric tensor $g = \sum_{i=1}^{n} \varepsilon_i\, (dx_i)^2$

[4]Hendrik Lorentz (1853–1928), Dutch physicist

[5]Felix Klein (1849–1925), German mathematician

where $\varepsilon_i = -1$ for $1 \leq i \leq \nu$ and $\varepsilon_i = +1$ for $\nu + 1 \leq i \leq n$. The resulting pseudo-Riemannian manifold $\mathbb{R}^n_\nu = (\mathbb{R}^n, g)$ is the ***pseudo-Euclidean space*** of index ν. In particular, the Lorentzian manifold $\mathbb{R}^n_1$ for $n \geq 2$ is called the ***Minkowski space***[6], and especially $\mathbb{R}^4_1$ is the simplest example of relativistic space-time. $\triangle$

Example 4.8. The ***Walker metric***[7] [117] is defined on some open subset M of $\mathbb{R}^{2n}$ with the usual coordinates $(x_1, x_2, \ldots, x_{2n})$, by the Gram matrix related to the natural global frame $(\partial_1, \partial_2, \ldots, \partial_{2n})$ with

$$g = \begin{pmatrix} f_{11} & f_{12} & \ldots & f_{1n} & 1 & 0 & \ldots & 0 \\ f_{12} & f_{22} & \ldots & f_{2n} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{1n} & f_{2n} & \ldots & f_{nn} & 0 & 0 & \ldots & 1 \\ 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \end{pmatrix}, \tag{4.6}$$

where $f_{ij} = f_{ij}(x_1, x_2, \ldots, x_{2n})$ for $1 \leq i \leq j \leq n$ are arbitrary smooth functions on M. The matrix from (4.6) is obviously symmetric and invertible (of determinant $(-1)^n \neq 0$). A common problem in a pseudo-Riemannian manifold construction is to ensure the constancy of the index for each particular scalar product. From the equation (4.6) it is clear that $\mathcal{W}_p = \operatorname{Span}\{(\partial_{n+1})_p, (\partial_{n+2})_p, \ldots, (\partial_{2n})_p\}$ is an n-dimensional totally isotropic subspace of the scalar product space $(T_p\mathbb{R}^{2n}, g_p)$ for each point $p \in M$. According to Theorem 4.11 we have $n = \dim \mathcal{W}_p \leq \operatorname{Ind}(g_p) \leq \dim M - \dim \mathcal{W}_p = n$, which is possible for $\operatorname{Ind}(g_p) = n$ only. Thus, g is a Kleinian metric on $\mathbb{R}^{2n}$, and we obtain a huge family of pseudo-Riemannian manifolds (M, g) of signature (n, n). $\triangle$

Often, instead of a metric tensor g we consider the corresponding squared norm given by $\varepsilon_X = g(X, X)$ for $X \in TM$, which by (4.2) completely determines the metric tensor. This squared form ε is called the ***line element*** of M, and denoted by ds^2. In coordinates we have $ds^2 = \sum_{i,j} g_{ij}\, dx_i\, dx_j$, which means

$$\varepsilon_X = \sum_{i,j} g_{ij}\, dx_i(X)\, dx_j(X) = \sum_{i,j} g_{ij} X^i X^j.$$

If p and p' are nearby points with coordinates $(x_1, \ldots, x_n)$ and $(x_1 + \Delta x_1, \ldots, x_n + \Delta x_n)$ in some chart, then the tangent vector $\Delta p = \sum_i \Delta x_i \partial_i$ at p points approximately to p'. Because of this, we expect that the square of the distance ds from p to p' to be approximately equal to $|\Delta p|^2 = g(\Delta p, \Delta p) = \sum_{i,j} g_{ij}(p) \Delta x_i\, \Delta x_j$, as in the formula for ds^2, which justifies this unusual notation.

In the presence of a metric tensor, we can talk about orthonormal vector fields. Let U be an open subset of a pseudo-Riemannian n-manifold (M, g). A ***local orthonormal frame*** for M over U is a local frame $(E_1, \ldots, E_n)$ for M over U that at each point $p \in M$ forms an orthonormal basis for the tangent space T_pM. For example, the coordinate frame $(\partial_1, \ldots, \partial_n)$ is a global orthonormal frame for $(\mathbb{R}^n, \overline{g})$.

Example 4.9. Consider the open subset $U = \mathbb{R}^2 \backslash \{0\} \subset \mathbb{R}^2$. Let us set the unit vector field E_1 tangent to radial lines and the unit vector field E_2 tangent to circles centred at the origin. Then

$$E_1 = \frac{\partial}{\partial r} = \frac{1}{\sqrt{x^2 + y^2}} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right), \quad E_2 = \frac{1}{r} \frac{\partial}{\partial \theta} = \frac{1}{\sqrt{x^2 + y^2}} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

[6] Hermann Minkowski (1864–1909), German mathematician
[7] Arthur Geoffrey Walker (1909–2001), British mathematician

make a local orthonormal frame for $\mathbb{R}^2$ over U (see Example 2.29 for calculations). However, (E_1, E_2) cannot be a coordinate frame with respect to any choice of local coordinates, since $[E_1, E_2] = -(\partial/\partial\theta)/r^2 \neq 0$. △

Orthonormal frames are very useful for the study of Riemannian and pseudo-Riemannian manifolds. For a Riemannian manifold, things are simple since the Gram–Schmidt process smoothly creates a local orthonormal frame starting from any local frame. The Riemannian case is easy because the vectors whose norms appear in the denominators are nowhere vanishing. However, if the metric is indefinite we need to avoid null vectors, so we have to be careful.

Theorem 4.13. *There is a local orthonormal frame over a neighbourhood of any point of a pseudo-Riemannian manifold.*

Proof. Let (M, g) be a pseudo-Riemannian n-manifold and $p \in M$. Consider a local coordinate frame $(\partial_1, \dots, \partial_n)$ over some coordinate neighbourhood $U \ni p$ and an arbitrary orthonormal basis $(V_1, \dots, V_n)$ in T_pM. Because $(V_1, \dots, V_n) = ((\partial_1)_p, \dots, (\partial_n)_p)A$ holds for the transition matrix $A \in \mathrm{GL}(n, \mathbb{R})$, a new local frame $(X_1, \dots, X_n)$ can be set with $((X_1)_q, \dots, (X_n)_q) = ((\partial_1)_q, \dots, (\partial_n)_q)A$ for all $q \in U$. Since $g_p((X_i)_p, (X_j)_p) = g_p(V_i, V_j) = \pm\delta_{ij}$ and $g(X_i, X_j) \in \mathfrak{F}(U)$, by continuity there exists a neighbourhood $W \subseteq U$ of p where for all $1 \leq i \neq j \leq n$ we have

$$|g(X_i, X_i)| > a = \frac{3}{4} + \frac{1}{3n} < 1 \quad \text{and} \quad |g(X_i, X_j)| < b = \frac{1}{6n}.$$

The Gram–Schmidt process allows us to inductively construct an orthogonal local frame $(Y_1, \dots, Y_n)$ on W such that for all $1 \leq j < k \leq n$ we have

$$|g(Y_j, Y_j)| > d = \frac{3}{4} \quad \text{and} \quad |g(Y_j, X_k)| < c = \frac{1}{2n}.$$

Let $Y_1 = X_1$. Suppose that we inductively set $(Y_1, \dots, Y_{j-1})$ for some $2 \leq j \leq n$ and define

$$Y_j = X_j - \sum_{i=1}^{j-1} \frac{g(Y_i, X_j)}{g(Y_i, Y_i)} Y_i.$$

It is clear that we obtain $g(Y_j, Y_i) = 0$ for $1 \leq i \leq j-1$. Additionally we provide

$$\begin{aligned}
|g(Y_j, Y_j)| &= \left| g(X_j, X_j) - \sum_{i=1}^{j-1} \frac{(g(Y_i, X_j))^2}{g(Y_i, Y_i)} \right| \\
&\geq |g(X_j, X_j)| - \sum_{i=1}^{j-1} \frac{|g(Y_i, X_j)|^2}{|g(Y_i, Y_i)|} > a - n\frac{c^2}{d} = d, \\
|g(Y_j, X_k)| &= \left| g(X_j, X_k) - \sum_{i=1}^{j-1} \frac{g(Y_i, X_j) g(Y_i, X_k)}{g(Y_i, Y_i)} \right| \\
&\leq |g(X_j, X_k)| + \sum_{i=1}^{j-1} \frac{|g(Y_i, X_j)||g(Y_i, X_k)|}{|g(Y_i, Y_i)|} < b + n\frac{c^2}{d} = c,
\end{aligned}$$

for all $1 \leq j < k \leq n$, which is what we wanted to show. Since $\|Y_i\| = \sqrt{|g(Y_i, Y_i)|} \in \mathfrak{F}(W)$ is non-vanishing we can set $E_i = Y_i/\|Y_i\| \in \mathfrak{X}(W)$ for each $1 \leq i \leq n$, and get a desired orthonormal local frame $(E_1, \dots, E_n)$ over $W \ni p$. □

It is worth noting that the constant index condition in the definition of a metric tensor is not too strong. Namely, since we did not use this condition in the proof of the previous theorem, where we get continuous functions $g(E_i, E_i) \in \mathfrak{F}(W)$ which must be constant (1 or -1), we can conclude that the index is constant on each connected component of the manifold.

4.4 Pullback of metric tensors

When we build new manifolds from the old ones, there is often a corresponding way to derive a metric on the new manifolds from the metric on the old. This is why we need the pullback of a covariant tensor field. Let $f\colon M \to N$ be a smooth map between a manifold M and a pseudo-Riemannian manifold (N, g). The pullback of $g \in \mathfrak{T}^0_2(N)$ is $f^*g \in \mathfrak{T}^0_2(M)$ such that at each point $p \in M$ for $X_p, Y_p \in T_pM$ we have

$$(f^*g)_p(X_p, Y_p) = g_{f(p)}(f_*(X_p), f_*(Y_p)),$$

and since g is symmetric then f^*g is also symmetric. However, if f is not an immersion then f_* is not injective and for some point $p \in M$ there exists $0 \neq X_p \in T_pM$ such that $f_*(X_p) = 0$, which follows $(f^*g)_p(X_p, Y_p) = 0$ for all $Y_p \in T_pM$, that is, $(f^*g)_p$ is not nondegenerate. Because of that, f^*g can be a metric only if f is an immersion, but in general this is not enough.

Example 4.10. Consider the unit circle $\mathbf{S}^1$ that is embedded into $\mathbb{R}^2_1$ by the natural embedding $f(\theta) = (\cos\theta, \sin\theta)$, where θ is a local coordinate. The pullback of the metric tensor is $f^*g = f^*(-dx^2 + dy^2) = -(-\sin\theta\, d\theta)^2 + (\cos\theta\, d\theta)^2 = \cos 2\theta\, d\theta^2$, which is degenerate at $\theta \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$, where it also changes the index. △

However, in the Riemannian case it is also the sufficient condition. If f_* is injective, then positive definite g implies that f^*g is positive definite, and thus f^*g is a Riemannian metric.

Theorem 4.14. *If (N, g) is a Riemannian manifold and $f\colon M \to N$ is an immersion, then (M, f^*g) is a Riemannian manifold.*

Let P be a submanifold of a pseudo-Riemannian manifold (M, g) where $\imath\colon P \hookrightarrow M$ is the appropriate inclusion. If $\imath^*g$ is a metric on P then we say that $(P, \imath^*g)$ is a ***pseudo-Riemannian submanifold*** of (M, g). Each tangent space T_pP is regarded as a subspace of T_pM, but there is no specific reason why it would have to be nondegenerate relative to g_p. Moreover, no one guarantees that if this is the case, we have the constancy of $\mathrm{Ind}(g_p)$.

Thanks to Theorem 4.14, things are much simpler in the Riemannian case. If P is a submanifold of a Riemannian manifold (M, g) with the inclusion $\imath\colon P \hookrightarrow M$, then $(P, \imath^*g)$ is a ***Riemannian submanifold*** of (M, g). Since at each point $p \in P$ holds $T_pP \leq T_pM$, the Riemannian metric $\imath^*g$ on P is obtained merely by applying the metric tensor g.

If we know the coordinate representation for an immersion, then the induced Riemannian metric is easy to compute. In some chart with the coordinate functions x_i holds

$$f^*g = f^*\left(\sum_{i,j} g_{ij}\, dx_i\, dx_j\right) = \sum_{i,j} f^*(g_{ij}) f^*(dx_i) f^*(dx_j) = \sum_{i,j} (g_{ij} \circ f)\, d(x_i \circ f)\, d(x_j \circ f),$$

which we apply in the following concrete examples.

Example 4.11. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ is given by $f(u, v) = (u\cos v, u\sin v, v)$, which is an immersion whose image is a helicoid. The induced metric $f^*\overline{g}$ we calculate by

$$\begin{aligned} f^*\overline{g} &= f^*(dx^2 + dy^2 + dz^2) = d(u\cos v)^2 + d(u\sin v)^2 + d(v)^2 \\ &= (\cos v\, du - u\sin v\, dv)^2 + (\sin v\, du + u\cos v\, dv)^2 + dv^2 = du^2 + (u^2+1)\, dv^2. \end{aligned}$$

△

To transform a Riemannian metric under a change of coordinates, we consider the identity map expressed in terms of different coordinates for the domain and codomain.

Example 4.12. The Euclidean metric $\bar{g} = dx^2 + dy^2$ on $\mathbb{R}^2$ in polar coordinates can be calculated as the pullback of the identity map. Because of $x = r\cos\theta$ and $y = r\sin\theta$ we have

$$\begin{aligned}\bar{g} &= dx^2 + dy^2 = d(r\cos\theta)^2 + d(r\sin\theta)^2\\ &= (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2 = dr^2 + r^2 d\theta^2.\end{aligned}$$

△

A Riemannian metric demands positive definiteness and it is the most commonly studied metric. One nice feature of Riemannian metrics is that there are a lot of them.

Theorem 4.15. *Every manifold admits a Riemannian metric.*

Proof. Let M be a manifold with a smooth atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$. By Theorem 1.20 there exists a partition of unity $(\psi_\alpha)_{\alpha\in\Lambda}$ subordinate to an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$. In each chart there is a Riemannian metric $g_\alpha = \varphi_\alpha^*\bar{g}$ induced by the standard Euclidean metric from (4.5). By local finiteness, there are only finitely many nonzero terms in a neighbourhood of each point, and therefore $g = \sum_{\alpha\in\Lambda}\psi_\alpha g_\alpha$ defines a symmetric covariant tensor field of order two. For $0 \neq X \in T_pM$ we have $g_p(X,X) = \sum_{\alpha\in\Lambda}\psi_\alpha(p)(g_\alpha)_p(X,X)$. Since each term is nonnegative, the sum is nonnegative, and since $\sum_{\alpha\in\Lambda}\psi_\alpha(p) = 1$, then at least one of $\alpha \in \Lambda$ has $\psi_\alpha(p) > 0$. Thus $g_p(X,X) > 0$ and g is a positive definite metric. □

It is important to note that there is a lot of choice in the construction of a metric g for a given manifold. In particular, various metrics on the same manifold can have highly different geometric properties. However, the same procedure does not work in an indefinite case, although we can construct a metric of index ν on each coordinate neighbourhood for any $0 < \nu < n$. The reason is that the sum of two metrics of index ν may be degenerate, and is not necessarily a metric.

Example 4.13. Let (M, g_M) and (N, g_N) be pseudo-Riemannian manifolds, and consider the natural projections $\pi_M\colon M\times N \to M$ and $\pi_N\colon M\times N\to N$. On the product manifold $M\times N$ we can define the ***product metric*** by $g_M \times g_N = \pi_M^*(g_M) + \pi_N^*(g_N)$. If we use the natural isomorphism $T_{(p,q)}(M\times N) \cong T_pM\times T_qN$ from Example 2.2 then the associated Gram matrix is a block diagonal matrix where the blocks are individual Gram matrices. The pseudo-Riemannian manifold $(M\times N, g_M\times g_N)$ is called the ***pseudo-Riemannian product*** of (M, g_M) and (N, g_N). △

Example 4.14. More generally than the previous example, for any strictly positive function $f \in \mathfrak{F}(M)$ we can set

$$g_M \times_f g_N = \pi_M^*(g_M) + (f\circ\pi_M)\pi_N^*(g_N).$$

This time, the Gram matrix at a point (p,q) is a block diagonal matrix where the first block is the Gram matrix for g_M at p, and the other is the Gram matrix of g_N at q multiplied by the positive constant $c = f(p) > 0$. Symmetry and nondegeneracy are obvious, while from $\mathrm{Ind}(cg_N) = \mathrm{Ind}(g_N)$ follows $\mathrm{Ind}(g) = \mathrm{Ind}(g_M) + \mathrm{Ind}(g_N)$. The pseudo-Riemannian manifold $M\times_f N = (M\times N, g_M\times_f g_N)$ is called the ***warped product***, while in the special case $f = 1$ we have the standard pseudo-Riemannian product. △

Let (M, g_M) and (N, g_N) be two pseudo-Riemannian manifolds. A smooth map $f\colon M\to N$ between the underlying manifolds is a ***pseudo-Riemannian immersion*** if it preserves the metric tensors, $f^*(g_N) = g_M$, which can be written explicitly by

$$(g_M)_p(X_p, Y_p) = (g_N)_{f(p)}(T_pf(X_p), T_pf(Y_p))$$

for all $p \in M$ and $X_p, Y_p \in T_pM$. Any such map is an immersion, which justifies the name and yields the inequalities $\dim M \leq \dim N$ and $\operatorname{Ind} g_M \leq \operatorname{Ind} g_N$.

An ***isometry*** from (M, g_M) to (N, g_N) is a pseudo-Riemannian immersion $f\colon M \to N$ which is also a diffeomorphism. We say that pseudo-Riemannian manifolds are ***isometric*** if there exist an isometry between them. If a pseudo-Riemannian immersion f is only a local diffeomorphism then it is called a ***local isometry***. Since such f is already an immersion, for a local isometry by Theorem 2.9 we additionally only have that $\dim M = \dim N$.

It is easy to see that the composition of isometries and the inverse of an isometry are isometries, as well as the identity map. Thus, being isometric is an equivalence relation, so we can say that an isometry is a special kind of map that provides a notion of isomorphism in the category of pseudo-Riemannian manifolds. ***Pseudo-Riemannian geometry*** is the study of properties of pseudo-Riemannian manifolds that are invariant under local or global isometries.

For a fixed pseudo-Riemannian manifold M, an isometry $f\colon M \to M$ is called an ***isometry of*** M. The set of all isometries of M is a group $\mathcal{I}(M)$, called the ***isometry group*** of M. An important deep theorem asserts that the isometry group $\mathcal{I}(M)$ of a pseudo-Riemannian manifold M has the structure of a Lie group with respect to the compact-open topology [80, Corollary 2]. For Riemannian manifolds this has been established by Myers[8] and Steenrod in 1939 [87].

Example 4.15. Let $f\colon \mathbf{S}^1 \to \mathbf{S}^1$ is given by $f(z) = z^2$, where $\imath\colon \mathbf{S}^1 \hookrightarrow \mathbb{C}$, so $\mathbf{S}^1$ has a Riemannian metric $g = \imath^*\bar{g}$. Since f is an immersion, f^*g is a Riemannian metric by Theorem 4.14. Thus, f is a Riemannian immersion, but it is not an isometry because f is not a diffeomorphism. △

Example 4.16. Let $(\mathcal{V}, g)$ be a scalar product space of dimension n. By choosing a basis on $\mathcal{V}$ we induce a bijective linear map which is a homeomorphism between $\mathcal{V}$ and $\mathbb{R}^n$, so $\mathcal{V}$ is a manifold. There exists a canonical linear isomorphism $X \mapsto X_Z$ from $\mathcal{V}$ onto each tangent space $T_Z\mathcal{V}$ given by the directional derivative

$$X_Z h = \left.\frac{d}{dt}\right|_{t=0} h(Z + tX),$$

where $g_Z(X_Z, Y_Z) = g(X, Y)$ gives rise to a metric tensor on the manifold $\mathcal{V}$, making it a pseudo-Riemannian manifold. If $f\colon \mathcal{V} \to \mathcal{W}$ is a linear isometry of scalar product spaces $(\mathcal{V}, g^{\mathcal{V}})$ and $(\mathcal{W}, g^{\mathcal{W}})$, then

$$T_Z f(X_Z)(h) = X_Z(h \circ f) = \left.\frac{d}{dt}\right|_{t=0} h(f(Z) + tf(X)) = f(X)_{f(Z)} h,$$

which gives $T_Z f(X_Z) = f(X)_{f(Z)}$, and hence

$$\begin{aligned}(f^* g^{\mathcal{W}})_Z(X_Z, Y_Z) &= g^{\mathcal{W}}_{f(Z)}(f_* X_Z, f_* Y_Z) = g^{\mathcal{W}}_{f(Z)}(f(X)_{f(Z)}, f(Y)_{f(Z)}) \\ &= g^{\mathcal{W}}(f(X), f(Y)) = g^{\mathcal{V}}(X, Y) = g^{\mathcal{V}}_Z(X_Z, Y_Z),\end{aligned}$$

so f is a pseudo-Riemannian immersion. Since linear maps are smooth, the linear isomorphism f is a diffeomorphism, and therefore an isometry between pseudo-Riemannian manifolds $\mathcal{V}$ and $\mathcal{W}$. It follows that if $\mathcal{V}$ is a scalar product space of dimension n and index ν, then as a pseudo-Riemannian manifold, $\mathcal{V}$ is isometric to $\mathbb{R}^n_\nu$. In fact, the coordinate isomorphism of any orthonormal basis in $\mathcal{V}$ is a isometry. △

[8] Sumner Byron Myers (1910–1955), American mathematician

4.5 Musical isomorphisms

A pseudo-Riemannian metric induces an isomorphism between vector and covector fields. For a given pseudo-Riemannian manifold (M,g) we can define the map $\flat\colon \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ called the ***flat***, that a vector field $X \in \mathfrak{X}(M)$ maps to a covector field $X^\flat \in \mathfrak{X}^*(M)$ defined by $X^\flat(Y) = g(X,Y)$ for each $Y \in \mathfrak{X}(M)$.

For an arbitrary $p \in M$ we can restrict the flat and get $\flat\colon TM \to T^*M$, or $\flat_p\colon T_pM \to T_p^*M$, that a vector $X_p \in T_pM$ maps to a covector $X_p^\flat \in T_p^*M$ given by $X_p^\flat(Y_p) = g_p(X_p, Y_p)$ for each $Y_p \in T_pM$.

Note that the nondegeneracy condition for a symmetric bilinear form $g_p \in \mathfrak{T}_2^0(T_pM)$ is equivalent to $\operatorname{Ker}(\flat_p) = 0$, or that $\flat_p$ is injective. As we deal with finite dimensions only, $\dim T_pM = \dim T_p^*M = \dim M$, which is equivalent to the condition that $\flat_p$ is an isomorphism. A metric is always nondegenerate, so the flat is an isomorphism between corresponding tangent and cotangent spaces, that is, between vector and covector fields.

Let us see what is happening in a chart (U,φ) with the coordinate functions $x_i = \pi_i \circ \varphi$. If $X = \sum_{i=1}^n X^i \partial_i$, then we have

$$X^\flat = \sum_{j=1}^n X^\flat\left(\frac{\partial}{\partial x_j}\right) dx_j = \sum_{j=1}^n g\left(\sum_{i=1}^n X^i \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) dx_j = \sum_{i,j=1}^n g_{ij} X^i \, dx_j = \sum_{j=1}^n X_j \, dx_j,$$

where $X_j = \sum_{i=1}^n g_{ij} X^i$. We can say that $X^\flat$ is obtained from X by lowering an index, which is why we call the operation the flat. As we can see, the matrix of flat in terms of coordinate basis is actually the Gram matrix of g.

On the other hand, the inverse isomorphism $\sharp = \flat^{-1}\colon \mathfrak{X}^*(M) \to \mathfrak{X}(M)$ we call the ***sharp***. The sharp, a covector field $\omega \in \mathfrak{X}^*(M)$ maps to a vector field $\omega^\sharp \in \mathfrak{X}(M)$ in such a way that $\omega(X) = g(\omega^\sharp, X)$ holds for each $X \in \mathfrak{X}(M)$. In coordinates, the matrix of sharp must be the inverse matrix of flat.

Since g is nondegenerate, the Gram matrix with entries g_{ij} is invertible at each point, so there exists its inverse matrix with entries g^{jk}, where $\sum_{j=1}^n g_{ij} g^{jk} = \delta_{ik}$. The components of the inverse matrix smoothly depend of the initial components, so the functions g^{ij} are smooth on U. Because g is symmetric, we have

$$g^{jk} = \sum_{i=1}^n g^{ik} \delta_{ij} = \sum_{i,l=1}^n g^{ik}(g_{il} g^{lj}) = \sum_{i,l=1}^n (g^{ik} g_{li}) g^{lj} = \sum_{l=1}^n \delta_{kl} g^{lj} = g^{kj},$$

so the inverse is also symmetric, $g^{jk} = g^{kj}$. Now for $\omega = \sum_{i=1}^n \omega_i \, dx_i$ we have $\omega^\sharp = \sum_{j=1}^n \omega^j \partial_j$, where $\omega^j = \sum_{i=1}^n g^{ij} \omega_i$ and we say that $\omega^\sharp$ is obtained from ω by raising an index.

Maps $\flat\colon \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ and $\sharp\colon \mathfrak{X}^*(M) \to \mathfrak{X}(M)$ are the ***musical isomorphisms***, a nice name propagated (and probably named) by Berger[9] [17].

Probably, the most important application of the sharp is an extension of the classical gradient to pseudo-Riemannian manifolds. If (M,g) is a pseudo-Riemannian manifold and $f \in \mathfrak{F}(M)$, the ***gradient*** of f is a vector field $\operatorname{grad} f = df^\sharp \in \mathfrak{X}(M)$ obtained from the differential $df \in \mathfrak{X}^*(M)$ by raising an index. The gradient can be expressed by the formula

$$g(\operatorname{grad} f, X) = df(X) = Xf,$$

which holds for all $X \in \mathfrak{X}(M)$, while in local coordinates we have

$$\operatorname{grad} f = df^\sharp = \sum_{j=1}^n \left(\sum_{i=1}^n g^{ij} \, df\left(\frac{\partial}{\partial x_i}\right)\right) \frac{\partial}{\partial x_j} = \sum_{i,j=1}^n g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}.$$

[9] Marcel Berger (1927–2016), French mathematician

Especially, for natural coordinates on pseudo-Euclidean space $\mathbb{R}^n_\nu$, the coordinate frame is orthonormal, and the previous formula is reduced to $\operatorname{grad} f = \sum_{i=1}^n \varepsilon_i \partial_i f \partial_i$, which gives the usual formula on Euclidean space where the gradient has the same components as df.

The flat and the sharp can be applied to tensor of arbitrary type and at any slot. In this way, tensor can be transferred from contravariant to covariant and vice versa. For example, in the tensor $A \in \mathfrak{T}^1_2(M)$ we can lower an upper index to obtain the covariant tensor $A^\flat \in \mathfrak{T}^0_3(M)$ with components $A_{ijk} = (A^\flat)_{ijk} = \sum_{l=1}^n g_{il} A^l_{jk}$.

In a general case, the flat can pass to the corresponding argument since from the equality (3.6) we have

$$\begin{aligned} A^\flat(\dots, X, \dots) &= \sum_i \sum \cdots X^i \cdots (A^\flat)^{\cdots}_{\dots i} = \sum_i \sum \cdots X^i \cdots \sum_j g_{ji} A^{\dots j}_{\dots} \\ A(\dots, X^\flat, \dots) &= \sum_j \sum \cdots (X^\flat)_j \cdots A^{\dots j}_{\dots} = \sum_j \sum \cdots \sum_i g_{ji} X^i \cdots A^{\dots j}_{\dots}, \end{aligned}$$

which also applies to the sharp. Concretely, $A^\flat(X, Y, Z) = A(X^\flat, Y, Z)$ holds for $A \in \mathfrak{T}^1_2(M)$ and $X, Y, Z \in \mathfrak{X}(M)$.

Another important implication of the flat and sharp is the extension of contraction. For example, for a symmetric covariant tensor $A \in \mathfrak{T}^0_2(M)$ we can raise one index (it does not matter which since A is symmetric) to get $A^\sharp \in \mathfrak{T}^1_1(M)$ and then apply the contraction to obtain $\operatorname{tr}_g(A) = C(A^\sharp) \in \mathfrak{F}(M)$, which is the ***trace*** of A related to g. In coordinates this is $\operatorname{tr}_g(A) = C(A^\sharp) = \sum_{i=1}^n A_i{}^i = \sum_{i,j=1}^n g^{ij} A_{ij}$, which in the case of an orthonormal basis becomes the ordinary trace of the matrix.

Another application of the musical isomorphisms allows us to investigate conditions under which some n-manifold admits a metric of fixed index $0 < \nu < n$.

Theorem 4.16. *A manifold admits a metric of index ν if and only if it admits a ν-dimensional distribution.*

Proof. Let (M, g) be a pseudo-Riemannian n-manifold of index $0 < \nu < n$. We can choose a Riemannian metric $\bar{g}$ on M and create $T \in \mathfrak{T}^1_1(M)$ by $T(X) = (X^\flat)^{\bar{\sharp}}$ for $X \in \mathfrak{X}(M)$, where $\bar{\sharp}$ is the sharp with respect to $\bar{g}$. In other words, we construct a linear operator $T\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that $g(X, Y) = \bar{g}(T(X), Y)$ holds for $X, Y \in \mathfrak{X}(M)$. The matrix of T with respect to $\bar{g}$ coincides with the Gram matrix of g, which gives the characteristic polynomial $\det(\lambda \mathbb{1} - T) = (\lambda - 1)^{n-\nu}(\lambda + 1)^\nu$. Thus, $T + \mathbb{1}$ is a tangent bundle homomorphism over M of constant rank $n - \nu$ which, according to Theorem 3.3, yields a ν-dimensional distribution $\operatorname{Ker}(T + \mathbb{1})$ on M. Moreover, we have $TM = \operatorname{Ker}(T + \mathbb{1}) \oplus \operatorname{Ker}(T - \mathbb{1})$.

On the other hand, the existence of a ν-dimensional distribution D on a Riemannian manifold (M, g) implies the existence of its complementary distribution $D^\perp$ such that we have $TM = D \oplus D^\perp$ and $T_pM = D_p \oplus D^\perp_p$ for $p \in M$. Then we can split the metric into the direct sum of bundle metrics $g = g_D \oplus g_{D^\perp}$, where g_D and $g_{D^\perp}$ are the corresponding restrictions, which allows us to create a metric $(-g_D) \oplus g_{D^\perp}$ of index ν. □

Although we obtain a one-dimensional distribution directly from a non-vanishing vector field on a manifold, the converse is not obvious, but their existences are closely related. Algebraic topology techniques can show that the condition for the existence of a non-vanishing vector field on a manifold M is equivalent to M being either noncompact or having the Euler[10] characteristic $\chi(M) = 0$, which is also equivalent to the existence of a one-dimensional distribution on M (see O'Neill [96, Proposition 5.37]). Therefore, any noncompact manifold admits a Lorentzian metric, while torus and Klein bottle are only compact two-dimensional manifolds with this property. A concrete construction of a Lorentzian metric from a vector field we can see in the following example.

[10] Leonhard Euler (1707–1783), Swiss mathematician

Example 4.17. Let us suppose that there exists a non-vanishing vector field $X \in \mathfrak{X}(M)$ on a manifold M. By Theorem 4.15 we can pick a Riemannian metric g on M. Consider a symmetric covariant tensor g_L of order two defined by

$$g_L = g - \frac{2}{g(X,X)} X^\flat \otimes X^\flat.$$

It is clear that $g_L(X,X) = -g(X,X) < 0$, while $g(X,Y) = 0$ implies $g_L(Y,Z) = g(Y,Z)$. Thus, an orthogonal basis for g that includes X is also an orthogonal basis for g_L which proves that g_L is a Lorentzian metric. △

4.6 Model spaces

The development of Riemannian and pseudo-Riemannian geometry has been very affected by certain highly symmetric spaces called model spaces. They are very good examples, so we can compare more abstract pseudo-Riemannian manifolds with these more simple spaces and look for common features. The main feature of model spaces is that they are highly symmetric, which means that they have a large group of isometries.

The most important application of Lie groups in geometry involves their action on manifolds. A (left) ***group action*** of a group G on a manifold M is a map $\theta\colon G \times M \to M$, with $\theta(h,p)$ often shortened to $h \cdot p$, that satisfies $h_1 \cdot (h_2 \cdot p) = (h_1 h_2) \cdot p$ and $e \cdot p = p$ for all $h_1, h_2 \in G$ and $p \in M$, where e is the identity element of G. Given an action of G on M, for every $p \in M$ the ***isotropy group*** (or ***stabilizer subgroup***) of G at p is the subgroup $G_p = \{h \in G : h \cdot p = p\}$ of G consisting of all elements that fix p. The action of G on M is ***free*** if all G_p are trivial, which means that $h \cdot p = p$ implies $h = e$. The ***orbit*** of a point $p \in M$ is the set $G \cdot p = \{h \cdot p : h \in G\}$ of points in M to which p can be moved by the elements of G. The action is ***transitive*** if it has exactly one orbit, which means that for every pair of points $p, q \in M$ there exists $h \in G$ such that $h \cdot p = q$.

If G is a Lie group, an action $\theta\colon G \times M \to M$ is ***smooth action*** if θ is smooth. In this case, for each $h \in G$, a map $\theta_h\colon M \to M$ given by $\theta_h(p) = h \cdot p$ is a diffeomorphism, because θ_h is smooth as well as its inverse $\theta_{h^{-1}}$.

To describe the symmetries of a pseudo-Riemannian manifold (M,g) we start from its isometry group $\mathcal{I}(M)$ which naturally define an action on M by $f \cdot p = f(p)$ for $f \in \mathcal{I}(M)$ and $p \in M$. Moreover, it is worth noting that for any $f \in \mathcal{I}(M)$, the global tangent map Tf maps TM to itself and restricts to a linear isometry $T_p f\colon T_p M \to T_{f(p)} M$ for each $p \in M$. A pseudo-Riemannian manifold M is ***homogeneous*** if the isometry group $\mathcal{I}(M)$ acts transitively on M, which means that for any two points $p, q \in M$ there exists an isometry $f \in \mathcal{I}(M)$ such that $f(p) = q$.

The ***isotropy group*** of $\mathcal{I}(M)$ at a point $p \in M$ is its subgroup $\mathcal{I}_p(M)$ that contains all $f \in \mathcal{I}(M)$ such that $f(p) = p$. For $p \in M$ and $f \in \mathcal{I}_p(M)$, the linear map $T_p f$ takes $T_p M$ to itself, so we obtain the ***isotropy representation*** $I_p\colon \mathcal{I}_p(M) \to \mathrm{GL}(T_p M)$ given by $I_p(f) = T_p f$. We say that M is ***isotropic at*** $p \in M$ if the isotropy representation of $\mathcal{I}_p(M)$ acts transitively on the set of unit vectors in $T_p M$, which means that for any unit vectors $X, Y \in T_p M$ there is an isometry $f \in \mathcal{I}_p(M)$ such that $f_*(X) = Y$. A pseudo-Riemannian manifold is ***isotropic*** if it is isotropic at every point.

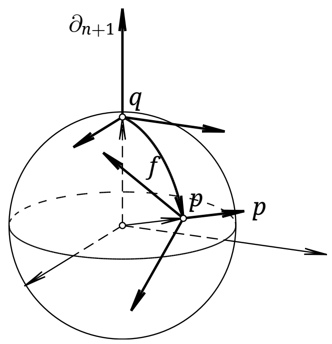
A homogeneous pseudo-Riemannian manifold looks geometrically the same when viewed from any point, while an isotropic one looks the same in every direction. Of course, a homogeneous pseudo-Riemannian manifold that is isotropic at one point is isotropic at every point. However, it turns out that an isotropic pseudo-Riemannian manifold is homogeneous (see Theorem 7.11).

The three model spaces of Riemannian manifolds are Euclidean spaces, spheres, and hyperbolic spaces. The basic and most important model Riemannian manifold is the Euclidean space $\mathbb{R}^n$ with the Euclidean metric $\overline{g}$ from (4.5) (Example 4.6). Let us remark that

Example 4.16 showed that any scalar product space of index 0 is isometric to $(\mathbb{R}^n, \overline{g})$. It is easy to construct isometries of the Riemannian manifold $(\mathbb{R}^n, \overline{g})$, since every orthogonal linear transformation on $\mathbb{R}^n$ preserves the Euclidean metric as does every translation.

The next model space is the sphere $\mathbf{S}^n$ whose inclusion $\imath\colon \mathbf{S}^n \hookrightarrow \mathbb{R}^{n+1}$ induces the Riemannian metric $\mathring{g} = \imath^*\overline{g}$ on $\mathbf{S}^n$ called the ***round metric***. The sphere $(\mathbf{S}^n, \mathring{g})$ is a Riemannian manifold that is homogeneous and isotropic.

Example 4.18. The linear action of the orthogonal group $\mathrm{O}(n+1)$ on $\mathbb{R}^{n+1}$ preserves both $\mathbf{S}^n$ and the Euclidean metric, and therefore its restriction to $\mathbf{S}^n$ acts isometrically on it. Consider the north pole $q(0,\ldots,0,1) \in \mathbf{S}^n$ with the standard basis $(\partial_1,\ldots,\partial_n)$ in $T_q\mathbf{S}^n$. Let $p \in \mathbf{S}^n$ be an arbitrary point with an orthonormal basis $(E_1,\ldots,E_n)$ in $T_p\mathbf{S}^n$. Since these basis vectors are tangent to the sphere, they are orthogonal to p, so $(E_1,\ldots,E_n,p)$ is an orthonormal basis in $T_p\mathbb{R}^{n+1}$. If $f \in O(n+1)$ is the matrix whose columns are these basis vectors, then $f\colon (\partial_1,\ldots,\partial_{n+1}) \mapsto (E_1,\ldots,E_n,p)$, and in particular $f(q) = f(\partial_{n+1}) = p$. Moreover, since f acts linearly on $\mathbb{R}^{n+1}$, its tangent map $T_qf\colon T_q\mathbb{R}^{n+1} \to T_p\mathbb{R}^{n+1}$ in standard coordinates has a representation which is given by the same matrix as f, and therefore we obtain $f_*\partial_i = E_i$ for $1 \le i \le n$.



Thus, we find an isometry f that takes q to p, while pushes the orthonormal basis at q forward to a chosen orthonormal basis at p, which proves that $\mathbf{S}^n$ is homogeneous and isotropic. △

A smooth map $f\colon M \to N$ between pseudo-Riemannian manifolds (M, g_M) and (N, g_N) is ***conformal*** if $f^*(g_N) = hg_M$ for some positive function $h \in \mathfrak{F}(M)$. If there exists a conformal diffeomorphism between pseudo-Riemannian manifolds we say that they are ***conformally equivalent***. Two metrics g_1 and g_2 on a manifold M are ***conformal*** if there is a positive function $h \in \mathfrak{F}(M)$ such that $g_2 = hg_1$, that is, if the identity map between (M, g_1) and (M, g_2) is conformal. In Riemannian geometry it is known that two metrics are conformal if and only if they define the same angles, while a diffeomorphism is conformal if and only if it preserves angles.

Example 4.19. The sphere is locally conformally equivalent to Euclidean space, while the stereographic projection is a conformal diffeomorphism between $\mathbf{S}^n \setminus \{(0,\ldots,0,1)\}$ and $\mathbb{R}^n$. Consider the stereographic projection $\varphi = \varphi_-$ from Example 1.15 given by

$$\varphi(x_1,\ldots,x_n,x_{n+1}) = \frac{1}{1-x_{n+1}}(x_1,\ldots,x_n), \tag{4.7}$$

with the inverse

$$\varphi^{-1}(y_1,\ldots,y_n) = \frac{1}{1+y_1^2+\cdots+y_n^2}(2y_1,\ldots,2y_n,-1+y_1^2+\cdots+y_n^2). \tag{4.8}$$

For an arbitrary point $s \in \mathbb{R}^n$ and a vector $V \in T_s\mathbb{R}^n$, from

$$\varphi_*^{-1}V = \sum_{i=1}^{n+1}(\varphi_*^{-1}V)(x_i)\frac{\partial}{\partial x_i} = \sum_{i=1}^{n+1} V(x_i \circ \varphi^{-1})\frac{\partial}{\partial x_i},$$

we can compute the pullback metric,

$$(\varphi^{-1})^*\mathring{g}(V,V) = \mathring{g}(\varphi_*^{-1}V, \varphi_*^{-1}V) = \bar{g}(\varphi_*^{-1}V, \varphi_*^{-1}V) = \sum_{i=1}^{n+1}(V(x_i \circ \varphi^{-1}))^2,$$

where

$$V(x_i \circ \varphi^{-1}) = V\left(\frac{2y_i}{1+y_1^2+\cdots+y_n^2}\right) = 2\frac{(1+y_1^2+\cdots+y_n^2)V(y_i) - y_iV(y_1^2+\cdots+y_n^2)}{(1+y_1^2+\cdots+y_n^2)^2}$$

for $1 \le i \le n$, with additional

$$V(x_{n+1} \circ \varphi^{-1}) = V\left(\frac{-1+y_1^2+\cdots+y_n^2}{1+y_1^2+\cdots+y_n^2}\right) = 2\frac{V(y_1^2+\cdots+y_n^2)}{(1+y_1^2+\cdots+y_n^2)^2}.$$

Hence, because of $V(y_1^2+\cdots+y_n^2) = \sum_{i=1}^n 2y_iV(y_i)$, we have

$$\begin{aligned}(\varphi^{-1})^*\mathring{g}(V,V) =& \frac{4}{(1+y_1^2+\cdots+y_n^2)^4}\Big((V(y_1^2+\cdots+y_n^2))^2 \\ &+ \sum_{i=1}^n((1+y_1^2+\cdots+y_n^2)V(y_i) - y_iV(y_1^2+\cdots+y_n^2))^2\Big) \\ =& 4\frac{(V(y_1))^2+\cdots+(V(y_n))^2}{(1+y_1^2+\cdots+y_n^2)^2} = \frac{4}{(1+y_1^2+\cdots+y_n^2)^2}\bar{g}(V,V).\end{aligned}$$

In other words,

$$(\varphi^{-1})^*\mathring{g} = \frac{4}{(1+y_1^2+\cdots+y_n^2)^2}\bar{g},$$

where $\bar{g}$ represents the Euclidean metric on $\mathbb{R}^n$, and therefore this stereographic projection φ is a conformal diffeomorphism. It follows that the sphere is ***locally conformally flat***, which means that each point of $\mathbf{S}^n$ has a neighbourhood that is conformally equivalent to an open subset in $\mathbb{R}^n$. △

The third class of model Riemannian manifolds consists of the hyperbolic space $\mathbf{H}^n$ defined by

$$\mathbf{H}^n = \{X = (x_0, \dots, x_n) \in \mathbb{R}_1^{n+1} : g(X,X) = -1,\ x_0 > 0\}$$

as a pseudo-Riemannian submanifold of the Minkowski space $\mathbb{R}_1^{n+1} = (\mathbb{R}^{n+1}, g)$. Geometrically, this is the upper sheet of the two-sheeted hyperboloid given by the equation $x_0^2 - x_1^2 - \cdots - x_n^2 = 1$. Consider the smooth function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $f(X) = g(X,X)$. Because of

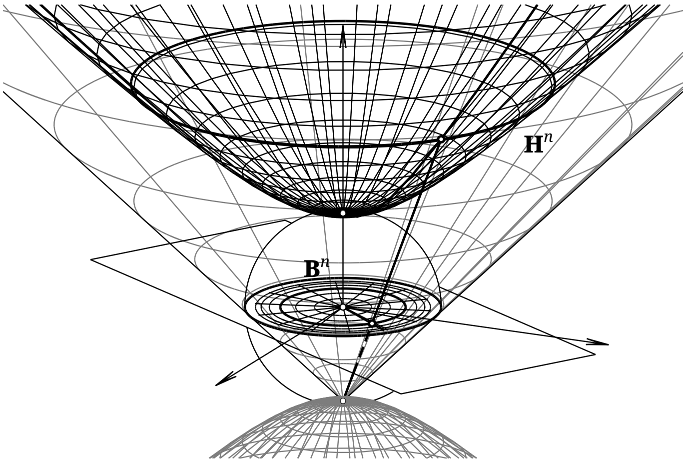
$$df_X(Y_X) = Y_Xf = \left.\frac{d}{dt}\right|_{t=0} f(X+tY) = \left.\frac{d}{dt}\right|_{t=0}(f(X) + 2tg(X,Y) + t^2f(Y)) = 2g(X,Y) = 2g_X(X_X, Y_X),$$

we have $df_X = 2(X_X)^\flat$, so T_Xf has rank 1 for any $X \neq 0$, and therefore -1 is a regular value of f. By Theorem 2.15, the regular level set $f^{-1}(-1)$ is a submanifold of codimension 1, which also holds for its open subset $\mathbf{H}^n$. Moreover, by Lemma 2.16 for each $X \in \mathbf{H}^n$ we have $T_X\mathbf{H}^n = \operatorname{Ker} T_Xf = \operatorname{Ker} X^\flat = X^\perp$. Because of $g(X,X) = -1 < 0$, the restriction of the ambient metric to $X^\perp$ is positive definite, so g induces the Riemannian metric $h = \imath^*g$ on $\mathbf{H}^n$.

Example 4.20. Consider the ***hyperbolic stereographic projection*** π from $\mathbf{H}^n$ to the hyperplane $x_0 = 0$ using the point $(-1, 0, \dots, 0)$. From the formula

$$\pi(x_0, x_1, \dots, x_n) = \frac{1}{1+x_0}(x_1, \dots, x_n),$$

we see that $\|\pi(x_0, x_1, \dots, x_n)\| = \sqrt{(x_0 - 1)/(x_0 + 1)} \nearrow 1$ when $x_0 \to +\infty$, so $\mathrm{Im}(\pi) = \mathbf{B}^n_r$ is the unit ball in $\mathbb{R}^n$. Thus, the diffeomorphism $\pi \colon \mathbf{H}^n \to \mathbf{B}^n$ give rise to another model called the ***Poincaré ball model*** $\mathbf{HB}^n = (\mathbf{B}^n, (\pi^{-1})^* h)$ where h is the metric on $\mathbf{H}^n$.



After we calculate the inverse,

$$\pi^{-1}(y_1, \dots, y_n) = \frac{1}{1 - y_1^2 - \dots - y_n^2}(1 + y_1^2 + \dots + y_n^2, 2y_1, \dots, 2y_n),$$

as before, we consider an arbitrary point $s \in \mathbf{B}^n$ and a vector $V \in T_s\mathbf{B}^n$. From

$$\pi_*^{-1} V = \sum_{i=0}^{n} (\pi_*^{-1} V)(x_i) \frac{\partial}{\partial x_i} = \sum_{i=0}^{n} V(x_i \circ \pi^{-1}) \frac{\partial}{\partial x_i},$$

we can compute,

$$(\pi^{-1})^* h(V, V) = h(\pi_*^{-1} V, \pi_*^{-1} V) = g(\pi_*^{-1} V, \pi_*^{-1} V) = -(V(x_0 \circ \pi^{-1}))^2 + \sum_{i=1}^{n} (V(x_i \circ \pi^{-1}))^2,$$

where

$$V(x_i \circ \pi^{-1}) = V\left(\frac{2y_i}{1 - y_1^2 - \dots - y_n^2}\right) = 2\frac{(1 - y_1^2 - \dots - y_n^2)V(y_i) + y_i V(y_1^2 + \dots + y_n^2)}{(1 - y_1^2 - \dots - y_n^2)^2}$$

for $1 \le i \le n$, with additional

$$V(x_0 \circ \pi^{-1}) = V\left(\frac{1 + y_1^2 + \dots + y_n^2}{1 - y_1^2 - \dots - y_n^2}\right) = 2\frac{V(y_1^2 + \dots + y_n^2)}{(1 - y_1^2 - \dots - y_n^2)^2}.$$

Thus,

$$\begin{aligned}(\pi^{-1})^* h(V, V) =& \frac{4}{(1 - y_1^2 - \dots - y_n^2)^4}\Big(- (V(y_1^2 + \dots + y_n^2))^2 \\ &+ \sum_{i=1}^{n} ((1 - y_1^2 - \dots - y_n^2)V(y_i) + y_i V(y_1^2 + \dots + y_n^2))^2\Big) \\ =& 4\frac{(V(y_1))^2 + \dots + (V(y_n))^2}{(1 - y_1^2 - \dots - y_n^2)^2} = \frac{4}{(1 - y_1^2 - \dots - y_n^2)^2}\overline{g}(V, V),\end{aligned}$$

which gives the metric of $\mathbf{HB}^n$,

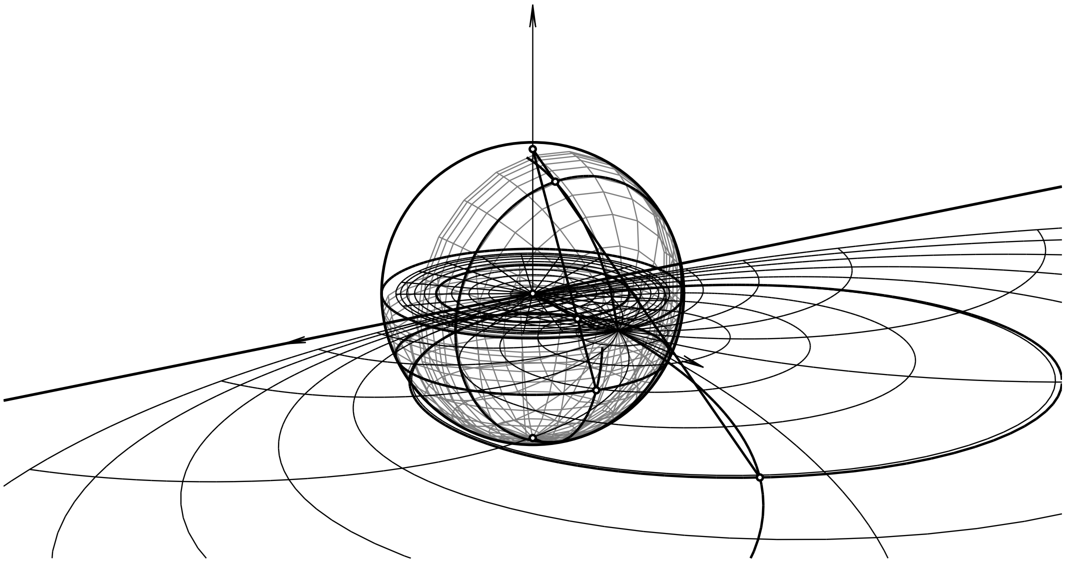
$$h_2 = (\pi^{-1})^* h = 4\frac{(dy_1)^2 + \dots + (dy_n)^2}{(1 - y_1^2 - \dots - y_n^2)^2}. \tag{4.9}$$

In this way, we obtained the Riemannian manifolds $\mathbf{H}^n = (\mathbf{H}^n, h)$ and $\mathbf{HB}^n = (\mathbf{B}^n, h_2)$ which are isometric. △

Example 4.21. Let $\varphi\colon \mathbf{S}^n \setminus \{q\} \to \mathbb{R}^n$ be the stereographic projection from Example 4.19 with the formulas (4.7) and (4.8), and $\rho\colon \mathbf{S}^n \to \mathbf{S}^n$ be the rotation defined by the equation $\rho(x_1, \dots, x_{n-1}, x_n, x_{n+1}) = (x_1, \dots, x_{n-1}, -x_{n+1}, x_n)$ taking the hemisphere $\{x_{n+1} < 0\}$ to the hemisphere $\{x_n > 0\}$. Consider the diffeomorphism

$$\psi = \varphi \circ \rho \circ \varphi^{-1}\colon \mathbb{R}^n \setminus \{(0, \dots, 0, 1)\} \to \mathbb{R}^n \setminus \{(0, \dots, 0, -1)\}$$

and its restriction from the ball $\mathbf{B}^n$ onto the half-space $\mathbf{U}^n = \{(z_1, \dots, z_n) \in \mathbb{R}^n : z_n > 0\}$. Then the map $\psi\restriction_{\mathbf{B}^n}\colon \mathbf{B}^n \to \mathbf{U}^n$ gives rise to the third model of our hyperbolic space, the ***Poincaré half-space model*** $\mathbf{HU}^n = (\mathbf{U}^n, (\psi^{-1})^* h_2)$.



Let us calculate the inverse $f = \psi^{-1} = \varphi \circ \rho^{-1} \circ \varphi^{-1}$ in coordinates,

$$\begin{aligned}
f(z_1, \dots, z_n) &= \varphi \circ \rho^{-1} \circ \varphi^{-1}(z_1, \dots, z_n) \\
&= \varphi \circ \rho^{-1} \left(\frac{1}{1 + z_1^2 + \dots + z_n^2} (2z_1, \dots, 2z_n, -1 + z_1^2 + \dots + z_n^2) \right) \\
&= \varphi \left(\frac{1}{1 + z_1^2 + \dots + z_n^2} (2z_1, \dots, 2z_{n-1}, -1 + z_1^2 + \dots + z_n^2, -2z_n) \right) \\
&= \frac{1}{(1 + z_n)^2 + z_1^2 + \dots + z_{n-1}^2} (2z_1, \dots, 2z_{n-1}, -1 + z_1^2 + \dots + z_n^2).
\end{aligned}$$

For an arbitrary point $p \in \mathbf{U}^n$ we obtain

$$\begin{aligned}
f_* \frac{\partial}{\partial z_n} &= \sum_{i=1}^{n} \frac{\partial (y_i \circ f)}{\partial z_n} \frac{\partial}{\partial y_i} \circ f \\
&= \frac{2((1 + z_n)^2 - z_1^2 - \dots - z_{n-1}^2) \frac{\partial}{\partial y_n} \circ f - 4(1 + z_n) \sum_{i=1}^{n-1} z_i \frac{\partial}{\partial y_i} \circ f}{((1 + z_n)^2 + z_1^2 + \dots + z_{n-1}^2)^2},
\end{aligned}$$

and hence for the induced metric $h_3 = f^* h_2$, with $z^2 = z_1^2 + \dots + z_{n-1}^2$, we have

$$\begin{aligned}
h_3 \left(\frac{\partial}{\partial z_n}, \frac{\partial}{\partial z_n} \right) &= h_2 \left(f_* \frac{\partial}{\partial z_n}, f_* \frac{\partial}{\partial z_n} \right) \\
&= \frac{4}{(1 - (y_1 \circ f)^2 - \dots - (y_n \circ f)^2)^2} \frac{4((1 + z_n)^2 - z^2)^2 + 16(1 + z_n)^2 z^2}{((1 + z_n)^2 + z^2)^4} \\
&= 16 \frac{((1 + z_n)^2 + z^2)^2}{(((1 + z_n)^2 + z^2)^2 - 4z^2 - (-1 + z^2 + z_n^2)^2)^2} = \frac{1}{z_n^2}.
\end{aligned}$$

However, from the formula (4.9) we see that h_2 and $\bar{g}$ are conformal on $\mathbf{B}^n$, while Example 4.19 shows that f^{-1} is a conformal diffeomorphism between $(\mathbf{B}^n, \bar{g})$ and $(\mathbf{U}^n, \bar{g})$, and therefore h_3 and $\bar{g}$ are conformal on $\mathbf{U}^n$. This is why we only need to calculate $h_3(V, V)$ for a single chosen vector field, for example for $V = \partial/\partial z_n$. Finally, it follows

$$h_3 = f^* h_2 = \frac{(dz_1)^2 + \cdots + (dz_n)^2}{z_n^2}. \tag{4.10}$$

Since the Riemannian manifolds $\mathbf{H}^n$, $\mathbf{HB}^n$, and $\mathbf{HU}^n$ are all mutually isometric, we often use the simple notation $\mathbb{R}\mathbf{H}^n$, to refer to any of these hyperbolic models that is the most convenient in a given context. △

Example 4.22. The symmetries of $\mathbb{R}\mathbf{H}^n$ are most easily seen in the hyperboloid model $\mathbf{H}^n$. The ***Lorentz group*** $\mathrm{O}(1, n)$ is the group of linear maps from $\mathbb{R}^{n+1}_1$ to itself that preserve the Minkowski metric. Each element of it preserves the hyperboloid $\{x_0^2 - x_1^2 - \cdots - x_n^2 = 1\}$, which has two components determined by $\{x_0 > 0\}$ and $\{x_0 < 0\}$. The subgroup $\mathrm{O}^+(1, n)$ that preserves the direction of time consists of maps that take the component $\{x_0 > 0\}$ to itself, and therefore it acts isometrically on $\mathbf{H}^n$. We can analogously follow the arguments from Example 4.18 to conclude that there is an isometry from $\mathrm{O}^+(1, n)$ which maps an arbitrary point q to an arbitrary point p and pushes forward an arbitrary orthonormal basis for $T_q\mathbf{H}^n$ into an arbitrary orthonormal basis for $T_p\mathbf{H}^n$, so the hyperbolic space $\mathbb{R}\mathbf{H}^n$ is homogeneous and isotropic. △

The Euclidean, spherical, and hyperbolic metrics can be adapted to give model pseudo-Riemannian manifolds. The first example is the pseudo-Euclidean space $\mathbb{R}^n_\nu$ of index $\nu > 0$ from Example 4.7.

For other examples we consider pseudo-Riemannian submanifolds of the pseudo-Euclidean space $(\mathbb{R}^{n+1}_\nu, g)$ given by $M = \{X \in \mathbb{R}^{n+1}_\nu : g(X, X) = c\}$ for some real $c = \pm 1$. Similar to the case of hyperbolic space $\mathbf{H}^n$ we can set the function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ by $f(X) = g(X, X)$. Then we have $df_X = 2X^\flat$, so c is a regular value of f and therefore $M = f^{-1}(c)$ is a submanifold of dimension n, while its metric is induced from the ambient metric g. Since $T_X M = \operatorname{Ker} T_X f = \operatorname{Ker} X^\flat = X^\perp$ and $g(X, X) = c$, it is easy to see that the index will decrease by 1 only if $c < 0$.

The pseudo-Riemannian manifold $(M, \imath^* g)$ for $c = 1$ has index ν and it is called the ***pseudosphere***, while for $c = -1$ has index $\nu - 1$ and it is called the ***pseudohyperbolic space***. Especially, ***de Sitter space***[11] is the pseudosphere $\mathbf{dS}^n$ embedded in $\mathbb{R}^{n+1}_1$, while ***anti-de Sitter space*** is the pseudohyperbolic space $\mathbf{AdS}^n$ embedded in $\mathbb{R}^{n+1}_2$. In this way, we get three models of the Lorentzian geometry, namely Minkowski, de Sitter, and anti-de Sitter spaces.

4.7 Length and distance

On Euclidean spaces there is an immediate idea of distance between points, so defining the lengths of curves is justified by the sums of distances for a fine polygonal approximation. Riemannian manifolds are not metric spaces in advance, while pseudo-Riemannian manifolds (with indefinite metrics) are not metric spaces at all. However, the positive definite scalar product structures on the tangent space of a Riemannian manifold give rise to a concept of lengths of tangent vectors. Thus, we naturally obtain the ability to measure lengths of curves, which allows us to define the distance between points on a connected Riemannian manifold which becomes a metric space.

[11] Willem de Sitter (1872–1934), Dutch mathematician, physicist, and astronomer

Let M be a smooth manifold. A ***curve*** (or ***smooth parametric curve***) in M is a smooth map $\gamma\colon I \to M$, where $I \subseteq \mathbb{R}$ is some open interval. Since I is an open submanifold of the real line $\mathbb{R}^1$, it has the identity chart consisting of the single coordinate function $u = \mathbb{1}_I$. The ***velocity*** of a curve $\gamma\colon I \to M$ is defined as the push-forward of the coordinate vector field, $\gamma' = T\gamma \circ (d/du)\colon I \to TM$. More precisely, a curve γ assigns to each $t \in I$ the ***velocity vector*** $\gamma'(t) \in T_{\gamma(t)}M$ defined by

$$\gamma'(t) = (T_t\gamma)\left(\frac{d}{du}\right)_t = \gamma_*\left(\frac{d}{du}\right)_t,$$

where $(d/du)_t \in T_tI$ is the canonical basis vector at t which maps $f \in \mathfrak{F}(I)$ to the old-fashioned derivative $f'(t)$ in the sense of calculus.

We say that a curve γ is ***regular*** if $\gamma'(t) \neq 0$ holds for all $t \in I$, which implies that γ is an immersion. The concept of regular curves can be extended to a closed bounded interval $[a, b] \subseteq \mathbb{R}$. A map $\gamma\colon [a, b] \to M$ is a ***regular curve segment*** if it extends to a regular curve defined on some open interval containing $[a, b]$. More generally, a continuous map $\gamma\colon [a, b] \to M$ is an ***admissible curve*** (or ***piecewise regular curve segment***) from $\gamma(a)$ to $\gamma(b)$ if there exists a finite sequence $a = t_0 < t_1 < \cdots < t_k = b$ of numbers $t_i \in \mathbb{R}$, called a ***partition*** of $[a, b]$, such that the restriction $\gamma\restriction_{[t_{i-1},t_i]}$ is a regular curve segment for each $1 \leq i \leq k$.

The usual calculus concept of the length of curves in Euclidean space generalises in a natural way for a pseudo-Riemannian manifold (M, g). The existence of metric g allows to measure the size of the velocity vectors. The ***speed*** of a curve $\gamma \in \mathfrak{F}(I)$ at some time $t \in I$ is the magnitude of its velocity vector at t,

$$\|\gamma'(t)\| = \sqrt{|g_{\gamma(t)}(\gamma'(t), \gamma'(t))|}.$$

Since the distance an object travels is the integral of speed over the time interval, we define the ***arc length*** of an admissible curve $\gamma\colon [a, b] \to M$ by

$$L(\gamma) = \int_a^b \|\gamma'(t)\|\, dt.$$

The speed function is bounded and continuous everywhere on $[a, b]$ except possibly at the finitely many points where γ is not smooth, so it is integrable in the Riemann (or Lebesgue[12]) integral sense, and therefore $L(\gamma)$ is a well-defined finite nonnegative number.

A ***reparametrization*** of a curve $\gamma\colon I \to M$ is a curve of the form $\widetilde{\gamma} = \gamma \circ \varphi\colon I' \to M$, where $I' \subseteq \mathbb{R}$ is another interval and $\varphi\colon I' \to I$ is a diffeomorphism. Since intervals are connected, φ is strictly monotone (either increasing or decreasing) on I'. We say that $\widetilde{\gamma}$ is a ***forward reparametrization*** if φ is increasing ($\varphi' > 0$), and a ***backward reparametrization*** if it is decreasing ($\varphi' < 0$).

More generally, a ***reparametrization*** of an admissible curve $\gamma\colon [a, b] \to M$ is an admissible curve of the form $\widetilde{\gamma} = \gamma \circ \varphi$, where $\varphi\colon [c, d] \to [a, b]$ is a homeomorphism for which there is a partition $c = t_0 < t_1 < \cdots < t_k = d$ of $[c, d]$ such that the restriction $\varphi\restriction_{[t_{i-1},t_i]}$ is a diffeomorphism on its image for each $1 \leq i \leq k$. If the derivative φ' does not change the sign we say that the corresponding reparametrization is ***monotone***.

It is important to notice that the arc length of an admissible curve is invariant under monotone reparametrizations, where for $\varphi' > 0$ we have

$$L(\gamma \circ \varphi) = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \|(\gamma \circ \varphi)'(t)\|\, dt = \sum_{i=1}^{k} \int_{\varphi(t_{i-1})}^{\varphi(t_i)} \|\gamma'(\varphi(t))\|\, d(\varphi(t)) = L(\gamma).$$

[12]Henri Lebesgue (1875–1941), French mathematician

Although the arc length on a Riemannian manifold behaves much as in Euclidean space, in the case of an indefinite metric the term length can be misleading. For example, a null curve has length zero. To avoid potential problems where a nonzero null $\gamma'(t)$ has speed zero we focus on Riemannian geometry.

Let (M, g) be a Riemannian manifold. Hence, an admissible curve $\gamma\colon [a,b] \to M$ has a positive speed $\|\gamma'(t)\| > 0$ for all t except the points from a corresponding partition of $[a,b]$. Consider a function $s\colon [a,b] \to \mathbb{R}$ called the ***arc-length function*** of γ and defined by

$$s(t) = L(\gamma\restriction_{[a,t]}) = \int_a^t \|\gamma'(u)\|\, du,$$

which is continuous everywhere, and by the fundamental theorem of calculus it is smooth wherever γ is, with the derivative $s'(t) = \|\gamma'(t)\|$. Since $s'(t) > 0$ holds for all smooth points $t \in [a,b]$, the function s is strictly increasing on $[a,b]$. Its inverse $\varphi = s^{-1}\colon [0, L(\gamma)] \to [a,b]$ determines a forward reparametrization $\widetilde{\gamma} = \gamma \circ \varphi$ of γ, where

$$\|\widetilde{\gamma}'(t)\| = \|\gamma'(s^{-1}(t)) \cdot (s^{-1})'(t)\| = \left\| \frac{\gamma'(s^{-1}(t))}{s'(s^{-1}(t))} \right\| = \left\| \frac{\gamma'(s^{-1}(t))}{\|\gamma'(s^{-1}(t))\|} \right\| = 1$$

holds wherever γ is smooth. Such reparametrization $\widetilde{\gamma}$ has ***unit speed***, while its arc-length function has the simple form $s(t) = t$. A unit-speed admissible curve whose parameter interval is of the form $[0,b]$ for some $b > 0$ is said to be ***parametrized by arc length***.

Lemma 4.17. *Every admissible curve in a Riemannian manifold has a unique forward reparametrization by arc length.*

Let us introduce the concept of distance between points on a connected Riemannian manifold (M, g). For each pair of points $p, q \in M$ we establish the ***path space*** $\Omega_{p,q}$ of all admissible curves from p to q. This space is not empty, which we see in the following lemma.

Lemma 4.18. *Any two points of a connected manifold can be joined by an admissible curve.*

Proof. A connected manifold M is path-connected, and therefore any two points $p, q \in M$ can be joined by a continuous path $\gamma\colon [a,b] \to M$. By compactness it follows that there is a partition $a = t_0 < \cdots < t_k = b$ of $[a,b]$ such that $\gamma\restriction_{[t_{i-1},t_i]}$ is contained in a coordinate neighbourhood that is diffeomorphic to a Euclidean ball, for each $1 \leq i \leq k$. Then, each $\gamma\restriction_{[t_{i-1},t_i]}$ can be replaced by a straight-line path in coordinates, which yields an admissible curve from p to q. $\square$

The previous lemma allows us to well define the ***Riemannian distance*** $d(p,q)$ between points $p, q \in M$ by

$$d(p,q) = \inf\{L(\gamma) : \gamma \in \Omega_{p,q}\}.$$

Example 4.23. The infimum of curve length in the definition of $d(p,q)$ can fail to be realized. Consider the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$ as a Riemannian submanifold of $\mathbb{R}^2$. The distance between points $p(-1,0)$ and $q(1,0)$ is $d(p,q) = 2$, but this distance is not realized by any admissible curve, since any such path in $\mathbb{R}^2$ passes through $(0,0)$. $\triangle$

It immediately follows from the definition that $d(p,q) = d(q,p) \geq 0$ and $d(p,p) = 0$. Additionally, the triangle inequality $d(p,q) \leq d(p,r) + d(r,q)$ for all $p, q, r \in M$ follows from the fact that an admissible curve from p to q can be obtained by combining one from p to r with one from r to q (possibly changing the starting time of the parametrization), whose length is the sum of the lengths of two starting admissible curves. In order for (M, d) to become a metric space, we only need to show the fact that $d(p,q) = 0$ implies $p = q$. To prove this we introduce the following topological statement.

Lemma 4.19. *Let X and Y be topological spaces and let Y be compact. For any continuous $f\colon X\times Y\to\mathbb{R}$, the function $h\colon X\to\mathbb{R}$ given by $h(x)=\min_{y\in Y}f(x,y)$ is well-defined and continuous.*

Proof. For any fixed $x\in X$, the function $f_x\colon Y\to\mathbb{R}$ defined by $f_x(y)=f(x,y)$ is continuous with a compact domain, so $f_x(Y)$ is a compact interval, which gives $h(x)=\min f_x(Y)$. For continuity of h it suffices to check that $h^{-1}(-\infty,a)$ and $h^{-1}(b,\infty)$ are open. It is easy to see that $h^{-1}(-\infty,a)=\pi_X\circ f^{-1}(-\infty,a)$ is open, where $\pi_X\colon X\times Y\to X$ is the (continuous and open) canonical projection. On the other hand for any $x\in h^{-1}(b,\infty)$ and $y\in Y$ there is a neighbourhood $(x,y)\in U_{x,y}\times V_{x,y}\subseteq f^{-1}(b,\infty)$. Since Y is compact there exist $y_1(x),\dots,y_{k(x)}(x)\in Y$ such that $Y\subseteq\bigcup_{i=1}^{k(x)}V_{x,y_i(x)}$, so we obtain

$$\{x\}\times Y\subseteq\left(\bigcap_{i=1}^{k(x)}U_{x,y_i(x)}\right)\times Y\subseteq\bigcup_{i=1}^{k(x)}(U_{x,y_i(x)}\times V_{x,y_i(x)})\subseteq f^{-1}(b,\infty),$$

and therefore $h^{-1}(b,\infty)=\bigcup_{x\in h^{-1}(b,\infty)}\bigcap_{i=1}^{k(x)}U_{x,y_i(x)}$ is open. $\square$

Let us fix an arbitrary point p of an n-manifold M, and take a chart (U,φ) centred at p with the coordinate functions $x_i=\pi_i\circ\varphi$. The Euclidean metric $\overline{g}$ on U, that is defined by $\overline{g}(\partial_i,\partial_j)=\delta_{ij}$, yields the Euclidean distance $\overline{d}$ given by

$$\overline{d}(q,r)=\sqrt{\sum_{i=1}^{n}(x_i(q)-x_i(r))^2}.$$

Without loss of generality, a coordinate neighbourhood U can be reduced to some ball $\{q\in U:\overline{d}(p,q)<\varepsilon\}$ for small $\varepsilon>0$. Let $f\colon U\times\mathbf{S}^{n-1}\to\mathbb{R}$ be the smooth function given by $f(q,V)=g(V,V)(q)=g_q(V_q,V_q)$, where $(\alpha_1,\dots,\alpha_n)\in\mathbf{S}^{n-1}$ and $\sum_{i=1}^n\alpha_i\partial_i\in\mathfrak{X}(U)$ are identified. Since $\mathbf{S}^{n-1}$ is compact we apply Lemma 4.19 which yields the well-defined continuous positive functions $\mu,\nu\colon U\to\mathbb{R}$ defined by

$$\mu(q)=\min_{V\in\mathbf{S}^{n-1}}f(q,V)=\min_{\overline{g}_q(V,V)=1}g_q(V,V)=\min_{0\neq V\in T_qM}\frac{g_q(V,V)}{\overline{g}_q(V,V)}>0,$$

$$\nu(q)=\max_{V\in\mathbf{S}^{n-1}}f(q,V)=\max_{\overline{g}_q(V,V)=1}g_q(V,V)=\max_{0\neq V\in T_qM}\frac{g_q(V,V)}{\overline{g}_q(V,V)}>0.$$

This implies

$$\mu(q)\overline{g}_q(V,V)\le g_q(V,V)\le\nu(q)\overline{g}_q(V,V)$$

for all $V\in T_qM$, which allows us to compare two metrics g and $\overline{g}$ for $q\in U$. Let $\gamma\colon[a,b]\to M$ be an admissible curve from p to $q\in U$ and

$$c=\min_{t\in[a,b]}\sqrt{\mu(\gamma(t))}>0,\quad C=\max_{t\in[a,b]}\sqrt{\nu(\gamma(t))}>0.$$

If γ lies entirely in U then

$$L(\gamma)=\int_a^b\sqrt{g_{\gamma(t)}(\gamma'(t),\gamma'(t))}\,dt\ge c\int_a^b\sqrt{\overline{g}_{\gamma(t)}(\gamma'(t),\gamma'(t))}\,dt\ge c\overline{d}(p,q).$$

Otherwise, γ lives U, and there exists the smallest s such that $\gamma(s)\notin U$, and again

$$L(\gamma)\ge\int_a^s\sqrt{g_{\gamma(t)}(\gamma'(t),\gamma'(t))}\,dt\ge c\int_a^s\sqrt{\overline{g}_{\gamma(t)}(\gamma'(t),\gamma'(t))}\,dt\ge c\varepsilon\ge c\overline{d}(p,q).$$

Anyway, taking the infimum it follows $d(p,q) \geq c\bar{d}(p,q)$. On the other hand, if γ is a straight-line path in the Euclidean metric $\bar{g}$, then it lies in U and we have

$$C\bar{d}(p,q) = C\int_a^b \sqrt{\bar{g}_{\gamma(t)}(\gamma'(t),\gamma'(t))}\,dt \geq \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t),\gamma'(t))}\,dt = L(\gamma) \geq d(p,q).$$

Consequently, we obtain

$$0 < c\bar{d}(p,q) \leq d(p,q) \leq C\bar{d}(p,q), \tag{4.11}$$

which proves that $d(p,q) > 0$ for $q \neq p$. Moreover, (4.11) shows that the Euclidean distance $\bar{d}$ and the Riemannian distance d are comparable on a neighbourhood of p, which means that the metric topology and the manifold topology (coming from the Euclidean distance) are equivalent.

Theorem 4.20. *A connected Riemannian manifold with the Riemannian distance is a metric space whose metric topology is the same as the manifold topology.*

4.8 Problems

Problem 4.1. Let $\mathcal{V}$ and $\mathcal{W}$ be inner product spaces of te same finite dimension. If $f\colon \mathcal{V} \to \mathcal{W}$ is a map that preserves the origin and distances ($f(0) = 0$ and $\|f(x) - f(y)\| = \|x - y\|$), then f is a linear isometry.

Problem 4.2. Let $(\mathcal{V}, g)$ be an indefinite scalar product space, and let b be a symmetric bilinear form on $\mathcal{V}$. Prove that $b = cg$ for some $c \in \mathbb{R}$ if and only if $b(N,N) = 0$ holds for every null $N \in \mathcal{V}$.

Problem 4.3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x,y) = (\sqrt{1 + 2x^2 + 2y^2}, x+y, x-y)$. Compute f^*g for the Minkowski metric $g = -dx_1^2 + dx_2^2 + dx_3^2 \in \mathfrak{T}_2^0(\mathbb{R}^3)$, and show that it is a Riemannian metric on $\mathbb{R}^2$.

Problem 4.4. Prove that the pseudo-sphere of dimension n, index ν and radius r is diffeomorphic to $\mathbb{R}^\nu \times \mathbf{S}^{n-\nu}$. Prove that the pseudo-hyperbolic space of dimension n, index ν and radius r is diffeomorphic to $\mathbb{R}^{n-\nu} \times \mathbf{S}^\nu$.

Chapter 5

CONNECTION

5.1 Covariant derivatives

In order to introduce a curvature on pseudo-Riemannian manifolds it is necessary to study geodesics, the pseudo-Riemannian generalisations of straight lines. In the original sense, a geodesic was the shortest route between two points on a surface, so it is reasonable to define geodesics as length minimizing curves, at least locally. However, pseudo-Riemannian manifolds whose metric is indefinite are not metric spaces and geodesics there are not distance minimizing. Even if we restrict observations on Riemannian manifolds only, the minimizing property as a definition has big technical difficulties.

This is why we choose a different property of straight lines and generalise that. A good candidate is the property that straight lines are the only curves in Euclidean space that have parametrizations with zero acceleration, and we generalise this to pseudo-Riemannian manifolds.

Let $\gamma\colon I \to M$ be a curve in a pseudo-Riemannian manifold (M, g). The velocity vector at time $t \in I$ represents the derivative $\gamma'(t)$, while another derivative $\gamma''(t)$ gives rise to the acceleration. Therefore, it is necessary to make the quotient by subtracting the vectors $\gamma'(t+h) \in T_{\gamma(t+h)}M$ and $\gamma'(t) \in T_{\gamma(t)}M$, which is not convenient because they live in different spaces. In the case of abstract manifolds this difference makes no sense, so we need a way to compare the values of vector field at different points, or to connect close tangent spaces. That connection is additional information on a manifold that allows differentiation of vector fields as well as interpretation of the curve acceleration.

Let us first consider the case of the Euclidean space $\mathbb{R}^n$. We can differentiate a function $f \in \mathfrak{F}(\mathbb{R}^n)$ at a point $p \in \mathbb{R}^n$ in the direction $X_p \in T_p\mathbb{R}^n$ with

$$D_{X_p}f = \lim_{t\to 0} \frac{f(p + tX_p) - f(p)}{t} = (f \circ \gamma)'(0) = \gamma'(0)f = X_p(f), \tag{5.1}$$

where $\gamma(t) = p + tX_p$. Similarly, we can differentiate a vector field $Y = \sum_{i=1}^{n} Y^i\partial_i \in \mathfrak{X}(\mathbb{R}^n)$ by

$$\begin{aligned} D_{X_p}Y &= \lim_{t\to 0} \frac{Y(p + tX_p) - Y(p)}{t} = \sum_{i=1}^{n} \lim_{t\to 0} \frac{Y^i(p + tX_p)\partial_i - Y^i(p)\partial_i}{t} \\ &= \sum_{i=1}^{n} D_{X_p}Y^i\partial_i = \sum_{i=1}^{n} X_p(Y^i)\partial_i. \end{aligned} \tag{5.2}$$

In the case of an arbitrary manifold M, which is not necessarily embedded in an Euclidean space, we can use the result from (5.1) to differentiate a function $f \in \mathfrak{F}(M)$ in the direction of vector $X_p \in T_pM$ by $\nabla_{X_p}f = X_p(f)$. However, since there is no canonical basis of tangent space T_pM, the result from (5.2) is not usable directly. Since we do not have a canonical way to define vector field differentiation on an abstract manifold, we use the formula (5.2) to establish the properties that $D_{X_p}Y$ has in $\mathbb{R}^n$, to base a general theory on them.

For a vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ we define $D_X Y$ by $(D_X Y)_p = D_{X_p} Y$ for each $p \in \mathbb{R}^n$. Because of (5.2) we have $(D_X Y)_p = \sum_{i=1}^n X_p(Y^i)(\partial_i)_p$, so vector fields X and Y bring a new vector field

$$D_X Y = \sum_{i=1}^n X(Y^i)\partial_i \tag{5.3}$$

and the $\mathbb{R}$-bilinear binary operation $D\colon \mathfrak{X}(\mathbb{R}^n) \times \mathfrak{X}(\mathbb{R}^n) \to \mathfrak{X}(\mathbb{R}^n)$. The map D is obviously $\mathfrak{F}(\mathbb{R}^n)$-linear by the first argument, while it is Leibnizian by the second argument, since for $X, Y \in \mathfrak{X}(\mathbb{R}^n)$ and $f \in \mathfrak{F}(\mathbb{R}^n)$ holds

$$D_X(fY) = \sum_{i=1}^n X(fY^i)\partial_i = \sum_{i=1}^n (Xf)Y^i\partial_i + \sum_{i=1}^n fX(Y^i)\partial_i = (Xf)Y + fD_X Y.$$

Observed properties motivate us to introduce the connection operation on an arbitrary manifold.

A ***connection*** (***affine connection***, ***linear connection***) on a manifold M is an $\mathbb{R}$-bilinear map $\nabla\colon \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ with the notation $(X, Y) \mapsto \nabla_X Y$, that is $\mathfrak{F}(M)$-linear in its first argument and Leibnizian in its second argument. The symbol ∇ is read "nabla"or "del" and $\nabla_X Y$ is called the ***covariant derivative*** of Y in the direction of X.

For arbitrary $X, Y, Z \in \mathfrak{X}(M)$ and $f, h \in \mathfrak{F}(M)$, we can write the covariant derivative formulas as follows. Since the connection is $\mathfrak{F}(M)$-linear in its first argument we have

$$\nabla_{fX+hY} Z = f\nabla_X Z + h\nabla_Y Z. \tag{5.4}$$

The connection is not $\mathfrak{F}(M)$-linear in its second argument, but only $\mathbb{R}$-linear, so

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z, \tag{5.5}$$

while a deviation from $\mathfrak{F}(M)$-linearity can be seen through the Leibniz rule

$$\nabla_X(fY) = f\nabla_X Y + (X(f))Y. \tag{5.6}$$

The motivational operation D defined by (5.3) surely satisfies the definition conditions and we call it the ***standard connection*** on $\mathbb{R}^n$.

A covariant derivative ∇_X with respect to a vector field $X \in \mathfrak{X}(M)$ can be extended to an arbitrary tensor field on M. We have naturally $\nabla_X f = Xf$ for $f \in \mathfrak{F}(M)$, while $\nabla_X Y$ by formulas (5.5) and (5.6) completes the conditions of Theorem 3.16, so there exists a unique tensor field derivation on M that generalises a covariant derivative.

For an arbitrary tensor field $A \in \mathfrak{T}^r_s(M)$ we have $\nabla_X A \in \mathfrak{T}^r_s(M)$, so additionally we define a map $\nabla A\colon \mathfrak{X}^*(M)^r \times \mathfrak{X}(M)^{s+1} \to \mathfrak{F}(M)$ by

$$\nabla A(\omega_1, \dots, \omega_r, Y_1, \dots, Y_s, X) = (\nabla_X A)(\omega_1, \dots, \omega_r, Y_1, \dots, Y_s).$$

Since $\nabla_X A \in \mathfrak{T}^r_s(M)$ then ∇A is $\mathfrak{F}(M)$-multilinear in the first $r + s$ arguments, while by (5.4) it is $\mathfrak{F}(M)$-linear in the last argument. In this way, we get a new tensor field $\nabla A \in \mathfrak{T}^r_{s+1}(M)$ called the ***total covariant derivative*** of A. We say that a tensor field $A \in \mathfrak{T}^r_s(M)$ is ***parallel*** if $\nabla A = 0$.

Example 5.1. For $f \in \mathfrak{F}(M)$ we have $(\nabla f)(X) = \nabla_X f = Xf = df(X)$, so the total covariant derivative of function is equal to its differential, $\nabla f = df$. Additionally, if $\nabla f = 0$, then $df = 0$, and as the differential is identified with the tangent map we have $Tf = 0$, so from the proof od Lemma 2.20 we see that f is a local constant. Hence, if $\nabla f = 0$, then f is locally constant, while for a connected M it is globally constant. △

Example 5.2. The covariant tensor of order two $\nabla^2 f = \nabla(\nabla f)$ is called the ***Hessian***[1] of f. If C is the $(1,1)$ contraction we have

$$\begin{aligned}\nabla_X(\nabla_Y f) &= \nabla_X(\nabla f(Y)) = \nabla_X(C(\nabla f \otimes Y)) = C(\nabla_X(\nabla f \otimes Y)) = C(\nabla_X \nabla f \otimes Y + \nabla f \otimes \nabla_X Y)\\ &= (\nabla_X \nabla f)(Y) + \nabla f(\nabla_X Y) = \nabla^2 f(Y,X) + \nabla_{\nabla_X Y} f,\end{aligned}$$

and therefore $\nabla^2 f(Y,X) = X(Y(f)) - (\nabla_X Y)f$. △

Although the connection is defined on global vector fields, it is actually a local operator, which we see in the following theorem.

Theorem 5.1. *Let ∇ be a connection on a manifold M. The value of a vector field $\nabla_X Y$ at $p \in M$ depends only on the value for $Y \in \mathfrak{X}(M)$ in a neighbourhood of p and the value for $X \in \mathfrak{X}(M)$ at p.*

Proof. We have the dependence on the component Y as a tensor derivative in Theorem 3.15, while the dependence on the component X can be seen in Theorem 3.13 if for a fixed Y we consider $\mathfrak{T}^1_1(M) \ni T\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $T(X) = \nabla_X Y$. □

Let $V \in T_pM$ be a tangent vector at a point $p = \pi(V) \in M$ and $Z \in \mathfrak{X}(U)$ be a vector field where $U \subseteq M$ is a neighbourhood of p. Theorem 5.1 allows us to well define $\nabla_V Z = (\nabla_X Y)_p$, where $X, Y \in \mathfrak{X}(M)$ are arbitrary vector fields such that $X_p = V$ and $Y\restriction_U = Z$.

Let ∇ be a connection on an n-manifold M, and $(E_1, \dots, E_n)$ is a local frame over an open subset $U \subseteq M$. Most often this is a coordinate frame with $E_i = \partial_i$, but it is useful to see general calculations. For any $1 \le i, j \le n$ the covariant derivative $\nabla_{E_i} E_j$ can be expressed in terms of the same frame by

$$\nabla_{E_i} E_j = \sum_{k=1}^{n} \Gamma^k_{ij} E_k.$$

In this way, we get n^3 functions $\Gamma^k_{ij} \in \mathfrak{F}(U)$ called ***Christoffel symbols***[2] of ∇ related to that frame. If $X = \sum_{i=1}^n X^i E_i$ and $Y = \sum_{j=1}^n Y^j E_j$, then

$$\begin{aligned}\nabla_X Y &= \nabla_X\Big(\sum_j Y^j E_j\Big) = \sum_j (XY^j)E_j + \sum_j Y^j \nabla_{\sum_i X^i E_i} E_j\\ &= \sum_j (XY^j)E_j + \sum_{i,j} X^i Y^j \nabla_{E_i} E_j = \sum_k (XY^k)E_k + \sum_{i,j,k} X^i Y^j \Gamma^k_{ij} E_k,\end{aligned}$$

which implies

$$\nabla_X Y = \sum_{k=1}^n \left(XY^k + \sum_{i,j=1}^n X^i Y^j \Gamma^k_{ij} \right) E_k,$$

so the connection on U is completely determined by its Christoffel symbols. In the case of coordinate frame, the previous formula is reduced to the equality

$$\nabla_X Y = \sum_{k=1}^n \left(X(Y(x_k)) + \sum_{i,j=1}^n X(x_i)Y(x_j)\Gamma^k_{ij} \right) \frac{\partial}{\partial x_k}, \tag{5.7}$$

from where it is easy to conclude the locality of the connection from Theorem 5.1.

Moreover, in the case that the manifold atlas consists of a single chart, $(\partial_1, \dots, \partial_n)$ is a global frame and it turns out that an arbitrary choice of n^3 smooth functions $\Gamma^k_{ij} \in \mathfrak{F}(M)$ defines a connection ∇ using the equality (5.7). First we notice that $X, Y \in \mathfrak{X}(M)$ obviously give $\nabla_X Y \in \mathfrak{X}(M)$. It is also obvious $\mathbb{R}$-linearity by Y and $\mathfrak{F}(M)$-linearity by X. It remains the Leibnizian property by Y, which is a simple straightforward calculation.

[1] Ludwig Otto Hesse (1811–1874), German mathematician

[2] Elwin Bruno Christoffel (1829–1900), German mathematician and physicist

Example 5.3. If we set all Christoffel symbols to be zero, then from the equation (5.7) we obtain $\nabla_X Y = \sum_{k=1}^n XY^k \partial_k$ which is the standard connection $D_X Y$ when $M = \mathbb{R}^n$. △

Example 5.4. Any choice of smooth functions $\Gamma_{ij}^k \in \mathfrak{F}(U_\alpha)$ creates a connection ∇^α on each coordinate neighbourhood U_α. For partitions of unity ψ_α subordinate to atlas we can set $\nabla_X Y = \sum_\alpha \psi_\alpha \nabla_X^\alpha Y$, which allows us to construct plentiful connections on a manifold M. It is easy to see the smoothness, $\mathbb{R}$-linearity by Y, and $\mathfrak{F}(M)$-linearity by X. For Leibnizian by Y, each linear combination of connections will not make a connection, but because of $\sum_\alpha \psi_\alpha = 1$ we have $\nabla_X(fY) = \sum_\alpha \psi_\alpha \nabla_X^\alpha(fY) = \sum_\alpha \psi_\alpha((Xf)Y + f\nabla_X^\alpha Y) = (Xf)Y + f\nabla_X Y$. △

5.2 Levi-Civita connection

Let (M, g) be a pseudo-Riemannian manifold and let ∇ be a connection on M. We say that a connection ∇ is metric (or preserves the metric g) if g is a parallel tensor field, $\nabla g = 0$. For a metric connection we have

$$0 = (\nabla g)(Y, Z, X) = (\nabla_X g)(Y, Z) = \nabla_X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z),$$

which gives the agreement between the metric g and the connection ∇,

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{5.8}$$

The map $\tau\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by $\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ we call the ***torsion***. The torsion is evidently $\mathfrak{F}(M)$-bilinear, and therefore $\tau \in \mathfrak{T}_2^1(M)$ (see Example 3.23). We say that a connection is ***symmetric*** if it is torsion-free, $\tau = 0$, which gives

$$[X, Y] = \nabla_X Y - \nabla_Y X. \tag{5.9}$$

Since the commutator of coordinate vector fields is equal to zero, the symmetry of the connection one can see in the equality $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$, that is the symmetry $\Gamma_{ij}^k = \Gamma_{ji}^k$ between Christoffel symbols related to the coordinate frame.

Example 5.5. In Example 5.2, by $\nabla^2 f(Y, X) = X(Y(f)) - (\nabla_X Y)f$ we expressed the Hessian of a function $f \in \mathfrak{F}(M)$, which implies $\nabla^2 f(X, Y) - \nabla^2 f(Y, X) = (\tau(X, Y))f$, so the symmetry of connection is equivalent to the symmetry of Hessian. △

We often require that a connection satisfies the last two conditions, which bring us a new concept of connection that includes the metric. A ***Levi-Civita connection***[3] on a pseudo-Riemannian manifold (M, g) is a symmetric metric connection ∇. In other words, a Levi-Civita connection is a map $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ that satisfies (5.4), (5.5), (5.6), (5.8), and (5.9) for every $f, h \in \mathfrak{F}(M)$ and all $X, Y, Z \in \mathfrak{X}(M)$. It turns out that all of them can be unified with one single general equality. Let us start with the following expression,

$$X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)).$$

Since ∇ is metric, we apply (5.8) three times to get

$$g(X, \nabla_Y Z - \nabla_Z Y) - g(Y, \nabla_Z X - \nabla_X Z) + g(Z, \nabla_X Y + \nabla_Y X).$$

Since ∇ is symmetric, we apply (5.9) three times to get

$$g(X, [Y, Z]) - g(Y, [Z, X]) - g(Z, [X, Y]) + 2g(\nabla_X Y, Z).$$

[3] Tullio Levi-Civita (1873–1941), Italian mathematician

Thus, we have the following equality,

$$2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]), \quad (5.10)$$

called the ***Koszul formula***[4], which is very useful because it determines a Levi-Civita connection.

Theorem 5.2. *A map $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a Levi-Civita connection if and only if for all $X, Y, Z \in \mathfrak{X}(M)$ the Koszul formula* (5.10) *holds.*

Proof. We already know that a Levi-Civita connection ∇ satisfies the Koszul formula, since it is derived from the properties of a Levi-Civita connection. Vice versa, it remains to prove the formulas (5.4), (5.5), (5.6), (5.8), and (5.9) under the assumption that (5.10) holds for all $X, Y, Z \in \mathfrak{X}(M)$.

We use the fact that $g(V,X) = g(W,X)$ for all $X \in \mathfrak{X}(M)$ implies $V^\flat = W^\flat$, and therefore $V = W$. In the following calculations functions $f \in \mathfrak{F}(M)$ go ahead of the tensor field g, while for commutators we can use special cases of the formula (2.6). For example, checking the Leibnizian property (5.6), for all $Z \in \mathfrak{X}(M)$ gives

$$\begin{aligned}
2g(\nabla_X fY, Z) =& X(g(fY,Z)) + fY(g(Z,X)) - Z(g(X,fY)) \\
&- g(X,[fY,Z]) + g(fY,[Z,X]) + g(Z,[X,fY]) \\
=& X(fg(Y,Z)) + fY(g(Z,X)) - Z(fg(X,Y)) \\
&- g(X, f[Y,Z] - (Zf)Y) + fg(Y,[Z,X]) + g(Z, f[X,Y] + (Xf)Y) \\
=& 2fg(\nabla_X Y, Z) + (Xf)g(Y,Z) - (Zf)g(X,Y) + g(X,(Zf)Y) + g(Z,(Xf)Y) \\
=& 2fg(\nabla_X Y, Z) + 2g((Xf)Y, Z) = 2g(f\nabla_X Y + (Xf)Y, Z).
\end{aligned}$$

The symmetry (5.9) follows from the simple calculation,

$$\begin{aligned}
&2g(\nabla_X Y - \nabla_Y X, Z) \\
&\quad = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \\
&\qquad - Y(g(X,Z)) - X(g(Z,Y)) + Z(g(Y,X)) + g(Y,[X,Z]) - g(X,[Z,Y]) - g(Z,[Y,X]) \\
&\quad = g(Z,[X,Y]) - g(Z,[Y,X]) = 2g([X,Y],Z).
\end{aligned}$$

In a similar way, we can prove that (5.10) implies the remaining formulas (5.4), (5.5), and (5.8). ☐

Let us create the map $F(X,Y)\colon \mathfrak{X}(M) \to \mathfrak{F}(M)$ by

$$F(X,Y)\colon Z \mapsto X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]).$$

It is easy to show that $F(X,Y)$ is $\mathfrak{F}(M)$-linear, which gives $F(X,Y) \in \mathfrak{X}^*(M)$. By Theorem 5.2, ∇ is a Levi-Civita connection if and only if $2g(\nabla_X Y, Z) = F(X,Y)Z$ holds for all $X, Y, Z \in \mathfrak{X}(M)$. This condition is equivalent to $2(\nabla_X Y)^\flat = F(X,Y) \in \mathfrak{X}^*(M)$, that is,

$$\nabla_X Y = \frac{1}{2} F(X,Y)^\sharp,$$

whence the existence and uniqueness of a Levi-Civita connection follow.

Theorem 5.3. *Any pseudo-Riemannian manifold admits a unique Levi-Civita connection.*

[4] Jean-Louis Koszul (1921–2018), French mathematician

We can express the unique Levi-Civita connection on a pseudo-Riemannian n-manifold (M, g) by computing the Christoffel symbols in an arbitrary chart (U, φ) on M. Applying the Koszul formula on the coordinate vector fields we have

$$F(\partial_i, \partial_j) = \sum_{l=1}^{n} F(\partial_i, \partial_j)\partial_l \, dx_l = \sum_{l=1}^{n} \left(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{li} - \frac{\partial}{\partial x_l} g_{ij} \right) dx_l,$$

which implies

$$\nabla_{\partial_i}\partial_j = \frac{1}{2} F(\partial_i, \partial_j)^\sharp = \frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} g^{lk} \left(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{li} - \frac{\partial}{\partial x_l} g_{ij} \right) \partial_k,$$

and therefore we obtain the explicit formula

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{n} g^{lk} \left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right). \tag{5.11}$$

Consider the behaviour of the unique Levi-Civita connection from Theorem 5.3 in the case of an isometry $f\colon M \to N$. Since f is a diffeomorphism, we use the formulas (2.9) and (2.10), so for every $X, Y, Z \in \mathfrak{X}(M)$ we obtain

$$\begin{aligned} X(g_M(Y, Z)) &= X((f^* g_N)(Y, Z)) = X(g_N(f_* Y, f_* Z) \circ f) = f_* X(g_N(f_* Y, f_* Z)) \circ f, \\ g_M(X, [Y, Z]) &= (f^* g_N)(X, [Y, Z]) = g_N(f_* X, f_*[Y, Z]) \circ f = g_N(f_* X, [f_* Y, f_* Z]) \circ f. \end{aligned}$$

In light of this, the Koszul formula (5.10) gives

$$2g_N(f_*(\nabla_X Y), f_* Z) \circ f = 2g_M(\nabla_X Y, Z) = 2g_N(\nabla_{f_* X} f_* Y, f_* Z) \circ f,$$

and consequently

$$f_*(\nabla_X Y) = \nabla_{f_* X} f_* Y, \tag{5.12}$$

which means that the Levi-Civita connection is preserved by isometries.

Lemma 5.4. *The total covariant derivative commutes with the musical isomorphisms.*

Proof. Consider the flat, which lowers the p th contravariant index of an arbitrary tensor $A \in \mathfrak{T}_s^r(M)$, where for components we have

$$(A^\flat)^{i_1 \dots i_{p-1} i_{p+1} \dots i_r}_{j_1 \dots j_s k} = \sum_l A^{i_1 \dots i_{p-1} l i_{p+1} \dots i_r}_{j_1 \dots j_s} g_{kl} = \sum_l (A \otimes g)^{i_1 \dots i_{p-1} l i_{p+1} \dots i_r}_{j_1 \dots j_s k l} = (C(A \otimes g))^{i_1 \dots i_{p-1} i_{p+1} \dots i_r}_{j_1 \dots j_s k},$$

from which follows $A^\flat = C(A \otimes g)$, where $C = C^p_{s+2}$ is the corresponding contraction. For $X \in \mathfrak{X}(M)$, since $\nabla g = 0$, we have

$$\nabla_X(A^\flat) = \nabla_X C(A \otimes g) = C\nabla_X(A \otimes g) = C(\nabla_X A \otimes g + A \otimes \nabla_X g) = C(\nabla_X A \otimes g) = (\nabla_X A)^\flat,$$

so we obtain

$$\nabla(A^\flat)(\dots, X) = \nabla_X(A^\flat)(\dots) = (\nabla_X A)^\flat(\dots) = (\nabla A)(\dots, Y^\flat, \dots, X) = (\nabla A)^\flat(\dots, X),$$

which means that the flat commutes with the covariant derivative, $\nabla(A^\flat) = (\nabla A)^\flat$. Since the sharp and flat are inverses of each other (applied to the same index position), we have

$$(\nabla A)^\sharp = (\nabla(A^{\sharp\flat}))^\sharp = (\nabla(A^\sharp))^{\flat\sharp} = \nabla(A^\sharp),$$

so the sharp also commutes with the covariant derivative $\nabla(A^\sharp) = (\nabla A)^\sharp$. $\square$

5.3 Parallel transport

Let $\gamma\colon I \to M$ be the curve in a manifold M, and $\gamma'\colon I \to TM$ is its velocity. In some chart (U,φ) at $\gamma(t) \in M$ with the coordinate functions $x_i = \pi_i \circ \varphi$, for a function $f \in \mathfrak{F}(M)$ holds

$$\gamma'(t)f = \frac{d(f\circ\gamma)}{dt}(t) = \sum_{i=1}^{n} \frac{\partial(f\circ\varphi^{-1})}{\partial\pi_i}(\varphi(\gamma(t)))\frac{d(\pi_i\circ\varphi\circ\gamma)}{dt}(t) = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\gamma(t))\frac{d(x_i\circ\gamma)}{dt}(t).$$

If $\gamma_i = x_i \circ \gamma$ for $1 \le i \le n$ denotes the curve components, then the velocity in coordinates can be expressed as

$$\gamma'(t) = \sum_{i=1}^{n} \gamma_i'(t)\left(\frac{\partial}{\partial x_i}\right)_{\gamma(t)}. \tag{5.13}$$

A ***vector field along a curve*** $\gamma\colon I \to M$ is a smooth map $V\colon I \to TM$ which satisfies $V(t) \in T_{\gamma(t)}M$ for all $t \in I$, that is, $\pi \circ V = \gamma$. Let $\mathfrak{X}(\gamma)$ denotes the set of all vector fields along γ, and it is a module under $\mathfrak{F}(I)$. A basic example of a vector field along a curve γ is its velocity $\gamma' \in \mathfrak{X}(\gamma)$, which is expressed in coordinates by the formula (5.13).

A large class of examples is obtained from an arbitrary vector field $X \in \mathfrak{X}(M)$ for a curve $\gamma\colon I \to M$, by placing $X_\gamma = X \circ \gamma \in \mathfrak{X}(\gamma)$, that is, $X_\gamma(t) = X_{\gamma(t)}$ for $t \in I$. For $V \in \mathfrak{X}(\gamma)$ is said to be ***extendible*** if there exists a vector field X on a neighbourhood of the image of γ such that $V = X_\gamma$. However, if $\gamma(t_1) = \gamma(t_2)$ and $\gamma'(t_1) \neq \gamma'(t_2)$ for some $t_1, t_2 \in I$, then $\gamma' \in X_\gamma$ is not extendible. Moreover, the velocity of the injective immersion $\gamma\colon (-\pi,\pi) \to \mathbb{R}^2$, $\gamma(t) = (\sin 2t, \sin t)$ from Example 2.16 is not extensible.

A connection ∇ on a manifold M yields an operator $\nabla/dt\colon \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$ whose natural properties are expressed by the equalities

$$\frac{\nabla}{dt}(V+W) = \frac{\nabla V}{dt} + \frac{\nabla W}{dt}, \tag{5.14}$$

$$\frac{\nabla}{dt}(fV) = \frac{df}{dt}V + f\frac{\nabla V}{dt}, \tag{5.15}$$

$$\frac{\nabla X_\gamma}{dt}(t) = \nabla_{\gamma'(t)}X, \tag{5.16}$$

for $V, W \in \mathfrak{X}(\gamma), X \in \mathfrak{X}(M), f \in \mathfrak{F}(I)$.

Theorem 5.5. *Let ∇ be a connection on a manifold M, and $\gamma\colon I \to M$ is a curve. Then, there exists a unique operator $\nabla/dt\colon \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$ satisfying the formulas* (5.14), (5.15), *and* (5.16) *for all $V, W \in \mathfrak{X}(\gamma), X \in \mathfrak{X}(M), f \in \mathfrak{F}(I)$.*

Proof. Suppose that an operator ∇/dt satisfies the desired conditions. For an arbitrary $t_0 \in I$ consider a chart (U,φ) at $\gamma(t_0) \in M$ with $x_i = \pi_i \circ \varphi$. In a neighbourhood I_0 of $t_0 \in I$, where $\gamma(I_0) \subseteq U$, we can locally express $V \in \mathfrak{X}(\gamma)$ by $V(t) = \sum_j v_j(t)(\partial_j)_{\gamma(t)}$, where $v_j(t) = (V(t))(x_j)$. Using the formulas (5.14) and (5.15) we get

$$\frac{\nabla V}{dt} = \sum_{j=1}^{n}\frac{\nabla}{dt}\left(v_j(\partial_j)_\gamma\right) = \sum_{j=1}^{n}\left(v_j'(\partial_j)_\gamma + v_j\frac{\nabla}{dt}(\partial_j)_\gamma\right).$$

Applying the formula (5.16) with locally expressed $\gamma'(t)$ by the components $\gamma_i = x_i \circ \gamma$ from (5.13), we have

$$\frac{\nabla(\partial_j)_\gamma}{dt}(t) = \nabla_{\gamma'(t)}\partial_j = \sum_{i=1}^{n}\gamma_i'(t)(\nabla_{\partial_i}\partial_j)_{\gamma(t)} = \sum_{i=1}^{n}\gamma_i'(t)\sum_{k=1}^{n}\Gamma_{ij}^k(\gamma(t))(\partial_k)_{\gamma(t)},$$

which finally gives

$$\frac{\nabla V}{dt}(t) = \sum_{k=1}^{n} \left(v_k'(t) + \sum_{i,j=1}^{n} v_j(t)\gamma_i'(t)\Gamma_{ij}^k(\gamma(t)) \right) (\partial_k)_{\gamma(t)}. \tag{5.17}$$

Therefore, a required operator must be unique, that is, at $t_0 \in I$ expressed by (5.17). For existence, in the general case we can cover $\gamma(I)$ with coordinate neighbourhoods and define the operator by (5.17) in each chart. Then uniqueness implies that various definitions agree whenever two or more charts overlap. Of course, we should prove that it satisfies the properties (5.14), (5.15), and (5.16). □

The ***covariant derivative*** of $V \in \mathfrak{X}(\gamma)$ ***along*** $\gamma\colon I \to M$ is $V' = \nabla V/dt \in \mathfrak{X}(\gamma)$. We say that $V \in \mathfrak{X}(\gamma)$ is ***parallel along*** γ with respect to ∇ if $V' \equiv 0$. A vector field $X \in \mathfrak{X}(M)$ turns out to be parallel ($\nabla X = 0$) if it is parallel along every curve in M.

Example 5.6. Consider the standard connection $D_X Y = \sum_{i=1}^n X(Y^i)\partial_i$ on $\mathbb{R}^n$ from the formula (5.3). For such D we have $\Gamma_{ij}^k \equiv 0$, so by (5.17) follows $V'(t) = \sum_k v_k'(t)(\partial_k)_{\gamma(t)}$. Thus, $V \in \mathfrak{X}(\gamma)$ is parallel along curve if and only if $v_k' \equiv 0$ for all $1 \leq k \leq n$, which means that all functions v_k are constant, that is, V is a constant vector field along curve. △

The fundamental fact about parallel vector fields is that every tangent vector at any point on a curve can be uniquely extended to a parallel vector field along the entire curve.

Theorem 5.6. *Given a curve $\gamma\colon I \to M$, $t_0 \in I$, and a vector $V_0 \in T_{\gamma(t_0)}M$, there exist a unique parallel vector field V along γ such that $V(t_0) = V_0$.*

Proof. First suppose that $\gamma(I)$ is contained in a single chart. In coordinates of that chart we use (5.17), so V is parallel along γ if and only if

$$v_k'(t) = -\sum_{i,j=1}^{n} v_j(t)\gamma_i'(t)\Gamma_{ij}^k(\gamma(t))$$

holds for all $1 \leq k \leq n$. On the other hand, the initial condition $V(t_0) = V_0$ transforms into $v_k(t_0) = V_0(x_k)$ for $1 \leq k \leq n$. This is a linear system of ordinary differential equations with the initial condition, so we have existence and uniqueness of a solution on all of I. If $\gamma(I)$ is not covered by a single chart we consider a as a supremum of all $b > t_0$ for which there exists a unique desired parallel vector field on $[t_0, b]$. For b close enough to t_0, the image $\gamma[t_0, b]$ is contained in a single chart, so we have $a > t_0$. If $a \in I$, we can choose a coordinate neighbourhood that contains $\gamma(a-\varepsilon, a+\varepsilon)$ for some $\varepsilon > 0$. Now, on $(a-\varepsilon, a+\varepsilon)$ there is a parallel vector field W with the initial condition $W(a-\varepsilon/2) = V(a-\varepsilon/2)$. By uniqueness on the common domain it follows that W is an extension of V that past a, which is a contradiction. Similarly, the proof works for values b less than t_0. □

The vector field V along a curve γ from the previous theorem we call the ***parallel transport*** of V_0 along γ. For $a, b \in I$ we define the ***parallel transport operator*** $P_a^b\colon T_{\gamma(a)}M \to T_{\gamma(b)}M$ with $P_a^b V_a = V(b)$, where V is the parallel transport of V_a along γ.

We say that a connection ∇ on a pseudo-Riemannian manifold (M, g) is ***compatible*** with the metric g if the parallel transport operator keeps the metric. In other words, compatibility of connection and metric, for every curve γ and parallel vector fields $V, W \in \mathfrak{X}(\gamma)$ yields the equality

$$g_{\gamma(t)}(V(t), W(t)) = g_{\gamma(t_0)}(V(t_0), W(t_0)) = \text{Const},$$

which is an obvious consequence of the more general equality

$$\frac{d}{dt}g(V, W) = g\left(\frac{\nabla V}{dt}, W\right) + g\left(V, \frac{\nabla W}{dt}\right). \tag{5.18}$$

Lemma 5.7. *A connection ∇ of a pseudo-Riemannian manifold (M, g) is compatible with g if and only if for any two vector fields V and W along a curve $\gamma\colon I \to M$ holds* (5.18)*.*

Proof. If we choose an orthonormal basis in $T_{\gamma(t_0)}M$, and then by Theorem 5.6 extend the basis vectors to the corresponding parallel $E_1, \dots, E_n \in \mathfrak{X}(\gamma)$, then the compatibility between ∇ and g for any $t \in I$ gives an orthonormal basis $(E_1(t), \dots, E_n(t))$ in $T_{\gamma(t)}M$. For arbitrary $V = \sum_i v_i E_i$ and $W = \sum_j w_j E_j$ we have $g(V, W) = \sum_{i,j} v_i w_j \delta_{ij} \varepsilon_i = \sum_i \varepsilon_i v_i w_i$. On the other hand, from $E_i' \equiv 0$ we have $V' = \sum_i v_i' E_i$ and $W' = \sum_j w_j' E_j$, and therefore we have $g(V, W) = \sum_{i,j} \varepsilon_i v_i w_j \delta_{ij} = \sum_i v_i w_i$, which implies (5.18). □

Theorem 5.8. *A connection ∇ of a pseudo-Riemannian manifold (M, g) is compatible with g if and only if it is a metric connection.*

Proof. Let ∇ is compatible with g and $X, Y, Z \in \mathfrak{X}(M)$. For an arbitrary curve γ such that $\gamma(0) = p$, $\gamma'(0) = X_p$ holds and vector fields $V = Y_\gamma$, $W = Z_\gamma$, from Lemma 5.7 we have $(g(Y_\gamma, Z_\gamma))' = g(Y_\gamma', Z_\gamma) + g(Y_\gamma, Z_\gamma')$. Because of

$$\frac{d}{dt} g(Y_\gamma, Z_\gamma) = \frac{d}{dt}(g(Y, Z) \circ \gamma) = \gamma_* \frac{d}{dt}(g(Y, Z)) = \gamma'(t)(g(Y, Z)), \tag{5.19}$$

for $t = 0$ we obtain

$$X_p(g(Y, Z)) = g_p(\nabla_{X_p} Y, Z_p) + g_p(Y_p, \nabla_{X_p} Z),$$

which proves the formula (5.8), at each point p, and therefore ∇ is a metric connection.

Conversely, it is enough to check the claim for curves γ that lie entirely in some coordinate neighbourhood. Let us set basis vector fields along a curve with $E_i = (\partial_i)_\gamma$, and express $V = \sum_i v_i E_i$ and $W = \sum_j w_j E_j$. On the left hand side we have

$$\frac{d}{dt} g(V, W) = \sum_{i,j=1}^{n} \left(v_i' w_j g(E_i, E_j) + v_i w_j' g(E_i, E_j) + v_i w_j \frac{d}{dt} g(E_i, E_j) \right),$$

while the right hand side gives

$$g(V', W) + g(V, W') = \sum_{i,j=1}^{n} \left(v_i' w_j g(E_i, E_j) + v_i w_j g(E_i', E_j) + v_i w_j' g(E_i, E_j) + v_i w_j g(E_i, E_j') \right),$$

so it is enough to prove the claim for basis vector fields $V = E_i$, $W = E_j$. Because of the formula (5.19), it remains to prove

$$\gamma'(t)(g(\partial_i, \partial_j)) = g_{\gamma(t)}(\nabla_{\gamma'(t)} \partial_i, (\partial_j)_{\gamma(t)}) + g_{\gamma(t)}((\partial_i)_{\gamma(t)}, \nabla_{\gamma'(t)} \partial_j).$$

However, since ∇ is metric, the required compatibility follows from the equality (5.8) for a vector field $X \in \mathfrak{X}(M)$ such that $X_p = \gamma'(0)$ with $Y = \partial_i$, $Z = \partial_j$. □

Let $V, W \in \mathfrak{X}(\gamma)$ be parallel and $\alpha, \beta \in \mathbb{R}$. Because of $(\alpha V + \beta W)' = \alpha V' + \beta W' = 0$ we have $P_a^b(\alpha V(a) + \beta W(a)) = \alpha V(b) + \beta W(b)$, which proves that P_a^b is linear. If $P_a^b(V(a)) = 0$ then by uniqueness $V = 0$ and therefore $V(a) = 0$, so P_a^b is injective hence bijective. Bearing in mind Theorem 5.8, we arrive at the following crucial property of parallel translation.

Theorem 5.9. *Parallel transport is a linear isometry in the case of Levi-Civita connection.*

The covariant differentiation along a curve γ can be recovered from the parallel transport. We can start with basis vectors in $T_{\gamma(a)}M$ to get the corresponding parallel vector fields $E_1, \dots, E_n \in \mathfrak{X}(\gamma)$, where $E_j' \equiv 0$ and $E_j(t) = P_a^t E_j(a)$ for $1 \le j \le n$. From $V = \sum_j v_j E_j$ we obtain

$$V'(a) = \sum_{j=1}^{n} (v_j E_j)'(a) = \sum_{j=1}^{n} v_j'(a) E_j(a) = \lim_{t \to a} \sum_{j=1}^{n} \frac{v_j(t) - v_j(a)}{t - a} E_j(a) = \lim_{t \to a} \frac{(P_a^t)^{-1} V(t) - V(a)}{t - a}.$$

Lemma 5.10. *For a vector field V along a curve holds*

$$\frac{\nabla V}{dt}(a) = \lim_{t\to a} \frac{(P_a^t)^{-1}V(t) - V(a)}{t-a}.$$

5.4 Geodesics

Let ∇ be a connection on a manifold M, and γ is a curve in M. The covariant derivative along γ allows to define the ***acceleration*** of γ by $(\gamma')' \in \mathfrak{X}(\gamma)$. A curve γ is called ***geodesic*** with respect to ∇ if its acceleration is zero, $(\gamma')' \equiv 0$. In other words, a geodesic can be characterized as a curve whose velocity vector field is parallel along the curve.

The covariant derivative along a curve for $V \in \mathfrak{X}(\gamma)$ we calculate in some coordinate neighbourhood according to the formula (5.17). After the substitution $V = \gamma'$ we have $v_j = \gamma_j'$ for $1 \le j \le n$, so

$$\frac{\nabla\gamma'}{dt}(t) = \sum_{k=1}^{n}\left(\gamma_k''(t) + \sum_{i,j=1}^{n}\gamma_j'(t)\gamma_i'(t)\Gamma_{ij}^k(\gamma(t))\right)(\partial_k)_{\gamma(t)}.$$

The geodesic condition $(\gamma')' \equiv 0$ establishes the system of ordinary differential equations of second order,

$$\gamma_k''(t) + \sum_{i,j=1}^{n}\gamma_j'(t)\gamma_i'(t)\Gamma_{ij}^k(\gamma(t)) = 0 \tag{5.20}$$

for all $1 \le k \le n$, called the ***local geodesics equations***. For each $t_0 \in I$, there exists $\varepsilon > 0$ such that $\gamma(t_0 - \varepsilon, t_0 + \varepsilon)$ is contained in some coordinate neighbourhood, so γ is geodesic if and only if the corresponding restriction satisfies the local geodesics equations in every chart whose domain intersects the image of γ.

Theorem 5.11. *Let ∇ be a connection on a manifold M. For any $p \in M$ and $V \in T_pM$ there exist an open interval $0 \in I \subseteq \mathbb{R}$ and a geodesic $\gamma\colon I \to M$ such that $\gamma(0) = p$, $\gamma'(0) = V$. Any two such geodesics agree on their common domain.*

Proof. An usual trick is to introduce auxiliary functions $\xi_k = \gamma_k'$ that converts the local geodesics equations (5.20) to the equivalent first-order system,

$$\begin{aligned}\gamma_k'(t) &= \xi_k(t),\\ \xi_k'(t) &= -\sum_{i,j=1}^{n}\xi_i(t)\xi_j(t)\Gamma_{ij}^k(\gamma(t)),\end{aligned}$$

in twice the number of variables and equations. According to Picard[5]–Lindelöf[6] theorem (for a system of ordinary differential equations of first order with initial condition) for some $\varepsilon > 0$ there exists a unique solution

$$\zeta\colon (-\varepsilon,\varepsilon) \to M\times\mathbb{R}^n, \quad \zeta(t) = (\gamma_1(t),\dots,\gamma_n(t),\xi_1(t),\dots,\xi_n(t))$$

satisfying the initial condition $\zeta(0) = (p, V)$, so the geodesic is $\gamma(t) = (\gamma_1(t),\dots,\gamma_n(t))$.

For the uniqueness part, let $\gamma_1, \gamma_2\colon I \to M$ be two geodesics such that $\gamma_1'(0) = \gamma_2'(0)$. For $a = \inf\{t \in I : t > 0,\ \gamma_1(t) \ne \gamma_2(t)\} > 0$ we have $\gamma_1'(t) = \gamma_2'(t)$ on $(0, a)$, and from continuity $\gamma_1'(a) = \gamma_2'(a)$ holds. Now, $t \mapsto \gamma_1(a+t)$ and $t \mapsto \gamma_2(a+t)$ are geodesics with the initial velocity $\gamma_1'(a) = \gamma_2'(a)$, so γ_1 and γ_2 agree on some open interval containing a, which is a contradiction. There is a similar approach for values $t < 0$ which completes the proof. □

[5] Charles Émile Picard (1856–1941), French mathematician

[6] Ernst Leonard Lindelöf (1870–1946), Finish mathematician

A geodesic $\gamma\colon I \to M$ is ***maximal*** if there is no other geodesic with an open domain that strictly contains I, such that agrees with γ on I. From Theorem 5.11 directly follows that for any $V \in TM$ there is a unique maximal geodesic γ_V with $\gamma_V'(0) = V$.

If the domain of every maximal geodesic that goes through $p \in M$ is the whole $\mathbb{R}$, we say that M is ***geodesically complete at point*** p. For a pseudo-Riemannian manifold is said to be ***geodesically complete*** if it is geodesically complete at every its point. Example 4.23 shows that there are manifolds which are not geodesically complete.

Example 5.7. Consider the pseudo-Riemannian space $\mathbb{R}^n_\nu$ of index ν. The Christoffel symbols of the Levi-Civita connection we calculate according to the formula (5.11), but since the coefficients of metric are constant it follows $\Gamma^k_{ij} \equiv 0$. For geodesics γ we have $\sum_k \gamma_k''(t)(\partial_k)_{\gamma(t)} = 0$, so $\gamma_k'' \equiv 0$ for all $1 \le k \le n$. Thus, geodesics have a form $t \mapsto p + tV$ for some $p, V \in \mathbb{R}^n_\nu$. △

Example 5.8. Consider the hyperbolic half-plane $\mathbf{HU}^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ with the Riemannian metric $g = (dx_1^2 + dx_2^2)/x_2^2$ from Example 4.21. The components of Riemannian metric one can see from the following matrices,

$$g = \begin{pmatrix} 1/x_2^2 & 0 \\ 0 & 1/x_2^2 \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} x_2^2 & 0 \\ 0 & x_2^2 \end{pmatrix}.$$

First, we calculate the Christoffel symbols of the Levi-Civita connection by (5.11), and because of $g^{ab} = x_2^2 \delta_{ab}$ we have

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{2} g^{lk}\left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right) = \frac{1}{2}x_2^2\left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right).$$

Since

$$\frac{\partial g_{ab}}{\partial x_c} = \delta_{ab}\delta_{c2}\frac{\partial(1/x_2^2)}{\partial x_2} = -\frac{2}{x_2^3}\delta_{ab}\delta_{c2},$$

we have

$$\Gamma^k_{ij} = -\frac{1}{x_2}(\delta_{jk}\delta_{i2} + \delta_{ki}\delta_{j2} - \delta_{ij}\delta_{k2}),$$

from where we get

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0, \quad \Gamma^2_{11} = -\Gamma^1_{12} = -\Gamma^2_{22} = \frac{1}{x_2}. \tag{5.21}$$

We want to find geodesics $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ using the local geodesic equations (5.20),

$$\gamma_1''(t) + \sum_{i,j}\gamma_j'(t)\gamma_i'(t)\Gamma^1_{ij}(\gamma(t)) = 0, \quad \gamma_2''(t) + \sum_{i,j}\gamma_j'(t)\gamma_i'(t)\Gamma^2_{ij}(\gamma(t)) = 0,$$

which by substitution (5.21) become

$$\gamma_1'' - 2\gamma_1'\gamma_2'\frac{1}{\gamma_2} = 0, \quad \gamma_2'' + ((\gamma_1')^2 - (\gamma_2')^2)\frac{1}{\gamma_2} = 0. \tag{5.22}$$

The equations (5.22) are solved by interpreting two cases, where in simpler one holds $\gamma_1' = 0$, that is, $\gamma_1 = C = \text{Const}$. This satisfies the first equation, while the second equation becomes $\gamma_2'' - (\gamma_2')^2/\gamma_2 = 0$, and after dividing by $\gamma_2 > 0$ we get

$$\frac{\gamma_2''\gamma_2 - \gamma_2'\gamma_2'}{(\gamma_2)^2} = \left(\frac{\gamma_2'}{\gamma_2}\right)' = 0.$$

Then $\gamma_2'/\gamma_2 = (\ln\gamma_2)' = D$, so $\ln\gamma_2 = Dt + E$ and finally $\gamma_2 = e^{Dt+E}$. Thus, the first type of geodesics are the curves

$$\gamma(t) = (C, e^{Dt+E}),$$

which are open rays perpendicular to the x_1-axis.

In the second case, we have $\gamma_1' \neq 0$, where from the first equation of (5.22) implies

$$\frac{\gamma_1''}{\gamma_1'} - 2\frac{\gamma_2'}{\gamma_2} = (\ln|\gamma_1'| - 2\ln\gamma_2)' = \left(\ln\frac{|\gamma_1'|}{\gamma_2^2}\right)' = 0,$$

so $\gamma_1' = C\gamma_2^2$. Then we have

$$\frac{\gamma_2''\gamma_2 - \gamma_2'\gamma_2'}{(\gamma_2)^2} + C^2\gamma_2^2 = \left(\frac{\gamma_2'}{\gamma_2}\right)' + C^2\gamma_2^2 = 0.$$

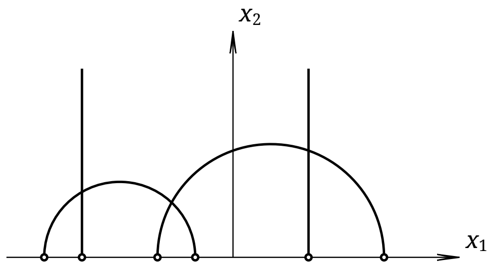
If we introduce $f = \gamma_2'/\gamma_2 = (\ln\gamma_2)'$, the equation becomes $f' + C^2\gamma_2^2 = 0$, and hence $f' < 0$. By differentiation we obtain $f'' + 2C^2\gamma_2\gamma_2' = f'' - 2f'f = (f' - f^2)' = 0$, so $f' = f^2 - A^2$ for some constant $A > 0$. Then we have,

$$B + \int dt = \int \frac{df}{f^2 - A^2} = -\frac{1}{A}\int \frac{d(f/A)}{1 - (f/A)^2} = -\frac{1}{A}\operatorname{artanh}\left(\frac{f}{A}\right),$$

that gives $f = -A\tanh(A(t+B)) = (\ln\gamma_2)'$. It follows $\gamma_2 = r/(\cosh(A(t+B)))$, where from $f' + C^2\gamma_2^2 = 0$ we get $A^2 = C^2r^2$. From $\gamma_1' = C\gamma_2^2$ we obtain $\gamma_1 = (Cr^2/A)\tanh(A(t+B)) + l$. Thus, the second type of geodesics are the curves of form

$$\gamma(t) = \left(\pm r\tanh(A(t+B)) + l, \frac{r}{\cosh(A(t+B))}\right),$$

for which $(\gamma_1 - l)^2 + \gamma_2^2 = r^2$ holds, so geometrically these are half-circles with centre $(l, 0)$ and radius r.



△

5.5 Exponential map

Let ∇ be an arbitrary connection on a manifold M. According to Theorem 5.11 each initial velocity vector $V \in TM$ determines a unique maximal geodesic γ_V with $\gamma_V'(0) = V$ and $\gamma_V(0) = \pi V$. For a deeper understanding of geodesics, it is necessary to figure out how they change when we vary the initial tangent vector. Geodesics with proportional initial velocities are closely related, which is discussed in the following rescaling lemma.

Lemma 5.12. *For any $V \in TM$ and $c, t \in \mathbb{R}$ holds $\gamma_{cV}(t) = \gamma_V(ct)$, whenever either side of the equality is defined.*

Proof. We assume $c \neq 0$ since for $c = 0$ both sides of the equality are equal to πV. It is sufficient to prove the lemma whenever the right-hand side is defined, since we obtain the converse when we replace V, t, c with $cV, ct, 1/c$ respectively. For $\gamma = \gamma_V\colon I \to M$ we define the new curve $\psi\colon (1/c)I \to M$ by $\psi(t) = \gamma(ct)$, where $\psi(0) = \gamma(0) = \pi V$ is immediately valid. In local coordinates $x_i = \pi_i \circ \varphi$ of some chart (U, φ) we have components $\gamma_i = x_i \circ \gamma$ and $\psi_i = x_i \circ \psi$. Then we obtain $\psi_i'(t) = (d/dt)\gamma_i(ct) = c\gamma_i'(ct)$, and especially $\psi'(0) = c\gamma'(0) = cV$. Since $(\gamma')'(ct) = 0$, we have

$$\psi_k''(t) + \sum_{i,j} \psi_j'(t)\psi_i'(t)\Gamma_{ij}^k(\psi(t)) = c^2\gamma_k''(ct) + c^2 \sum_{i,j} \gamma_j'(ct)\gamma_i'(ct)\Gamma_{ij}^k(\gamma(ct)) = 0,$$

so ψ is a geodesic with the initial condition $\psi'(0) = cV$ and therefore $\psi = \gamma_{cV}$. □

Let $\mathcal{E} \subseteq TM$ be the set of all tangent vectors $V \in TM$ such that the maximal geodesic γ_V is defined on an interval containing $[0, 1]$. The map $\exp\colon \mathcal{E} \to M$ defined by the equation $\exp V = \gamma_V(1)$ is called the ***exponential map***, and we use the same name for its restriction $\exp_p\colon \mathcal{E}_p = \mathcal{E} \cap T_pM \to M$ for a point $p \in M$.

Lemma 5.12 allows to obtain from an existing maximal geodesic a new geodesic whose domain interval will be arbitrary large, by sufficiently reducing the initial vector. If $tV \in \mathcal{E}$, then $\gamma_{tV}(1)$ is defined, which implies that the geodesics γ_V has a form

$$\gamma_V(t) = \gamma_{tV}(1) = \exp(tV).$$

Therefore, the maximal geodesics through $p \in M$ with initial velocity $V \in T_pM$ has the form $t \mapsto \exp_p tV$. Thus, it follows that straight lines of the tangent space T_pM through the origin $0_p \in T_pM$ are mapped by $\exp_p$ onto geodesics. These geodesics are called the ***radial geodesics*** through p, and analogously we can speak of radial geodesic segments and radial geodesic rays emanating from p.

Standard results for solutions of ordinary differential equations ensure that $\exp_p(V)$ depends smoothly on both V and p, so $\exp_p$ is well defined and smooth in some neighbourhood of the origin $0_p \in T_pM$.

Let $\tau\colon I \to T_pM$ be the curve in the tangent space defined by $\tau(t) = tV$. It is easy to see that for $V_{0_p} = \tau'(0) \in T_{0_p}(T_pM)$ we have

$$(T_{0_p}\exp_p)(V_{0_p}) = (T_{0_p}\exp_p)(\tau'(0)) = (\exp_p \circ \tau)'(0) = \gamma_V'(0) = V,$$

and therefore $T_{0_p}\exp_p\colon T_{0_p}(T_pM) \to T_pM$ is a canonical map $V_{0_p} \mapsto V$, which is the identity map under the identification $T_{0_p}(T_pM) \cong T_pM$.

Lemma 5.13. *Let p be an arbitrary point of a pseudo-Riemannian manifold M. The tangent map of the exponential map $\exp_p$ at the origin $0_p \in T_pM$ is the identity map under the canonical identification.*

Since the tangent map $T_{0_p}\exp_p$ is an isomorphism, the inverse function theorem (Theorem 2.8) implies that $\exp_p$ is a local diffeomorphism at the origin 0_p.

Theorem 5.14. *Let ∇ be a connection on a manifold M and $p \in M$. Then, there is a neighbourhood $0_p \in \mathcal{U} \subseteq T_pM$ and a neighbourhood $p \in U \subseteq M$, such that $\exp_p \restriction_{\mathcal{U}}\colon \mathcal{U} \to U$ is a diffeomorphism.*

A subset $\mathcal{U} \ni 0$ of a vector space $\mathcal{V}$ is called ***star-shaped*** at 0 if $V \in \mathcal{U}$ implies $tV \in \mathcal{U}$ for all $t \in [0, 1]$. Let us notice that $\mathcal{E}_p$ is star-shaped at $0_p \in T_pM$. If the domain $\mathcal{U}$ of a diffeomorphism from Theorem 5.14 is star-shaped at 0_p then its image $U = \exp_p(\mathcal{U})$ is called a ***normal neighbourhood*** of p.

Let (M, g) be a pseudo-Riemannian manifold of dimension n and let $p \in M$ be an arbitrary point. A choice of orthonormal basis $(E_1, \ldots E_n)$ in the scalar product space (T_pM, g_p)

is equivalent to an isometry $L\colon T_pM \to \mathbb{R}^n_\nu$ given by $L(\sum_{i=1}^n a_i E_i) = (a_1, \dots, a_n)$. If U is a normal neighbourhood of $p \in M$, then $\varphi\colon U \to \mathbb{R}^n$ given by $\varphi = L \circ \exp_p^{-1}\restriction_U$ is a chart at $p \in M$. Coordinates $x_i = \pi_i \circ \varphi$ are called the ***normal coordinates*** centred at $p \in M$. Normal coordinates are extremely useful since they yield a very simple representation for geodesics.

Theorem 5.15. *Let (M, g) be a pseudo-Riemannian n-manifold endowed with its Levi-Civita connection ∇, and let (U, φ) be a chart with normal coordinates centred at $p \in M$. Then for all $1 \leq i, j, k \leq n$ we have*

$$\Gamma^k_{ij}(p) = 0, \quad g_{ij}(p) = \varepsilon_i \delta_{ij}, \quad \partial_k g_{ij}(p) = 0.$$

Proof. Every vector $0 \neq V = \sum_{i=1}^n v_i E_i \in T_pM$ determines the radial geodesic γ_V through p of form $\gamma_V(t) = \exp_p(tV)$ for $t \in I$ where $0 \in I \subseteq \mathbb{R}$ is some open interval. In normal coordinates centred at p we have $\varphi \circ \gamma_V(t) = L(tV) = (tv_1, \dots tv_n)$, so $\gamma_i = x_i \circ \gamma_V = tv_i$ holds for $1 \leq i \leq n$, which must satisfy the local geodesic equations (5.20), and we obtain $\sum_{i,j} v_i v_j \Gamma^k_{ij}(\gamma_V(t)) = 0$ for $1 \leq k \leq n$. This is certainly true for $t = 0$ where $\gamma_V(0) = p$, whence every choice of $(v_1, \dots, v_n) \in \mathbb{R}^n$ yields

$$\sum_{i,j=1}^n v_i v_j \Gamma^k_{ij}(p) = 0$$

for all $1 \leq k \leq n$. A suitable choice is $V = E_i$ from which we get $\Gamma^k_{ii}(p) = 0$, and after that for $V = E_i + E_j$ we have $\Gamma^k_{ii}(p) + \Gamma^k_{ij}(p) + \Gamma^k_{ji}(p) + \Gamma^k_{jj}(p) = 0$, which implies $\Gamma^k_{ij}(p) + \Gamma^k_{ji}(p)$, and since ∇ is symmetric, we obtain $\Gamma^k_{ij}(p) = 0$, so all Christoffel symbols vanish at p.

Consider the remaining properties. From $V(x_i) = (\gamma'_V(0))(x_i) = (x_i \circ \gamma_V)'(0) = \gamma'_i(0) = v_i$ we have $V = \sum_{i=1}^n v_i (\partial_i)_p$, which for $V = E_i = (\partial_i)_p$ yields $g_p(\partial_i, \partial_j) = \varepsilon_i \delta_{ij}$. Since ∇ is metric we have $\partial_k g_{ij} = g(\nabla_{\partial_k} \partial_i, \partial_j) + g(\partial_i, \nabla_{\partial_k} \partial_j) = \sum_l (\Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il})$, and therefore $\partial_k g_{ij}(p) = 0$ because ∇ is also symmetric. $\square$

In Theorem 5.14 we showed that every point of a pseudo-Riemannian manifold has a normal neighbourhood. We say that a normal neighbourhood is ***totally normal*** if it is a normal neighbourhood of each of its points.

Theorem 5.16. *Every point of a pseudo-Riemannian manifold has a totally normal neighbourhood.*

Proof. We start from a normal coordinate neighbourhood $U \subseteq M$ of a point $p \in M$. We define the map $E\colon \mathcal{E}_U \to U \times U$ by $E(q, V) = (q, \exp_q V)$, where $\mathcal{E}_U = \mathcal{E} \cap TU \subseteq U \times \mathbb{R}^n$. The Jacobian matrix of the tangent map of E at the point $0_p = (p, 0)$ is a block lower triangular matrix with the identical matrices on the diagonal (the second of them comes from Lemma 5.13) and therefore it is invertible. According to the inverse function theorem (Theorem 2.8) there is a neighbourhood $W \subseteq TU$ of the point 0_p such that $E\restriction_W$ is a diffeomorphism onto a neighbourhood of $(p, p) \in U \times U$. Since E is continuous, we can assume that $E\restriction_W(W) = U_\epsilon \times U_\epsilon$ for some $\epsilon > 0$, where $U_\epsilon = \{q \in U : \sum_{i=1}^n (x_i(q))^2 < \epsilon\}$ is a Euclidean ball related to the normal coordinates $(x_1, \dots, x_n)$. Let the tensor $b \in \mathfrak{T}^0_2(U_\epsilon)$ be defined by its components related to the normal coordinates by

$$b_{ij} = \delta_{ij} - \sum_{k=1}^n \Gamma^k_{ij} x_k.$$

It is obviously symmetric and positive definite at p, so we can take a smaller $\epsilon > 0$ such that b is positive definite on the whole U_ϵ. For an arbitrary point q from such U_ϵ and

$W_q = W \cap T_qM$ we know that $E\restriction_{W_q}$ is a diffeomorphism onto $\{q\} \times U_\epsilon$, which means that $\exp_q \restriction_{W_q}$ is a diffeomorphism onto U_ϵ, and U_ϵ is a normal neighbourhood of each of its points q if we prove that W_q is star-shaped at 0_q.

Let $q \neq s \in U_\epsilon$ and $E(q, V) = (q, s)$, which means that $\gamma_V\colon [0, 1] \to M$ is a geodesic from from q to s. If we set its components by $\gamma_i = x_i \circ \gamma_V$ for $1 \leq i \leq n$, using the geodesic equations (5.20) we can calculate

$$0 < 2b(\gamma_V', \gamma_V') = 2\sum_{i,j=1}^{n} b_{ij}\gamma_i'\gamma_j' = 2\sum_{i,j=1}^{n}\left(\delta_{ij} - \sum_{k=1}^{n}\gamma_k(\Gamma_{ij}^k \circ \gamma_V)\right)\gamma_i'\gamma_j'$$

$$= 2\sum_{i=1}^{n}(\gamma_i')^2 - 2\sum_{i,j,k=1}^{n}\gamma_k\gamma_i'\gamma_j'(\Gamma_{ij}^k \circ \gamma_V) = 2\sum_{k=1}^{n}(\gamma_k')^2 + 2\sum_{k=1}^{n}\gamma_k\gamma_k'' = \sum_{k=1}^{n}(2\gamma_k\gamma_k')' = \sum_{k=1}^{n}(\gamma_k^2)'',$$

from which it follows that the function $\sum_{k=1}^{n}\gamma_k^2$ is convex and cannot have a maximum on the interval $(0, 1)$. Therefore $\gamma_V([0, 1]) \subseteq U_\epsilon$, so W_q is star-shaped at 0_q and U_ϵ is a normal neighbourhood of the point q which completes the proof. □

5.6 Geodesics and minimizing curves

A ***two-parameter map*** is a smooth map $f\colon I \times J \to M$, where $I, J \subseteq \mathbb{R}$ are open intervals. Vector fields along a curve can be generalised to vector fields along a two-parameter map. Thus, a vector field $Z \in \mathfrak{X}(f)$ along f means a smooth map $Z\colon I \times J \to TM$ such that $\pi \circ Z = f$. The partial velocities $(\partial f/\partial t)$ and $(\partial f/\partial s)$ are vector fields along $f = f(t, s)$ defined by

$$\frac{\partial f}{\partial t}(t, s) = T_{(t,s)}f\left(\frac{\partial}{\partial t}\right)_{(t,s)} = (f_s)'(t), \quad \frac{\partial f}{\partial s}(t, s) = T_{(t,s)}f\left(\frac{\partial}{\partial s}\right)_{(t,s)} = (f^t)'(s),$$

where $f_s\colon I \to M$ and $f^t\colon J \to M$ are given with $f_s(t) = f(t, s) = f^t(s)$. In some coordinate neighbourhood, from (2.2) we have

$$\frac{\partial f}{\partial t} = f_*\frac{\partial}{\partial t} = \sum_j \frac{\partial f_j}{\partial t}\frac{\partial}{\partial x_j} \circ f,$$

where $f_j = x_j \circ f$, so from the formula (5.17) follows

$$\frac{\nabla}{ds}\frac{\partial f}{\partial t} = \sum_k \left(\frac{\partial^2 f_k}{\partial s \partial t} + \sum_{i,j}\frac{\partial f_j}{\partial t}\frac{\partial f_i}{\partial s}(\Gamma_{ij}^k \circ f)\right)\partial_k \circ f.$$

In the case of symmetric connection, $\Gamma_{ij}^k = \Gamma_{ji}^k$ holds, and we obtain

$$\frac{\nabla}{ds}\frac{\partial f}{\partial t} = \frac{\nabla}{dt}\frac{\partial f}{\partial s}. \tag{5.23}$$

Let (M, g) be a pseudo-Riemannian manifold, and ∇ is its Levi-Civita connection. Note that since ∇ is symmetric, the formula (5.23) holds. Let us fix $p \in M$, and let $V, W \in T_pM$ be such that $0 \neq V \in \mathcal{E}_p$. Consider the two-parameter map $\tau\colon I \times J \to T_pM$, $\tau(t, s) = tV + tsW$ and its exponential image in M, $f\colon I \times J \to M$, $f(t, s) = \exp_p(tV + tsW)$. It is not hard to see that there exists a small $\varepsilon > 0$ such that for intervals $I = (-\varepsilon, 1 + \varepsilon)$ and $J = (-\varepsilon, \varepsilon)$ we have $tV + tsW \in \mathcal{E}_p$. For any $s \in J$, the curve $f_s\colon I \to M$, $f_s(t) = f(t, s)$ is the restriction of the maximal geodesic γ_{V+sW}, and therefore

$$g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right)(t, s) = g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right)(0, 0) = g_p(V + sW, V + sW), \quad \frac{\nabla}{dt}\frac{\partial f}{\partial t} \equiv 0.$$

However, using $\nabla g = 0$ and the symmetry (5.23) we have

$$\begin{aligned}\frac{\partial}{\partial t} g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) &= g\left(\frac{\nabla}{dt}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) + g\left(\frac{\partial f}{\partial t}, \frac{\nabla}{dt}\frac{\partial f}{\partial s}\right) \\ &= g\left(\frac{\partial f}{\partial t}, \frac{\nabla}{ds}\frac{\partial f}{\partial t}\right) = \frac{1}{2}\frac{\partial}{\partial s} g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) = g_p(V + sW, W),\end{aligned}$$

which gives

$$g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) = t g_p(V + sW, W) + g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)(0,0) = t g_p(V + sW, W).$$

Since

$$\frac{\partial \tau}{\partial t}(t,s) = (V + sW)_{tV+tsW}, \quad \frac{\partial \tau}{\partial s}(t,s) = (tW)_{tV+tsW},$$

we have

$$\begin{aligned}T_V \exp_p(V_V) &= T_V \exp_p \circ T_{(1,0)}\tau\left(\frac{\partial}{\partial t}\right) = T_{(1,0)}(\exp_p \circ \tau)\left(\frac{\partial}{\partial t}\right) = \frac{\partial f}{\partial t}(1,0), \\ T_V \exp_p(W_V) &= T_V \exp_p \circ T_{(1,0)}\tau\left(\frac{\partial}{\partial s}\right) = T_{(1,0)}(\exp_p \circ \tau)\left(\frac{\partial}{\partial s}\right) = \frac{\partial f}{\partial s}(1,0),\end{aligned}$$

and therefore

$$g_{\exp_p V}(T_V \exp_p(V_V), T_V \exp_p(W_V)) = g_p(V, W). \tag{5.24}$$

This result is known as the ***Gauss lemma***.

Lemma 5.17. *Let (M, g) be a pseudo-Riemannian manifold, $p \in M$, and $V, W \in T_pM$ such that $0 \neq V \in \mathcal{E}_p$, then the equality* (5.24) *holds.*

The most important case consider a connected Riemannian manifold (M, g), which is a metric space whose metric topology is the same as the manifold topology (Theorem 4.20). Let us fix a point $p \in M$. In the inner product space (T_pM, g_p) we define ***tangent spheres*** and ***tangent balls*** of radii $r > 0$ by

$$S_r(0_p) = \{X \in T_pM : \|X\| = r\}, \quad B_r(0_p) = \{X \in T_pM : \|X\| < r\}.$$

For r small enough, $B_r(p) = \exp_p B_r(0_p)$ is a normal neighbourhood of p, and then we say that $B_r(p)$ is a ***geodesic ball*** (or ***normal ball***) in M centred at p, while for all $\varepsilon < r$, $S_\varepsilon(p) = \exp_p S_\varepsilon(0_p)$ is a ***geodesic sphere*** (or ***normal sphere***) in M centred at p. Gauss lemma (Lemma 5.17) asserts that a geodesic sphere $S_\varepsilon(p)$ is perpendicular to all geodesics emanating from p, and allows the exponential map to be understood as a radial isometry.

An admissible curve $\gamma\colon [a,b] \to M$ in a Riemannian manifold M is said to be ***minimizing*** if $L(\gamma) \leq L(\psi)$ for every admissible curve ψ from $\gamma(a)$ to $\gamma(b)$. Of course, γ is minimizing if and only if $L(\gamma) = d(\gamma(a), \gamma(b))$ (provided that M is connected, which is what we assume here).

Theorem 5.18. *Let $B_\varepsilon(p)$ is a geodesic ball in a Riemannian manifold M centred at $p \in M$, and $p \neq q \in B_\varepsilon(p)$. The radial geodesic from p to q is up to reparametrization the unique minimizing curve from p to q in M.*

Proof. For the unique radial geodesic $\gamma\colon [0,1] \to M$ from p to q ($\gamma'(0) = \exp_p^{-1}(q)$) we have $L(\gamma) = \|\exp_p^{-1}(q)\| = r < \varepsilon$. Let $\psi\colon [a,b] \to M$ be an admissible curve from p to q. Let $a_0 \in [a,b]$ be the largest t such that $\psi(t) = p$ and let $b_0 \in [a_0, b]$ be the smallest t such that $\psi(t) \in S_r(p)$. For $t \in [a_0, b_0]$ we can express

$$\psi(t) = \exp_p(\rho(t)V(t)),$$

where $\rho\colon [a_0, b_0] \to [0,1]$ and $V\colon [a_0, b_0] \to S_r(0_p)$. Hence

$$\psi' = T_{\rho V}\exp_p \circ (\rho V)' = T_{\rho V}\exp_p(\rho' V_{\rho V} + \rho V'_{\rho V}) = \rho' T_{\rho V}\exp_p(V_{\rho V}) + \rho T_{\rho V}\exp_p(V'_{\rho V}),$$

where $V' = \nabla V/dt$ has the property $2g(V, V') = (g(V,V))' = (r^2)' = 0$. The Gauss lemma implies

$$g(\psi', \psi') = (\rho')^2 g(V,V) + 2\rho'\rho g(V, V') + \rho^2 g(V', V') = (\rho' r)^2 + \rho^2 g(V', V') \geq (\rho' r)^2,$$

and therefore

$$L(\psi) \geq \int_{a_0}^{b_0} \|\psi'(t)\|\, dt \geq \int_{a_0}^{b_0} |\rho' r|\, dt \geq r\int_{a_0}^{b_0} \rho'\, dt = r\rho(q) - r\rho(p) = r,$$

which proves $L(\psi) \geq L(\gamma) = d(p,q)$.

In order to have the equality $L(\psi) = r$, it is necessary that $a_0 = a$ and $b_0 = b$ hold, and then $V' = 0$ and $\rho' > 0$. This means that V is constant while ρ is increasing, so ψ is the exponential image of the segment connecting 0_p and $\exp_p^{-1}(q)$. Thus, ψ is a monotone reparametrization of γ, and they have the same image, $\gamma([a,b]) = \psi([a,b])$. □

Theorem 5.19. *Every geodesic in a Riemannian manifold is locally minimizing.*

Proof. Let $\gamma\colon I \to M$ be a geodesic in a Riemannian manifold M defined for some open interval I. For $t \in I$, according to Theorem 5.16, there exists a totally normal neighbourhood $U \ni \gamma(t)$. For $a, b \in \gamma^{-1}(U)$, the point $\gamma(b)$ is contained in the normal neighbourhood U centred at $\gamma(a)$, so by Theorem 5.18, the radial geodesic from $\gamma(a)$ to $\gamma(b)$ is minimizing between these two points. However, the restriction of γ to $[a,b]$ is also a geodesic segment from $\gamma(a)$ to $\gamma(b)$ contained in U, and therefore $\gamma\restriction_{[a,b]}$ coincides with this minimizing geodesic. □

From the previous theorems, we see that locally, a geodesic (or its reparametrization) connecting two points is the shortest such curve. However, globally this may not be the case.

Example 5.9. Consider a sphere, where we know that the images of geodesics are the great circles of the sphere, and two points from a great circle determine two arcs, one of which is longer (unless they are antipodal points).

More interesting is the case of a cylinder, where the images of geodesics are helices, so for two nearby points there are infinitely many geodesics (which have different images) connecting them, but there is only one (up to reparametrization) whose image is in a normal neighbourhood of one of those points. △

Theorem 5.20. *Every minimizing curve in a Riemannian manifold is up to reparametrization geodesic.*

Proof. Let $\psi\colon [a,b] \to M$ be a minimizing admissible curve parametrized by arc length. For each $t \in [a,b]$ there exists a totally normal neighbourhood $U_t \ni \psi(t)$, so the image $\psi([a,b])$ can be covered by finitely many totally normal neighbourhoods. In every neighbourhood U_t, the unique minimizing curve is a radial geodesic, so ψ satisfies the geodesic equations, and since it holds for each point, ψ is a geodesic. □

5.7 Completeness

A pseudo-Riemannian manifold (M, g) is geodesically complete at a point $p \in M$ if every geodesic through p extends indefinitely. Since the maximal geodesic for some $V \in T_pM$ is of the form $\gamma_V(t) = \exp_p(tV)$, it is clear that the previous condition is equivalent to the fact that T_pM is the domain of $\exp_p$. Thus, M is geodesically complete if and only if the whole TM is the domain of the exponential map. For example, an open submanifold $U \subset \mathbb{R}^n_\nu$ of pseudo-Euclidean space is not geodesically complete beacause there are geodesics that reach the boundary in finite time.

Any two different points of a totally normal neighbourhood can be connected by a minimizing geodesic, which is a property that does not hold for two arbitrary points of a Riemannian manifold. However, if a manifold is geodesically complete at one of these points, we have the following lemma.

Lemma 5.21. *If $p \in M$ is a point of a connected Riemannian manifold M such that $\exp_p$ is defined on the whole T_pM, then for every $q \in M$ there is a minimizing geodesic from p to q.*

Proof. Let x be the point of a (compact) geodesic sphere $S_\epsilon(p)$ where the (continuous) function $S_\epsilon(p) \ni s \mapsto d(s,q)$ attains its minimum. Then $x = \exp_p(\epsilon V)$, for some unit $V \in T_pM$ and we can consider the geodesic γ_V which for each $t \in \mathbb{R}$ is defined by $\gamma_V(t) = \exp_p(tV)$. Every, admissible curve from p to q must pass through $S_\epsilon(p)$, so

$$r = d(p,q) = \min_{s \in S_\epsilon(p)} (d(p,s) + d(s,q)) = \epsilon + \min_{s \in S_\epsilon(p)} d(s,q) = \epsilon + d(x,q),$$

which implies $d(x,q) = r - \epsilon$. The set $I = \{t \in [\epsilon, r] : d(\gamma_V(t), q) = r - t\}$ is nonempty (due to $\epsilon \in I$), closed and bounded, and therefore reaches the maximum $t_0 \in I$. For $y = \gamma_V(t_0)$ there is a point z of a geodesic sphere $S_\delta(y)$ for $0 < \delta < r - t_0$ where the function $S_\delta(y) \ni s \mapsto d(s,q)$ attains its minimum. We obtain $d(z,q) = r - t_0 - \delta$, and therefore

$$d(p,z) \geq d(p,q) - d(z,q) = r - (r - t_0 - \delta) = t_0 + \delta = d(p,y) + d(y,q),$$

so $d(p,z) = d(p,y) + d(y,q)$, which means that the minimizing curve from p to z goes through y. As a minimizing curve according to Theorem 5.20 is geodesic, we have $z = \exp_p((t_0+\delta)V)$ and therefore $t_0 + \delta \in I$, which is impossible, unless $t_0 = r$. Hence we obtain $\gamma_V(r) = q$, and γ_V is a minimizing geodesic from p to q. □

Let (M,g) be a connected Riemannian manifold. According to Theorem 4.20, M with the Riemannian distance forms a metric space whose metric topology is equal to the topology of M. Naturally the question arises whether M is complete as a metric space.

Assume that $K \subseteq M$ is closed and bounded subset of the metric space (M,d). If M is geodesically complete at a point $p \in M$, by Lemma 5.21 for every $q \in M$ there is a minimizing geodesic from p to q, so bounded K is contained in some sufficiently large geodesic sphere $B_r(p)$. Thus, there exists $r > 0$ such that $K \subset \exp_p(\overline{B_r(0_p)})$, so a closed subset of a compact is compact. The set of points of Cauchy[7] sequence is bounded, so its closure is compact, which means that a Cauchy sequence has a convergent subsequence and is therefore convergent.

In this way, we prove that if M is geodesically complete, then M is complete as a metric space. However, the converse also holds, which means that geodesic completeness and metric completeness are equivalent concepts, which is known as Hopf[8]–Rinow[9] theorem from 1931 [68].

Theorem 5.22. *A connected Riemannian manifold is geodesically complete if and only if it is complete as a metric space.*

[7] Augustin-Louis Cauchy (1789–1857), French mathematician
[8] Heinz Hopf (1894–1971), German mathematician
[9] Willi Ludwig August Rinow (1907–1979), German mathematician

Proof. Let $p \in M$ and $V \in T_pM$ be a unit vector. Consider the maximal geodesic $\gamma_V : I \to M$. Thanks to Theorem 5.11, the interval I is open. Let $\{t_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence such that $\lim_{n\to\infty} t_n = b = \sup I < \infty$. Since $d(\gamma_V(t_i), \gamma_V(t_j)) \leq |t_i - t_j|$, we see that $\{\gamma_V(t_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in M, so using the fact that M is complete we have the point $q = \lim_{n\to\infty} \gamma_V(t_n) \in M$. There exists $\epsilon > 0$ such that $B_{2\epsilon}(q)$ is geodesic ball. Let us choose $a \in \mathbb{R}$ such that $0 < b - a < \varepsilon$, so for $W = \gamma'_V(a)$ consider the geodesic γ_W. Then $\gamma_W(t) = \gamma_V(a+t)$, so γ_V is defined for $a + \epsilon > b$, which is a contradiction. Therefore the interval I is closed, hence $I = \mathbb{R}$, which proves that M is geodesically complete at the point p, but it holds for every point, and therefore M is geodesically complete. $\square$

5.8 Problems

Problem 5.1. Let (M, g) be a Riemannian manifold, and let $P, Q \in \mathfrak{T}^0_2(M)$ be parallel tensor fields. if $T \in \mathfrak{T}^0_4(M)$ is defined by $T(X, Y, Z, W) = P(X, Y)Q(Z, W)$ for all $X, Y, Z, W \in \mathfrak{X}(M)$, prove that T is also parallel.

Problem 5.2. Consider $\mathbb{R}^3$ with the standard Euclidean metric and introduce a connection ∇ defined in standard coordinates by $\Gamma^3_{12} = \Gamma^1_{23} = \Gamma^2_{31} = 1$ and $\Gamma^3_{21} = \Gamma^1_{32} = \Gamma^2_{13} = -1$, with all other Christoffel symbols equal to zero. Is ∇ compatible with the Euclidean metric? Determine the geodesics with respect to ∇. Is ∇ the Levi-Civita connection?

Problem 5.3. Construct the (Beltrami-Klein) model of hyperbolic space using the central projection $c\colon \mathbf{H}^n_r \to \mathbf{K}^n_r$ that maps a point $P \in \mathbf{H}^n_r \subset \mathbb{R}^{n+1}_1$ to the intersection of the line OP with the hyperplane $x_0 = r$. Show that c is a diffeomorphism and compute the induced metric in the natural coordinates of $\mathbf{K}^n_r$.

Problem 5.4. Determine the geodesics in the Beltrami–Klein disc model (M, g), where $M = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$,

$$g = \frac{dx_1^2 + dx_2^2}{1 - x_1^2 - x_2^2} + \frac{(x_1 dx_1 + x_2 dx_2)^2}{(1 - x_1^2 - x_2^2)^2}.$$

CHAPTER 6

CURVATURE

6.1 Curvature tensor fields

An important question in pseudo-Riemannian geometry is whether there are local invariants that are preserved by isometries. Some useful structures in differential geometry do not have local invariants. For example, any non-vanishing vector field can be written locally as a partial derivative, and all of them are locally equivalent. Also, Riemannian 1-manifolds are all mutually locally isometric to $\mathbb{R}$. However, the sphere $\mathbf{S}^2$ and the plane $\mathbb{R}^2$ are not locally isometric.

Example 6.1. If we introduce the spherical coordinates on the sphere $\mathbf{S}^2 \subset \mathbb{R}^3$ without the closed set $\{(x, y, z) : x \leq 0, y = 0\}$ by

$$x = \sin\varphi \cos\theta, \quad y = \sin\varphi \sin\theta, \quad z = \cos\varphi,$$

for the inclination $0 < \varphi < \pi$ and the azimuth $-\pi < \theta < \pi$, its metric inherited from $\mathbb{R}^3$ is

$$\mathring{g} = dx^2 + dy^2 + dz^2 = d\varphi^2 + \sin^2\varphi \, d\theta^2.$$

Calculations for the Christoffel symbols of the Levi-Civita connection yield

$$\Gamma^{\varphi}_{\varphi\varphi} = \Gamma^{\theta}_{\varphi\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\theta}_{\theta\theta} = 0, \quad \Gamma^{\theta}_{\varphi\theta} = \cot\varphi, \quad \Gamma^{\varphi}_{\theta\theta} = -\sin\varphi\cos\varphi,$$

from where it can be seen that meridians $\theta = \mathrm{Const}$ are geodesics on the sphere, because the curves of the form $\gamma(t) = (t, \theta)$ obviously satisfy the geodesic equations. Covariant derivatives of the coordinate vector field ∂_φ are

$$\nabla_{\partial_\varphi}\partial_\varphi = 0, \quad \nabla_{\partial_\theta}\partial_\varphi = \cot\varphi \, \partial_\theta,$$

which means that it is parallel along each meridian, but also along the equator ($\varphi = \pi/2$). Consider the point p given by coordinates $(\varphi, \theta) = (\pi/2, 0)$ and the vector $(\partial_\varphi)_p$. If there exists a parallel extension of this vector to a neighbourhood of p, it can only be ∂_φ (parallel transport along the equator, and then along the corresponding meridian), but we have $\nabla\partial_\varphi \not\equiv 0$. In Euclidean space each tangent vector has a parallel extension to the entire space, which is not valid for the sphere, so the sphere and the plane are not locally isometric. △

For a pseudo-Riemannian 2-manifold (M, g) there is an obvious attempt to extend a vector $Z_p \in T_pM$ to a parallel vector field $Z \in \mathfrak{X}(M)$. Choose local coordinates (x_1, x_2) centred at p, construct the parallel vector field along x_1-axis, and then parallel translate the resulting vectors along the coordinate lines parallel to the x_2-axis. Such vector field Z is parallel along any x_2 coordinate line, and along the x_1-axis. The question is whether $\nabla_{\partial_1} Z \equiv 0$, or whether Z is parallel along x_1-coordinate lines other than the x_1-axis itself. The condition $\nabla_{\partial_2}\nabla_{\partial_1}Z \equiv 0$ determines a unique parallel transport along x_2-coordinate lines for some initial vector, but since $\nabla_{\partial_1}Z$ vanishes at points $x_2 = 0$, this implies $\nabla_{\partial_1}Z \equiv 0$ On the other hand we have $\nabla_{\partial_2}Z \equiv 0$, and also $\nabla_{\partial_1}\nabla_{\partial_2}Z \equiv 0$, so the problem would be solved if ∇_{∂_1} and ∇_{∂_2} commute.

Direct calculations give $D_{\partial_2}D_{\partial_1}Z = D_{\partial_2}(\sum_i \partial_1 Z^i \partial_i) = \sum_i \partial_2\partial_1 Z^i \partial_i$, for the standard connection $\nabla = D$ in $\mathbb{R}^2$, and because partial derivatives commute we get $D_{\partial_2}D_{\partial_1}Z = D_{\partial_1}D_{\partial_2}Z$. However, this does not hold in a general case, since the non-commutativity of such covariant derivatives allows us to distinguish locally the sphere and the plane in Example 6.1. In order to express non-commutativity in a coordinate invariant way, consider $\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z$, where in $\mathbb{R}^n$ holds $D_X D_Y Z = D_X(\sum_i YZ^i\partial_i) = \sum_i XYZ^i\partial_i$, and therefore we obtain $D_X D_Y Z - D_Y D_X Z = \sum_i (XY - YX)Z^i\partial_i = D_{[X,Y]}Z$.

This analysis motivates us to define the ***curvature operator*** $\mathcal{R}\colon \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ on a pseudo-Riemannian manifold (M, g) by

$$\mathcal{R}(X,Y)(Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for $X, Y, Z \in \mathfrak{X}(M)$, where ∇ is the associated Levi-Civita connection. In other words, the curvature operator is the map $\mathcal{R}(X,Y)\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $\mathcal{R}(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. Let us remark that some authors define the curvature operator with the opposite sign, which is a difference that is not important, but we should be careful about that.

From the very definition, the anti-symmetry

$$\mathcal{R}(Y,X) = -\mathcal{R}(X,Y) \tag{6.1}$$

is evident, as well as its special case $\mathcal{R}(X,X) = 0$. Since both the connection and the commutator are $\mathbb{R}$-bilinear, the curvature operator is $\mathbb{R}$-multilinear with additivity by all three arguments. Moreover, $\mathcal{R}$ is $\mathfrak{F}(M)$-multilinear. For $\mathfrak{F}(M)$-linearity by the first argument we have,

$$\begin{aligned}\mathcal{R}(fX,Y) &= [\nabla_{fX}, \nabla_Y] - \nabla_{[fX,Y]} = [f\nabla_X, \nabla_Y] - \nabla_{f[X,Y]-(Yf)X}\\ &= f[\nabla_X, \nabla_Y] - (Yf)\nabla_X - f\nabla_{[X,Y]} + (Yf)\nabla_X = f\mathcal{R}(X,Y),\end{aligned}$$

which also works by the second argument because of (6.1). For the third argument we use properties (5.6) and (5.5) to obtain

$$\begin{aligned}\mathcal{R}(X,Y)(fZ) =&\nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ)\\ =&\nabla_X(f\nabla_Y Z + (Yf)Z) - \nabla_Y(f\nabla_X Z + (Xf)Z) - (f\nabla_{[X,Y]}Z + ([X,Y]f)Z)\\ =&f\nabla_X\nabla_Y Z + (Xf)\nabla_Y Z + (Yf)\nabla_X Z + (XYf)Z - f\nabla_Y\nabla_X Z\\ &- (Yf)\nabla_X Z - (Xf)\nabla_Y Z - (YXf)Z - f\nabla_{[X,Y]}Z - ([X,Y]f)Z\\ =&f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - f\nabla_{[X,Y]}Z = f\mathcal{R}(X,Y)Z.\end{aligned}$$

Thus, the curvature operator $\mathcal{R}\colon \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ is $\mathfrak{F}(M)$-multilinear and we usually treat it as a tensor field $\mathcal{R} \in \mathfrak{T}^1_3(M)$. By multiple use of (5.9), which is the fact that the connection is symmetric, we get

$$\begin{aligned}&\mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y\\ &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y\\ &= \nabla_X[Y,Z] + \nabla_Y[Z,X] + \nabla_Z[X,Y] - \nabla_{[X,Y]}Z - \nabla_{[Y,Z]}X - \nabla_{[Z,X]}Y\\ &= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]],\end{aligned}$$

and therefore the Jacobi identity (2.7) implies

$$\mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y = 0, \tag{6.2}$$

which is the formula well known as the ***Bianchi identity***[1].

Theorem 6.1. *For the curvature operator $\mathcal{R} \in \mathfrak{T}^1_3(M)$ of a pseudo-Riemannian manifold M, the formulas* (6.1) *and* (6.2) *hold for all $X, Y, Z \in \mathfrak{X}(M)$.*

In local coordinates we have $\mathcal{R}(\partial_i, \partial_j)\partial_k = \sum_l \mathcal{R}^l_{ijk}\partial_l$ (see Example 3.25), so the components of the curvature operator can be expressed through the Christoffel symbols. From

$$\begin{aligned}
\mathcal{R}(\partial_i, \partial_j)\partial_k &= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k - \nabla_{[\partial_i,\partial_j]}\partial_k \\
&= \nabla_{\partial_i}\sum_m \Gamma^m_{jk}\partial_m - \nabla_{\partial_j}\sum_m \Gamma^m_{ik}\partial_m \\
&= \sum_m (\partial_i\Gamma^m_{jk})\partial_m + \sum_m \Gamma^m_{jk}\nabla_{\partial_i}\partial_m - \sum_m (\partial_j\Gamma^m_{ik})\partial_m - \sum_m \Gamma^m_{ik}\nabla_{\partial_j}\partial_m \\
&= \sum_m (\partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik})\partial_m + \sum_{m,l}(\Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm})\partial_l,
\end{aligned}$$

we obtain

$$\mathcal{R}(\partial_i, \partial_j)\partial_k = \sum_l \left(\left(\partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik}\right) + \sum_m (\Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}) \right)\partial_l,$$

and hence

$$\mathcal{R}^l_{ijk} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \sum_m (\Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}). \tag{6.3}$$

By lowering an index we get the ***curvature tensor*** $R = \mathcal{R}^\flat \in \mathfrak{T}^0_4(M)$, which is a covariant tensor field

$$R = \sum_{i,j,k,l} R_{ijkl}\, dx_i \otimes dx_j \otimes dx_k \otimes dx_l$$

with components $R_{ijkl} = \sum_m g_{lm}\mathcal{R}^m_{ijk}$, and therefore for all $X, Y, Z, W \in \mathfrak{X}(M)$ holds

$$R(X, Y, Z, W) = g(\mathcal{R}(X, Y)Z, W).$$

From (6.1) directly follows the identity

$$R(X, Y, Z, W) = -R(Y, X, Z, W), \tag{6.4}$$

while as a consequence of (6.2) we have the ***first Bianchi identity***,

$$R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0. \tag{6.5}$$

Since the Levi-Civita connection ∇ is metric, from (5.8) follows

$$\begin{aligned}
g(\nabla_X\nabla_Y Z, Z) + g(\nabla_Y Z, \nabla_X Z) &= Xg(\nabla_Y Z, Z), \\
g(\nabla_Y\nabla_X Z, Z) + g(\nabla_X Z, \nabla_Y Z) &= Yg(\nabla_X Z, Z), \\
2g(\nabla_W Z, Z) = g(\nabla_W Z, Z) + g(Z, \nabla_W Z) &= Wg(Z, Z),
\end{aligned}$$

where the last equality is interesting in the cases $W \in \{Y, X, [X, Y]\}$. Thus, we have

$$\begin{aligned}
R(X, Y, Z, Z) &= g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, Z) \\
&= Xg(\nabla_Y Z, Z) - Yg(\nabla_X Z, Z) - g(\nabla_{[X,Y]}Z, Z) \\
&= \frac{1}{2}XYg(Z, Z) - \frac{1}{2}YXg(Z, Z) - \frac{1}{2}[X, Y]g(Z, Z) = 0,
\end{aligned}$$

so by polarization from

$$R(X, Y, Z + W, Z + W) = R(X, Y, Z, Z) + R(X, Y, Z, W) + R(X, Y, W, Z) + R(X, Y, W, W)$$

we obtain

$$R(X, Y, Z, W) = -R(X, Y, W, Z). \tag{6.6}$$

[1]Luigi Bianchi (1856–1928), Italian mathematician

Theorem 6.2. *For the curvature tensor $R \in \mathfrak{T}^0_4(M)$ of a pseudo-Riemannian manifold M, the formulas* (6.4)*,* (6.6)*, and* (6.5) *hold for all $X, Y, Z, W \in \mathfrak{X}(M)$.*

Similar properties can be seen for the total covariant derivative of curvature tensor ∇R. Since

$$\begin{aligned}\nabla R(X,Y,Z,W,V) =& V(R(X,Y,Z,W)) - R(\nabla_V X,Y,Z,W) - R(X,\nabla_V Y,Z,W)\\ &- R(X,Y,\nabla_V Z,W) - R(X,Y,Z,\nabla_V W),\end{aligned}$$

the property (6.4) immediately gives

$$\nabla R(X,Y,Z,W,V) = -\nabla R(Y,X,Z,W,V), \tag{6.7}$$

while

$$\nabla R(X,Y,Z,W,V) = -\nabla R(X,Y,W,Z,V) \tag{6.8}$$

is a straightforward consequence of (6.6). From multiple application (6.2) and (6.5),

$$\begin{aligned}&\nabla R(X,Y,Z,W,V) + \nabla R(Y,Z,X,W,V) + \nabla R(Z,X,Y,W,V)\\ =&\ \ V(R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W))\\ &- g(\mathcal{R}(\nabla_V X,Y)Z + \mathcal{R}(Z,\nabla_V X)Y + \mathcal{R}(Y,Z)\nabla_V X, W)\\ &- g(\mathcal{R}(\nabla_V Y,Z)X + \mathcal{R}(X,\nabla_V Y)Z + \mathcal{R}(Z,X)\nabla_V Y, W)\\ &- g(\mathcal{R}(\nabla_V Z,X)Y + \mathcal{R}(Y,\nabla_V Z)X + \mathcal{R}(X,Y)\nabla_V Z, W)\\ &- g(\mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y, \nabla_V W) = 0,\end{aligned}$$

we obtain the covariant derivative of the first Bianchi identity

$$\nabla R(X,Y,Z,W,V) + \nabla R(Y,Z,X,W,V) + \nabla R(Z,X,Y,W,V) = 0. \tag{6.9}$$

If we use (5.8) to calculate $\nabla R(X,Y,Z,W,V)$, we have

$$\begin{aligned}\nabla R(X,Y,Z,W,V) =& V(g(\mathcal{R}(X,Y)Z,W)) - R(X,Y,Z,\nabla_V W)\\ &- R(X,Y,\nabla_V Z,W) - R(\nabla_V X,Y,Z,W) - R(X,\nabla_V Y,Z,W)\\ =&\ \ g(\nabla_V\nabla_X\nabla_Y Z - \nabla_V\nabla_Y\nabla_X Z - \nabla_V\nabla_{[X,Y]}Z, W)\\ &- g(\nabla_X\nabla_Y\nabla_V Z - \nabla_Y\nabla_X\nabla_V Z - \nabla_{[X,Y]}\nabla_V Z, W)\\ &- g(\nabla_{\nabla_V X}\nabla_Y Z - \nabla_Y\nabla_{\nabla_V X}Z - \nabla_{[\nabla_V X,Y]}Z, W)\\ &- g(\nabla_X\nabla_{\nabla_V Y}Z - \nabla_{\nabla_V Y}\nabla_X Z - \nabla_{[X,\nabla_V Y]}Z, W),\end{aligned}$$

which can be written as

$$\nabla R(X,Y,Z,W,V) = g((\mathcal{T}_1(X,Y,V) + \mathcal{T}_2(X,Y,V) + \mathcal{T}_3(X,Y,V))Z, W),$$

where $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ are defined by

$$\begin{aligned}\mathcal{T}_1(X,Y,V) &= \nabla_V\nabla_X\nabla_Y - \nabla_V\nabla_Y\nabla_X - \nabla_X\nabla_Y\nabla_V + \nabla_Y\nabla_X\nabla_V = [\nabla_V,[\nabla_X,\nabla_Y]];\\ \mathcal{T}_2(X,Y,V) &= \nabla_Y\nabla_{\nabla_V X} - \nabla_{\nabla_V X}\nabla_Y + \nabla_{\nabla_V Y}\nabla_X - \nabla_X\nabla_{\nabla_V Y} + \nabla_{[X,Y]}\nabla_V - \nabla_V\nabla_{[X,Y]}\\ &= [\nabla_Y,\nabla_{\nabla_V X}] + [\nabla_{\nabla_V Y},\nabla_X] + [\nabla_{[X,Y]},\nabla_V];\\ \mathcal{T}_3(X,Y,V) &= \nabla_{[X,\nabla_V Y]} + \nabla_{[\nabla_V X,Y]}.\end{aligned}$$

The Jacobi identity from Lemma A.31 applied to the algebra of all linear endomorphisms $\mathrm{End}(\mathfrak{X}(M))$ of the module $\mathfrak{X}(M)$ over the ring $\mathfrak{F}(M)$, gives

$$\mathcal{T}_1(X,Y,V) + \mathcal{T}_1(Y,V,X) + \mathcal{T}_1(V,X,Y) = [\nabla_V,[\nabla_X,\nabla_Y]] + [\nabla_X,[\nabla_Y,\nabla_V]] + [\nabla_Y,[\nabla_V,\nabla_X]] = 0.$$

From the equality (5.9) we have

$$\begin{aligned}&\mathcal{T}_2(X,Y,V)+\mathcal{T}_2(Y,V,X)+\mathcal{T}_2(V,X,Y)\\&\qquad=[\nabla_{\nabla_V Y-\nabla_Y V-[V,Y]},\nabla_X]+[\nabla_{\nabla_X V-\nabla_V X-[X,V]},\nabla_Y]+[\nabla_{\nabla_Y X-\nabla_X Y-[Y,X]},\nabla_V]=0.\end{aligned}$$

The equality (5.9) and the Jacobi identity (2.7) yield

$$\begin{aligned}&\mathcal{T}_3(X,Y,V)+\mathcal{T}_3(Y,V,X)+\mathcal{T}_3(V,X,Y)\\&\quad=\nabla_{[X,\nabla_V Y]}+\nabla_{[\nabla_V X,Y]}+\nabla_{[Y,\nabla_X V]}+\nabla_{[\nabla_X Y,V]}+\nabla_{[V,\nabla_Y X]}+\nabla_{[\nabla_Y V,X]}\\&\quad=\nabla_{[X,\nabla_V Y-\nabla_Y V]}+\nabla_{[Y,\nabla_X V-\nabla_V X]}+\nabla_{[V,\nabla_Y X-\nabla_X Y]}\\&\quad=\nabla_{[X,[V,Y]]+[Y,[X,V]]+[V,[Y,X]]}=0.\end{aligned}$$

Finally, by combining the previous results, we obtain the ***second Bianchi identity***

$$\nabla R(X,Y,Z,W,V)+\nabla R(Y,V,Z,W,X)+\nabla R(V,X,Z,W,Y)=0. \tag{6.10}$$

Theorem 6.3. *For the total covariant derivative of curvature tensor $\nabla R\in\mathfrak{T}^0_5(M)$ of a pseudo-Riemannian manifold M, the formulas* (6.7), (6.8), (6.9), *and* (6.10) *hold for all $X,Y,Z,W,V\in\mathfrak{X}(M)$.*

Long and painful calculations used in the proof of Theorem 6.3 can be simplified more intelligently. Namely, as far as tensor equalities are concerned, by Theorem 3.13 it is suffice to show the statement at an arbitrary point $p\in M$. Due to the multilinear nature of tensors it is enough to show the formula for the basis elements related to some frame. If we introduce, by Theorem 5.14, normal coordinates centred at p, next to standard $[\partial_i,\partial_j]\equiv 0$, according to Theorem 5.15 additionally we have

$$(\nabla_{\partial_i}\partial_j)_p=\sum_k\Gamma^k_{ij}(p)(\partial_k)_p=0.$$

For example, in normal coordinates centred at p for coordinate vector fields X,Y,Z,W,V in some neighbourhood of p, for (6.9) we have immediately

$$\begin{aligned}&(\nabla R(X,Y,Z,W,V)+\nabla R(Y,Z,X,W,V)+\nabla R(Z,X,Y,W,V))(p)\\&\qquad=V_p(R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W))=0.\end{aligned}$$

The second Bianchi identity (6.10) is getting more incomparably faster and easier,

$$\begin{aligned}(\nabla R(X,Y,Z,W,V))(p)&=(V(R(X,Y,Z,W)))(p)\\&=g(\nabla_V(\mathcal{R}(X,Y)Z),W)(p)+g(\mathcal{R}(X,Y)Z,\nabla_V W)(p)\\&=g(\nabla_V\nabla_X\nabla_Y Z-\nabla_V\nabla_Y\nabla_X Z-\nabla_V\nabla_{[X,Y]}Z,W)(p)\\&=g(\nabla_V\nabla_X\nabla_Y Z-\nabla_V\nabla_Y\nabla_X Z,W)(p),\end{aligned}$$

and therefore

$$\begin{aligned}&(\nabla R(X,Y,Z,W,V)+\nabla R(Y,V,Z,W,X)+\nabla R(V,X,Z,W,Y))(p)\\&\quad=g((\nabla_V\nabla_X\nabla_Y-\nabla_V\nabla_Y\nabla_X+\nabla_X\nabla_Y\nabla_V-\nabla_X\nabla_V\nabla_Y+\nabla_Y\nabla_V\nabla_X-\nabla_Y\nabla_X\nabla_V)Z,W)(p)\\&\quad=R(V,X,\nabla_Y Z,W)(p)+R(X,Y,\nabla_V Z,W)(p)+R(Y,V,\nabla_X Z,W)(p)=0.\end{aligned}$$

Let $f\colon M\to N$ be an isometry between pseudo-Riemannian manifolds. Since the Levi-Civita connection is preserved in the sense of the formula (5.12), so is the curvature it induces,

$$f_*(\mathcal{R}(X,Y)Z)=\mathcal{R}(f_*X,f_*Y)f_*Z.$$

It is important to understand the behaviour of isometries and local isometries. The various concepts of pseudo-Riemannian geometry that we have defined so far are preserved in an appropriate sense by isometries. As we have constructed the concepts from the metric tensor using the tools of manifolds theory, while isometries preserve both the tools and the tensor, it is natural to expect that they are isometric invariants.

Example 6.2. Let $f\colon M \to N$ be a local isometry between pseudo-Riemannian manifolds. The covariant derivative along a curve γ is preserved with

$$T_{\gamma(t)}f \frac{\nabla V}{dt} = \frac{\nabla f_* V}{dt},$$

for $V \in \mathfrak{X}(\gamma)$ where $(f_* V)(t) = T_{\gamma(t)}f(V(t))$ for all t. Hence, the parallel translation along a curve is preserved by

$$T_{\gamma(b)}f \circ P_\gamma = P_{f\circ\gamma} \circ T_{\gamma(a)}f,$$

where $P_\gamma = P_{ab}$ denotes the parallel transport from $\gamma(a)$ to $\gamma(b)$ along γ in M and $P_{f\circ\gamma}$ denotes the parallel transport from $f(\gamma(a))$ to $f(\gamma(b))$ along $f\circ\gamma$ in N. Thus, if γ is a geodesic in M, then $f\circ\gamma$ is a geodesic in N, or more concretely, the geodesics are preserved by

$$f \circ \gamma_V = \gamma_{T_p f(V)}, \tag{6.11}$$

whenever both sides are defined (the domain of $\gamma_{(T_p f)V}$ may be larger than the domain of γ_V), where $\gamma_V(0) = p$ and $\gamma_V'(0) = V$. Therefore, the exponential maps are preserved by

$$f \circ \exp_p = \exp_{f(p)} \circ T_p f \tag{6.12}$$

whenever both sides are defined. △

6.2 Algebraic curvature tensors

In the theory of pseudo-Riemannian manifolds, it is convenient to work in a purely algebraic setting. Using the reduction of a pseudo-Riemannian manifold (M, g) to an arbitrary point $p \in M$, we obtain the vector space $\mathcal{V} = T_pM$ with a scalar product g_p, while natural tensors yield the concept of algebraic curvature tensors. A tensor $R \in \mathfrak{T}^0_4(\mathcal{V})$ over a scalar product space $(\mathcal{V}, g)$ is said to be an ***algebraic curvature tensor*** if it for all $X, Y, Z, W \in \mathcal{V}$ satisfies the symmetries

$$R(X,Y,Z,W) = -R(Y,X,Z,W), \tag{6.4 revisited}$$
$$R(X,Y,Z,W) = -R(X,Y,W,Z), \tag{6.6 revisited}$$
$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0. \tag{6.5 revisited}$$

It is important to notice that this definition nicely agrees with Theorem 6.2. The curvature tensor of a pseudo-Riemannian manifold, reduced to any point, is an algebraic curvature tensor. Thus, any result derived from an algebraic curvature tensor can be applied to the curvature tensor of a pseudo-Riemannian manifold.

Applying (6.6), then (6.5) and finally (6.4) and (6.6), we get

$$\begin{aligned}
&2R(X,Y,Z,W) - 2R(Z,W,X,Y)\\
&\quad= R(X,Y,Z,W) - R(X,Y,W,Z) - R(Z,W,X,Y) + R(Z,W,Y,X)\\
&\quad= (-R(Y,Z,X,W) - R(Z,X,Y,W)) - (-R(Y,W,X,Z) - R(W,X,Y,Z))\\
&\qquad - (-R(W,X,Z,Y) - R(X,Z,W,Y)) + (-R(W,Y,Z,X) - R(Y,Z,W,X))\\
&\quad= -R(Y,Z,X,W) - R(Y,Z,W,X) - R(Z,X,Y,W) + R(X,Z,W,Y)\\
&\qquad + R(Y,W,X,Z) - R(W,Y,Z,X) + R(W,X,Y,Z) + R(W,X,Z,Y) = 0,
\end{aligned}$$

and therefore we obtain the symmetry by pairs

$$R(X,Y,Z,W) = R(Z,W,X,Y), \tag{6.13}$$

which holds for all $X, Y, Z, W \in \mathcal{V}$ and consequently it can be transferred to the curvature tensor on a manifold. With this in mind, let us remark that the equality (6.6) from the definition of algebraic curvature tensor, can be replaced by (6.13), since (6.6) is a direct consequence of (6.4) and (6.13).

Let R be an algebraic curvature tensor on a scalar product space $(\mathcal{V}, g)$ of dimension n. For simplicity, the properties of the scalar product will be relayed to the tensor. This allows us to say that R is Riemannian or positive definite (if $\operatorname{Ind} g = 0$), definite ($\operatorname{Ind} g = 0$ or $\operatorname{Ind} g = n$), indefinite ($1 \leq \operatorname{Ind} g \leq n-1$), Lorentzian ($\operatorname{Ind} g = 1$), n-dimensional ($\dim \mathcal{V} = n$), and so on.

Example 6.3. The basic example of an algebraic curvature tensor on a scalar product space $(\mathcal{V}, g)$ is certainly $R^1 \in \mathfrak{T}_4^0(\mathcal{V})$ given by

$$R^1(X, Y, Z, W) = g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \tag{6.14}$$

for all $X, Y, Z, W \in \mathcal{V}$. It is easy to see that R^1 satisfies the $\mathbb{Z}_2$ symmetries (6.4) and (6.6) as well as the first Bianchi identity (6.5). △

Let $J\colon \mathcal{V} \to \mathcal{V}$ be a linear operator on a scalar product space $(\mathcal{V}, g)$. We say that J is ***self-adjoint*** or ***symmetric*** if $g(JX, Y) = g(X, JY)$ holds for all $X, Y \in \mathcal{V}$. Similarly, J is ***skew-adjoint*** or ***skew-symmetric*** if $g(JX, Y) = -g(X, JY)$ holds for all $X, Y \in \mathcal{V}$. Self-adjoint and skew-adjoint endomorphisms allow us to obtain new examples of algebraic curvature tensors.

Example 6.4. A self-adjoint endomorphism $J\colon \mathcal{V} \to \mathcal{V}$ on a scalar product space $(\mathcal{V}, g)$ generates the algebraic curvature tensor $R^J \in \mathfrak{T}_4^0(\mathcal{V})$ defined by

$$R^J(X, Y, Z, W) = g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W),$$

for all $X, Y, Z, W \in \mathcal{V}$, which can be easily checked. Let us notice that R^1 is the special case for $J = 1 \cdot \mathbb{1}$. △

Example 6.5. A skew-adjoint endomorphism $J\colon \mathcal{V} \to \mathcal{V}$ on a scalar product space $(\mathcal{V}, g)$ generates the algebraic curvature tensor $R^J \in \mathfrak{T}_4^0(\mathcal{V})$ defined by

$$R^J(X, Y, Z, W) = g(JX, Z)g(JY, W) - g(JY, Z)g(JX, W) + 2g(JX, Y)g(JZ, W), \tag{6.15}$$

for all $X, Y, Z, W \in \mathcal{V}$, which can be easily checked. △

Example 6.6. Let $R_1, \dots, R_k \in \mathfrak{T}_4^0(\mathcal{V})$ be algebraic curvature tensors on a scalar product space $(\mathcal{V}, g)$ and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$. Then

$$\sum_{i=1}^{k} \alpha_i R_i \in \mathfrak{T}_4^0(\mathcal{V})$$

is also an algebraic curvature tensor, since obviously satisfies the equalities (6.4), (6.6), and (6.5). Thus, a new algebraic curvature tensor can be made as a linear combination of existing tensors. In particular, kR^1 is an algebraic curvature tensors for any $k \in \mathbb{R}$. △

Raising the last index we obtain the ***affine curvature operator*** $\mathcal{R} = R^\sharp \in \mathfrak{T}_3^1(\mathcal{V})$ for which the equality

$$R(X, Y, Z, W) = g(\mathcal{R}(X, Y)Z, W),$$

holds for all $X, Y, Z, W \in \mathcal{V}$. Through the restriction, we intentionally kept the notation from a manifold, R for a curvature tensor and $\mathcal{R}$ for a curvature operator. In this way we can take over the global terminology.

A common way to express an algebraic curvature tensor R is via coordinate curvature tensor components $R_{ijkl} = R(E_i, E_j, E_k, E_l)$ related to some basis $(E_1, E_2, \dots, E_n)$ in $\mathcal{V}$. It turns out that we need only $n^2(n^2-1)/12$ components, which is given in the following theorem (see Weinberg[2] [119, pp.142–143]).

[2]Steven Weinberg (1933), American theoretical physicist

Theorem 6.4. *The dimension of the space of algebraic curvature tensors on a scalar product space of dimension n is equal to* $n^2(n^2-1)/12$*.*

Proof. Consider components R_{ijkl} as two pairs (i,j) and (k,l). Since (6.4) holds we have $\binom{n}{2}$ independent pairs on the first place. Since (6.6) holds we have the same on the second place. However, the pairs are related with (6.13), which gives $\binom{n}{2}+\cdots+2+1$ possibilities. The first Bianchi identity (6.5) adds $\binom{n}{4}$ dependent components, for example these are R_{ijkl} for $i<j<k<l$. Therefore, the number of independent components R_{ijkl} is equal to

$$\begin{aligned}\binom{n}{2}+\cdots+2+1-\binom{n}{4} &= \frac{1}{2}\binom{n}{2}\left(\binom{n}{2}+1\right)-\binom{n}{4}\\ &= \frac{n(n-1)(n^2-n+2)}{8}-\frac{n(n-1)(n-2)(n-3)}{24}=\frac{n^2(n^2-1)}{12}.\end{aligned}$$

□

The Jacobi operator and its related operators play important roles in our theory. The ***polarized Jacobi operator*** is the linear map $\mathcal{J}\colon \mathcal{V}^3\to\mathcal{V}$ defined by

$$\mathcal{J}(X,Y)(Z)=\frac{1}{2}\left(\mathcal{R}(Z,X)Y+\mathcal{R}(Z,Y)X\right), \tag{6.16}$$

for all $X,Y,Z\in\mathcal{V}$. The ***Jacobi operator*** for $X\in\mathcal{V}$ is the linear map $\mathcal{J}_X\colon\mathcal{V}\to\mathcal{V}$ defined by $\mathcal{J}_X=\mathcal{J}(X,X)$, which is often expressed by

$$\mathcal{J}_X(Y)=\mathcal{R}(Y,X)X. \tag{6.17}$$

Since (6.6) implies $g(\mathcal{J}_X(Y),X)=R(Y,X,X,X)=0$, then we have $\mathcal{J}_X(Y)\perp X$, so the codomain of the Jacobi operator $\mathcal{J}_X$ is $X^\perp$. In the case of nonnull X ($\varepsilon_X\neq 0$), the orthogonal $X^\perp$ is a nondegenerate hypersurface in $\mathcal{V}$, while (6.4) gives $\mathcal{J}_X(X)=\mathcal{R}(X,X)X=0$. Thus, the Jacobi operator for nonnull $X\in\mathcal{V}$ is completely determined by its restriction

$$\widetilde{\mathcal{J}}_X=\mathcal{J}_X\restriction_{X^\perp}: X^\perp\to X^\perp,$$

called the ***reduced Jacobi operator***.

From (6.13), (6.4), and (6.6) follow $R(Y,X,X,Z)=R(Z,X,X,Y)$, and therefore the Jacobi operator is self-adjoint,

$$g(\mathcal{J}_X(Y),Z)=g(\mathcal{J}_X(Z),Y).$$

Similarly, (6.4) and (6.6) imply $R(Y,X,X,Y)=R(X,Y,Y,X)$, so we obtain the ***compatibility*** of the Jacobi operators,

$$g(\mathcal{J}_X(Y),Y)=g(\mathcal{J}_Y(X),X). \tag{6.18}$$

Thus, the Jacobi operators are self-adjoint endomorphisms on $\mathcal{V}$ for which the compatibility (6.18) holds.

Let $(E_1,\dots,E_n)$ be an arbitrary orthonormal basis in a scalar product space $(\mathcal{V},g)$ of dimension n. For any vector $X=\sum_{i=1}^n x_iE_i$ we have $\mathcal{J}_X(E_j)=\sum_{i=1}^n\varepsilon_{E_i}g(\mathcal{J}_X(E_j),E_i)E_i$, so we can calculate the entries of the matrix $\mathcal{J}_X$ related to the given basis,

$$(\mathcal{J}_X)_{ij}=\varepsilon_{E_i}g(\mathcal{J}_X(E_j),E_i)=\varepsilon_{E_i}R(E_j,X,X,E_i)=\varepsilon_{E_i}\sum_{p,q=1}^n R_{jpqi}x_px_q,$$

which clearly shows that they are homogeneous polynomials of degree 2 (quadratic forms) in n variables $x_1,\dots,x_n$. Of course, this also applies to bases that are not orthonormal, because $M^{-1}\mathcal{J}_XM$ is a new matrix of $\mathcal{J}_X$, where M is the transition matrix.

Lemma 6.5. *The entries of the matrix $\mathcal{J}_X$ are homogeneous polynomials of degree 2 in coefficients of X.*

Although we defined the Jacobi operator using a scalar product space $\mathcal{V}$, it is important to notice that we can always substitute $\mathcal{V} = T_pM$, where p is a point in a pseudo-Riemannian manifold M. Also, we can extend the Jacobi operator on the whole $\mathfrak{X}(M)$ (or at least on the tangent bundle TM) and keep our terminology.

6.3 Sectional curvature

The curvature tensor R of a pseudo-Riemannian manifold (M, g) is reasonably complicated, so we introduce commonly used quantity called the sectional curvature. A ***tangent plane*** σ to M at $p \in M$ is a two-dimensional subspace of the tangent space T_pM. The ***sectional curvature*** κ of a nondegenerate tangent plane $\sigma = \operatorname{Span}\{X, Y\}$ in T_pM is given by

$$\kappa(\sigma) = \kappa(X, Y) = \frac{R(X, Y, Y, X)}{\varepsilon_X\varepsilon_Y - (g(X, Y))^2}.$$

Let us notice that the denominator $\varepsilon_X\varepsilon_Y - (g(X, Y))^2 = g(X, X)g(Y, Y) - g(X, Y)g(Y, X)$ is the determinant of the Gram matrix for $g\restriction_\sigma$ related to basis $X, Y \in T_pM$ in $\sigma = \operatorname{Span}\{X, Y\}$. Since by definition we consider only nondegenerate planes σ, our denominator according to Lemma 4.1 is not zero. Moreover, for Riemannian manifolds, it represents the square of the area of the parallelogram determined by the pair of vectors $X, Y \in T_pM$.

We should check that the value $\kappa(X, Y)$ depends only on the (nondegenerate) plane spanned by the vectors X and Y. Let $X_1 = \alpha X + \beta Y$ and $Y_1 = \gamma X + \delta Y$ form another basis in σ. The change of basis for a bilinear form g gives

$$\begin{pmatrix} g(X_1, X_1) & g(X_1, Y_1) \\ g(Y_1, X_1) & g(Y_1, Y_1) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} g(X, X) & g(X, Y) \\ g(Y, X) & g(Y, Y) \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix},$$

and therefore $\varepsilon_{X_1}\varepsilon_{Y_1} - (g(X_1, Y_1))^2 = (\alpha\delta - \beta\gamma)^2 \left(\varepsilon_X\varepsilon_Y - (g(X, Y))^2\right)$. On the other hand, by using the symmetry properties of R, we have

$$\begin{aligned} R(X_1, Y_1, Y_1, X_1) &= R(\alpha X + \beta Y, \gamma X + \delta Y, Y_1, X_1) \\ &= (\alpha\delta - \beta\gamma)R(X, Y, \gamma X + \delta Y, \alpha X + \beta Y), \\ &= (\alpha\delta - \beta\gamma)^2 R(X, Y, Y, X). \end{aligned}$$

Since the determinant of the change of basis is not zero we have $\alpha\delta - \beta\gamma \neq 0$, and therefore $\kappa(X_1, Y_1) = \kappa(X, Y)$, which means that the sectional curvature is well defined.

Note that for a two-dimensional Riemannian manifold, there is only one sectional curvature at each point, which is the well-known Gaussian curvature. The sectional curvature is a real-valued function defined on the 2-Grassmannian bundle over M. However, although the sectional curvature seems simpler than the curvature tensor R, it contains the same information.

First, we can see that the curvature operator depends only on Jacobi operators, because using (6.1) and (6.2) we have

$$\begin{aligned} 3\mathcal{R}(X, Y)Z &= \mathcal{R}(X, Y)Z + (-\mathcal{R}(Y, X)Z) + (-\mathcal{R}(Y, Z)X - \mathcal{R}(Z, X)Y) \\ &= (\mathcal{R}(X, Y)Z + \mathcal{R}(X, Z)Y) - (\mathcal{R}(Y, X)Z + \mathcal{R}(Y, Z)X) \\ &= \mathcal{R}(X, Y + Z)(Y + Z) - \mathcal{R}(X, Y)Y - \mathcal{R}(X, Z)Z \\ &\quad - \mathcal{R}(Y, X + Z)(X + Z) + \mathcal{R}(Y, X)X + \mathcal{R}(Y, Z)Z \\ &= \mathcal{J}_{Y+Z}X - \mathcal{J}_Y X - \mathcal{J}_Z X - \mathcal{J}_{X+Z}Y + \mathcal{J}_X Y + \mathcal{J}_Z Y. \end{aligned}$$

Since $R(W,Z,Y,X) = R(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z,W)$, we have

$$R(X,Y,Z,W) = \frac{1}{3}g((\mathcal{J}_{Y+Z} - \mathcal{J}_Y - \mathcal{J}_Z)W - (\mathcal{J}_{Y+W} - \mathcal{J}_Y - \mathcal{J}_W)Z, X), \tag{6.19}$$

which shows that the Jacobi operators completely determine the curvature tensor.

Also, the curvature tensor can be expressed only by values $\mu(X,Y) = R(X,Y,Y,X)$. From the polarization

$$2R(Y,X,X,W) = R(Y+W,X,X,Y+W) - R(Y,X,X,Y) - R(W,X,X,W)$$

follows

$$2g(\mathcal{J}_X Y, W) = \mu(Y+W,X) - \mu(Y,X) - \mu(W,X),$$

and therefore $R(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z,W)$ is expressed with 18 terms

$$\begin{aligned}
6R(X,Y,Z,W) = \ & \mu(X+W,Y+Z) - \mu(X,Y+Z) - \mu(W,Y+Z) \\
& - \mu(X+W,Y) + \mu(X,Y) + \mu(W,Y) \\
& - \mu(X+W,Z) + \mu(X,Z) + \mu(W,Z) \\
& - \mu(Y+W,X+Z) + \mu(Y,X+Z) + \mu(W,X+Z) \\
& + \mu(Y+W,X) - \mu(Y,X) - \mu(W,X) \\
& + \mu(Y+W,Z) - \mu(Y,Z) - \mu(W,Z),
\end{aligned}$$

where 4 terms vanish in pairs for the final result,

$$\begin{aligned}
6R(X,Y,Z,W) = \ & \mu(X+W,Y+Z) - \mu(X,Y+Z) - \mu(W,Y+Z) \\
& - \mu(X+W,Y) + \mu(W,Y) - \mu(X+W,Z) + \mu(X,Z) \\
& - \mu(Y+W,X+Z) + \mu(Y,X+Z) + \mu(W,X+Z) \\
& + \mu(Y+W,X) - \mu(W,X) + \mu(Y+W,Z) - \mu(Y,Z).
\end{aligned} \tag{6.20}$$

Alternatively, we have

$$\begin{aligned}
&\left.\frac{\partial}{\partial s}\right|_{s=0} R(X+sW,Y+tZ,Y+tZ,X+sW) \\
&\quad= \lim_{s\to 0} \big(R(X,Y+tZ,Y+tZ,W) + R(W,Y+tZ,Y+tZ,X) + 2sR(W,Y+tZ,Y+tZ,W)\big) \\
&\quad= 2R(X,Y+tZ,Y+tZ,W),
\end{aligned}$$

whence it implies

$$\begin{aligned}
&\left.\frac{\partial^2}{\partial t \partial s}\right|_{t=0,s=0} R(X+sW,Y+tZ,Y+tZ,X+sW) \\
&\qquad= 2\lim_{t\to 0} \big(R(X,Y,Z,W) + R(X,Z,Y,W) + 2tR(X,Z,Z,W)\big) \\
&\qquad= 2\big(R(X,Y,Z,W) + R(X,Z,Y,W)\big),
\end{aligned}$$

so using some symmetries we can express R by μ in a different way,

$$\begin{aligned}
6R(X,Y,Z,W) &= 2(R(X,Y,Z,W) + (-R(X,Y,W,Z)) + (-R(Y,Z,X,W) - R(Z,X,Y,W))) \\
&= 2\big(R(X,Y,Z,W) + R(X,Z,Y,W)\big) - 2\big(R(X,Y,W,Z) + R(X,W,Y,Z)\big) \\
&= \left.\frac{\partial^2}{\partial s \partial t}\right|_{s=0,t=0} \big(R(X+sW,Y+tZ,Y+tZ,X+sW) - R(X+sZ,Y+tW,Y+tW,X+sZ)\big) \\
&= \left.\frac{\partial^2}{\partial s \partial t}\right|_{s=0,t=0} \big(\mu(X+sW,Y+tZ) - \mu(X+sZ,Y+tW)\big),
\end{aligned}$$

and therefore

$$R(X,Y,Z,W) = \frac{1}{6}\,\frac{\partial^2}{\partial s \partial t}\bigg|_{s=0,t=0} \big(\mu(X+sW,Y+tZ) - \mu(X+sZ,Y+tW)\big), \tag{6.21}$$

which is known from Lee [76, Proposition 13.27].

We have shown (in two ways) that the overall values for μ completely determine the curvature tensor. In the case of a nondegenerate plane $\mathrm{Span}\{X,Y\}$, its sectional curvature determines $\mu(X,Y) = R(X,Y,Y,X) = (\varepsilon_X\varepsilon_Y - (g(X,Y))^2)\kappa(X,Y)$. However, the values μ in a degenerate case $\varepsilon_X\varepsilon_Y = (g(X,Y))^2$ follow from continuity. Namely, for a null X we can find an arbitrary vector Z for which $g(X,Z) \neq 0$ holds, and for $0 \neq t \in \mathbb{R}$ observe the nondegenerate plane $\mathrm{Span}\{X, Y+tZ\}$ (since $\varepsilon_X\varepsilon_{Y+tZ} - (g(X,Y+tZ))^2 = -t^2(g(X,Z))^2 \neq 0$), where

$$\mu(X,Y) = \lim_{t\to 0}(\mu(X,Y) + 2tR(X,Y,Z,X) + t^2\mu(X,Z)) = \lim_{t\to 0}\mu(X,Y+tZ).$$

The last case is a degenerate plane with a nonnull X, where we set $Z = Y - \theta X$ for $\theta = g(Y,X)/\varepsilon_X$ to get $Z \perp X$, and therefore $\mu(X,Y) = \mu(X,Z)$, where the value $\mu(X,Z)$ is already calculated because either Z is null or the plane $\mathrm{Span}\{X,Z\}$ is nondegenerate. Thus, two algebraic curvature tensors that have the same sectional curvatures must be equal.

Theorem 6.6. *The values $\mu(X,Y) = R(X,Y,Y,X)$ completely determine the curvature tensor. The sectional curvature completely determines the curvature tensor at points where the scalar product is known.*

Although the sectional curvature looks simpler than the curvature tensor, its importance arises from the fact that knowledge of all sectional curvatures completely determines the curvature tensor (at points where the scalar product is known). This refers to the Jacobi operators, which also contain the very same information as the curvature tensor. These well known results give the uniqueness of the curvature tensor and are purely algebraic in nature.

The question of existence of a curvature tensor for given Jacobi operators naturally arises. This problem was observed and solved by Andrejić[3] in 2022 [11, Theorem 1], and we follow the arguments from [13], where omissions from the original work were corrected, and the theorem was generalised to a pseudo-Riemannian case.

Let us suppose that we know self-adjoint endomorphisms $\mathcal{K}_X$ on $\mathcal{V}$ for each nonnull $X \in \mathcal{V}$, such that they are compatible in the sense that (6.18) holds. Is there an algebraic curvature tensor R on $(\mathcal{V}, g)$, such that $\mathcal{J}_X = \mathcal{K}_X$ holds for all nonnull $X \in \mathcal{V}$?

Consider the condition $\mathcal{J}_X X = 0$ which holds for all Jacobi operators. However, the construction $\mathcal{K}_X = \varepsilon_X \mathbb{1}$ gives self-adjoint endomorphisms on $\mathcal{V}$ that are compatible, $g(\varepsilon_X \mathbb{1}\, Y, Y) = \varepsilon_X\varepsilon_Y = g(\varepsilon_Y \mathbb{1}\, X, X)$, but $\varepsilon_X \mathbb{1}\, X = \varepsilon_X X \neq 0$ holds for a nonnull $X \in \mathcal{V}$. Therefore, we add the natural condition that

$$\mathcal{K}_X X = 0, \tag{6.22}$$

holds for any nonnull $X \in \mathcal{V}$.

The first step is to extend the family $\mathcal{K}_X$ for all $X \in \mathcal{V}$. The natural extension is $\mathcal{K}_0 = 0$, which completes the family in the Riemannian setting. However, if the scalar product is indefinite, then we need to define $\mathcal{K}_X$ for any null $X \in \mathcal{V}$.

Let $X, Y, X+Y, X-Y \in \mathcal{V}$ be nonnull. Using (6.18), from

$$\begin{aligned} g(\mathcal{K}_{X\pm Y}Z, Z) = g(\mathcal{K}_Z(X\pm Y), X\pm Y) &= g(\mathcal{K}_Z X, X) \pm 2g(\mathcal{K}_Z X, Y) + g(\mathcal{K}_Z Y, Y) \\ &= g(\mathcal{K}_X Z, Z) \pm 2g(\mathcal{K}_Z X, Y) + g(\mathcal{K}_Y Z, Z), \end{aligned}$$

it follows $g(\mathcal{K}_{X+Y}Z, Z) + g(\mathcal{K}_{X-Y}Z, Z) = 2g(\mathcal{K}_X Z, Z) + 2g(\mathcal{K}_Y Z, Z)$ for any nonnull $Z \in \mathcal{V}$. The polarization $Z = V + W$ gives $g((\mathcal{K}_{X+Y} + \mathcal{K}_{X-Y} - 2\mathcal{K}_X - 2\mathcal{K}_Y)V, W) = 0$, when $\varepsilon_V\varepsilon_W\varepsilon_{V+W} \neq 0$.

[3] Vladica Andrejić (1978), Serbian mathematician

If $(E_1, E_2, \dots, E_n)$ is an orthonormal basis in $(\mathcal{V}, g)$, then we can consider the orthogonal basis $(E_1, 2E_2, \dots, nE_n)$, which provides $\varepsilon_{iE_i} = i^2\varepsilon_{E_i} \neq 0$ and $\varepsilon_{iE_i+jE_j} = i^2\varepsilon_{E_i} + j^2\varepsilon_{E_j} \neq 0$ for all $1 \leq i \neq j \leq n$. Hence,

$$\mathcal{K}_{X+Y} + \mathcal{K}_{X-Y} = 2\mathcal{K}_X + 2\mathcal{K}_Y \tag{6.23}$$

holds, whenever $\varepsilon_X\varepsilon_Y\varepsilon_{X+Y}\varepsilon_{X-Y} \neq 0$.

The equality (6.23) motivates us to define $\mathcal{K}_N$ for a null $N \in \mathcal{V}$ by

$$2\mathcal{K}_N = \mathcal{K}_{N+X} + \mathcal{K}_{N-X} - 2\mathcal{K}_X,$$

whenever the right hand side is defined, which immediately shows that $\mathcal{K}_N$ is a self-adjoint endomorphism on $\mathcal{V}$. Because of (6.23), if $\varepsilon_X\varepsilon_Y\varepsilon_{N+X}\varepsilon_{N-X}\varepsilon_{N+Y}\varepsilon_{N-Y} \neq 0$, then we have

$$\begin{aligned} 2(\mathcal{K}_{N+X} + \mathcal{K}_{N-X} - 2\mathcal{K}_X) &= (\mathcal{K}_{N+X+Y} + \mathcal{K}_{N+X-Y} - 2\mathcal{K}_Y) + (\mathcal{K}_{N-X+Y} + \mathcal{K}_{N-X-Y} - 2\mathcal{K}_Y) - 4\mathcal{K}_X \\ &= (\mathcal{K}_{N+X+Y} + \mathcal{K}_{N-X+Y} - 2\mathcal{K}_X) + (\mathcal{K}_{N+X-Y} + \mathcal{K}_{N-X-Y} - 2\mathcal{K}_X) - 4\mathcal{K}_Y \\ &= 2(\mathcal{K}_{N+Y} + \mathcal{K}_{N-Y} - 2\mathcal{K}_Y), \end{aligned}$$

whenever $\varepsilon_{N+X+Y}\varepsilon_{N+X-Y}\varepsilon_{N-X+Y}\varepsilon_{N-X-Y} \neq 0$. Otherwise, we can use

$$\mathcal{K}_{N+X} + \mathcal{K}_{N-X} - 2\mathcal{K}_X = \mathcal{K}_{N+tX} + \mathcal{K}_{N-tX} - 2\mathcal{K}_{tX} = \mathcal{K}_{N+Y} + \mathcal{K}_{N-Y} - 2\mathcal{K}_Y,$$

where t is not a root of

$$\varepsilon_{tX}\varepsilon_{N+tX}\varepsilon_{N-tX}\varepsilon_{N+X+tX}\varepsilon_{N-X+tX}\varepsilon_{N+X-tX}\varepsilon_{N-X-tX}\varepsilon_{N+Y+tX}\varepsilon_{N-Y+tX}\varepsilon_{N+Y-tX}\varepsilon_{N-Y-tX} = 0,$$

which is a polynomial equation of degree 22. This proves that $\mathcal{K}_N$ does not depend on the choice of X, and therefore $\mathcal{K}_N$ is well-defined.

If we use $Z \in \mathcal{V}$ that satisfies $\varepsilon_{N+Z}\varepsilon_{N-Z}\varepsilon_Z \neq 0$, then the equality

$$\begin{aligned} 2g(\mathcal{K}_N X, X) &= g((\mathcal{K}_{N+Z} + \mathcal{K}_{N-Z} - 2\mathcal{K}_Z)X, X) \\ &= g(\mathcal{K}_X(N+Z), N+Z) + g(\mathcal{K}_X(N-Z), N-Z) - 2g(\mathcal{K}_X Z, Z) = 2g(\mathcal{K}_X N, N) \end{aligned}$$

holds for any null N and any nonnull X, which means that endomorphisms $\mathcal{K}_N$ and $\mathcal{K}_X$ are compatible. With that in mind, we see that the very same equality holds when both N and X are null. In this way, we obtain $\mathcal{K}_X$ for any $X \in \mathcal{V}$, and this extended family remains compatible.

Let $N \in \mathcal{V}$ be null. For any $X \in \mathcal{V}$ such that $\varepsilon_X\varepsilon_{N+X}\varepsilon_{N-X} \neq 0$ holds, using the properties (6.22) and (6.18) we have

$$\begin{aligned} 2g(\mathcal{K}_N N, X) &= g(\mathcal{K}_{N+X}N + \mathcal{K}_{N-X}N - 2\mathcal{K}_X N, X) = -g(\mathcal{K}_{N+X}X, X) + g(\mathcal{K}_{N-X}X, X) - 2g(\mathcal{K}_X X, N) \\ &= -g(\mathcal{K}_X(N+X), N+X) + g(\mathcal{K}_X(N-X), N-X) = 0. \end{aligned}$$

If $(E_1, E_2, \dots, E_n)$ is an orthonormal basis in $(\mathcal{V}, g)$, then we create an orthogonal basis $(mE_1, mE_2, \dots, mE_n)$, where $m > \max_{1\leq i\leq n} |2g(N, E_i)|$ to provide $\varepsilon_{mE_i} \neq 0$ and $\varepsilon_{N\pm mE_i} \neq 0$ for $1 \leq i \leq n$, which yields $K_N N = 0$.

The second step considers a compatible family of self-adjoint endomorphisms $\mathcal{K}_X$ for all $X \in \mathcal{V}$ such that $\mathcal{K}_X X = 0$ holds. For all $X, Y \in \mathcal{V}$ and $t \in \mathbb{R}$ we have

$$g(\mathcal{K}_{tX}Y, Y) = g(\mathcal{K}_Y tX, tX) = t^2 g(\mathcal{K}_Y X, X) = g(t^2\mathcal{K}_X Y, Y),$$

where the polarization $Y = V + W$ gives $g(\mathcal{K}_{tX}V, W) = g(t^2\mathcal{K}_X V, W)$ for $V, W \in \mathcal{V}$, so since g is nondegenerate, it follows $\mathcal{K}_{tX} = t^2\mathcal{K}_X$, which is a natural property of Jacobi operators.

For all $X, Y, Z \in \mathcal{V}$ and $t \in \mathbb{R}$, using the compatibility we have

$$\begin{aligned} g(\mathcal{K}_{X+tY}Z, Z) &= g(\mathcal{K}_Z(X+tY), X+tY) = g(\mathcal{K}_Z X, X) + 2tg(\mathcal{K}_Z X, Y) + t^2 g(\mathcal{K}_Z Y, Y) \\ &= g(\mathcal{K}_X Z, Z) + 2tg(\mathcal{K}_Z X, Y) + t^2 g(\mathcal{K}_Y Z, Z), \end{aligned}$$

which implies

$$g((\mathcal{K}_{X+tY} - \mathcal{K}_X - t^2\mathcal{K}_Y)Z, Z) = 2tg(\mathcal{K}_Z X, Y) = tg((\mathcal{K}_{X+Y} - \mathcal{K}_X - \mathcal{K}_Y)Z, Z).$$

After the polarization $Z = V + W$ we get $g((\mathcal{K}_{X+tY} - \mathcal{K}_X - t^2\mathcal{K}_Y)V, W) = tg((\mathcal{K}_{X+Y} - \mathcal{K}_X - \mathcal{K}_Y)V, W)$ for all $V, W \in \mathcal{V}$, and therefore, since g is nondegenerate, we obtain a generalisation of (6.23),

$$\mathcal{K}_{X+tY} - \mathcal{K}_X - t^2\mathcal{K}_Y = t(\mathcal{K}_{X+Y} - \mathcal{K}_X - \mathcal{K}_Y), \tag{6.24}$$

which yields $\mathcal{K}_{X+tY} = t\mathcal{K}_{X+Y} + (1-t)\mathcal{K}_X + (t^2 - t)\mathcal{K}_Y$. Using the property (6.22), we get

$$\begin{aligned} 0 = \mathcal{K}_{X+tY}(X + tY) &= t\mathcal{K}_{X+Y}(X + tY) + (1-t)\mathcal{K}_X(X + tY) + (t^2 - t)\mathcal{K}_Y(X + tY) \\ &= t\mathcal{K}_{X+Y}((t-1)Y) + (1-t)\mathcal{K}_X(tY) + t(t-1)\mathcal{K}_Y(X) \\ &= t(t-1)(\mathcal{K}_{X+Y}Y - \mathcal{K}_X Y + \mathcal{K}_Y X), \end{aligned}$$

which for $t \in \mathbb{R} \setminus \{0, 1\}$ implies

$$\mathcal{K}_{X+Y}Y - \mathcal{K}_X Y + \mathcal{K}_Y X = 0 \tag{6.25}$$

for all $X, Y \in \mathcal{V}$.

Let us use a compatible family of self-adjoint endomorphisms $\mathcal{K}_X$ on $\mathcal{V}$ that satisfies $\mathcal{K}_X X = 0$ to define a map $R\colon \mathcal{V}^4 \to \mathbb{R}$, resembling the formula (6.19) by

$$R(X, Y, Z, W) = \frac{1}{3} g((\mathcal{K}_{Y+Z} - \mathcal{K}_Y - \mathcal{K}_Z)W - (\mathcal{K}_{Y+W} - \mathcal{K}_Y - \mathcal{K}_W)Z, X), \tag{6.26}$$

for all $X, Y, Z, W \in \mathcal{V}$. From (6.24), if we take the limit where t tends to zero, then it follows

$$g((\mathcal{K}_{Y+Z} - \mathcal{K}_Y - \mathcal{K}_Z)W, X) = \frac{g(\mathcal{K}_{Y+tZ}W, X) - g(\mathcal{K}_Y W, X)}{t} - tg(\mathcal{K}_Z W, X) = \left.\frac{\partial}{\partial t}\right|_{t=0} g(\mathcal{K}_{Y+tZ}W, X),$$

and therefore the equality (6.26) is equivalent to

$$R(X, Y, Z, W) = \frac{1}{3} \left.\frac{\partial}{\partial t}\right|_{t=0} g(\mathcal{K}_{Y+tZ}W - \mathcal{K}_{Y+tW}Z, X). \tag{6.27}$$

However, from

$$\begin{aligned} 2g(\mathcal{K}_{Y+tZ}W, X) &= \left.\frac{\partial}{\partial s}\right|_{s=0} \Big(g(\mathcal{K}_{Y+tZ}X, X) + 2sg(\mathcal{K}_{Y+tZ}W, X) + s^2 g(\mathcal{K}_{Y+tZ}W, W)\Big) \\ &= \left.\frac{\partial}{\partial s}\right|_{s=0} \Big(g(\mathcal{K}_{Y+tZ}W, W) + 2sg(\mathcal{K}_{Y+tZ}W, X) + s^2 g(\mathcal{K}_{Y+tZ}X, X)\Big), \end{aligned}$$

we obtain

$$2g(\mathcal{K}_{Y+tZ}W, X) = \left.\frac{\partial}{\partial s}\right|_{s=0} \mu(X + sW, Y + tZ) = \left.\frac{\partial}{\partial s}\right|_{s=0} \mu(W + sX, Y + tZ), \tag{6.28}$$

where $\mu(X, Y) = g(\mathcal{K}_Y X, X) = g(\mathcal{K}_X Y, Y)$ for all $X, Y \in \mathcal{V}$. The equality on the left side in (6.28) shows that (6.27) is equivalent to our formula (6.21),

$$R(X, Y, Z, W) = \frac{1}{6} \left.\frac{\partial^2}{\partial s \partial t}\right|_{s=0, t=0} (\mu(X + sW, Y + tZ) - \mu(X + sZ, Y + tW)). \tag{6.21 revisited}$$

The definition (6.26) of R is equivalent to (6.21), but the latter is easier to prove that R is an algebraic curvature tensor. The property (6.6) follows directly from (6.21), while (6.4) is a consequence of commutativity $\partial_s \partial_t = \partial_t \partial_s$. The equality on the right side in (6.28) helps us to easily see (6.5). From (6.27), $R(X,Y,Z,W)$ is obviously linear by X, but due to the already proven symmetries, where (6.13) automatically follows, R is multi-linear, which proves that R is an algebraic curvature tensor.

It remains to show that $\mathcal{K}_X$ for $X \in \mathcal{V}$ are the Jacobi operators for R. From (6.26), using $g(\mathcal{J}_Y W, X) = R(W,Y,Y,X)$ we have $3g(\mathcal{J}_Y W, X) = g(2\mathcal{K}_Y W - \mathcal{K}_{Y+W}Y + \mathcal{K}_W Y, X)$, which implies

$$3\mathcal{J}_Y W = 2\mathcal{K}_Y W - \mathcal{K}_{Y+W}Y + \mathcal{K}_W Y.$$

According to (6.25) we have $-\mathcal{K}_{Y+W}Y = \mathcal{K}_{Y+W}W = \mathcal{K}_Y W - \mathcal{K}_W Y$, so $3\mathcal{J}_Y W = 3\mathcal{K}_Y W$ holds for all $Y, W \in \mathcal{V}$, which gives $\mathcal{J}_Y = \mathcal{K}_Y$. This finally proves the following theorem (see [13, Theorem 1] and [11, Theorem 1]).

Theorem 6.7. *Let $\mathcal{K}_X$ for all nonnull $X \in \mathcal{V}$ be a compatible family of self-adjoint endomorphisms on a scalar product space $\mathcal{V}$ that satisfies $\mathcal{K}_X X = 0$. Then there exists a unique algebraic curvature tensor on $\mathcal{V}$ such that $\mathcal{K}_X$ are its Jacobi operators.*

6.4 Constant sectional curvature

The most simple case of a pseudo-Riemannian manifold is a space of constant sectional curvature. Let (M,g) be a pseudo-Riemannian manifold such that $\kappa(\sigma) = k(p)$ for any nondegenerate tangent plane $\sigma \leq T_pM$ and some fixed $k(p) \in \mathbb{R}$. According to Theorem 6.6 this uniquely determines the curvature tensor R. For any point $p \in M$ we have the restriction that gives the scalar product space $\mathcal{V} = T_pM$, and the associated algebraic curvature tensor. Consider an algebraic curvature tensor $R = kR^1 \in \mathfrak{T}^0_4(\mathcal{V})$ from Example 6.6, where R^1 is defined in (6.14). Since it satisfies $R(X,Y,Y,X) = k(\varepsilon_X\varepsilon_Y - (g(X,Y)^2)$, we have $\kappa(X,Y) = k$, so the curvature tensor on (M,g) has a form

$$R(X,Y,Z,W) = k\big(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big),$$

where $k \in \mathfrak{F}(M)$. Moreover, in dimension $n \geq 3$, we have the following surprising theorem of Schur[4] [108].

Theorem 6.8. *Let (M,g) be a connected pseudo-Riemannian manifold of dimension $n \geq 3$. If the sectional curvature $\kappa(\sigma)$ does not depend on the plane $\sigma \subseteq T_pM$, but only on the point $p \in M$, then κ is constant.*

Proof. The sectional curvature at any point $p \in M$ is constant $k(p)$, so the curvature tensor must be $R = kR^1$. Since ∇ is a metric connection, from $\nabla g = 0$ we have

$$\begin{aligned}(\nabla_V R^1)(X,Y,Z,W) = \;& V(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)) \\ & - (g(Y,Z)g(\nabla_V X,W) - g(\nabla_V X,Z)g(Y,W)) \\ & - (g(\nabla_V Y,Z)g(X,W) - g(X,Z)g(\nabla_V Y,W)) \\ & - (g(Y,\nabla_V Z)g(X,W) - g(X,\nabla_V Z)g(Y,W)) \\ & - (g(Y,Z)g(X,\nabla_V W) - g(X,Z)g(Y,\nabla_V W)) = 0,\end{aligned}$$

and therefore R^1 is a parallel tensor field, $\nabla_V R^1 = 0$. Thus

$$\nabla_V R = \nabla_V(k \otimes R^1) = \nabla_V k \cdot R^1 + k \cdot \nabla_V R^1 = (Vk)R^1,$$

and therefore

$$\begin{aligned}(\nabla_V R)(X,Y,Z,W) &= (Vk)(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)), \\ (\nabla_X R)(Y,V,Z,W) &= (Xk)(g(V,Z)g(Y,W) - g(Y,Z)g(V,W)), \\ (\nabla_Y R)(V,X,Z,W) &= (Yk)(g(X,Z)g(V,W) - g(V,Z)g(X,W)).\end{aligned}$$

[4] Friedrich Heinrich Schur (1856–1932), German mathematician

The sum of the previous equalities is zero by the second Bianchi identity (6.10), so

$$g(((Vk)g(Y,Z)-(Yk)g(V,Z))X+((Xk)g(V,Z)-(Vk)g(X,Z))Y \\ +((Yk)g(X,Z)-(Xk)g(Y,Z))V,W)=0$$

for all W, and therefore from the nondegeneracy of metric

$$((Vk)g(Y,Z)-(Yk)g(V,Z))X+((Xk)g(V,Z)-(Vk)g(X,Z))Y+((Yk)g(X,Z)-(Xk)g(Y,Z))V=0$$

holds for all $X,Y,Z,V\in\mathfrak{X}(M)$. By restricting the previous equality to a point $p\in M$, thanks to Example 2.24, we see that

$$g_p((V_pk)Y_p-(Y_pk)V_p,Z_p)X_p+g_p((X_pk)V_p-(V_pk)X_p,Z_p)Y_p+g_p((Y_pk)X_p-(X_pk)Y_p,Z_p)V_p=0$$

holds for all $X_p,Y_p,Z_p,V_p\in T_pM$. Let us fix $X_p\in T_pM$. Since $n\geq 3$, we can find $Y_p,V_p\in T_pM$ such that vectors X_p,Y_p,V_p are linearly independent. This linear independence yields zero coefficients in the previous equality, which gives

$$g_p((Y_pk)X_p-(X_pk)Y_p,Z_p)=0.$$

This equality holds for every Z_p, so the nondegeneracy of g_p gives $(Y_pk)X_p-(X_pk)Y_p=0$, and by linear independence we have $X_pk=0$. Since this hold for each $p\in M$, we have $Xk=0$ for $X\in\mathfrak{X}(M)$, so by Example 5.1, k is locally constant. Finally, since M is connected, k is constant. □

A ***space of constant sectional curvature*** is a pseudo-Riemannian manifold (M,g) for which the sectional curvature $\kappa(\sigma)$ is constant, that is equivalent to $\mathcal{R}=\kappa\mathcal{R}^1$ for $\kappa\in\mathbb{R}$. The scaling is the process in which a metric g is changed by a metric $\widetilde{g}=\lambda g$ for some constant $\lambda>0$ (the multiplication of a metric by a negative number is not convenient since it turns its signature which essentially changes the manifold). After the scaling we get $\widetilde{\mathcal{R}}^1(X,Y)Z=\lambda\mathcal{R}^1(X,Y)Z$, while the corresponding relations are

$$\widetilde{\Gamma}^k_{ij}=\Gamma^k_{ij},\quad \widetilde{\nabla}_XY=\nabla_XY,\quad \widetilde{\mathcal{R}}(X,Y)Z=\mathcal{R}(X,Y)Z,\quad \widetilde{\kappa}=\kappa/\lambda.$$

Thus, there are three essential cases for a curvature tensor of constant sectional curvature, $R=R^1$ (with $\kappa=1$), $R=0$ (with $\kappa=0$), and $R=-R^1$ (with $\kappa=-1$). The corresponding model spaces for Riemannian manifolds are, respectively, a sphere $\mathbf{S}^n$, an Euclidean space $\mathbb{R}^n$, and a hyperbolic space $\mathbf{H}^n$.

Example 6.7. For the hyperbolic half-plane $\mathbf{HU}^2$ from Example 5.8 we had calculated $g_{11}=g_{22}=1/(x_2)^2$ as well as $\Gamma^1_{11}=\Gamma^2_{12}=\Gamma^1_{22}=0$, $\Gamma^2_{11}=-\Gamma^1_{12}=-\Gamma^2_{22}=1/x_2$. From the formula (6.3) we have

$$\begin{aligned}\mathcal{R}^1_{121}&=\partial_1\Gamma^1_{21}-\partial_2\Gamma^1_{11}+\Gamma^1_{21}\Gamma^1_{11}-\Gamma^1_{11}\Gamma^1_{21}+\Gamma^2_{21}\Gamma^1_{12}-\Gamma^2_{11}\Gamma^1_{22}=0,\\ \mathcal{R}^2_{121}&=\partial_1\Gamma^2_{21}-\partial_2\Gamma^2_{11}+\Gamma^1_{21}\Gamma^2_{11}-\Gamma^1_{11}\Gamma^2_{21}+\Gamma^2_{21}\Gamma^2_{12}-\Gamma^2_{11}\Gamma^2_{22}=(1-1+1)/(x_2)^2=1/(x_2)^2,\end{aligned}$$

that is, $\mathcal{R}(\partial_1,\partial_2)\partial_1=(1/(x_2)^2)\partial_2$. Then $R(\partial_1,\partial_2,\partial_1,\partial_2)=1/(x_2)^4$ and finally $\kappa=-1$. △

In the Riemannian setting, the sectional curvature κ may be viewed as a continuous real-valued function on the Grassmann bundle of 2-planes of a Riemannian manifold M. From this it will follow that κ on a compact subset of M is bounded, that is, κ attains its minimum and maximum values [21, Section 9.3]. Thus, in Riemannian geometry, lower and upper bounds of the sectional curvature have been intensively studied.

However, in a pseudo-Riemannian setting, some bounds of κ usually imply that κ is constant. Let us examine the sectional curvature at some fixed point of a pseudo-Riemannian manifold. Consider an algebraic curvature tensor R on an indefinite scalar product space $(\mathcal{V}, g)$ of dimension $n \geq 3$.

Let $A, B, X \in \mathcal{V}$ be orthonormal vectors such that $\varepsilon_A = \varepsilon_B = -\varepsilon_X$. Then, we have

$$\kappa(X+\theta B, A) = \frac{R(X+\theta B, A, A, X+\theta B)}{\varepsilon_{X+\theta B}\varepsilon_A} = \frac{\theta^2\kappa(B,A) - \kappa(X,A) + 2\theta R(X,A,A,B)}{\theta^2 - 1}. \tag{6.29}$$

If $\kappa \geq m$ is bounded from below, then $\theta^2\kappa(B,A) - \kappa(X,A) + 2\theta R(X,A,A,B)$ is greater than $m(\theta^2-1)$ for $|\theta| > 1$ and less than $m(\theta^2-1)$ for $|\theta| < 1$, so by continuity it is zero for $\theta = 1$ and $\theta = -1$, which implies $\kappa(B,A) = \kappa(X,A)$ and $R(X,A,A,B) = 0$.

Changing the orthonormal vectors to $A\cosh t + X\sinh t, B, A\sinh t + X\cosh t$, for an arbitrary $t \neq 0$, we have

$$\begin{aligned} 0 &= R(A\sinh t + X\cosh t, A\cosh t + X\sinh t, A\cosh t + X\sinh t, B) \\ &= R(X,A,A,B)\cosh t - R(A,X,X,B)\sinh t, \end{aligned}$$

where $R(X,A,A,B) = 0$ implies $R(A,X,X,B) = 0$. This argument was originally given by Kulkarni[5] [74], and the rest is not difficult to finish.

For example, we can use an orthogonal basis $(E_1, \dots, E_n)$ in $(\mathcal{V}, g)$ with $E_1 = X$ and $E_2 = A$ to get $\mathcal{J}_X(A) = \sum_{i=1}^n \varepsilon_{E_i} R(A,X,X,E_i)E_i = \varepsilon_A R(A,X,X,A)A$. Hence, if X is spacelike (timelike) then any timelike (spacelike) vector in $X^\perp$ is an eigenvector of $\mathcal{J}_X$. Especially, A and $E_i + 2A$ for $i > 2$ are non-orthogonal eigenvectors and therefore share the same eigenvalue (Lemma A.18), which yields $\widetilde{\mathcal{J}}_X = \varepsilon_X\mu(X)\,\mathbb{1}_{X^\perp}$ for some function μ. However, the equation (6.18) implies $\varepsilon_X\varepsilon_Y\mu(X) = \varepsilon_Y\varepsilon_X\mu(Y)$ for orthogonal X and Y, so for definite vectors holds $\mu(X) = \mu(Y) = \mathrm{Const}$. This ensures a constant sectional curvature which proves the following theorem given by Kulkarni in 1979 [74].

Theorem 6.9. *Let κ be the sectional curvature function of an indefinite algebraic curvature tensor. If κ is either bounded from above or bounded from below, then κ is constant.*

Consequently, using the Schur's theorem (Theorem 6.8), if the sectional curvature function of a connected indefinite manifold M of dimension $n \geq 3$ is either bounded from above or bounded from below, then M is a space of constant sectional curvature. This Kulkarni's result was the starting point for a wide research on the sectional curvature of indefinite (usually Lorentzian) metrics.

Let us suppose that the sectional curvature of indefinite planes are bounded both above and below. Again, we start with orthonormal vectors $A, B, X \in \mathcal{V}$ such that $\varepsilon_A = \varepsilon_B = -\varepsilon_X$ and use the equality (6.29). For $|\theta| < 1$, the plane $\mathrm{Span}\{X+\theta B, A\}$ is indefinite and therefore $-m \leq \kappa(X+\theta B, A) \leq m$ for some constant $m > 0$, which gives

$$-m(1-\theta^2) \leq \theta^2\kappa(B,A) - \kappa(X,A) + 2\theta R(X,A,A,B) \leq m(1-\theta^2).$$

By continuity at $\theta = 1$ and $\theta = -1$, we have $\kappa(B,A) - \kappa(X,A) \pm 2R(X,A,A,B) = 0$, and the rest of proof is the same as in the previous theorem. Thus, we have the following theorem given by Dajczer and Nomizu[6] in 1980 [39].

Theorem 6.10. *Let κ be the sectional curvature function of an indefinite algebraic curvature tensor. If κ of indefinite planes is bounded both above and below, then κ is constant.*

Of course, as before, if the sectional curvature function of indefinite planes of a connected indefinite manifold M of dimension $n \geq 3$ is bounded both above and below, then M is a space of constant sectional curvature. However, one-side boundedness on indefinite planes does not imply that κ is a constant function.

[5] Ravindra Shripad Kulkarni (1942), Indian mathematician

[6] Katsumi Nomizu (1924–2008), Japanese-American mathematician

6.5 Ricci curvature

The contraction is an important operation that allows us to get new tensors from existing ones. Often we need to write the value of a new tensor at some point of a manifold as simple as possible, what happens if we use a local orthonormal frame $(E_1,\dots,E_n)$ from Theorem 4.13. We denote its dual coframe by $(E_1^*,\dots,E_n^*)$, and it is easily verified that $E_i^* = \varepsilon_{E_i} E_i^\flat$ for $1 \le i \le n$.

For some tensor field A, the contraction of the new covariant argument of total covariant derivative ∇A with some of the original arguments is called the ***divergence*** and we use the notation $\operatorname{div} A$. It is mostly used in two special cases where there is a unique divergence.

In the first case we consider an arbitrary vector field $V \in \mathfrak{X}(M) = \mathfrak{T}_0^1(M)$, where we set

$$\operatorname{div} V = C(\nabla V) \in \mathfrak{F}(M).$$

In an orthonormal frame for $V = \sum_j V^j E_j$ we have

$$\begin{aligned}\operatorname{div} V &= \sum_i (\nabla V)_i^i = \sum_i (\nabla_{E_i} V)(E_i^*) = \sum_i E_i^*(\nabla_{E_i} V) = \sum_i \varepsilon_{E_i} g(\nabla_{E_i} V, E_i)\\ &= \sum_i \varepsilon_{E_i} g\left(E_i, \sum_j E_i(V^j)E_j + \sum_j V^j \sum_k \Gamma_{ij}^k E_k\right) = \sum_i E_i(V^i) + \sum_{i,j} V^j \Gamma_{ij}^i.\end{aligned}$$

In natural coordinates on $\mathbb{R}_\nu^n$ holds $\operatorname{div} V = \sum_i \partial V^i/\partial \pi_i$, which is the usual formula on $\mathbb{R}^3$.

Example 6.8. The ***Laplacian***[7] Δf of a function $f \in \mathfrak{F}(M)$ is the divergence of its gradient, $\Delta f = \operatorname{div}(\operatorname{grad} f) \in \mathfrak{F}(M)$. According to Lemma 5.4 the sharp commute with the covariant derivative, so

$$\Delta f = \operatorname{div}(\operatorname{grad} f) = C(\nabla(df^\sharp)) = C((\nabla df)^\sharp) = C((\nabla\nabla f)^\sharp) = \operatorname{tr}_g(\nabla^2 f),$$

which means that the Laplacian is the trace of Hessian. In an orthonormal frame we have

$$\begin{aligned}\Delta f = \operatorname{div}(\operatorname{grad} f) &= \sum_i \varepsilon_{E_i} E_i E_i f + \sum_{i,j} \varepsilon_{E_j}(E_j f)\Gamma_{ij}^i\\ &= \operatorname{tr}_g(\nabla^2 f) = \sum_i \varepsilon_{E_i} E_i E_i f - \sum_{i,j} \varepsilon_{E_i}(E_j f)\Gamma_{ii}^j,\end{aligned}$$

so in natural coordinates on $\mathbb{R}_\nu^n$ we have $\Delta f = \sum_i \varepsilon_{E_i} \partial^2 f/\partial \pi_i^2$, which is reduced to the usual formula on $\mathbb{R}^3$. △

In the second case we consider a symmetric covariant tensor field A of order 2, where we define

$$\operatorname{div} A = C((\nabla A)^\sharp) \in \mathfrak{T}_1^0(M) = \mathfrak{X}^*(M),$$

which can be calculated in an orthonormal frame with

$$(\operatorname{div} A)(X) = \sum_i ((\nabla A)^\sharp)(E_i^*, X, E_i) = \sum_{i,j} g^{ij}(\nabla A)(X, E_j, E_i) = \sum_i \varepsilon_{E_i}(\nabla_{E_i} A)(X, E_i).$$

Let (M,g) be a pseudo-Riemannian manifold, $\operatorname{Sym} A = A \in \mathfrak{T}_2^0(M)$, and $f \in \mathfrak{F}(M)$. Then, in an orthonormal frame,

$$\begin{aligned}(\operatorname{div} fA)(X) &= \sum_i \varepsilon_{E_i}(\nabla_{E_i} fA)(X, E_i) = \sum_i \varepsilon_{E_i}\big((\nabla_{E_i} f)A(X,E_i) + f(\nabla_{E_i} A)(X,E_i)\big)\\ &= A\left(X, \sum_i \varepsilon_{E_i}(E_i f)E_i\right) + f\sum_i \varepsilon_{E_i}(\nabla_{E_i} A)(X,E_i) = A(\operatorname{grad} f, X) + f(\operatorname{div} A)(X).\end{aligned}$$

[7] Pierre-Simon Laplace (1749–1827), French mathematician, physicist and astronomer

Especially, in the case $A = g$ for $f \in \mathfrak{F}(M)$ it yields $(\mathrm{div}(fg))(X) = g(\mathrm{grad} f, X) = df(X)$, which gives the following useful formula

$$\mathrm{div}(fg) = df. \tag{6.30}$$

Since covariant tensors of higher order are quite complicated, it is often useful to construct simpler tensors that compress some information. Let us start with covariant tensors, like a curvature tensor R, where we can consider the trace, that is, the contraction of sharp. We already had $R = \mathcal{R}^\flat$, so we need $\mathrm{tr}_g R = C(R^\sharp) = C\mathcal{R}$. However, it is necessary to emphasize which covariant index we should pair with the covariant index. If we try with the third covariant index, then we have

$$\sum_k \mathcal{R}^k_{ijk} = \sum_k \sum_l g^{kl} R_{ijkl} = -\sum_l \sum_k g^{lk} R_{ijlk} = -\sum_l \mathcal{R}^l_{ijl},$$

which is possible only if $\sum_k \mathcal{R}^k_{ijk} = 0$, so the third index is not suitable for the contraction. We have a choice between the first and the second index, but from $R_{ijkl} = -R_{jikl}$ we have $\sum_i \mathcal{R}^i_{jik} = -\sum_i \mathcal{R}^i_{ijk}$, so these possibilities of non-zero contractions differ in the sign only. If we choose the first index, we get the ***Ricci tensor***[8], $\mathrm{Ric} = \mathrm{tr}_g R = C\mathcal{R} \in \mathfrak{T}^0_2(M)$ where we have

$$\mathrm{Ric}(X, Y) = \mathrm{Tr}(Z \mapsto \mathcal{R}(Z, X)Y) = \mathrm{Tr}(\mathcal{J}(X, Y)).$$

The Ricci tensor is obviously symmetric, $\mathrm{Ric}(X, Y) = \mathrm{Ric}(Y, X)$, and if

$$R_{ij} = \sum_l \mathcal{R}^l_{lij} = \sum_{l,k} g^{lk} R_{lijk}$$

denote its components, we have $\mathrm{Ric} = \sum_{i,j} R_{ij}\, dx_i \otimes dx_j$, which in an orthonormal frame becomes

$$\mathrm{Ric}(X, Y) = \sum_i \varepsilon_{E_i} g(\mathcal{R}(E_i, X)Y, E_i) = \sum_i \varepsilon_{E_i} R(E_i, X, Y, E_i). \tag{6.31}$$

The ***scalar curvature*** is the trace of the Ricci tensor, $\mathrm{Sc} = \mathrm{tr}_g \mathrm{Ric} = C(\mathrm{Ric}^\sharp) \in \mathfrak{F}(M)$, so $\mathrm{Sc} = \sum_{i,j} g^{ij} R_{ij} = \sum_{i,j,k,l} g^{ij} g^{lk} R_{lijk}$ holds, while in an orthonormal frame we have

$$\mathrm{Sc} = \sum_{i,j} \varepsilon_{E_i} \varepsilon_{E_j} R_{ijji}.$$

If we consider a local orthonormal frame at a point $p \in M$, that is, an orthonormal basis $(E_1, \ldots, E_n)$ in T_pM, we can use the sectional curvature to describe the Ricci tensor and the scalar curvature,

$$\varepsilon_{E_k} \mathrm{Ric}_p(E_k, E_k) = \varepsilon_{E_k} \sum_{i=1}^n \varepsilon_{E_i} R_p(E_i, E_k, E_k, E_i) = \sum_i^{i \neq k} \kappa(E_i, E_k),$$
$$\mathrm{Sc}(p) = \sum_{i,j=1}^n \varepsilon_{E_i} \varepsilon_{E_j} R_p(E_i, E_j, E_j, E_i) = \sum_{i,j}^{i \neq j} \kappa(E_i, E_j).$$

Example 6.9. The Ricci tensor is determined by the sectional curvatures, but generally contains less information. However, in small dimensions ($n = 2, 3$) the Ricci tensor determines the full curvature tensor. In dimension $n = 3$ for a nondegenerate plane σ we can take an orthonormal basis (E_1, E_2, E_3) such that $\sigma = \mathrm{Span}\{E_1, E_2\}$. Then $\varepsilon_k R_{kk} = \sum_{i \neq k} \kappa(E_k, E_i)$, and therefore $\kappa(\sigma) = \kappa(E_1, E_2) = (\varepsilon_1 R_{11} + \varepsilon_2 R_{22} - \varepsilon_3 R_{33})/2$. △

[8] Gregorio Ricci-Curbastro (1853–1925), Italian mathematician

Let us start with the second Bianchi identity that can be expressed through the components of the new covariant tensor $T \in \mathfrak{T}^0_5(M)$ given by

$$T_{ijklm} = \nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0,$$

where $\nabla_i R_{jklm} = (\nabla R)_{jklmi}$. We can contract T twice using the trace on the indices i and m and the trace on the indices j and l, to get $\sum_{i,m} g^{im} \sum_{j,l} g^{jl} T_{ijklm} = 0$. We perform the calculation at some point $p \in M$ using the normal coordinates centred at that point. Since $\nabla g = 0$, the trace commutes with the covariant derivative,

$$\sum_{i,m} g^{im} \nabla_{E_i} \sum_{j,l} g^{jl} R_{jklm} + \sum_{j,l} g^{jl} \nabla_{E_j} \sum_{i,m} g^{im} R_{kilm} + \nabla_{E_k} \sum_{i,m} g^{im} \sum_{j,l} g^{jl} R_{ijlm} = 0,$$

and therefore

$$\nabla_{E_k} \operatorname{Sc} = 2 \sum_{i,m} g^{im} \nabla_{E_i} R_{km}.$$

However, we have

$$\operatorname{div} \operatorname{Ric}(E_k) = \sum_{i,m} g^{im} (\nabla_{E_i} \operatorname{Ric})(E_k, E_m) = \sum_{i,m} g^{im} \nabla_{E_i} (\operatorname{Ric}(E_k, E_m)),$$

which yields the equality $\nabla_{E_k} \operatorname{Sc} = 2 \operatorname{div} \operatorname{Ric}(E_k)$. Hence the identity $2 \operatorname{div} \operatorname{Ric} = \nabla \operatorname{Sc}$ holds in normal coordinates, but this is a proper tensor equality and therefore stands in any basis. It is the fundamental equation in general relativity called the ***contracted Bianchi identity***.

Theorem 6.11. *For a pseudo-Riemannian manifolds holds* $2 \operatorname{div} \operatorname{Ric} = d \operatorname{Sc}$.

We say that a pseudo-Riemannian manifold (M, g) is ***Einstein***[9] if its Ricci tensor is proportional to the metric, that is, if $\operatorname{Ric} = \lambda g$ for some constant $\lambda \in \mathbb{R}$. If we take the trace, we get $\operatorname{Sc} = \operatorname{tr}_g \operatorname{Ric} = \operatorname{tr}_g(\lambda g) = \lambda \operatorname{tr}_g g = \lambda n$, and therefore an Einstein manifold has constant scalar curvature.

Example 6.10. Let (M, g) be a space of constant sectional curvature κ. Then we have,

$$R_{ij} = \sum_{l,k} g^{lk} R_{lijk} = \sum_{l,k} \kappa g^{lk} (g_{lk} g_{ij} - g_{lj} g_{ik}) = (n-1)\kappa g_{ij},$$

and therefore $\operatorname{Ric} = (n-1)\kappa g$, so (M, g) is an Einstein manifold and $\operatorname{Sc} = n(n-1)\kappa$ holds. △

More relaxed, we can say that (M, g) is Einstein at a point $p \in M$ if the corresponding algebraic curvature tensor at that point is Einstein, that is, if the Ricci tensor is proportional to the metric at p. Using such definitions we have the following well known statement for Einstein manifolds (see Besse[10] [19, Theorem 1.97]).

Theorem 6.12. *If a connected pseudo-Riemannian manifold of dimension* $n \geq 3$ *is Einstein at each point, then it is Einstein.*

Proof. Since (M, g) is Einstein at each point, $\operatorname{Ric} = \lambda g$ holds for some function $\lambda \in \mathfrak{F}(M)$. Taking the trace with respect to g, we get $\operatorname{Sc} = n\lambda$, while applying the divergence, using Theorem 6.11 and the formula (6.30), we obtain

$$n \nabla \lambda = \nabla \operatorname{Sc} = 2 \operatorname{div} \operatorname{Ric} = 2 \operatorname{div}(\lambda g) = 2 \nabla \lambda.$$

Hence $\nabla \lambda = 0$ holds for $n \neq 2$, so λ is a local constant on a connected manifold, and therefore it is a global constant, which also holds for Sc. □

[9] Albert Einstein (1879–1955), German theoretical physicist

[10] Arthur Besse is a pseudonym of French differential geometers, led by Marcel Berger

Example 6.11. In the previous proof we can avoid use of Theorem 6.11, and calculate directly in an orthonormal frame ($g_{ij} = g^{ij} = \varepsilon_i \delta_{ij}$),

$$\begin{aligned}\nabla_{E_k} \mathrm{Sc} &= \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j \nabla_k R_{ijji} = \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j (-\nabla_i R_{jkji} - \nabla_j R_{kiji}) = \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j \nabla_i R_{jkij} + \sum_{j,i=1}^{n} \varepsilon_j \varepsilon_i \nabla_j R_{ikji} \\ &= 2 \sum_{i=1}^{n} \varepsilon_i \sum_{j=1}^{n} \varepsilon_j \nabla_i R_{jkij} = 2 \sum_{i=1}^{n} \varepsilon_i \sum_{j,l=1}^{n} g^{jl} \nabla_i R_{jkil} = 2 \sum_{i=1}^{n} \varepsilon_i \nabla_{E_i} R_{ki} = 2 \nabla_{E_k} \lambda,\end{aligned}$$

and therefore $n \nabla \lambda = \nabla \mathrm{Sc} = 2 \nabla \lambda$. △

By Theorem 6.4, an algebraic curvature tensor of dimension $n = 3$ has only six independent components, while for $n = 2$ there is only one. Thus, in small dimensions we get the following interesting examples.

Example 6.12. Every two-dimensional algebraic curvature tensor is Einstein. All Ricci tensor components $R_{ij} = \sum_{k,l} g^{kl} R_{kijl}$ for $n = 2$ can be expressed only by R_{1221}. We have $R_{11} = g^{22} R_{1221}$, $R_{12} = -g^{12} R_{1221}$, $R_{22} = g^{11} R_{1221}$, and therefore $\mathrm{Ric} = (R_{1221}/\det g) g$. △

Example 6.13. In dimension $n = 3$, by Example 6.9, the Ricci tensor completely determines the sectional curvature, and consequently also determines the curvature tensor. In particular, if a three-dimensional algebraic curvature tensor is Einstein, then it has constant sectional curvature. Additionally, a three-dimensional connected Einstein manifold is a space of constant sectional curvature. △

6.6 The Ricci identities

Let M be a pseudo-Riemannian manifold and ∇ its arbitrary connection. Comparing the components, for an arbitrary tensor field $A \in \mathfrak{T}^r_s(M)$ and $X, Y \in \mathfrak{X}(M)$ we have

$$(\nabla_X A)^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \sum_k (\nabla A)^{i_1 \ldots i_r}_{j_1 \ldots j_s k} X^k = (C^{r+1}_{s+1}(\nabla A \otimes X))^{i_1 \ldots i_r}_{j_1 \ldots j_s},$$

from where we get

$$\nabla_X A = C^{r+1}_{s+1}(\nabla A \otimes X),$$

and similarly we have

$$(\nabla^2_{X,Y} A)^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \sum_{k,l} (\nabla^2 A)^{i_1 \ldots i_r}_{j_1 \ldots j_s kl} Y^k X^l = C^{r+1}_{s+1}(C^{r+1}_{s+2}(\nabla^2 A \otimes X) \otimes Y)^{i_1 \ldots i_r}_{j_1 \ldots j_s},$$

which gives

$$\nabla^2_{X,Y} A = C^{r+1}_{s+1}(C^{r+1}_{s+2}(\nabla^2 A \otimes X) \otimes Y).$$

Because of

$$\begin{aligned}\nabla_X \nabla_Y A =& \nabla_X C^{r+1}_{s+1}(\nabla A \otimes Y) = C^{r+1}_{s+1} \nabla_X (\nabla A \otimes Y) = C^{r+1}_{s+1}(\nabla_X \nabla A \otimes Y + \nabla A \otimes \nabla_X Y) \\ =& C^{r+1}_{s+1}(C^{r+1}_{s+2}(\nabla^2 A \otimes X) \otimes Y) + C^{r+1}_{s+1}(\nabla A \otimes \nabla_X Y) = \nabla^2_{X,Y} A + \nabla_{\nabla_X Y} A,\end{aligned}$$

we obtain

$$\nabla^2_{X,Y} A - \nabla^2_{Y,X} A = \nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{\nabla_X Y - \nabla_Y X} A,$$

or more simply written $\nabla^2_{X,Y} - \nabla^2_{Y,X} = [\nabla_X, \nabla_Y] - \nabla_{\nabla_X Y - \nabla_Y X}$.

The simplest case of a tensor field $A \in \mathfrak{T}^r_s(M)$ is a smooth function $f \in \mathfrak{F}(M)$, and then the derived formula gives

$$\nabla^2 f(X,Y) - \nabla^2 f(Y,X) = \nabla^2_{Y,X} f - \nabla^2_{X,Y} f = (-[X,Y] + \nabla_X Y - \nabla_Y X)f = \tau(X,Y)f,$$

where τ is the torsion, which we have already seen in Example 5.5, related to the Hessian of a function f.

In the context of curvature, a connection ∇ is always Levi-Civita. Then the symmetry of ∇ gives symmetry of the Hessian, $\nabla^2_{X,Y} f = \nabla^2_{Y,X} f$, while in the case of an arbitrary tensor field we obtain

$$\nabla^2_{X,Y} - \nabla^2_{Y,X} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{6.32}$$

The next simplest case considers a tensor field $A \in \mathfrak{T}^r_s(M)$ as a vector field $Z \in \mathfrak{X}(M)$, and then the formula (6.32), by the definition of $\mathcal{R}$, becomes

$$\nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z = \mathcal{R}(X,Y)Z. \tag{6.33}$$

In the case of a covector field $\omega \in \mathfrak{X}^*(M)$ we have

$$\begin{aligned}(\nabla_X \nabla_Y \omega)Z =& X((\nabla_Y \omega)Z) - (\nabla_Y \omega)(\nabla_X Z)\\ =& X(Y(\omega(Z)) - \omega(\nabla_Y Z)) - (Y(\omega(\nabla_X Z)) - \omega(\nabla_Y \nabla_X Z))\\ =& XY(\omega(Z)) - (\nabla_X \omega)(\nabla_Y Z) - \omega(\nabla_X \nabla_Y Z) - (\nabla_Y \omega)(\nabla_X Z),\end{aligned}$$

from where, using the formula (6.32), we obtain

$$\begin{aligned}(\nabla^2_{X,Y}\omega - \nabla^2_{Y,X}\omega)Z =& (\nabla_X \nabla_Y \omega)Z - (\nabla_Y \nabla_X \omega)Z - (\nabla_{[X,Y]}\omega)Z\\ =& (XY - YX)(\omega(Z)) - \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z) - [X,Y](\omega(Z)) + \omega(\nabla_{[X,Y]} Z)\\ =& -\omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) = -\omega(\mathcal{R}(X,Y)Z).\end{aligned}$$

If $\mathcal{R}^*(X,Y) \colon \mathfrak{X}^*(M) \to \mathfrak{X}^*(M)$ is the map defined by

$$(\mathcal{R}^*(X,Y)\omega)Z = \omega(\mathcal{R}(X,Y)Z),$$

then our formula yields

$$\nabla^2_{X,Y}\omega - \nabla^2_{Y,X}\omega = -\mathcal{R}^*(X,Y)\omega. \tag{6.34}$$

Let us now consider the general case. First, let us note that if A and B are arbitrary tensor fields then $\nabla_X \nabla_Y (A \otimes B) = \nabla_X \nabla_Y A \otimes B + \nabla_Y A \otimes \nabla_X B + \nabla_X A \otimes \nabla_Y B + A \otimes \nabla_X \nabla_Y B$ holds, where from (6.32) we get

$$\begin{aligned}(\nabla^2_{X,Y} - \nabla^2_{Y,X})(A \otimes B) =& (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(A \otimes B)\\ =& (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})A \otimes B + A \otimes (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})B\\ =& (\nabla^2_{X,Y} - \nabla^2_{Y,X})A \otimes B + A \otimes (\nabla^2_{X,Y} - \nabla^2_{Y,X})B.\end{aligned}$$

An arbitrary tensor field $A \in \mathfrak{T}^r_s(M)$ can be written as a sum of tensor products composed of vector and covector fields, so for $V_1, \ldots, V_r \in \mathfrak{X}(M)$ and $\lambda_1, \ldots, \lambda_s \in \mathfrak{X}^*(M)$, using (6.33) and (6.34), we calculate

$$\begin{aligned}&(\nabla^2_{X,Y} - \nabla^2_{Y,X})(V_1 \otimes \cdots \otimes V_r \otimes \lambda_1 \otimes \cdots \otimes \lambda_s)\\ &\quad= \mathcal{R}(X,Y)V_1 \otimes V_2 \otimes \cdots \otimes \lambda_s + \cdots + V_1 \otimes \cdots \otimes V_{r-1} \otimes \mathcal{R}(X,Y)V_r \otimes \lambda_1 \otimes \cdots \otimes \lambda_s\\ &\qquad - V_1 \otimes \cdots \otimes V_r \otimes \mathcal{R}^*(X,Y)\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_s - \cdots - V_1 \otimes \cdots \otimes \lambda_{s-1} \otimes \mathcal{R}^*(X,Y)\lambda_s.\end{aligned}$$

Since $(\mathcal{R}(X,Y)V_i)\omega_i = V_i(\mathcal{R}^*(X,Y)\omega_i)$ for $1 \le i \le r$, while $(\mathcal{R}^*(X,Y)\lambda_j)Z_j = \lambda_j(\mathcal{R}(X,Y)Z_j)$ for $1 \le j \le s$, where $\omega_i \in \mathfrak{X}^*(M)$ and $Z_j \in \mathfrak{X}(M)$, we obtain

$$\begin{aligned}&((\nabla^2_{X,Y} - \nabla^2_{Y,X})(V_1 \otimes \cdots \otimes V_r \otimes \lambda_1 \otimes \cdots \otimes \lambda_s))(\omega_1, \ldots, \omega_r, Z_1, \ldots, Z_s)\\ &= V_1(\mathcal{R}^*(X,Y)\omega_1)V_2(\omega_2)\cdots\lambda_s(Z_s) + \cdots + V_1(\omega_1)\cdots V_{r-1}(\omega_{r-1})V_r(\mathcal{R}^*(X,Y)\omega_r)\lambda_1(Z_1)\cdots\lambda_s(Z_s)\\ &\quad - V_1(\omega_1)\cdots V_r(\omega_r)\lambda_1(\mathcal{R}(X,Y)Z_1)\lambda_2(Z_2)\cdots\lambda_s(Z_s) - \cdots - V_1(\omega_1)\cdots\lambda_{s-1}(Z_{s-1})\lambda_s(\mathcal{R}(X,Y)Z_s).\end{aligned}$$

From here follows the final formula for an arbitrary tensor field $A \in \mathfrak{T}^r_s(M)$,

$$\begin{aligned}(\nabla^2_{X,Y}A - \nabla^2_{Y,X}A)&(\omega_1, \dots, \omega_r, Z_1, \dots, Z_s)\\ =&A(\mathcal{R}^*(X,Y)\omega_1, \omega_2, \dots, Z_s) + \cdots + A(\omega_1, \dots, \omega_{r-1}, \mathcal{R}^*(X,Y)\omega_r, Z_1, \dots, Z_s)\\ &- A(\omega_1, \dots, \omega_r, \mathcal{R}(X,Y)Z_1, Z_2, \dots Z_s) - \cdots - A(\omega_1, \dots, Z_{s-1}, \mathcal{R}(X,Y)Z_s),\end{aligned} \tag{6.35}$$

called the ***Ricci identity***.

It is usual to observe special cases of the Ricci identity in which $A \in \mathfrak{T}^r_s(M)$ are simpler tensor fields. In the equalities (6.33) and (6.34) we had cases when A is a vector or covector field. For $J \in \mathfrak{T}^1_1(M)$ the formula (6.35) gives

$$(\nabla^2_{X,Y}J - \nabla^2_{Y,X}J)(\omega, Z) = J(\mathcal{R}^*(X,Y)\omega, Z) - J(\omega, \mathcal{R}(X,Y)Z),$$

and considering J as the $\mathfrak{F}(M)$-linear operator $J\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$, we get the formula

$$(\nabla_X\nabla_Y J - \nabla_Y\nabla_X J - \nabla_{[X,Y]}J)Z = \mathcal{R}(X,Y)(JZ) - J(\mathcal{R}(X,Y)Z). \tag{6.36}$$

Chapter 7

MORE PSEUDO-RIEMANNIAN GEOMETRY

7.1 Jacobi fields

For an arbitrary vector field along a two-parameter map we have the following relation with the curvature.

Lemma 7.1. *Let (M, g) be a pseudo-Riemannian manifold, and $f\colon L \times I \to M$ is a two-parameter map. Then for every $V \in \mathfrak{X}(f)$ we have*

$$\frac{\nabla}{ds}\frac{\nabla}{dt}V - \frac{\nabla}{dt}\frac{\nabla}{ds}V = \mathcal{R}\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V. \tag{7.1}$$

Proof. This is a local result, so for each $(s, t) \in L \times I$ we use local coordinates $(x_1, \dots, x_n)$ defined on some neighbourhood of the point $f(s, t)$. If we denote $V_k(s, t) = (V(s, t))(x_k)$ and $f_k = x_k \circ f$ for $1 \le k \le n$, then we calculate

$$\frac{\nabla}{dt}V = \sum_{k=1}^{n}\left(\frac{\partial}{\partial t}V_k\frac{\partial}{\partial x_k} + V_k\frac{\nabla}{dt}\frac{\partial}{\partial x_k}\right)\circ f,$$

and therefore

$$\frac{\nabla}{ds}\frac{\nabla}{dt}V = \sum_{k=1}^{n}\left(\frac{\partial}{\partial s}\frac{\partial}{\partial t}V_k\frac{\partial}{\partial x_k} + \frac{\partial}{\partial t}V_k\frac{\nabla}{ds}\frac{\partial}{\partial x_k} + \frac{\partial}{\partial s}V_k\frac{\nabla}{dt}\frac{\partial}{\partial x_k} + V_k\frac{\nabla}{ds}\frac{\nabla}{dt}\frac{\partial}{\partial x_k}\right)\circ f,$$

which gives

$$\frac{\nabla}{ds}\frac{\nabla}{dt}V - \frac{\nabla}{dt}\frac{\nabla}{ds}V = \sum_{k=1}^{n}V_k\left(\frac{\nabla}{ds}\frac{\nabla}{dt}\frac{\partial}{\partial x_k} - \frac{\nabla}{dt}\frac{\nabla}{ds}\frac{\partial}{\partial x_k}\right)\circ f.$$

On the other hand, we have

$$\begin{aligned}\frac{\nabla}{ds}\frac{\nabla}{dt}\frac{\partial}{\partial x_k} = \frac{\nabla}{ds}\left(\nabla_{\frac{\partial f}{\partial t}}\frac{\partial}{\partial x_k}\right) &= \frac{\nabla}{ds}\sum_{j=1}^{n}\frac{\partial f_j}{\partial t}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k} = \sum_{j=1}^{n}\frac{\partial}{\partial s}\frac{\partial f_j}{\partial t}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k} + \sum_{j=1}^{n}\frac{\partial f_j}{\partial t}\frac{\nabla}{ds}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k}\\ &= \sum_{j=1}^{n}\frac{\partial^2 f_j}{\partial s\partial t}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k} + \sum_{i,j=1}^{n}\frac{\partial f_j}{\partial t}\frac{\partial f_i}{\partial s}\nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k},\end{aligned}$$

which implies

$$\begin{aligned}\frac{\nabla}{ds}\frac{\nabla}{dt}\frac{\partial}{\partial x_k} - \frac{\nabla}{dt}\frac{\nabla}{ds}\frac{\partial}{\partial x_k} &= \sum_{i,j=1}^{n}\frac{\partial f_j}{\partial t}\frac{\partial f_i}{\partial s}\left(\nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k}\right)\\ &= \sum_{i,j=1}^{n}\frac{\partial f_j}{\partial t}\frac{\partial f_i}{\partial s}\mathcal{R}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \mathcal{R}\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial}{\partial x_k},\end{aligned}$$

and finally

$$\frac{\nabla}{ds}\frac{\nabla}{dt}V - \frac{\nabla}{dt}\frac{\nabla}{ds}V = \sum_{k=1}^{n} V_k \left(\mathcal{R}\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) \frac{\partial}{\partial x_k} \right) \circ f = \mathcal{R}\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) V.$$

□

Consider a two-parameter map $f\colon L \times I \to M$ for a pseudo-Riemannian manifold (M, g), where $L = (-\epsilon, \epsilon)$ for some $\epsilon > 0$ and $I \subseteq \mathbb{R}$ is an interval. It determines the longitudinal curves $f_s\colon t \mapsto f(s, t)$, and the smooth curve $\gamma = f_0$ has special importance as the central longitudinal curve. The ***variation vector field*** along γ is $V \in \mathfrak{X}(\gamma)$ given by

$$V(t) = \frac{\partial f}{\partial s}(0, t).$$

We are interested in the special case when the partial maps f_s are geodesics for each $s \in L$ and we say that f is a ***variation through geodesics*** of γ. In that case, for the variation vector field $V \in \mathfrak{X}(\gamma)$ we use the equality (5.23) and Lemma 7.1 to compute

$$\frac{\nabla}{dt}\frac{\nabla}{dt}V = \frac{\nabla}{dt}\frac{\nabla}{dt}\frac{\partial f}{\partial s} = \frac{\nabla}{dt}\frac{\nabla}{ds}\frac{\partial f}{\partial t} = \frac{\nabla}{ds}\frac{\nabla}{dt}\frac{\partial f}{\partial t} - \mathcal{R}\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t} = -\mathcal{R}\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}.$$

Thus, if $\gamma\colon I \to M$ is a geodesic in a pseudo-Riemannian manifold M, then its variation vector field $J \in \mathfrak{X}(\gamma)$ satisfies

$$J'' + \mathcal{R}\left(J, \gamma'\right)\gamma' = 0. \tag{7.2}$$

The equation (7.2) is called the ***Jacobi equation***, while a vector field $J \in \mathfrak{X}(\gamma)$ along a geodesic γ that satisfies it is called a ***Jacobi field***.

It turns out that a Jacobi field J is determined by its initial conditions $J(a)$ and $J'(a)$ for some $a \in I$. Let us choose a parallel orthonormal frame $(E_1, \dots, E_n)$ along γ. An arbitrary $J \in \mathfrak{X}(\gamma)$ can be written as $J(t) = \sum_i J_i(t)E_i(t)$ for $J_i \in \mathfrak{F}(I)$, $1 \le i \le n$, so we can express the Jacobi equation (7.2) as

$$\sum_{i=1}^{n} J_i''(t)E_i(t) + \sum_{i=1}^{n} g_{\gamma(t)}(\mathcal{R}(J(t), \gamma'(t))\gamma'(t), E_i(t))E_i(t) = 0,$$

which gives

$$J_i'' + \sum_{j=1}^{n} J_j R(E_j, \gamma', \gamma', E_i) \circ \gamma = 0,$$

for all $1 \le i \le n$. This is a system of n second order ordinary differential equations for the n functions $J_i \in \mathfrak{F}(I)$. Making the usual substitution $V_i = J_i'$ converts it to an equivalent first order linear system for the $2n$ unknown functions,

$$J_i' = V_i, \quad V_i' = -\sum_{j=1}^{n}(R(E_j, \gamma', \gamma', E_i) \circ \gamma)J_j.$$

This guarantees the existence and uniqueness of a solution on the whole interval I with arbitrary initial conditions $J_i(a) = (E_i(a))^*(J(a))$ and $V_i(a) = (E_i(a))^*(J'(a))$.

Theorem 7.2. *Let $\gamma\colon I \to M$ be a geodesic in a pseudo-Riemannian manifold M. For $a \in I$ and every pair of vectors $X, Y \in T_{\gamma(a)}M$ there exists a unique Jacobi field $J \in \mathfrak{X}(\gamma)$ satisfying the initial conditions $J(a) = X$ and $J'(a) = Y$.*

Theorem 7.2 is the existence and uniqueness theorem for Jacobi fields, so we introduce the notation $J_X^Y \in \mathfrak{X}(\gamma)$ for the Jacobi field such that $J_X^Y(a) = X$ and $(J_X^Y)'(a) = Y$ hold. The set of all Jacobi fields along a geodesic $\gamma\colon I \to M$,

$$\mathcal{J}(\gamma) = \{J_X^Y : X, Y \in T_{\gamma(a)}M\} \subseteq \mathfrak{X}(\gamma),$$

can be identified with $T_{\gamma(a)}M \times T_{\gamma(a)}M$. Moreover, for $X, Y, Z, V \in T_{\gamma(a)}M$ and $\alpha, \beta \in \mathbb{R}$ we have

$$(\alpha J_X^Y + \beta J_Z^V)'' = \alpha (J_X^Y)'' + \beta (J_Z^V)'' = -\alpha \mathcal{R}(J_X^Y, \gamma')\gamma' - \beta \mathcal{R}(J_Z^V, \gamma')\gamma' = -\mathcal{R}(\alpha J_X^Y + \beta J_Z^V, \gamma')\gamma',$$

which proves that $\alpha J_X^Y + \beta J_Z^V$ is a Jacobi field, while

$$(\alpha J_X^Y + \beta J_Z^V)(a) = \alpha X + \beta Z = J_{\alpha X + \beta Z}^{\alpha Y + \beta V}(a), \quad (\alpha J_X^Y + \beta J_Z^V)'(a) = \alpha Y + \beta V = (J_{\alpha X + \beta Z}^{\alpha Y + \beta V})'(a),$$

yields

$$\alpha J_X^Y + \beta J_Z^V = J_{\alpha X + \beta Z}^{\alpha Y + \beta V}.$$

Hence, $\mathcal{J}(\gamma)$ becomes a vector space of dimension $2n$, isomorphic to $T_{\gamma(a)}M \times T_{\gamma(a)}M$.

Let us remark that the previous theorem works for an arbitrary $a \in I$, but to avoid ambiguity, unless otherwise stated we assume $a = 0 \in I$. Of course, the translation $t \mapsto t - a$ does not violate the fact that γ is a geodesic nor that J is a Jacobi field.

We have already seen that a variation vector field along γ is a Jacobi field. However, the converse also works, because every Jacobi field along γ is some variation vector field along γ.

Theorem 7.3. *Let $\gamma\colon I \to M$ be a geodesic in a pseudo-Riemannian manifold M, where I is a compact interval. Every Jacobi field along γ is the variation vector field of some variation through geodesics of γ.*

Proof. Without loss of generality $0 \in I$. We suppose $J_X^Y \in \mathcal{J}(\gamma)$. Let us choose a smooth curve $\psi\colon (-\epsilon, \epsilon) \to M$ such that $\psi(0) = \gamma(0)$ and $\psi'(0) = X$. Let us choose $V \in \mathfrak{X}(\gamma)$ such that $V(0) = \gamma'(0)$ and $(\nabla V/ds)(0) = Y$. We define a variation of γ by $f(s,t) = \exp_{\psi(s)}(tV(s))$. Since I is compact, the domain of exp is an open subset of TM that contains the compact set $\{(\psi(0), t\gamma'(0)) : t \in I\}$), so there is $0 < \delta \le \epsilon$ such that f is defined for $(s,t) \in (-\delta, \delta) \times I$.

Since $f(0,t) = \exp_{\psi(0)}(tV(0)) = \exp_{\gamma(0)}(t\gamma'(0)) = \gamma(t)$, we see that f is a variation of γ, while $f(s,0) = \exp_{\psi(s)}(0) = \psi(s)$. Let $W(t) = (\partial f/\partial s)(0,t)$ be the variation vector field. Then $W(0) = \psi'(0) = X$, each longitudinal curve f_s is a geodesic with $(f_s)'(0) = V(s)$, while we have

$$\frac{\nabla W}{dt}(0) = \frac{\nabla}{dt}\frac{\partial f}{ds}(0,0) = \frac{\nabla}{ds}\frac{\partial f}{dt}(0,0) = \frac{\nabla V}{ds}(0) = Y,$$

which proves that $J_X^Y = W$. □

The most obvious example of a Jacobi field is $\gamma' \in \mathfrak{X}(\gamma)$, since it satisfies (7.2) because of $(\gamma')' = 0$ and $\mathcal{R}(\gamma', \gamma')\gamma' = 0$, which is the variation field of the variation $f(s,t) = \gamma(s+t)$. Another simple example is $J(t) = t\gamma'(t)$, which is the variation field of $f(s,t) = \gamma((1+s)t)$. However, these Jacobi field examples are reparametrizations of γ and they do not give us anything new about the behaviour of the geodesic γ.

To exclude such unimportant examples, whenever $\gamma'(t)$ is not null we can use the decomposition $T_{\gamma(t)}M = \mathrm{Span}\{\gamma'(t)\} \oplus \mathrm{Span}\{\gamma'(t)\}^\perp$. If γ' is nowhere null, any $V \in \mathfrak{X}(\gamma)$ decomposes into an orthogonal sum $V^\top + V^\perp$, where $V^\top$ is a multiple of γ' and $V^\perp$ is normal to γ'. We say that $V^\top$ is a ***tangential vector field*** along γ, while $V^\perp$ is a ***normal vector field*** along γ. Using the compatibility with the metric

$$0 = \frac{d}{dt}g(V^\perp, \gamma') = g((V^\perp)', \gamma') + g(V^\perp, (\gamma')') = g((V^\perp)', \gamma'),$$

we see that $(V^{\perp})'$ is also normal to γ', and similarly $(V^{\top})'$ is parallel to γ'.

A Jacobi field J satisfying $J(t) \perp \gamma'(t)$ for all $t \in I$ is called a ***normal Jacobi field***, while we have the following theorem.

Theorem 7.4. *Let $\gamma\colon I \to M$ be a geodesic in a pseudo-Riemannian manifold M. If $J \in \mathcal{J}(\gamma)$ then the following are equivalent: J is a normal Jacobi field; J is orthogonal to γ' at two distinct points; both J and J' are orthogonal to γ' at one point; both J and J' are orthogonal to γ' everywhere along γ.*

Proof. Consider the function $f\colon I \to \mathbb{R}$ given by $f(t) = g(J(t), \gamma'(t))$. A Jacobi field is normal if and only if $f \equiv 0$. For $J \in \mathcal{J}(\gamma)$ we have $f'' = (g(J', \gamma'))' = g(J'', \gamma') = -g(\mathcal{R}(J, \gamma')\gamma', \gamma') = 0$, which means that $f(t) = ct + d$ for some $c, d \in \mathbb{R}$. Since $g(J', \gamma') = c$, we have $J' \perp \gamma'$ if and only if $c = 0$, while $J \perp \gamma'$ at $t \in I$ if and only if $ct + d = 0$, which is enough to complete this proof easily. □

We are interested in Jacobi fields that vanish at a concrete point. Let $\gamma\colon I \to M$ be a geodesic in a pseudo-Riemannian manifold M, where I is a compact interval containing 0. According to Theorem 7.2, an arbitrary Jacobi field $J \in \mathcal{J}(\gamma)$ such that $J(0) = 0$ is $J = J_0^X$ for some $X \in T_{\gamma(0)}M$. Theorem 7.3 claims that J is the variation field of a variation through geodesics of γ, where in the proof we set $\psi(s) = \gamma(0)$ and $V(s) = \gamma'(0) + sX$, to obtain the variation

$$f(s,t) = \exp_{\gamma(0)}(t(\gamma'(0) + sX)).$$

This allows us to calculate

$$J(t) = \frac{\partial f}{\partial s}(0,t) = \left(T_{t(\gamma'(0)+sX)} \exp_{\gamma(0)}\right)(tX)_{t(\gamma'(0)+sX)}\Big|_{s=0} = \left(T_{t\gamma'(0)} \exp_{\gamma(0)}\right)(tX)_{t\gamma'(0)},$$

which gives an explicit formula

$$J_0^X(t) = \left(T_{t\gamma'(0)} \exp_{\gamma(0)}\right)(tX)_{t\gamma'(0)}, \tag{7.3}$$

where $(tX)_{t\gamma'(0)} \in T_{t\gamma'(0)}(T_{\gamma(0)}M)$ canonically corresponds to $tX \in T_{\gamma(0)}M$.

In normal coordinates, the coordinate representation of the exponential map is the identity, so on a normal neighbourhood of $\gamma(0)$ containing the image of γ, f can be written explicitly in coordinates as $f(s,t) = (t(\gamma_1'(0) + sJ_1'(0)), \ldots, t(\gamma_n'(0) + sJ_n'(0)))$, so we obtain

$$J_0^X(t) = t\sum_{i=1}^{n} X_i(\partial_i)_{\gamma(t)},$$

where $\sum_{i=1}^{n} X_i(\partial_i)_{\gamma(0)}$ is the coordinate representation of X.

Example 7.1. Let (M, g) be a space of constant sectional curvature κ, which implies that $\mathcal{R} = \kappa\mathcal{R}^1$. In that case the Jacobi equation (7.2) becomes $J'' + \kappa(g(\gamma', \gamma')J - g(J, \gamma')\gamma') = 0$, so for a unit-speed geodesic γ in M, a normal Jacobi field J satisfies

$$J'' + \kappa J = 0.$$

Let E be an arbitrary parallel unit normal vector field along γ. It is reasonable to try $J(t) = \alpha(t)E(t)$ for some $\alpha \in \mathfrak{F}(I)$. After the substitution we obtain that α is a solution of the differential equation $\alpha'' + \kappa\alpha = 0$. We suppose $J(0) = 0$, which gives the initial condition $\alpha(0) = 0$, and the solutions are $\alpha(t) = c \cdot h_\kappa(t)$, where

$$h_\kappa(t) = \begin{cases} t & \text{if } \kappa = 0, \\ \frac{\sin(t\sqrt{\kappa})}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\ \frac{\sinh(t\sqrt{-\kappa})}{\sqrt{-\kappa}} & \text{if } \kappa < 0, \end{cases}$$

and c is an arbitrary constant. Thus, we obtain $J = ch_\kappa E$. The simple calculations show $J'(0) = ch'_\kappa(0)E(0) = cE(0)$, since $h'_\kappa(0) = 1$ always holds, while $\|J'(0)\| = |\kappa|$, because E is a unit vector field. $\triangle$

Consider $J = J_0^X \in \mathcal{J}(\gamma)$ where we have $J(0) = 0$ and $J'(0) = X$. Because of $J'' = -\mathcal{R}(J, \gamma')\gamma'$, we also have $J''(0) = 0$. Let us calculate the Taylor expansion of $\|J(t)\|^2$ about $t = 0$. For the first coefficients we have

$$\begin{aligned}
g(J,J)(0) &= g(J(0),J(0)) = 0,\\
(g(J,J))'(0) &= (g(J',J) + g(J,J'))(0) = 2g(J',J)(0) = 0,\\
(g(J,J))''(0) &= 2g(J'',J)(0) + 2g(J',J')(0) = 2g(X,X) = 2,\\
(g(J,J))'''(0) &= 2g(J''',J)(0) + 6g(J'',J')(0) = 0.
\end{aligned}$$

For an arbitrary $Y \in \mathfrak{X}(\gamma)$ we calculate

$$\begin{aligned}
g(J''', Y)(0) &= (g(J'', Y))'(0) - g(J''(0), Y'(0)) = (g(-\mathcal{R}(J, \gamma')\gamma', Y))'(0) = -(g(\mathcal{R}(Y, \gamma')\gamma', J))'(0)\\
&= -g((\mathcal{R}(Y, \gamma')\gamma')'(0), J(0)) - g(\mathcal{R}(Y, \gamma')\gamma', J')(0) = -g(\mathcal{R}(J', \gamma')\gamma', Y)(0),
\end{aligned}$$

which implies $J'''(0) = -(\mathcal{R}(J', \gamma')\gamma')(0)$. This helps us to calculate the next Taylor coefficient,

$$\begin{aligned}
(g(J,J))''''(0) &= 2g(J'''',J)(0) + 8g(J''',J')(0) + 6g(J'',J'')(0) = 8g(J''',J')(0)\\
&= -8g(\mathcal{R}(J', \gamma')\gamma', X)(0) = -8g_{\gamma(0)}(\mathcal{R}(X, \gamma'(0)), \gamma'(0), X) = -8\kappa(\gamma'(0), X),
\end{aligned}$$

where $\kappa(\gamma'(0), X)$ is the sectional curvature of the plane $\sigma = \operatorname{Span}\{\gamma'(0), J'(0)\} \leq T_{\gamma(0)}M$. The Taylor expansion shows

$$\|J(t)\|^2 = t^2 - \frac{1}{3}\kappa(\sigma)t^4 + \underset{t\to 0}{O}(t^5),$$

which implies

$$\|J(t)\| = t - \frac{1}{6}\kappa(\sigma)t^3 + \underset{t\to 0}{O}(t^4). \tag{7.4}$$

Now we are able to give the relation between geodesics and curvature. An arbitrary Jacobi field $J_0^X \in \mathcal{J}(\gamma)$ is the variation field of the variation $f(s, t) = \exp_{\gamma(0)}(t(\gamma'(0) + sX))$, and it is given in the formula (7.3). Intuitively, the formula (7.4) shows how fast the geodesics that start from $\gamma(0)$ and are tangent to the the plane $\sigma = \operatorname{Span}\{\gamma'(0), X\}$ spread apart. Locally, for $\kappa(\sigma) > 0$ the radial geodesics spread apart less than the rays in T_pM, while for $\kappa(\sigma) < 0$ they spread apart more than the rays in T_pM (see the details in Do Carmo [31, Remark 5.2.11]).

It is easy to check that the space $\mathcal{J}_a(\gamma)$ of all Jacobi fields $J \in \mathfrak{X}(\gamma)$ such that $J(a) = 0$ is an n-dimensional subspace of $\mathcal{J}(\gamma)$, which is based on the choice of linearly independent vectors $J'(a) \in T_{\gamma(a)}M$. Since the tangent map of the exponential map is linear it is easy to see that vectors $J'(a) \in T_{\gamma(a)}M$ are linearly independent if and only if Jacobi fields $J \in \mathcal{J}_a(\gamma)$ are linearly independent.

Lemma 7.5. *Let $\gamma\colon I \to M$ be a geodesic in a pseudo-Riemannian manifold M and $a \in I$. If $J_i \in \mathcal{J}_a(\gamma)$ for $1 \leq i \leq k \in \mathbb{N}$, then $J_1'(a), \ldots, J_k'(a) \in T_{\gamma(a)}M$ are linearly independent if and only if $J_1, \ldots, J_k \in \mathcal{J}_a(\gamma)$ are linearly independent.*

Proof. Suppose that J_i are linearly independent and $\sum_i \alpha_i J_i'(a) = 0$ for some $\alpha_i \in \mathbb{R}$. We create $J = \sum_i \alpha_i J_i \in \mathcal{J}_a(\gamma)$ satisfying $J'(a) = 0$, so Theorem 7.2 implies $J = 0$, and therefore $0 = J(a) = \sum_i \alpha_i J_i(a)$ gives $\alpha_i = 0$. Conversely, suppose that $J_i'(a)$ are linearly independent and $\sum_i \alpha_i J_i = 0$ for some $\alpha_i \in \mathbb{R}$. Differentiating, we obtain $\sum_i \alpha_i J_i' = 0$, which implies $\sum_i \alpha_i J_i'(a) = 0$, and therefore $\alpha_i = 0$. $\square$

The next application of Jacobi fields can allow us to examine when the exponential map is a local diffeomorphism, that is, to find the relation between the singularities of the exponential map and Jacobi fields.

Let $\gamma\colon I \to M$ be a geodesic in a pseudo-Riemannian manifold M of dimension n. We say that two distinct parameter values $a, b \in I$ are ***conjugate parameters*** along γ if there exists a Jacobi field $0 \not\equiv J \in \mathfrak{X}(\gamma)$ such that $J(a) = 0 = J(b)$. The ***multiplicity*** of conjugacy is the maximum number of such linearly independent Jacobi fields, which is actually the dimension of the subspace $\mathcal{J}_{ab}(\gamma) = \mathcal{J}_a(\gamma) \cap \mathcal{J}_b(\gamma)$.

It is often said that the points $\gamma(a)$ and $\gamma(b)$ are ***conjugate*** along γ if a and b are conjugate parameters along γ. However, if γ has self-intersections, this definition becomes ambiguous since we do not know which space $J_a(\gamma)$ should be taken for a point $\gamma(a)$ in the case $\gamma^{-1}(\gamma(a)) \neq \{a\}$.

Example 7.2. Consider the pseudo-Euclidean space $\mathbb{R}^n_\nu$, where $\mathcal{R} = 0$ yields the Jacobi equation $J'' = 0$. The case $J(a) = 0 = J(b)$ implies $J \equiv 0$, so there are no conjugate points. $\triangle$

Consider the tangential Jacobi field J given by $J(t) = (t-a)\gamma'(t)$. Since γ' vanishes nowhere we see $J \in \mathcal{J}_a(\gamma)$ and $J \notin \mathcal{J}_b(\gamma)$ for $b \neq a$. Therefore, $\dim \mathcal{J}_{ab}(\gamma) < \dim \mathcal{J}_a(\gamma) = n$, which implies that the multiplicity of conjugacy does not exceed $n-1$. The upper limit can be reached, as we see in the following example.

Example 7.3. Consider the unit sphere $\mathbf{S}^n \subset \mathbb{R}^{n+1}$, which is a space of constant sectional curvature 1. According to Example 7.1 we know that $J(t) = (\sin t)E(t)$ satisfies $J \in \mathcal{J}_0(\gamma)$ for an arbitrary parallel unit normal vector field $E \in \mathfrak{X}(\gamma)$. Moreover, such J satisfies $J(\pi) = 0$, and therefore $J \in \mathcal{J}_{0\pi}(\gamma)$, which means that $\gamma(0)$ and its antipodal point $\gamma(\pi)$ are conjugate along γ with multiplicity $n-1$. $\triangle$

Theorem 7.6. *Let $\gamma\colon [0,1] \to M$ be a geodesic segment in a pseudo-Riemannian manifold M such that $p = \gamma(0)$ and $V = \gamma'(0) \in \mathcal{E}_p \subseteq T_pM$. The point $\gamma(1)$ is conjugate to p along γ if and only if V is a critical point of $\exp_p$. Moreover, the multiplicity of conjugation is equal to $\dim \operatorname{Ker}(T_V \exp_p)$.*

Proof. Consider an arbitrary $J \in \mathcal{J}_{01}(\gamma)$. Since $\mathcal{J}_{01}(\gamma) \subseteq \mathcal{J}_0(\gamma)$ we have $J = J_0^X$ for some $X = J'(0) \in T_pM$, where the equation (7.3) yields $(T_V \exp_p)(X_V) = J(1) = 0$. Therefore $J_0^X \in \mathcal{J}_{01}(\gamma)$ if and only if $X_V \in \operatorname{Ker}(T_V \exp_p)$, where Lemma 7.5 completes the proof. $\square$

Let us remark that by reparametrizing $\gamma_V(t) = \gamma_{tV}(1)$ from Lemma 5.12 we have that $\gamma(0)$ and $\gamma(t)$ are conjugate along γ if and only if $t\gamma'(0)$ is a critical point of $\exp_p$.

According to Theorem 7.2, a unique Jacobi field is naturally determined by giving its initial value and initial derivative. However, if points $\gamma(a)$ and $\gamma(b)$ are not conjugate, then we can determine a unique Jacobi field by giving $J(a)$ and $J(b)$.

Theorem 7.7. *Let $\gamma\colon I \to M$ be a geodesic in a pseudo-Riemannian manifold M. If $\gamma(a)$ and $\gamma(b)$ for distinct $a, b \in I$ are not conjugate along γ, then for every $X \in T_{\gamma(a)}M$ and $Y \in T_{\gamma(b)}M$ there exists a unique $J \in \mathcal{J}(\gamma)$ such that $J(a) = X$ and $J(b) = Y$.*

Proof. Define a linear map $f\colon \mathcal{J}_a(\gamma) \to T_{\gamma(b)}$ by $f(J) = J(b)$. Since a and b are not conjugate parameters, $J(b) = 0$ implies $J \equiv 0$, which means that f is injective. Both the domain and the codomain of f are of the same dimension n, so f is an isomorphism. Hence, there exists $J_1 \in \mathcal{J}_a(\gamma)$ such that $J_1(b) = Y$. In a similar fashion, there exist $J_2 \in \mathcal{J}_b(\gamma)$ such that $J_2(a) = X$, so $J = J_1 + J_2 \in \mathcal{J}(\gamma)$ satisfies $J(a) = X$ and $J(b) = Y$ as claimed. If $K \in \mathcal{J}(\gamma)$ is another such Jacobi field, then $J - K \in \mathcal{J}(\gamma)$ vanishes at both a and b, and therefore $J \equiv K$, which proves uniqueness. $\square$

7.2 Pseudo-Riemannian submanifolds

Let $(\overline{M}, \overline{g})$ be a pseudo-Riemannian manifold and (M, g) is its pseudo-Riemannian submanifold. If $\imath\colon M \hookrightarrow \overline{M}$ is the corresponding inclusion, then it is a pseudo-Riemannian immersion and M is endowed with the induced metric $g = \imath^*\overline{g}$. We want to investigate the relations between the pseudo-Riemannian geometry of a submanifold and that of the ambient manifold.

For each $p \in M$, the tangent space T_pM is a nondegenerate subspace of the scalar product space $(T_p\overline{M}, \overline{g}_p)$ as well as its orthogonal $N_pM = (T_pM)^\perp$ called the ***normal space*** at p. Hence, the ambient tangent space splits as an orthogonal direct sum

$$T_p\overline{M} = T_pM \oplus N_pM$$

with the corresponding projections $\pi^\top\colon T_p\overline{M} \to T_pM$ and $\pi^\perp\colon T_p\overline{M} \to N_pM$, so every vector $X \in T_p\overline{M}$ has a unique expression $X = X^\top + X^\perp$, where $X^\top = \pi^\top X \in T_pM$, $X^\perp = \pi^\perp X \in N_pM$.

Consider the ***ambient tangent bundle***

$$T\overline{M}{\restriction_M} = \bigsqcup_{p\in M} T_p\overline{M},$$

which is a vector bundle as the restriction of $T\overline{M}$ (Example 3.3). Given any point $p \in M$, there is a neighbourhood U of p in M that is embedded in $\overline{M}$ and we can use slice coordinates from Theorem 2.13 to create an orthonormal frame $(\sigma_1, \ldots, \sigma_{\dim \overline{M}})$ that is adapted to U in the sense that the restrictions of σ_i to U for $1 \le i \le \dim M$ form a local orthonormal frame for M. The restriction of the last $\dim \overline{M} - \dim M$ vector fields σ_i to M for $\dim M < i \le \dim \overline{M}$ form a local frame for $NM = \bigsqcup_{p\in M} N_p\overline{M}$, which by Theorem 3.2 becomes a subbundle called the ***normal bundle*** of M.

Applying $\pi^\top$ and $\pi^\perp$ at each point of M, we obtain two bundle homomorphisms, the ***tangential projection*** $\pi^\top\colon T\overline{M}{\restriction_M} \to TM$ and the ***normal projection*** $\pi^\perp\colon T\overline{M}{\restriction_M} \to NM$. On the other hand, the smooth sections $\overline{\mathfrak{X}}(M) = \mathfrak{X}(\overline{M}){\restriction_M} = \Gamma(T\overline{M}{\restriction_M})$ of the ambient tangent bundle can be decomposed by

$$\overline{\mathfrak{X}}(M) = \mathfrak{X}(M) \oplus \mathfrak{X}(M)^\perp, \tag{7.5}$$

where $\mathfrak{X}(M)^\perp = \Gamma(NM)$ is the set of smooth sections of the normal bundle. Thus, $\mathfrak{X}(M)$ and $\mathfrak{X}(M)^\perp$ can be seen as submodules of $\overline{\mathfrak{X}}(M)$, while the tangential and normal projections yields the appropriate module projections that decompose $X \in \overline{\mathfrak{X}}(M)$ as $X = X^\top + X^\perp$ with $X^\top \in \mathfrak{X}(M)$ and $X^\perp \in \mathfrak{X}(M)^\perp$.

Let $\overline{\nabla}$ be the Levi-Civita connection on $(\overline{M}, g)$, and ∇ be the Levi-Civita connection on (M, g). For any $p \in M$ and $X \in \mathfrak{X}(M)$ there is a neighbourhood $p \in U \subseteq M$ and an extension $\overline{X} \in \mathfrak{X}(\overline{M})$ of $X{\restriction_U}$. Because of the local properties of a connection (Theorem 5.1), the appropriate restriction along M of $\overline{\nabla}_X Y = (\overline{\nabla}_{\overline{X}}\overline{Y}){\restriction_M} \in \overline{\mathfrak{X}}(M)$ does not depend on extensions $\overline{X}, \overline{Y} \in \mathfrak{X}(\overline{M})$ for $X, Y \in \mathfrak{X}(M)$. An important question is how $\overline{\nabla}_X Y$ is decomposed by (7.5) into a tangential and a normal component.

Let us start with $X, Y, Z \in \mathfrak{X}(M)$ and their extensions $\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\overline{M})$. Applying the Koszul formula (5.10) for $\overline{\nabla}$, along M for every $Z \in \mathfrak{X}(M)$ we have

$$2g((\overline{\nabla}_X Y)^\top, Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]).$$

Hence, the tangential term is a Levi-Civita connection, and since it is unique we obtain

$$(\overline{\nabla}_X Y)^\top = \nabla_X Y.$$

On the other hand, the symmetry of the connection $\overline{\nabla}$ implies that $(\overline{\nabla}_X Y)^\perp$ is symmetric because of $(\overline{\nabla}_X Y - \overline{\nabla}_Y X)^\perp = [X,Y]^\perp = 0$. Since $(\overline{\nabla}_X Y)^\perp$ is $\mathfrak{F}(M)$-linear in X, it also must be

$\mathfrak{F}(M)$-linear in Y. We define the ***second fundamental form*** (or ***shape tensor***) as a map $\mathrm{II}\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^\perp$ given by $\mathrm{II}(X,Y) = (\overline{\nabla}_X Y)^\perp$. It is a symmetric $\mathfrak{F}(M)$-bilinear map such that for any $X, Y \in \mathfrak{X}(M)$ we have the decomposition

$$\overline{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X,Y), \tag{7.6}$$

called the ***Gauss formula***.

For any $N \in \mathfrak{X}(M)^\perp$, the vector-valued second fundamental form II can be reduced to a simpler scalar-valued form $h_N \in \mathfrak{T}^0_2(M)$ given by

$$h_N(X,Y) = g(\mathrm{II}(X,Y), N),$$

and called the ***scalar second fundamental form*** of M with respect to N. Raising an index we get the tensor field $A_N = (h_N)^\sharp \in \mathfrak{T}^1_1(M)$ which can be seen as an $\mathfrak{F}(M)$-linear map $A_N\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by $A_N(X)f = (h_N)^\sharp(df, X)$, which means that $(h_N)^\sharp(\omega, X) = \omega(s(X))$ naturally holds. The map A_N is called the ***shape operator*** with respect to $N \in \mathfrak{X}(M)^\perp$, and because of

$$h_N(X,Y) = (h_N)^\sharp(X^\flat, Y) = \sum_i X_i (h_N)^\sharp(dx_i, Y) = \sum_{i,j} g_{ij}\, dx_j(X)\, dx_i(A_N Y) = g(X, A_N Y),$$

for all $X, Y \in \mathfrak{X}(M)$ we have

$$g(X, A_N Y) = h_N(X,Y) = g(\mathrm{II}(X,Y), N).$$

Since h_N is symmetric, A_N is self-adjoint, and therefore $g(A_N X, Y) = g(X, A_N Y)$.

In a more general case, we consider the connection $\overline{\nabla}$ restricted to M as a map $\overline{\nabla}\restriction_M\colon \mathfrak{X}(M) \times \overline{\mathfrak{X}}(M) \to \overline{\mathfrak{X}}(M)$. For $X, Y \in \mathfrak{X}(M)$ and $N \in \mathfrak{X}(M)^\perp$, along M we have $\overline{g}(Y, N) = 0$, so

$$0 = X\overline{g}(Y,N) = \overline{g}(\overline{\nabla}_X Y, N) + \overline{g}(Y, \overline{\nabla}_X N) = \overline{g}(\mathrm{II}(X,Y), N) + \overline{g}(\overline{\nabla}_X N, Y),$$

which implies the equality

$$\overline{g}(\overline{\nabla}_X N, Y) = -\overline{g}(\mathrm{II}(X,Y), N) = -h_N(X,Y) = -g(A_N X, Y). \tag{7.7}$$

called the ***Weingarten formula***[1]. Hence, the tangent component of $\overline{\nabla}_X N$ is $-A_N X$. On the other hand, the normal component $\nabla^\perp_X N = (\overline{\nabla}_X N)^\perp$ defines a compatible connection on the normal bundle NM, called the ***normal connection***. Thus, the Weingarten formula can be written as

$$\overline{\nabla}_X N = -A_N X + \nabla^\perp_X N. \tag{7.8}$$

It is interesting to consider the relation between the corresponding curvature operators $\overline{\mathcal{R}}$ and $\mathcal{R}$, or between the curvature tensors $\overline{R}$ and R. We use the Gauss formula (7.6) and the Weingarten formula (7.8) to write

$$\overline{\nabla}_X \overline{\nabla}_Y Z = \overline{\nabla}_X \nabla_Y Z + \overline{\nabla}_X \mathrm{II}(Y,Z) = \nabla_X \nabla_Y Z + \mathrm{II}(X, \nabla_Y Z) + \nabla^\perp_X \mathrm{II}(Y,Z) - A_{\mathrm{II}(Y,Z)} X$$

for all $X, Y, Z \in \mathfrak{X}(M)$, and therefore

$$\begin{aligned}
\overline{\mathcal{R}}(X,Y)Z =& \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z \\
=& \nabla_X \nabla_Y Z + \mathrm{II}(X, \nabla_Y Z) + \nabla^\perp_X(\mathrm{II}(Y,Z)) - A_{\mathrm{II}(Y,Z)} X \\
&- \nabla_Y \nabla_X Z - \mathrm{II}(Y, \nabla_X Z) - \nabla^\perp_Y(\mathrm{II}(X,Z)) + A_{\mathrm{II}(X,Z)} Y \\
&- \nabla_{[X,Y]} Z - \mathrm{II}(\nabla_X Y, Z) - \mathrm{II}(\nabla_Y X, Z) \\
=& \mathcal{R}(X,Y)Z + A_{\mathrm{II}(X,Z)} Y - A_{\mathrm{II}(Y,Z)} X + (\nabla^\perp_X \mathrm{II})(Y,Z) - (\nabla^\perp_Y)\, \mathrm{II}(X,Z),
\end{aligned}$$

[1] Julius Weingarten (1836–1910), German mathematician

where

$$(\nabla^\perp_X \mathrm{II})(Y,Z) = \nabla^\perp_X(\mathrm{II}(Y,Z)) - \mathrm{II}(\nabla_X Y, Z) - \mathrm{II}(Y, \nabla_X Z).$$

The tangential component is

$$(\overline{\mathcal{R}}(X,Y)Z)^\top = \mathcal{R}(X,Y)Z + A_{\mathrm{II}(X,Z)}Y - A_{\mathrm{II}(Y,Z)}X,$$

which after taking the scalar product by $W \in \mathfrak{X}(M)$ yields the ***Gauss equation*** (or ***Gauss curvature equation***),

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + \overline{g}(\mathrm{II}(X,Z),\mathrm{II}(Y,W)) - \overline{g}(\mathrm{II}(X,W),\mathrm{II}(Y,Z)). \tag{7.9}$$

The normal component is called the ***Codazzi equation***[2] (or ***Codazzi–Mainardi equation***[3]),

$$(\overline{\mathcal{R}}(X,Y)Z)^\perp = (\nabla^\perp_X \mathrm{II})(Y,Z) - (\nabla^\perp_Y \mathrm{II})(X,Z).$$

On the other hand, for $N \in \mathfrak{X}(M)^\perp$ we have

$$\overline{\nabla}_X\overline{\nabla}_Y N = \overline{\nabla}_X\nabla^\perp_Y N - \overline{\nabla}_X A_N Y = \nabla^\perp_X\nabla^\perp_Y N - A_{\nabla^\perp_Y N}X - \nabla_X A_N Y - \mathrm{II}(X, A_N Y),$$

and therefore

$$\begin{aligned}\overline{\mathcal{R}}(X,Y)N =& \overline{\nabla}_X\overline{\nabla}_Y N - \overline{\nabla}_Y\overline{\nabla}_X N - \overline{\nabla}_{[X,Y]}N\\ =& \nabla^\perp_X\nabla^\perp_Y N - A_{\nabla^\perp_Y N}X - \nabla_X A_N Y - \mathrm{II}(X,A_N Y)\\ &- \nabla^\perp_Y\nabla^\perp_X N + A_{\nabla^\perp_X N}Y + \nabla_Y A_N X + \mathrm{II}(Y, A_N X)\\ &- \nabla^\perp_{[X,Y]}N + A_N\nabla_X Y - A_N\nabla_Y X\\ =& \mathcal{R}^\perp(X,Y)N + \mathrm{II}(Y,A_N X) - \mathrm{II}(X,A_N Y) + (\nabla_Y A)(X,N) - (\nabla_X A)(Y,N),\end{aligned}$$

where

$$(\nabla_Y A)(X,N) = \nabla_Y A_N X - A_N\nabla_Y X - A_{\nabla^\perp_Y N}X.$$

The tangential component yields

$$(\overline{\mathcal{R}}(X,Y)N)^\top = (\nabla_Y A)(X,N) - (\nabla_X A)(Y,N),$$

which is an equivalent form of the Codazzi equation. The normal component is called the ***Ricci equation***,

$$(\overline{\mathcal{R}}(X,Y)N)^\perp = \mathcal{R}^\perp(X,Y)N - \mathrm{II}(X,A_N Y) + \mathrm{II}(Y,A_N X).$$

Taking the scalar product by $P \in \mathfrak{X}(M)^\perp$, we have

$$\overline{R}(X,Y,N,P) = R^\perp(X,Y,N,P) - g([A_N,A_P]X,Y), \tag{7.10}$$

which is also the Ricci equation.

The Gauss equation is a tensor equation, so it remains valid when we replace vector fields by individual tangent vectors. Using the Gauss equation in the definition of a sectional curvature gives the following relation between the sectional curvatures $\overline{\kappa}$ of $\overline{M}$ and κ of M, also called the ***Gauss equation***,

$$\kappa(X,Y) = \overline{\kappa}(X,Y) + \frac{\overline{g}(\mathrm{II}(X,X),\mathrm{II}(Y,Y)) - \overline{g}(\mathrm{II}(X,Y),\mathrm{II}(X,Y))}{g(X,X)g(Y,Y) - g(X,Y)g(X,Y)}. \tag{7.11}$$

[2]Delfino Codazzi (1824–1873), Italian mathematician
[3]Gaspare Mainardi (1800–1879), Italian mathematician

Example 7.4. The position vector field $N = \sum_{i=1}^n x_i\partial_i$ on the sphere $\mathbf{S}_r^{n-1} \hookrightarrow \mathbb{R}^n$ is normal at each point $(x_1, \ldots, x_n) \in \mathbf{S}_r^{n-1}$ as a consequence of Lemma 2.16. On the other hand, for the Levi-Civita connection $\overline{\nabla}$ in $\mathbb{R}^n$, for any $X \in \mathfrak{X}(\mathbf{S}_r^{n-1})$ we have $\overline{\nabla}_X N = \sum_{i=1}^n X(x_i)\partial_i = X$. The formula (7.7) implies $g(\mathrm{II}(X,Y), N) = -g(\overline{\nabla}_X N, Y) = -g(X,Y)$, so since $\varepsilon_N = r^2$ we obtain $\mathrm{II}(X,Y) = -g(X,Y)N/r^2$. Finally, the equation (7.11) for $\overline{\kappa} = 0$, yields $\kappa = 1/r^2$. △

The Gauss formula (7.6) can be adapted to compare intrinsic and extrinsic covariant derivatives along curves. For a curve $\gamma \colon I \to M$ and $Y \in \mathfrak{X}(M)$ we have

$$\overline{\nabla}_{\gamma'(t)} Y = \nabla_{\gamma'(t)} Y + \mathrm{II}(\gamma'(t), Y \circ \gamma(t)),$$

which implies

$$\frac{\overline{\nabla} Y_\gamma}{dt}(t) = \frac{\nabla Y_\gamma}{dt}(t) + \mathrm{II}(\gamma'(t), Y_\gamma(t)).$$

The special case $Y_\gamma = \gamma'$ implies the formula for the acceleration of γ,

$$\frac{\overline{\nabla}\gamma'}{dt} = \frac{\nabla\gamma'}{dt} + \mathrm{II}(\gamma', \gamma'). \tag{7.12}$$

As a consequence of the previous equation, a curve γ in $M \subset \overline{M}$ is a geodesic of M if and only if its $\overline{M}$ acceleration is everywhere normal to M.

Example 7.5. A ***great circle*** of the sphere $\mathbf{S}_r^n$ is a circle $\Pi \cap \mathbf{S}_r^n$ obtained as the intersection by a two-dimensional plane Π through the origin of $\mathbb{R}^{n+1}$. If γ is a constant speed parametrization of $\Pi \cap \mathbf{S}_r^n$, then γ' and γ'' are mutually orthogonal and tangent to the plane Π. However, the position vector field is also tangent to Π and orthogonal to $\gamma' \neq 0$, so the position vector field and γ'' are collinear at each point of the curve, which means that γ'' is normal to $\mathbf{S}_r^n$ and therefore γ is a geodesic of the sphere. Conversely, any nonconstant geodesic γ can obatained in this way. If Π is a plane through the origin and $\gamma(0)$ to which $\gamma'(0)$ is tangent, then there is a suitable constant speed parametrization of $\Pi \cap \mathbf{S}_r^n$ and the rest follows from the uniqueness of geodesics. △

A pseudo-Riemannian submanifold M of a pseudo-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be ***totally geodesic*** if every geodesic in M is also a geodesic in $\overline{M}$. The formula (7.12) provides that M is totally geodesic if and only if the second fundamental form II vanishes.

Let (M, g) be a pseudo-Riemannian manifold. Let us consider the set of all fixed points $\Phi = \{p \in M : f(p) = p\}$ of an isometry $f \in \mathcal{I}(M)$. Since Φ is the inverse image of the diagonal in $M \times M$ under the smooth map $p \mapsto (p, f(p))$, the set $\Phi \subseteq M$ is closed. If $p \in \Phi$ is not isolated, then there exists a normal neighbourhood $U = \exp_p(\mathcal{U})$ of p which contains some $p \neq q \in \Phi$. If γ_V is a unique radial geodesic from p to q, then it is also $\gamma_{T_pf(V)} = f \circ \gamma_V$ from (6.11), and therefore $\gamma_{T_pf(V)} = \gamma_V$. Hence $\Psi = \{V \in T_pM : T_pf(V) = V\}$ is a closed non-trivial subspace of T_pM.

For $q \in \Phi \cap \mathcal{U}$, the radial geodesic from p to q is γ_V where $V = \exp_p^{-1} q$ and thus $V \in \Psi$. Conversely, for $V \in \Psi \cap U$ we have $f \circ \gamma_V = \gamma_{T_pf(V)} = \gamma_V$, so $f \circ \gamma_V(1) = \gamma_V(1) = \exp_p(V) \in \Phi$. This proves $\exp_p(\Psi \cap \mathcal{U}) = \Phi \cap U$, which means that Φ is a submanifold of M, and since every radial geodesic γ through $p \in \Phi$ satisfies $f \circ \gamma = \gamma$, Φ is totally geodesic.

Theorem 7.8. *Every connected component of the fixed point set of an isometry of a pseudo-Riemannian manifold M is a totally geodesic submanifold of M.*

A ***pseudo-Riemannian hypersurface*** is a pseudo-Riemannian submanifold of codimension one. Let (M, g) be a pseudo-Riemannian hypersurface in $(\overline{M}, \overline{g})$. By definition, each tangent space T_pM is a nondegenerate subspace of $T_p\overline{M}$ that has constant index, so its complementary space N_pM is also nondegenerate and its constant index $\operatorname{Ind} N_pM$ is called the ***co-index*** of M. The co-index of a pseudo-Riemannian hypersurface M is either 0 or 1 and it determines the ***sign*** of M, which is the sign of any nonzero normal vector. Hence, the co-index 0 ($\operatorname{Ind} M = \operatorname{Ind} \overline{M}$) gives $\operatorname{sgn} M = 1$, while the co-index 1 ($\operatorname{Ind} M = \operatorname{Ind} \overline{M} - 1$) implies $\operatorname{sgn} M = -1$. Of course, any hypersurface in a Riemannian manifold is Riemannian with sign 1.

Example 7.6. Consider the case $\emptyset \neq M = f^{-1}(c)$ for some $f \in \mathfrak{F}(\overline{M})$ and $c \in \mathbb{R}$ such that $\operatorname{grad} f$ never vanishes on M. Since $\operatorname{grad} f \neq 0$ on M implies $df \neq 0$ on M, we see that M is a regular level set and by Theorem 2.15 we conclude that M is a hypersurface of $\overline{M}$. Since $g(\operatorname{grad} f, X) = df(X) = Xf$, while $f = c$ on M we conclude that $\operatorname{grad} f$ is normal to M. Therefore, M is a pseudo-Riemannian hypersurface $\overline{M}$ if and only if $g(\operatorname{grad} f, \operatorname{grad} f)$ has constant sign on M. In this case the sign of M is the constant sign of $g(\operatorname{grad} f, \operatorname{grad} f)$ and $\operatorname{grad} f / \|\operatorname{grad} f\|$ is a unit normal vector field on M. △

Example 7.7. Basic examples of pseudo-Riemannian hypersurfaces in $\mathbb{R}^{n+1}_\nu$ are pseudo-spheres and pseudo-hyperbolic spaces defined by $M = \{X \in \mathbb{R}^{n+1}_\nu : g(X, X) = c\} = f^{-1}(\{c\})$ for some $c \neq 0$, where $f(X) = g(X, X)$ (see Section 4.6). We already know that $df_X = 2X^\flat$ which gives $\operatorname{grad} f = 2P$, where $P = \sum_i x_i \partial_i$ is the position vector field. Consequently, $g(\operatorname{grad} f, \operatorname{grad} f) = 4g(P, P) = 4f$, which gives $g(\operatorname{grad} f, \operatorname{grad} f) = 4c$ on M and M is a pseudo-Riemannian hypersurface. △

For a hypersurface $M \subset \overline{M}$, we use a unit normal vector field $N \in \mathfrak{X}(M)^\perp$, which is locally unique up to sign. Once it has been fixed, we use the notation $A = A_N$ for the shape operator with respect to N. We have $\mathrm{II}(X, Y) = \varepsilon_N h_N(X, Y) N$, while the Gauss formula (7.6) becomes

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y) N.$$

On the other hand, since $2g(\overline{\nabla}_X N, N) = \overline{\nabla}_X g(N, N) = 0$, it follows that $\overline{\nabla}_X N$ is tangent to M, so the Weingarten formula (7.7) reduces to

$$\overline{\nabla}_X N = -AX.$$

Let M be a hypersurface of the Euclidean space. At each point $p \in M$, the shape operator s is a self-adjoint endomorphism of the tangent space T_pM. Such an operator (definiteness of metric is important) has real eigenvalues $\lambda_1, \ldots, \lambda_n$, and there exists an orthonormal basis $(E_1, \ldots E_n)$ in T_pM consists of eigenvectors of s, that is, $sE_i = \lambda_i E_i$ for $1 \leq i \leq n$. In this basis, both h and s are diagonal and hence $h(X, Y) = \sum_i \lambda_i X^i Y^i$.

The eigenvalues of s are called the ***principal curvatures*** of M at p, while the corresponding eigenspaces are called the ***principal directions***. They are independent on the basis choice, but the principal curvatures change sign if we reverse the normal vector. The principal curvatures give a description of the local shape of the embedded surface.

There are two combinations of the principle curvatures which play important roles for Euclidean hypersurfaces. The ***Gaussian curvature*** is defined as $K = \det s$, while the ***mean curvature*** is $H = \operatorname{tr} s/n = \operatorname{tr}_g h/n$. The determinant and trace of a linear endomorphism are invariants, while in terms of the principal curvatures we obtain $K = \lambda_1 \lambda_2 \cdots \lambda_n$ and $H = (\lambda_1 + \cdots + \lambda_n)/n$.

7.3 Symmetric spaces

The main feature of the most beautiful and most important pseudo-Riemannian manifolds is that they are highly symmetric (they have large groups of isometries). A connected pseudo-Riemannian manifold (M, g) is called a ***locally symmetric space*** if each $p \in M$ is an isolated fixed point of some involutive local isometry of M. It is called a (globally) ***symmetric space*** if each $p \in M$ is an isolated fixed point of an involutive isometry of M.

Suppose that there is an involutive isometry $s \in \mathcal{I}_p(U)$ with $p \in U \subseteq M$ to which p is an isolated fixed point. Then $s^2 = \mathbb{1}_U$ implies $(T_p s)^2 = \mathbb{1}_{T_pM}$, which after subtracting $\lambda^2\,\mathbb{1}$ becomes

$$(T_p s - \lambda\,\mathbb{1})(T_p s + \lambda\,\mathbb{1}) = (1-\lambda^2)\,\mathbb{1},$$

and by taking the determinant we see that the characteristic polynomial of $T_p s$ by λ divides the polynomial $(1-\lambda)^n(1+\lambda)^n$. However, the case $\det(T_p s - \mathbb{1}) = 0$ is not possible because then exists a nonzero $V \in T_pM$ such that $T_p s(V) = V$, from where the equation (6.12) shows that $\gamma_V(t) = \exp_p tV$ is a fixed point of s for every sufficiently small t, which contradicts the fact that p is an isolated fixed point. This is why -1 is the only eigenvalue of $T_p s$, which gives $\det(T_p s - \lambda\,\mathbb{1}) = (1+\lambda)^n$, and therefore $(T_p s + \mathbb{1})^n = 0$. Moreover,

$$0 = (T_p s + \mathbb{1})^n = (T_p s + \mathbb{1})^{n-2}((T_p s)^2 + 2T_p s + \mathbb{1}) = 2(T_p s + \mathbb{1})^{n-1} = 2^{n-1}(T_p s + \mathbb{1}),$$

from where we obtain $T_p s = -\,\mathbb{1}$.

The condition $T_p s = -\,\mathbb{1}$ by (6.12) gives $s(\exp_p(V)) = \exp_{s(p)}(T_p s(V)) = \exp_p(-V)$ for $V \in T_pM$, which uniquely determines s. We see that s reverses geodesics emanating from p because of $s(\gamma_V(t)) = \gamma_V(-t)$, which motivates us to introduce the following definition. A ***geodesic symmetry*** at $p \in M$ is a diffeomorphism f of some neighbourhood of p that fixes that point ($f(p) = p$) and reverses geodesics through it, that is, $f(\gamma(t)) = \gamma(-t)$ holds for a geodesic γ with $\gamma(0) = p$. So far, we have shown that our involutive isometry s is a geodesic symmetry.

Conversely, suppose that f is a geodesic symmetry at $p \in M$. Immediately $T_p f = -\,\mathbb{1}$ because we have $T_p f(V) = (f \circ \gamma_V)'(0) = (\gamma_{-V})'(0) = -V$. If f is an isometry on some neighbourhood $U \subseteq M$ of the point p, then according to Theorem 7.8 the set of all fixed points of f is a totally geodesic submanifold P of M whose tangent space at p is the fixed point set of $T_p f$ in T_pM. From $T_p f = -\,\mathbb{1}$ we obtain $T_pP = \{0\}$ and hence P is discrete. Moreover, f^2 is an isometry with $f^2(p) = p$ and $T_p f^2 = \mathbb{1}$, so it must be $f^2 = \mathbb{1}$. Hence f is an involutive isometry to which p is an isolated fixed point, and we obtain the following statement.

Theorem 7.9. *A pseudo-Riemannian manifold is a locally symmetric space if and only if the geodesic symmetry at each point is a local isometry.*

A symmetric space has the property that the geodesic symmetry at any point extends to an isometry of the whole space onto itself. Symmetric spaces can be observed from plenty different points of view. The algebraic description allowed Cartan[4] to develop the theory of Riemannian symmetric spaces merged with the theory of semisimple Lie groups which led to a complete classification in 1926 [29, 30].

One refined invariant of a symmetric space is the ***rank***, which is the maximal dimension of a totally geodesic flat submanifold. The rank is always at least one, with equality when the maximal flat submanifolds are geodesics, in which case the sectional curvature is positive (compact type) or negative (noncompact type). Among the pseudo-Riemannian symmetric spaces, those of rank one are of special importance.

On the other hand, we can consider the cosmological principle which says that the spatial distribution of matter in the universe is homogeneous and isotropic at a sufficiently large scale. A homogeneous pseudo-Riemannian manifolds looks geometrically the same when viewed from any point, while an isotropic one has the geometry that does not depend on directions.

Theorem 7.10. *Every homogeneous Riemannian manifold is complete.*

Proof. Let $\gamma_V\colon I \to M$ be a maximal unit-speed geodesic that cannot be extended, which means: $0 \in I$, $\gamma_V'(0) = V$, $\|V\| = 1$, and $a = \sup I < \infty$. For an arbitrary point $p \in M$,

[4]Élie Cartan (1869–1951), French mathematician

there exists a sufficiently small $0 < r < a$ such that $B_{2r}(p)$ is a geodesic ball. Consider an isometry $f\colon M \to M$ that maps $\gamma_V(a-r)$ to p, and let $W = (T_{\gamma(a-r)}f)(\gamma'(a-r))$. Isometries preserve geodesics, and according to the formula (6.11) we have $f \circ \gamma_{\gamma_V'(a-r)} = \gamma_W$, which is defined at least on $(-2r, 2r)$. Hence, $(-2r, 2r)$ is in the domain of $\gamma_{\gamma_V'(a-r)} = f^{-1} \circ \gamma_W$, but since $\gamma_V(t) = \gamma_{\gamma_V'(a-r)}(t + r - a)$ holds for $t \in [0, a)$, γ_V extends to $I \cup [a, a + r)$, which is a contradiction. We have shown that a homogeneous Riemannian manifold is geodesically complete, and is complete according to the Hopf-Rinow theorem. □

However, a homogeneous pseudo-Riemannian manifold does not have to be complete, as we see in the following example.

Example 7.8. Let M be the right half-plane $\{(u, v) \in \mathbb{R}^2 : u > 0\}$ with Lorentzian metric $g = 2du\,dv$. An arbitrary $(a, b) \in M$ is mapped to $(1, ab)$ by the isometry $(u, v) \mapsto (u/a, av)$, which is further mapped to $(1, 0)$ by the isometry $(u, v) \mapsto (u, v - ab)$, which proves that (M, g) is homogeneous. However, the null geodesic $\gamma(t) = (t, 0)$ has the maximal domain $(0, \infty)$, so (M, g) is not complete. △

A Riemannian manifold M is ***two-point homogeneous*** if any pair of points can be transformed by means of an appropriate isometry to any other pair of points with the same distance between them. This means that the isometry group $\mathcal{I}(M)$ acting transitively on equidistant pairs of points. The special case when the first point in a pair is fixed proves that a two-point homogeneous M is homogeneous and in particular complete. However, we have the following theorem of Wolf[5] [120, Lemma 8.12.1].

Theorem 7.11. *A connected Riemannian manifold is isotropic if and only if it is two-point homogeneous.*

Proof. Let M be two-point homogeneous, $B_r(p)$ is a geodesic ball centred at $p \in M$, and $0 < \varepsilon < r$. For arbitrary $X, Y \in S_1(0_p)$ we have $d(p, \exp_p \varepsilon X) = d(p, \exp_p \varepsilon Y)$, so there exists $f \in \mathcal{I}(M)$ such that $f(p) = p$ and $f(\exp_p \varepsilon X) = \exp_p \varepsilon Y$, but from the equation (6.12) holds $f(\exp_p \varepsilon X) = \exp_p(f_*(\varepsilon X))$, which implies $f_*(\varepsilon X) = \varepsilon Y$, and therefore $f_* X = Y$, which means that M is isotropic.

Let M be isotropic. For an arbitrary geodesic γ_V with $p \in M, V \in T_pM$ there exists an isometry $f \in \mathcal{I}_p$ that reverses it, $f_*V = -V$, so M is complete and homogeneous. Consider $x_1, x_2, y_1, y_2 \in M$ with $d(x_1, y_1) = d(x_2, y_2)$. By homogeneity, there exists $f_1 \in \mathcal{I}(M)$ such that $f_1(x_2) = x_1$. By completeness, we have minimizing geodesics $\gamma_{X_1}, \gamma_{X_2} \colon [0, 1] \to M$ such that $y_1 = exp_{x_1}(X_1), f_1(y_2) = \exp_{x_1}(X_2)$. Since $\|X_1\| = d(x_1, y_1) = d(x_2, y_2) = d(x_1, f_1 y_2) = \|X_2\|$, by isotropy we have $f_2 \in \mathcal{I}_{x_1}(M)$ with $f_{2*}X_2 = X_1$. Thus, we obtain $f_2 \circ f_1 \in \mathcal{I}(M)$ that satisfies $f_2 f_1(x_2) = f_2(x_1) = x_1$ and $f_2 f_1(y_2) = f_2 \exp_{x_1}(X_2) = \exp_{f_2 x_1}(f_{2*}X_2) = \exp_{x_1}(X_1) = y_1$, which proves that M is two-point homogeneous. □

Two-point homogeneous spaces were first studied by Busemann[6] in 1942 [28] and by Birkhoff[7] in 1944 [20]. We have a complete classification of these spaces, the compact ones were classified by Wang[8] in 1952 [118], while the noncompact ones by Tits[9] in 1955 [113]. As a consequence of the classification, it is known that any locally two-point homogeneous Riemannian manifold is either flat or locally isometric to a rank one symmetric space, see Helgason[10] [67, p.535]. For the details about two-point homogeneous spaces we recommend Wolf [120, pp.293–300] and Helgason [67, p.535].

[5] Joseph Albert Wolf (1936), American mathematician
[6] Herbert Busemann (1905–1994), German-American mathematician
[7] Garrett Birkhoff (1911–1996), American mathematician
[8] Hsien Chung Wang (1918–1978), Chinese-American mathematician
[9] Jacques Tits (1930), Belgium-born French mathematician
[10] Sigurdur Helgason (1927), Icelandic mathematician

Theorem 7.12. *A two-point homogeneous connected Riemannian manifold is isometric (up to a homothety) to one of the following: a Euclidean space; a sphere; a real, complex or quaternionic, projective or hyperbolic space; or the Cayley*[11] *projective or hyperbolic plane.*

More precisely, the classification of two-point homogeneous spaces is as follows:

- $\mathbb{R}^n = \mathbf{E}(n)/\mathbf{O}(n)$ for $n \geq 1$, Euclidean space;
- $\mathbf{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$ for $n \geq 1$, sphere;
- $\mathbb{R}\mathbf{P}^n = \mathbf{SO}(n+1)/\mathbf{O}(n)$ for $n \geq 2$, real projective space;
- $\mathbb{C}\mathbf{P}^n = \mathbf{SU}(n+1)/\mathbf{U}(n)$ for $n \geq 2$, complex projective space;
- $\mathbb{H}\mathbf{P}^n = \mathbf{Sp}(n+1)/\mathbf{Sp}(n) \times \mathbf{Sp}(1)$ for $n \geq 2$, quaternionic projective space;
- $\mathbb{O}\mathbf{P}^2 = \mathbf{F}_4/\mathbf{Spin}(9)$, Cayley projective plane;
- $\mathbb{R}\mathbf{H}^n = \mathbf{SO}^1(n+1)/\mathbf{SO}(n)$ for $n \geq 2$, real hyperbolic space;
- $\mathbb{C}\mathbf{H}^n = \mathbf{SU}^1(n+1)/\mathbf{U}(n)$ for $n \geq 2$, complex hyperbolic space;
- $\mathbb{H}\mathbf{H}^n = \mathbf{Sp}^1(n+1)/\mathbf{Sp}(n) \times \mathbf{Sp}(1)$ for $n \geq 2$, quaternionic hyperbolic space;
- $\mathbb{O}\mathbf{H}^2 = \mathbf{F}_4^*/\mathbf{Spin}(9)$, Cayley hyperbolic plane.

However, it is convenient to notice that there are following isomorphisms (isometries up to a homothety) in low dimensions: $\mathbb{R}\mathbf{P}^1 \cong \mathbf{S}^1$, $\mathbb{C}\mathbf{P}^1 \cong \mathbf{S}^2$, $\mathbb{H}\mathbf{P}^1 \cong \mathbf{S}^4$, $\mathbb{O}\mathbf{P}^1 \cong \mathbf{S}^8$, $\mathbb{C}\mathbf{H}^1 \cong \mathbb{R}\mathbf{H}^2$, $\mathbb{H}\mathbf{H}^1 \cong \mathbb{R}\mathbf{H}^4$, $\mathbb{O}\mathbf{H}^1 \cong \mathbb{R}\mathbf{H}^8$.

[11] Arthur Cayley (1821–1895), British mathematician

Chapter 8

OSSERMAN CONDITIONS

8.1 Osserman conditions

Let (M, g) be a pseudo-Riemannian manifold and $p \in M$ is an arbitrary point. The set of all unit vectors in T_pM,

$$S_pM = \{X \in T_pM : |g_p(X,X)| = 1\},$$

in the Riemannian setting corresponds to the tangent sphere $S_1(0_p)$. However, if g is indefinite, S_pM is not even compact (a hyperbola in dimension 2, a hyperboloid in dimension 3), and it is convenient to observe spacelike and timelike vectors separately (geometric difference between one-sheet and two-sheets hyperboloids). Thus, we introduce pseudo-spheres

$$S_p^-M = \{X \in T_pM : g_p(X,X) = -1\},$$
$$S_p^+M = \{X \in T_pM : g_p(X,X) = 1\},$$

where $S_pM = S_p^-M \cup S_p^+M$. Moreover, on a manifold, we often consider pseudo-sphere bundles $\{X \in TM : g_{\pi X}(X,X) = \text{Const}\}$, which are hypersurfaces of the tangent bundle TM. In this way we have the ***timelike unit tangent bundle*** $S^-M = \bigsqcup_{p\in M} S_p^-M$ and the ***spacelike unit tangent bundle*** $S^+M = \bigsqcup_{p\in M} S_p^+M$, as well as their union $SM = S^-M \cup S^+M$.

According to Theorem 7.12, the classification of two-point homogeneous connected Riemannian manifolds includes: $\mathbb{R}^n$, $\mathbf{S}^n$, $\mathbb{R}\mathbf{P}^n$, $\mathbb{C}\mathbf{P}^n$, $\mathbb{H}\mathbf{P}^n$, $\mathbb{O}\mathbf{P}^2$, $\mathbb{R}\mathbf{H}^n$, $\mathbb{C}\mathbf{H}^n$, $\mathbb{H}\mathbf{H}^n$, and $\mathbb{O}\mathbf{H}^2$. Local isometries of a locally two-point homogeneous space M act transitively on the sphere bundle SM, and therefore fix the characteristic polynomial of the Jacobi operator there. In this way we get a generalisation of locally two-point homogeneous Riemannian manifolds, called the ***Riemannian Osserman manifolds***, in which the characteristic polynomial (or equivalently, the eigenvalues and their multiplicities) of a Jacobi operator $\mathcal{J}_X$ is independent of X from the unit tangent bundle.

The lack of other examples led Osserman [97] to conjecture that the converse might also be true. The question of whether the converse is true (every Osserman Riemannian manifold is locally two-point homogeneous) is known as the ***Osserman conjecture***. The first results on the Osserman conjecture were given by Chi[1] [34], who established the affirmative answer for manifolds of dimension not divisible by four. The largest progress in solving the conjecture was made by Nikolayevsky[2] [88, 89, 90, 91], who proved it in all cases, except the manifolds of dimension 16 whose reduced Jacobi operator has an eigenvalue of multiplicity 7 or 8.

We can extend this concept to pseudo-Riemannian manifolds. We say that a pseudo-Riemannian manifold (M, g) is ***Osserman*** or ***globally Osserman***, if the characteristic

[1] Quo-Shin Chi (1955), Taiwanese-American mathematician
[2] Yuri Nikolayevsky, Australian mathematician

polynomial

$$\omega_X(\lambda) = \det(\lambda \mathbb{1} - \mathcal{J}_X)$$

of a Jacobi operator $\mathcal{J}_X$ is independent of a choice of $X \in TM$ on both S^-M and S^+M.

It is often convenient to study certain geometric problems in a purely algebraic setting. Reducing a pseudo-Riemannian manifold (M, g) to a point $p \in M$ allows us to deal with an algebraic curvature tensor R_p on the scalar product space (T_pM, g_p). In this context we introduce some notions related to an algebraic curvature tensor R on a scalar product space $(\mathcal{V}, g)$.

We say that R is ***timelike Osserman*** if ω_X is independent of unit timelike $X \in \mathcal{V}$. We say that R is ***spacelike Osserman*** if ω_X is independent of unit spacelike $X \in \mathcal{V}$. Naturally, R is called ***Osserman*** if it is both timelike and spacelike Osserman. However, it turns out that timelike Osserman and spacelike Osserman conditions are equivalent (see Theorem 8.1).

This algebraic approach brings us a less restrictive concept of manifolds in which the characteristic polynomial of Jacobi operator is constant on both S_p^-M and S_p^+M for all $p \in M$, but can vary from point to point. We say that a pseudo-Riemannian manifold M is ***pointwise Osserman*** if the associated algebraic curvature tensor R_p is Osserman at each point $p \in M$. Of course, globally Osserman manifolds are necessarily pointwise Osserman, while the converse is not true (see [23]).

The Osserman condition is equivalent to the constancy of (possibly complex) eigenvalues of the Jacobi operators counting multiplicities. It was originally established in the positive definite case, which enables us to diagonalize the Jacobi operator, as a self-adjoint operator. However, for indefinite scalar product, the eigen-structure of a self-adjoint operator is not determined by its characteristic polynomial (in general, the Jacobi operators are not diagonalisable), so the Jordan normal form plays a crucial role (see Section A.3).

Further generalisations of Osserman conditions at a point concern the eigen-structure of Jacobi operators. We say that R is ***timelike Jordan-Osserman*** if the Jordan normal form of $\mathcal{J}_X$ is independent of unit timelike $X \in \mathcal{V}$. We say that R is ***spacelike Jordan-Osserman*** if the Jordan normal form of $\mathcal{J}_X$ is independent of unit spacelike $X \in \mathcal{V}$. As before, R is called ***Jordan-Osserman*** if it is both timelike and spacelike Jordan-Osserman. Timelike and spacelike Jordan-Osserman conditions, unlike the original Osserman conditions, are not equivalent (see Theorem 9.7).

The simplest case of Jordan-Osserman R for indefinite g considers diagonalisable Jacobi operators, which means that the corresponding Jordan normal form consists of $\dim \mathcal{V}$ blocks of size one. We say that R is ***Jacobi-diagonalisable*** if for each nonnull $X \in \mathcal{V}$ there exists an orthonormal eigenbasis in $\mathcal{V}$ related to $\mathcal{J}_X$. Jacobi-diagonalisable algebraic curvature tensors are closest to the definite case. Moreover, this condition is natural in some way, because a Jordan-Osserman R that is not Kleinian ($\dim \mathcal{V} \neq 2 \operatorname{Ind} g$) have to be Jacobi-diagonalisable (see Gilkey and Ivanova[3] [57]).

We say that a pseudo-Riemannian manifold (M, g) is ***pointwise Jordan-Osserman*** if its corresponding algebraic curvature tensor R_p is Jordan-Osserman for each point $p \in M$, while it is ***globally Jordan-Osserman*** if the Jordan normal form of $\mathcal{J}_X$ is constant on both S^-M and S^+M. We say that (M, g) is ***Jacobi-diagonalisable*** if R_p is Jacobi-diagonalisable for each point $p \in M$.

Example 8.1. As we have already pointed out, two-point homogeneous connected Riemannian manifolds ($\mathbb{R}^n$, $\mathbf{S}^n$, $\mathbb{R}\mathbf{P}^n$, $\mathbb{C}\mathbf{P}^n$, $\mathbb{H}\mathbf{P}^n$, $\mathbb{O}\mathbf{P}^2$, $\mathbb{R}\mathbf{H}^n$, $\mathbb{C}\mathbf{H}^n$, $\mathbb{H}\mathbf{H}^n$, $\mathbb{O}\mathbf{H}^2$) are globally Osserman. $\triangle$

Example 8.2. Spaces of constant sectional curvature of dimension $n \geq 3$ (from Section 6.4) have the corresponding algebraic curvature tensor of form $R = \kappa R^1$ for some global

[3] Raina B. Ivanova

constant $\kappa \in \mathbb{R}$. Hence $\mathcal{R}(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y)$, so for a nonnull X and $Y \in X^{\perp}$ we have

$$\mathcal{J}_X Y = \mathcal{R}(Y,X)X = \kappa(g(X,X)Y - g(Y,X)X) = \varepsilon_X \kappa Y.$$

Thus $\widetilde{\mathcal{J}}_X = \varepsilon_X \kappa \, \mathbb{1}_{X^{\perp}}$ holds for any nonnull X, which shows that the reduced Jacobi operator $\widetilde{\mathcal{J}}_X$ has the single eigenvalue $\varepsilon_X \kappa$ on pseudo-spheres $\varepsilon_X = \mathrm{Const} \neq 0$. Therefore the characteristic polynomial of $\mathcal{J}_X$ is $\omega_X(\lambda) = \lambda(\lambda - \varepsilon_X \kappa)^{n-1}$, which is constant on both unit tangent bundles. As a corollary, spaces of constant sectional curvature are globally Osserman. In particular, apart from the Riemannian spaces already seen ($\mathbb{R}^n$, $\mathbf{S}^n$, $\mathbb{R}\mathbf{P}^n$, $\mathbb{R}\mathbf{H}^n$), we have pseudo-Riemannian model spaces from Section 4.6 such as $\mathbb{R}^n_\nu$, $\mathbf{dS}^n$, and $\mathbf{AdS}^n$. △

Example 8.3. Let $J \colon \mathcal{V} \to \mathcal{V}$ be a skew-adjoint endomorphism on a scalar product space $(\mathcal{V}, g)$ of dimension n. Such J generates an algebraic curvature tensor $R^J \in \mathfrak{T}^0_4(\mathcal{V})$ by (6.15) from Example 6.5. The corresponding curvature operator has

$$\mathcal{R}^J(X,Y)Z = g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ,$$

and consequently the Jacobi operator satisfies

$$\mathcal{J}^J_X Y = \mathcal{R}^J(Y,X)X = 3g(JY,X)JX - g(JX,X)JY = -3g(Y,JX)JX,$$

that is,

$$\mathcal{J}^J_X = \begin{cases} -3\varepsilon_{JX} \mathbb{1} & \text{on } \mathrm{Span}\{JX\} \\ 0 & \text{on } \mathrm{Span}\{JX\}^{\perp} \end{cases}.$$

If we additionally suppose that $J^2 = c\,\mathbb{1}$ holds for some constant $c \in \mathbb{R}$, then we have $\varepsilon_{JX} = g(JX,JX) = -g(X,J^2X) = -c\varepsilon_X$. It is clear that the case $c \neq 0$ implies

$$\omega_X(\lambda) = \det(\lambda \mathbb{1} - \mathcal{J}^J_X) = \lambda^{n-1}(\lambda - 3c\varepsilon_X). \tag{8.1}$$

The case $c = 0$ gives $\varepsilon_{JX} = 0$, so $\mathcal{J}^J_X$ vanishes on $\mathrm{Span}\{JX\} + \mathrm{Span}\{JX\}^{\perp} = \mathrm{Span}\{JX\}^{\perp}$. If $JX = 0$ then we have $\mathcal{J}^J_X = 0$. Otherwise, JX is null, so $\mathrm{Span}\{JX\}^{\perp} \leq \mathrm{Ker}((\mathcal{J}^J_X)^n)$ is degenerate subspace of dimension $n-1$, and therefore Lemma A.22 implies $\dim \mathrm{Ker}((\mathcal{J}^J_X)^n) > n-1$, which yields $\mathrm{Ker}((\mathcal{J}^J_X)^n) = \mathcal{V}$. This proves that all the eigenvalues are zero in the case $c = 0$, so (8.1) holds anyway. The formula (8.1) shows that $\omega_X(\lambda)$ is constant on pseudo-spheres, and therefore R^J is an Osserman algebraic curvature tensor.

Thus, any skew-adjoint endomorphism J such that $J^2 = c\,\mathbb{1}$ holds, generates an Osserman algebraic curvature tensor R^J. In particular any orthogonal skew-adjoint endomorphism ($c = -1$, a complex structure that preserves g) and any anti-orthogonal skew-adjoint endomorphism ($c = 1$, a product structure) yield an Osserman algebraic curvature tensor. △

8.2 Osserman manifolds examples

The following concrete examples allow us to illustrate our definitions in a nice way. García-Río, Kupeli, and Vázquez-Lorenzo [62] are constructed the collection of pseudo-Riemannian manifolds (M,g) with $M = \mathbb{R}^4$ and the usual coordinates (x_1, x_2, x_3, x_4), by the following Gram matrix

$$g = \begin{pmatrix} x_3 f_1 & a & 1 & 0 \\ a & x_4 f_2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{8.2}$$

related to the natural global frame $(\partial_1, \partial_2, \partial_3, \partial_4)$ for M, where smooth real-valued functions $f_1 = f_1(x_1, x_2)$ and $f_2 = f_2(x_1, x_2)$ depend on x_1 and x_2 only, while $a \in \mathbb{R}$. This is a special case of the Walker metric from Example 4.8, and therefore (M, g) is a pseudo-Riemannian manifold of index two, that is, a Kleinian manifold of dimension 4. The Christoffel symbols are given by the formula

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{n} g^{lk}\left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right), \tag{5.11 revisited}$$

where g_{ij} are entries of the matrix g from (8.2), while g^{ij} are entries of the inverse matrix,

$$g^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -x_3 f_1 & -a \\ 0 & 1 & -a & -x_4 f_2 \end{pmatrix}.$$

Concrete calculations give all nonzero Christoffel symbols:

$$\begin{aligned}
\Gamma_{11}^1 &= -\frac{1}{2}f_1, & \Gamma_{11}^3 &= \frac{1}{2}x_3\frac{\partial f_1}{\partial x_1} + \frac{1}{2}x_3 f_1^2, \\
\Gamma_{11}^4 &= -\frac{1}{2}x_3\frac{\partial f_1}{\partial x_2} + \frac{1}{2}a f_1, & \Gamma_{12}^3 &= \Gamma_{21}^3 = \frac{1}{2}x_3\frac{\partial f_1}{\partial x_2}, \\
\Gamma_{12}^4 &= \Gamma_{21}^4 = \frac{1}{2}x_4\frac{\partial f_2}{\partial x_1}, & \Gamma_{22}^2 &= -\frac{1}{2}f_2, \\
\Gamma_{22}^3 &= -\frac{1}{2}x_4\frac{\partial f_2}{\partial x_1} + \frac{1}{2}a f_2, & \Gamma_{22}^4 &= \frac{1}{2}x_4\frac{\partial f_2}{\partial x_2} + \frac{1}{2}x_4 f_2^2, \\
\Gamma_{13}^3 &= \Gamma_{31}^3 = \frac{1}{2}f_1, & \Gamma_{24}^4 &= \Gamma_{42}^4 = \frac{1}{2}f_2.
\end{aligned}$$

Using the formula

$$\mathcal{R}_{ijk}^l = \partial_i\Gamma_{jk}^l - \partial_j\Gamma_{ik}^l + \sum_m(\Gamma_{jk}^m\Gamma_{im}^l - \Gamma_{ik}^m\Gamma_{jm}^l), \tag{6.3 revisited}$$

we find all nonzero curvature operator components:

$$\begin{aligned}
\mathcal{R}(\partial_1, \partial_2)\partial_1 = {} & \frac{1}{2}\frac{\partial f_1}{\partial x_2}\partial_1 - \frac{1}{2}x_3 f_1\frac{\partial f_1}{\partial x_2}\partial_3 + \\
& + \frac{1}{4}\left(2x_3\frac{\partial^2 f_1}{\partial {x_2}^2} + 2x_4\frac{\partial^2 f_2}{\partial {x_1}^2} + (x_3 f_2 - 2a)\frac{\partial f_1}{\partial x_2} + x_4 f_1\frac{\partial f_2}{\partial x_1} - a f_1 f_2\right)\partial_4,
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}(\partial_1, \partial_2)\partial_2 = {} & -\frac{1}{2}\frac{\partial f_2}{\partial x_1}\partial_2 + \frac{1}{2}x_4 f_2\frac{\partial f_2}{\partial x_1}\partial_4 - \\
& - \frac{1}{4}\left(2x_3\frac{\partial^2 f_1}{\partial {x_2}^2} + 2x_4\frac{\partial^2 f_2}{\partial {x_1}^2} + x_3 f_2\frac{\partial f_1}{\partial x_2} + (x_4 f_1 - 2a)\frac{\partial f_2}{\partial x_1} - a f_1 f_2\right)\partial_3,
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}(\partial_1, \partial_2)\partial_3 &= -\frac{1}{2}\frac{\partial f_1}{\partial x_2}\partial_3, & \mathcal{R}(\partial_1, \partial_2)\partial_4 &= \frac{1}{2}\frac{\partial f_2}{\partial x_1}\partial_4, \\
\mathcal{R}(\partial_1, \partial_3)\partial_1 &= \frac{1}{2}\frac{\partial f_1}{\partial x_2}\partial_4, & \mathcal{R}(\partial_1, \partial_3)\partial_2 &= -\frac{1}{2}\frac{\partial f_1}{\partial x_2}\partial_3, \\
\mathcal{R}(\partial_2, \partial_4)\partial_1 &= -\frac{1}{2}\frac{\partial f_2}{\partial x_1}\partial_4, & \mathcal{R}(\partial_2, \partial_4)\partial_2 &= \frac{1}{2}\frac{\partial f_2}{\partial x_1}\partial_3.
\end{aligned}$$

The previous calculations give the matrix of Jacobi operator $\mathcal{J}_X$ for $X = \sum_{i=1}^{4} \alpha_i \partial_i$ as the block matrix

$$\mathcal{J}_X = \begin{pmatrix} A & 0 \\ B & A^{\intercal} \end{pmatrix},$$

where

$$A = \begin{pmatrix} \frac{1}{2}\alpha_1\alpha_2\frac{\partial f_1}{\partial x_2} & -\frac{1}{2}\alpha_1^2\frac{\partial f_1}{\partial x_2} \\ -\frac{1}{2}\alpha_2^2\frac{\partial f_2}{\partial x_1} & \frac{1}{2}\alpha_1\alpha_2\frac{\partial f_2}{\partial x_1} \end{pmatrix}.$$

Since the determinant of a block triangular matrix is the product of the determinants of its diagonal blocks, the characteristic polynomial of $\mathcal{J}_X$ does not depend on the matrix B,

$$\omega_X(\lambda) = \det(\lambda\,\mathbb{1} - \mathcal{J}_X) = \det(\lambda\,\mathbb{1} - A)\det(\lambda\,\mathbb{1} - A^{\intercal}) = (\det(\lambda\,\mathbb{1} - A))^2.$$

Since

$$\det(\lambda\,\mathbb{1} - A) = \lambda^2 - \frac{1}{2}\lambda\alpha_1\alpha_2\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\right),$$

it is not hard to conclude that the characteristic polynomial $\omega_X(\lambda)$ is constant on pseudo-spheres only if

$$\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} = 0 \tag{8.3}$$

holds at each point of M. Assuming (8.3), we have $\omega_X(\lambda) = \lambda^4$, and (M, g) is a globally Osserman pseudo-Riemannian manifold.

Let us consider Jordan-Osserman conditions for these manifolds. Thanks to a small dimension of M, the Jordan normal form of $\mathcal{J}_X$ can be described in terms of its minimal polynomial μ_X. If the more restrictive condition

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1} = 0 \tag{8.4}$$

holds at some point of M, then the curvature operator formulas are much simpler, and the curvature operator components at this point are nonzero only for

$$\begin{aligned} \mathcal{R}(\partial_1, \partial_2)\partial_1 &= \frac{1}{4}\left(2x_3\frac{\partial^2 f_1}{\partial {x_2}^2} + 2x_4\frac{\partial^2 f_2}{\partial {x_1}^2} - af_1f_2\right)\partial_4, \\ \mathcal{R}(\partial_1, \partial_2)\partial_2 &= -\frac{1}{4}\left(2x_3\frac{\partial^2 f_1}{\partial {x_2}^2} + 2x_4\frac{\partial^2 f_2}{\partial {x_1}^2} - af_1f_2\right)\partial_3. \end{aligned}$$

Hence

$$\mathcal{J}_X = \frac{1}{4}\left(2x_3\frac{\partial^2 f_1}{\partial {x_2}^2} + 2x_4\frac{\partial^2 f_2}{\partial {x_1}^2} - af_1f_2\right)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha_2^2 & \alpha_1\alpha_2 & 0 & 0 \\ \alpha_1\alpha_2 & -\alpha_1^2 & 0 & 0 \end{pmatrix}$$

and ${\mathcal{J}_X}^2 = 0$. At points satisfying (8.4), the condition

$$2x_3\frac{\partial^2 f_1}{\partial {x_2}^2} + 2x_4\frac{\partial^2 f_2}{\partial {x_1}^2} - af_1f_2 = 0$$

gives $\mu_X(\lambda) = \lambda$. Otherwise, $\mathcal{J}_X = 0$ holds only for $\alpha_1 = \alpha_2 = 0$, but any $X = \alpha_3\partial_3 + \alpha_4\partial_4$ is null, and therefore $\mu_X(\lambda) = \lambda^2$ holds for a nonnull X. Concrete Osserman manifolds can be given by a suitable choice for f_1, f_2, and a.

Example 8.4. For $f_1(x_1,x_2)=1, f_2(x_1,x_2)=1$, and $a=1$ hold

$$\frac{\partial f_1}{\partial x_2}=\frac{\partial f_2}{\partial x_1}=0,\quad 2x_3\frac{\partial^2 f_1}{\partial {x_2}^2}+2x_4\frac{\partial^2 f_2}{\partial {x_1}^2}-af_1f_2=-1\neq 0.$$

For a nonnull X we have $\mu_X(\lambda)=\lambda^2$ at each point of M. This manifold is globally Jordan-Osserman, but it is not Jacobi-diagonalisable. △

Example 8.5. For $f_1(x_1,x_2)=x_1, f_2(x_1,x_2)=1$, and $a=1$ hold

$$\frac{\partial f_1}{\partial x_2}=\frac{\partial f_2}{\partial x_1}=0,\quad 2x_3\frac{\partial^2 f_1}{\partial {x_2}^2}+2x_4\frac{\partial^2 f_2}{\partial {x_1}^2}-af_1f_2=-x_1.$$

In this case, for a nonnull X, we have $\mu_X(\lambda)=\lambda$ at points with $x_1=0$ and $\mu_X(\lambda)=\lambda^2$ at points with $x_1\neq 0$. This manifold is globally Osserman and pointwise Jordan-Osserman, but it is not globally Jordan-Osserman. △

For completeness and some possible examples we can calculate the matrix B. Under the condition (8.3) we have

$$A=\frac{1}{2}\frac{\partial f_1}{\partial x_2}\begin{pmatrix}\alpha_1\alpha_2 & -\alpha_1^2\\ \alpha_2^2 & -\alpha_1\alpha_2\end{pmatrix},\quad B=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix},$$

where

$$\begin{aligned}
b_{11}=&-\frac{1}{2}\alpha_2^2x_3\frac{\partial^2 f_1}{\partial x_2^2}+\frac{1}{2}\alpha_2^2x_4\frac{\partial^2 f_1}{\partial x_1\partial x_2}+\frac{1}{4}a\alpha_2^2f_1f_2\\
&-\frac{1}{4}\left(2\alpha_1\alpha_2x_3f_1+\alpha_2^2x_3f_2-\alpha_2^2x_4f_1+4\alpha_2\alpha_3+2a\alpha_2^2\right)\frac{\partial f_1}{\partial x_2},\\
b_{12}=&\frac{1}{2}\alpha_1\alpha_2x_3\frac{\partial^2 f_1}{\partial x_2^2}-\frac{1}{2}\alpha_1\alpha_2x_4\frac{\partial^2 f_1}{\partial x_1\partial x_2}-\frac{1}{4}a\alpha_1\alpha_2f_1f_2\\
&+\frac{1}{4}\left(2\alpha_1^2x_3f_1+\alpha_1\alpha_2x_3f_2-\alpha_1\alpha_2x_4f_1+2\alpha_1\alpha_3+2a\alpha_1\alpha_2-2\alpha_2\alpha_4\right)\frac{\partial f_1}{\partial x_2},\\
b_{21}=&\frac{1}{2}\alpha_1\alpha_2x_3\frac{\partial^2 f_1}{\partial x_2^2}-\frac{1}{2}\alpha_1\alpha_2x_4\frac{\partial^2 f_1}{\partial x_1\partial x_2}-\frac{1}{4}a\alpha_1\alpha_2f_1f_2\\
&-\frac{1}{4}\left(\alpha_1\alpha_2x_4f_1+2\alpha_2^2x_4f_2-\alpha_1\alpha_2x_3f_2-2\alpha_1\alpha_3+2a\alpha_1\alpha_2+2\alpha_2\alpha_4\right)\frac{\partial f_1}{\partial x_2},\\
b_{22}=&-\frac{1}{2}\alpha_1^2x_3\frac{\partial^2 f_1}{\partial x_2^2}+\frac{1}{2}\alpha_1^2x_4\frac{\partial^2 f_1}{\partial x_1\partial x_2}+\frac{1}{4}a\alpha_1^2f_1f_2\\
&+\frac{1}{4}\left(2\alpha_1\alpha_2x_4f_2+\alpha_1^2x_4f_1-\alpha_1^2x_3f_2+4\alpha_1\alpha_4+2a\alpha_1^2\right)\frac{\partial f_1}{\partial x_2}.
\end{aligned}$$

The straightforward calculation gives

$$\mathcal{J}_X{}^2=\begin{pmatrix}A^2 & 0\\ BA+A^{\mathsf{T}}B & (A^{\mathsf{T}})^2\end{pmatrix}=\begin{pmatrix}0 & 0\\ BA+A^{\mathsf{T}}B & 0\end{pmatrix}=\frac{1}{4}\varepsilon_X\left(\frac{\partial f_1}{\partial x_2}\right)^2\begin{pmatrix}0&0&0&0\\0&0&0&0\\-\alpha_2^2&\alpha_1\alpha_2&0&0\\ \alpha_1\alpha_2&-\alpha_1^2&0&0\end{pmatrix},$$

and

$$\mathcal{J}_X{}^3=\begin{pmatrix}0 & 0\\ (BA+A^{\mathsf{T}}B)A & 0\end{pmatrix}=0,$$

for all $X\in TM$. Therefore, for all nonnull X, the condition

$$\frac{\partial f_1}{\partial x_2}=-\frac{\partial f_2}{\partial x_1}\neq 0$$

implies $\mu_X(\lambda)=\lambda^3$, which allows new concrete examples.

Example 8.6. For $f_1(x_1, x_2) = x_2, f_2(x_1, x_2) = -x_1$, and $a = 1$ hold

$$\frac{\partial f_1}{\partial x_2} = -\frac{\partial f_2}{\partial x_1} = 1 \neq 0.$$

For a nonnull X we have $\mu_X(\lambda) = \lambda^3$ at each point of M, which gives another example of a globally Jordan-Osserman manifold. △

Example 8.7. For $f_1(x_1, x_2) = x_1 x_2, f_2(x_1, x_2) = -\frac{1}{2}x_1^2$, and $a = 1$ hold

$$\frac{\partial f_1}{\partial x_2} = -\frac{\partial f_2}{\partial x_1} = x_1, \quad 2x_3\frac{\partial^2 f_1}{\partial {x_2}^2} + 2x_4\frac{\partial^2 f_2}{\partial {x_1}^2} - af_1 f_2 = -2x_4 + \frac{1}{2}x_1^3 x_2.$$

Here, $\mu_X(\lambda)$ for a nonnull X varies from point to point. It can be λ^3 (for points with $x_1 \neq 0$), or λ^2 (for $x_1 = 0, x_4 \neq 0$), or λ (for $x_1 = x_4 = 0$). The constructed manifold is globally Osserman and pointwise Jordan-Osserman. △

8.3 Osserman algebraic curvature tensor

Let R be a spacelike Osserman algebraic curvature tensor on a scalar product space $(\mathcal{V}, g)$ of dimension n. The characteristic polynomial $\omega_X(\lambda) = \det(\lambda\,\mathbb{1} - \mathcal{J}_X)$ is independent of a unit spacelike X, so the coefficients of ω_X are constant, and according to (A.5) from Lemma A.17, there exist constants $C_1, C_2, \dots, C_n$ such that $\mathrm{Tr}(\mathcal{J}_X{}^j) = C_j$ holds for each $1 \le j \le n$ and any unit spacelike X. Evidently it works vice-versa, so R is spacelike Osserman if and only if such constants exist. Since

$$\mathrm{Tr}(\mathcal{J}_X)^j = \mathrm{Tr}\left(\sqrt{|\varepsilon_X|}^{2j}\left(\mathcal{J}_{X/\sqrt{|\varepsilon_X|}}\right)^j\right) = |\varepsilon_X|^j\,\mathrm{Tr}\left(\left(\mathcal{J}_{X/\sqrt{|\varepsilon_X|}}\right)^j\right),$$

our equations can be extended for a spacelike X to

$$\mathrm{Tr}(\mathcal{J}_X)^j = (\varepsilon_X)^j C_j, \tag{8.5}$$

for each $1 \le j \le n$.

The equation $\mathcal{R}(Z, X + tY)(X + tY) = \mathcal{R}(Z, X)X + t(\mathcal{R}(Z, X)Y + \mathcal{R}(Z, Y)X) + t^2\mathcal{R}(Z, Y)Y$ can be written using the polarized Jacobi operator as

$$\mathcal{J}_{X+tY} = \mathcal{J}_X + 2t\mathcal{J}(X, Y) + t^2\mathcal{J}_Y$$

for all $X, Y \in \mathcal{V}$ and $t \in \mathbb{R}$.

If X is timelike, $Y \in X^\perp$ is spacelike, and $|t| > \sqrt{-\varepsilon_X/\varepsilon_Y}$, then $\varepsilon_{X+tY} = \varepsilon_X + t^2\varepsilon_Y > 0$ which implies that $X + tY$ is spacelike. In that case for a spacelike Osserman R we have

$$\mathrm{Tr}((\mathcal{J}_X + 2t\mathcal{J}(X, Y) + t^2\mathcal{J}_Y)^j) = \mathrm{Tr}((\mathcal{J}_{X+tY})^j) = (\varepsilon_{X+tY})^j C_j = (\varepsilon_X + t^2\varepsilon_Y)^j C_j,$$

which yields

$$\sum_{i=0}^{2j}(\mathrm{Tr}(L_i))t^i = \sum_{i=0}^{j}\binom{j}{i}(\varepsilon_X)^{j-i}(\varepsilon_Y)^i C_j t^{2i}, \tag{8.6}$$

for some concrete linear operators $L_0, L_1, \dots, L_{2j}$. Since a polynomial of degree d has at most d roots, while we have infinitely many t such that (8.6) holds, the polynomials in (8.6) are equal, so their coefficients must also be equal. Evidently $L_0 = (\mathcal{J}_X)^j$ holds and therefore we have $\mathrm{Tr}(\mathcal{J}_X)^j = (\varepsilon_X)^j C_j$ for any timelike X.

In a similar way, it is easy to see that R is timelike Osserman if and only if there exist constants C_j such that (8.5) holds for each $1 \le j \le n$ and any timelike X. Thus, we have already proven that a spacelike Osserman R is timelike Osserman, while the converse can be obtained similarly. Therefore, spacelike Osserman and timelike Osserman are equivalent conditions in the case of indefinite g, which is originally proved by García-Río, Kupeli, Vázquez-Abal[4], and Vázquez-Lorenzo in 1999 [61].

Theorem 8.1. *An indefinite algebraic curvature tensor is spacelike Osserman if and only if it is timelike Osserman.*

Consequently, an algebraic curvature tensor is Osserman if and only if there exist constants $C_1, \dots, C_n$ such that $\operatorname{Tr}(\mathcal{J}_X)^j = (\varepsilon_X)^j C_j$ holds whenever X is nonnull. If we look at the coefficients of the characteristic polynomial,

$$\omega_X(\lambda) = \det(\lambda \mathbb{1} - \mathcal{J}_X) = \lambda^n + \sigma_1 \lambda^{n-1} + \cdots + \sigma_{n-1}\lambda + \sigma_n,$$

then the equation (A.5) from Lemma A.17 gives

$$m\sigma_m + \sigma_{m-1} \operatorname{Tr}(\mathcal{J}_X) + \sigma_{m-2} \operatorname{Tr}(\mathcal{J}_X)^2 + \cdots + \sigma_1 \operatorname{Tr}(\mathcal{J}_X)^{m-1} + \operatorname{Tr}(\mathcal{J}_X)^m = 0,$$

for each $1 \le m \le n$. After the substitution we have

$$\frac{\sigma_m}{(\varepsilon_X)^m} = -\frac{1}{m}\left(\frac{\sigma_{m-1}}{(\varepsilon_X)^{m-1}} C_1 + \frac{\sigma_{m-2}}{(\varepsilon_X)^{m-2}} C_2 + \cdots + \frac{\sigma_1}{\varepsilon_X} C_1 + C_m\right),$$

and by induction $\sigma_m/(\varepsilon_X)^m$ are constant for $1 \le m \le n$. Hence,

$$\det\left(\lambda \mathbb{1} - \frac{1}{\varepsilon_X}\mathcal{J}_X\right) = \frac{\det(\varepsilon_X \lambda \mathbb{1} - \mathcal{J}_X)}{(\varepsilon_X)^n} = \frac{\omega_X(\varepsilon_X \lambda)}{(\varepsilon_X)^n} = \lambda^n + \frac{\sigma_1}{\varepsilon_X}\lambda^{n-1} + \cdots + \frac{\sigma_{n-1}}{(\varepsilon_X)^{n-1}}\lambda + \frac{\sigma_n}{(\varepsilon_X)^n}$$

is a polynomial with constant coefficients which proves the following theorem.

Theorem 8.2. *An algebraic curvature tensor is Osserman if and only if the polynomial* $\det(\lambda \mathbb{1} - \mathcal{J}_X/\varepsilon_X)$ *is independent of a nonnull vector X.*

Thanks to Theorem 8.2, we no longer need to look at the Osserman condition through two independent constancy of different polynomials, one for spacelike unit and the other for timelike unite vectors, but we united all this through a unique polynomial that is constant for each definite vector.

So far, (8.5) holds for a nonnull X, and obviously for $X = 0$. Any null X, by Lemma 4.10, can be decomposed as $X = S + T$, with $\varepsilon_S = -\varepsilon_T > 0$. For $n > 2$ there exists a nonnull $Y \in \operatorname{Span}\{S, T\}^\perp$. Because of $Y \perp X$ we have $\varepsilon_{X+tY} = g(X + tY, X + tY) = t^2 \varepsilon_Y \neq 0$ for $t \neq 0$. According to Lemma 6.5, the entries of the matrix $\mathcal{J}_X$ are homogeneous polynomials of degree 2 in coefficients of X, so $\operatorname{Tr}(\mathcal{J}_{X+tY})^j$ is a polynomial of degree $2j$, and thus continuous by t. If t decreases to zero, then we obtain $\operatorname{Tr}(\mathcal{J}_{X+tY})^j = (\varepsilon_{X+tY})^j C_j = t^{2j} (\varepsilon_Y)^j C_j$ which implies

$$\operatorname{Tr}(\mathcal{J}_X)^j = \lim_{t \searrow 0} \operatorname{Tr}(\mathcal{J}_{X+tY})^j = \lim_{t \searrow 0} t^{2j} (\varepsilon_Y)^j C_j = 0.$$

Hence $\operatorname{Tr}(\mathcal{J}_X)^j = 0 = (\varepsilon_X)^j C_j$ holds, which extends the equation (8.5) for all $X \in \mathcal{V}$, and motivate us to introduce the following definition.

An algebraic curvature tensor R on a scalar product space $(\mathcal{V}, g)$ is called ***k-stein*** if there exist constants $C_1, \dots, C_k$ such that

$$\operatorname{Tr}(\mathcal{J}_X)^j = (\varepsilon_X)^j C_j \qquad \text{(8.5 revisited)}$$

holds for each $1 \le j \le k$ and any $X \in \mathcal{V}$. The previous calculations prove the following theorem (see Gilkey [54, Lemma 1.7.3]).

[4] María Elena Vázquez-Abal, Spanish mathematician

Theorem 8.3. *An algebraic curvature tensor of dimension n is Osserman if and only if it is n-stein.*

As a consequence, an Osserman algebraic curvature tensor has $\operatorname{Tr}(\mathcal{J}_X)^j = 0$ for a null X and each $1 \leq j \leq n$, which implies $\sigma_1 = \sigma_2 = \cdots = \sigma_n = 0$ for the coefficients of ω_X. In light of this $\omega_X(\lambda) = \det(\lambda \mathbb{1} - \mathcal{J}_X) = \lambda^n$ holds for all null X and we have the following theorem (see Gilkey [54, Lemma 1.7.4]).

Theorem 8.4. *If R is an Osserman algebraic curvature tensor, then all eigenvalues of $\mathcal{J}_X$ for a null X are equal to zero.*

8.4 Einstein, zwei-stein, ...

An Osserman algebraic curvature tensor R on a scalar product space $(\mathcal{V}, g)$ of dimension $n \geq 3$ is n-stein. Let us recall the previous idea starting with $\mathcal{J}_{X+tY}$, where the k-stein condition implies that

$$\operatorname{Tr}(\mathcal{J}_X + 2t\mathcal{J}(X,Y) + t^2\mathcal{J}_Y)^j = \operatorname{Tr}(\mathcal{J}_{X+tY})^j = (\varepsilon_{X+tY})^j C_j = (\varepsilon_X + 2tg(X,Y) + t^2\varepsilon_Y)^j C_j \tag{8.7}$$

holds for each $1 \leq j \leq k$, and all $X, Y \in \mathcal{V}$, $t \in \mathbb{R}$. The simplest case is $k = 1$, which yields the equal polynomials,

$$\operatorname{Tr}(\mathcal{J}_X) + 2t\operatorname{Tr}(\mathcal{J}(X,Y)) + t^2\operatorname{Tr}(\mathcal{J}_Y) = \varepsilon_X C_1 + 2tC_1 g(X,Y) + t^2\varepsilon_Y C_1,$$

and therefore $C_1 g(X,Y) = \operatorname{Tr}(\mathcal{J}(X,Y)) = \operatorname{Ric}(X,Y)$, which gives $\operatorname{Ric} = C_1 g$. Hence, an algebraic curvature tensor is Einstein if and only if it is 1-stein, which is the reason why Carpenter[5], Gray[6], and Willmore[7] [33] wittily choose the name k-stein for a generalisation. As the name of the famous physicist Einstein means one-stone in German, by generalisation we get zwei-stein (two stones), drei-stein (three stones), and so on.

Consider an orthonormal basis $(E_1, E_2, \ldots, E_n)$ in $(\mathcal{V}, g)$, where we introduce the shortcuts $\varepsilon_i = \varepsilon_{E_i} \in \{-1, 1\}$ for $1 \leq i \leq n$. Using the equality (6.31), the Einstein condition gives $C_1 g(X,Y) = \operatorname{Ric}(X,Y) = \sum_{i=1}^n \varepsilon_i R(E_i, X, Y, E_i)$, and therefore $\sum_{i=1}^n \varepsilon_i R_{ixyi} = C_1 \varepsilon_x \delta_{xy}$ holds for all $1 \leq x, y \leq n$, where $R_{ijkl} = R(E_i, E_j, E_k, E_l)$ for $1 \leq i, j, k, l \leq n$ are components of R. Hence, for an Einstein R we have

$$\sum_{1 \leq i \leq n} \varepsilon_i \varepsilon_x R_{ixxi} = \text{Const} = C_1, \tag{8.8}$$

$$\sum_{1 \leq i \leq n} \varepsilon_i R_{ixyi} = 0. \tag{8.9}$$

Theorem 8.5. *If R is an Einstein algebraic curvature tensor of dimension n, then in an orthonormal basis the equalities* (8.8) *and* (8.9) *hold for all* $1 \leq x \neq y \leq n$.

Let us recall the equation (8.7) and consider the next simplest case $k = 2$. Without loss of generality, we assume $X \perp Y$, so the equation becomes

$$\operatorname{Tr}(\mathcal{J}_X + 2t\mathcal{J}(X,Y) + t^2\mathcal{J}_Y)^2 = \operatorname{Tr}(\mathcal{J}_{X+tY})^2 = (\varepsilon_{X+tY})^2 C_2 = (\varepsilon_X + t^2\varepsilon_Y)^2 C_2.$$

The equal polynomials have the same coefficients, so comparing the coefficients of 1, t, and t^2 we obtain

$$\begin{aligned} \operatorname{Tr}(\mathcal{J}_X)^2 &= (\varepsilon_X)^2 C_2, \\ \operatorname{Tr}(\mathcal{J}_X\mathcal{J}(X,Y)) + \operatorname{Tr}(\mathcal{J}(X,Y)\mathcal{J}_X) &= 0, \\ \operatorname{Tr}(\mathcal{J}_X\mathcal{J}_Y) + 4\operatorname{Tr}(\mathcal{J}(X,Y))^2 + \operatorname{Tr}(\mathcal{J}_Y\mathcal{J}_X) &= 2\varepsilon_X\varepsilon_Y C_2, \end{aligned}$$

[5] Paul Carpenter, British mathematician
[6] Alfred Gray (1939–1998), American mathematician
[7] Thomas James Willmore (1919–2005), English mathematician

while due to the symmetry between X and Y, the equations obtained by comparing the coefficients of t^3 and t^4 are redundant.

For an arbitrary endomorphism $\mathcal{J}$ on $\mathcal{V}$, from (4.3) follows $\mathcal{J}(E_q) = \sum_i \varepsilon_i g(\mathcal{J}(E_q), E_i)E_i$, so the matrix of $\mathcal{J}$ related to our basis has the entries $(\mathcal{J})_{pq} = \varepsilon_p g(\mathcal{J}(E_q), E_p)$ for $1 \le p, q \le n$. Additionally, for endomorphisms $\mathcal{J}$ and $\mathcal{K}$ holds $\mathrm{Tr}(\mathcal{J}\mathcal{K}) = \mathrm{Tr}(\mathcal{K}\mathcal{J})$, which allows us to calculate the traces of concrete endomorphisms,

$$\begin{aligned} C_2 = (\varepsilon_x)^2 C_2 = \mathrm{Tr}(\mathcal{J}_{E_x})^2 &= \sum_{i,j} (\mathcal{J}_{E_x})_{ij}(\mathcal{J}_{E_x})_{ji} = \sum_{i,j} \varepsilon_i g(\mathcal{J}_{E_x}(E_j), E_i)\varepsilon_j g(\mathcal{J}_{E_x}(E_i), E_j) \\ &= \sum_{i,j} \varepsilon_i R_{jxxi}\varepsilon_j R_{ixxj} = \sum_{i,j} \varepsilon_i\varepsilon_j (R_{ixxj})^2, \end{aligned}$$

$$\begin{aligned} 0 &= \mathrm{Tr}(\mathcal{J}_{E_x}\mathcal{J}(E_x, E_y)) + \mathrm{Tr}(\mathcal{J}(E_x, E_y)\mathcal{J}_{E_x}) = 2\,\mathrm{Tr}(\mathcal{J}_{E_x}\mathcal{J}(E_x, E_y)) \\ &= 2\sum_{i,j} \varepsilon_i R_{jxxi}\varepsilon_j \frac{1}{2}(R_{ixyj} + R_{iyxj}) = 2\sum_{i,j} \varepsilon_i\varepsilon_j R_{ixxj}R_{ixyj}, \end{aligned}$$

$$\begin{aligned} 2\varepsilon_x\varepsilon_y C_2 &= \mathrm{Tr}(\mathcal{J}_{E_x}\mathcal{J}_{E_y}) + 4\,\mathrm{Tr}(\mathcal{J}(E_x, E_y))^2 + \mathrm{Tr}(\mathcal{J}_{E_y}\mathcal{J}_{E_x}) = 2\,\mathrm{Tr}(\mathcal{J}_{E_x}\mathcal{J}_{E_y}) + 4\,\mathrm{Tr}(\mathcal{J}(E_x, E_y))^2 \\ &= 2\sum_{i,j} \varepsilon_i R_{jxxi}\varepsilon_j R_{iyyj} + 4\sum_{i,j} \varepsilon_i \frac{1}{2}(R_{jxyi} + R_{jyxi})\varepsilon_j \frac{1}{2}(R_{ixyj} + R_{iyxj}) \\ &= 2\sum_{i,j} \varepsilon_i\varepsilon_j R_{ixxj}R_{iyyj} + \sum_{i,j} \varepsilon_i\varepsilon_j (R_{ixyj} + R_{iyxj})^2. \end{aligned}$$

Hence, we obtain the equalities that are characteristic of a zwei-stein R,

$$\sum_{1\le i,j\le n} \varepsilon_i\varepsilon_j (R_{ixxj})^2 = \mathrm{Const} = C_2, \tag{8.10}$$

$$\sum_{1\le i,j\le n} \varepsilon_i\varepsilon_j R_{ixxj}R_{ixyj} = 0, \tag{8.11}$$

$$2\sum_{1\le i,j\le n} \varepsilon_i\varepsilon_j R_{ixxj}R_{iyyj} + \sum_{1\le i,j\le n} \varepsilon_i\varepsilon_j (R_{ixyj} + R_{iyxj})^2 = 2\varepsilon_x\varepsilon_y C_2. \tag{8.12}$$

Theorem 8.6. *If R is a zwei-stein algebraic curvature tensor of dimension n, then in an orthonormal basis* (8.8), (8.9), (8.10), (8.11), *and* (8.12) *hold for all* $1 \le x \ne y \le n$.

Let R be an Osserman algebraic curvature tensor on a scalar product space $(\mathcal{V}, g)$ of dimension n. By Theorem 8.3, R is n-stein. In light of this, constant eigenvalues (with multiplicities) of $\mathcal{J}_X$ additionally give constant $\mathrm{Tr}(\mathcal{J}_X)^k$ for $k > n$, which extends R to be k-stein for any $k \in \mathbb{N}$. Hence,

$$\mathrm{Tr}(\mathcal{J}_X + 2t\mathcal{J}(X, Y) + t^2\mathcal{J}_Y)^k = \mathrm{Tr}(\mathcal{J}_{X+tY})^k = (\varepsilon_{X+tY})^k C_k = (\varepsilon_X + t^2\varepsilon_Y)^k C_k$$

holds for all $k \in \mathbb{N}$, $t \in \mathbb{R}$, and mutually orthogonal $X, Y \in \mathcal{V}$. That recalls the equation (8.6),

$$\sum_{i=0}^{2k} t^i\,\mathrm{Tr}(L_i) = \sum_{i=0}^{k} \binom{k}{i} (\varepsilon_X)^{k-i}(\varepsilon_Y)^i C_k t^{2i},$$

which holds for concrete linear operators $L_0, L_1, \dots, L_{2k}$. Especially, $\mathrm{Tr}(L_i) = 0$ for an odd i, as well as

$$\mathrm{Tr}(L_{2p}) = \binom{k}{p} (\varepsilon_X)^{k-p}(\varepsilon_Y)^p C_k, \tag{8.13}$$

for all $0 \le p \le k$. We have already seen $L_0 = (\mathcal{J}_X)^k$ which implies $\mathrm{Tr}(\mathcal{J}_X)^k = (\varepsilon_X)^k C_k$. The next simplest is

$$L_1 = 2((\mathcal{J}_X)^{k-1}\mathcal{J}(X,Y) + (\mathcal{J}_X)^{k-2}\mathcal{J}(X,Y)\mathcal{J}_X + \cdots + \mathcal{J}(X,Y)(\mathcal{J}_X)^{k-1}).$$

Since

$$\sum_{i=0}^{k-1} \mathrm{Tr}((\mathcal{J}_X)^{k-1-i}\mathcal{J}(X,Y)(\mathcal{J}_X)^i) = \sum_{i=0}^{k-1} \mathrm{Tr}((\mathcal{J}_X)^i(\mathcal{J}_X)^{k-1-i}\mathcal{J}(X,Y)) = k\,\mathrm{Tr}((\mathcal{J}_X)^{k-1}\mathcal{J}(X,Y)),$$

we obtain

$$\mathrm{Tr}((\mathcal{J}_X)^{k-1}\mathcal{J}(X,Y)) = \frac{1}{2k}\mathrm{Tr}(L_1) = 0.$$

The general case, for $X = E_x$, $Y = E_y$ with $x \neq y$, is rather complicated,

$$\sum_{i_1,\dots,i_k} \varepsilon_{i_1}\cdots\varepsilon_{i_k} R_{i_2xxi_1} R_{i_3xxi_2}\cdots R_{i_kxxi_{k-1}} \frac{1}{2}(R_{i_1xyi_k} + R_{i_1yxi_k}) = 0. \tag{8.14}$$

If we assume that R is Jacobi-diagonalisable, or at least that $\mathcal{J}_{E_x}$ is diagonalisable, then there exists an orthonormal eigenbasis $(E_1,\dots,E_n)$ related to $\mathcal{J}_{E_x}$. In this basis $R_{pxxq} = 0$ holds for all $p \neq q$, so the general equality (8.14) has sense only for $i_1 = i_2 = \cdots = i_k$, and therefore

$$0 = \frac{1}{2}\sum_i (\varepsilon_i)^k (R_{ixxi})^{k-1}(R_{ixyi} + R_{iyxi}) = \sum_i \varepsilon_i(\varepsilon_i R_{ixxi})^{k-1} R_{ixyi}. \tag{8.15}$$

If we denote the sectional curvatures by $\kappa_{ij} = \varepsilon_i\varepsilon_j R_{ijji} = \kappa(E_i, E_j)$, then we have

$$\mathcal{J}_{E_x}E_i = \sum_p \varepsilon_p g(\mathcal{J}_{E_x}(E_i), E_p)E_p = \sum_p \varepsilon_p R_{ixxp}E_p = \varepsilon_i R_{ixxi}E_i = \varepsilon_x \kappa_{xi} E_i.$$

Let $\{\lambda_1, \lambda_2, \dots, \lambda_m\} = \{\lambda : \det(\varepsilon_x\lambda\,\mathbb{1} - \mathcal{J}_{E_x}) = 0\}$ be the spectrum of the operator $\varepsilon_x\mathcal{J}_{E_x}$, then for any $1 \le a \le m$ we can combine together such $1 \le i \le n$ that $\kappa_{xi} = \lambda_a$ holds. Multiplying (8.15) by $(\varepsilon_x)^{k-1}$ we have

$$0 = \sum_i \varepsilon_i(\varepsilon_i\varepsilon_x R_{ixxi})^{k-1} R_{ixyi} = \sum_{1\le a\le m}\sum_{\kappa_{xi}=\lambda_a} \varepsilon_i(\lambda_a)^{k-1}R_{ixyi} = \sum_{1\le a\le m}(\lambda_a)^{k-1}W_a,$$

where $W_a = \sum_{\kappa_{xi}=\lambda_a}\varepsilon_i R_{ixyi}$. The equalities $\sum_{a=1}^m (\lambda_a)^{k-1}W_a = 0$ for $1 \le k \le m$ can be written as

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_m^{m-1} \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ \vdots \\ W_m \end{pmatrix} = 0.$$

This system of equations has a well known Vandermonde[8] matrix, generated by distinct numbers from the spectrum, and therefore its determinant is

$$\Delta = V(\lambda_1,\dots,\lambda_m) = \prod_{1\le a<b\le m}(\lambda_b - \lambda_a) \neq 0.$$

Thus, our homogeneous system is invertible and it has the unique trivial solution, which gives $W_a = 0$ for all $1 \le a \le m$ and the following theorem holds.

[8] Alexandre-Théophile Vandermonde (1735–1796), French mathematician, musician and chemist

Theorem 8.7. *Let R be an Osserman algebraic curvature tensor and let $(E_1, \dots, E_n)$ be an orthonormal eigenbasis related to $\mathcal{J}_{E_x}$. Then for any $\lambda \in \mathbb{R}$ and $y \neq x$ holds*

$$\sum_{\kappa_{xi}=\lambda} \varepsilon_i R_{ixyi} = 0.$$

Let us keep the previous notations ($X = E_x$, $Y = E_y$ with $x \neq y$, $R_{pxxq} = 0$ for $p \neq q$) and consider the next simplest L_2. It is not hard to see that

$$L_2 = \sum_{p+q=k-1} (\mathcal{J}_X)^p \mathcal{J}_Y (\mathcal{J}_X)^q + 4 \sum_{p+q+r=k-2} (\mathcal{J}_X)^p \mathcal{J}(X,Y)(\mathcal{J}_X)^q \mathcal{J}(X,Y)(\mathcal{J}_X)^r = P + S.$$

We calculate the trace of $L_2 = P + S$. The trace of the first term is simple,

$$\operatorname{Tr} P = \sum_{p+q=k-1} \operatorname{Tr}\left((\mathcal{J}_X)^p \mathcal{J}_Y (\mathcal{J}_X)^q\right) = k \operatorname{Tr}((\mathcal{J}_X)^{k-1}\mathcal{J}_Y) = k(\varepsilon_x)^{k-1}\varepsilon_y \sum_{1\le i\le n} \kappa_{yi}(\kappa_{xi})^{k-1}.$$

However, the trace of the second term is rather complicated,

$$\begin{aligned}
\operatorname{Tr} S =& 4 \sum_{p+q+r=k-2} \operatorname{Tr}\left((\mathcal{J}_X)^p \mathcal{J}(X,Y)(\mathcal{J}_X)^q \mathcal{J}(X,Y)(\mathcal{J}_X)^r\right) \\
=& 4 \sum_{s+q=k-2} (s+1) \operatorname{Tr}((\mathcal{J}_X)^s \mathcal{J}(X,Y)(\mathcal{J}_X)^q \mathcal{J}(X,Y)) \\
=& 2k \sum_{p+q=k-2} \operatorname{Tr}((\mathcal{J}_X)^p \mathcal{J}(X,Y)(\mathcal{J}_X)^q \mathcal{J}(X,Y)) \\
=& \frac{1}{2}k \sum_{p+q=k-2} \sum_{i,j} (\varepsilon_i)^{p+1}(\varepsilon_j)^{q+1}(R_{ixxi})^p(R_{jxyi}+R_{jyxi})(R_{jxxj})^q(R_{ixyj}+R_{iyxj}) \\
=& \frac{1}{2}k(\varepsilon_x)^{k-2} \sum_{1\le i,j\le n} \varepsilon_i\varepsilon_j(R_{ixyj}+R_{iyxj})^2 \sum_{p+q=k-2} (\kappa_{xi})^p(\kappa_{xj})^q.
\end{aligned}$$

It is necessary to discuss cases, depending on whether κ_{xi} and κ_{xj} are equal,

$$\sum_{p+q=k-2} (\kappa_{xi})^p(\kappa_{xj})^q = \begin{cases} (k-1)(\kappa_{xi})^{k-2} & \text{if } \kappa_{xi} = \kappa_{xj} \\ \frac{(\kappa_{xi})^{k-1}}{\kappa_{xi}-\kappa_{xj}} + \frac{(\kappa_{xj})^{k-1}}{\kappa_{xj}-\kappa_{xi}} & \text{if } \kappa_{xi} \neq \kappa_{xj} \end{cases}.$$

Using the symmetry we have

$$\sum_{1\le i,j\le n} \varepsilon_i\varepsilon_j(R_{ixyj}+R_{iyxj})^2 \frac{(\kappa_{xi})^{k-1}}{\kappa_{xi}-\kappa_{xj}} = \sum_{1\le j,i\le n} \varepsilon_i\varepsilon_j(R_{ixyj}+R_{iyxj})^2 \frac{(\kappa_{xj})^{k-1}}{\kappa_{xj}-\kappa_{xi}},$$

and therefore

$$\begin{aligned}
\operatorname{Tr} S =& k(\varepsilon_x)^{k-2} \sum_{\kappa_{xi}\neq\kappa_{xj}} \varepsilon_i\varepsilon_j(R_{ixyj}+R_{iyxj})^2 \frac{(\kappa_{xi})^{k-1}}{\kappa_{xi}-\kappa_{xj}} \\
&+ \frac{1}{2}k(\varepsilon_x)^{k-2} \sum_{\kappa_{xi}=\kappa_{xj}} \varepsilon_i\varepsilon_j(R_{ixyj}+R_{iyxj})^2(k-1)(\kappa_{xi})^{k-2}.
\end{aligned}$$

On the other hand, the equality (8.13) for $p = 1$ yields

$$\operatorname{Tr}(L_2) = k(\varepsilon_X)^{k-1}\varepsilon_Y C_k = k\frac{\varepsilon_Y}{\varepsilon_X} \operatorname{Tr}(\mathcal{J}_X)^k = k(\varepsilon_x)^{k-1}\varepsilon_y \sum_{1\le i\le n} (\kappa_{xi})^k.$$

This equality, together with the previous calculations, after dividing by $k(\varepsilon_x)^{k-2}$, gives

$$\varepsilon_x\varepsilon_y \sum_{1\le i\le n} (\kappa_{xi})^k = \varepsilon_x\varepsilon_y \sum_{1\le i\le n} \kappa_{yi}(\kappa_{xi})^{k-1} + \sum_{\kappa_{xi}\neq\kappa_{xj}} \varepsilon_i\varepsilon_j(R_{ixyj}+R_{iyxj})^2 \frac{(\kappa_{xi})^{k-1}}{\kappa_{xi}-\kappa_{xj}}$$
$$+\frac{1}{2}\sum_{\kappa_{xi}=\kappa_{xj}} \varepsilon_i\varepsilon_j(R_{ixyj}+R_{iyxj})^2(k-1)(\kappa_{xi})^{k-2}.$$

We can combine together relate to the spectrum $\{\lambda_1,\dots,\lambda_m\}$ of $\varepsilon_x\mathcal{J}_{E_x}$ to get

$$\sum_{1\le a\le m} \lambda_a^{k-1}\left(\varepsilon_x\varepsilon_y \sum_{\kappa_{xi}=\lambda_a}(\kappa_{yi}-\lambda_a) + \sum_{b\neq a}\frac{1}{\lambda_a-\lambda_b}\sum_{\kappa_{xi}=\lambda_a}\sum_{\kappa_{xj}=\lambda_b}\varepsilon_i\varepsilon_j(R_{ixyj}+R_{iyxj})^2\right)$$
$$+\sum_{1\le a\le m}(k-1)\lambda_a^{k-2}\left(\frac{1}{2}\sum_{\kappa_{xi}=\lambda_a}\sum_{\kappa_{xj}=\lambda_a}\varepsilon_i\varepsilon_j(R_{ixyj}+R_{iyxj})^2\right) = 0.$$

If we set

$$U_a = \varepsilon_x\varepsilon_y \sum_{\kappa_{xi}=\lambda_a}(\kappa_{yi}-\lambda_a) + \sum_{b\neq a}\frac{1}{\lambda_a-\lambda_b}\sum_{\kappa_{xi}=\lambda_a}\sum_{\kappa_{xj}=\lambda_b}\varepsilon_i\varepsilon_j(R_{ixyj}+R_{iyxj})^2,$$
$$W_a = \frac{1}{2}\sum_{\kappa_{xi}=\lambda_a}\sum_{\kappa_{xj}=\lambda_a}\varepsilon_i\varepsilon_j(R_{ixyj}+R_{iyxj})^2,$$

then our equality becomes

$$\sum_{1\le a\le m}\lambda_a^{k-1}U_a + \sum_{1\le a\le m}(k-1)\lambda_a^{k-2}W_a = 0,$$

while for $1\le k\le 2m$ we have the matrix form as follows,

$$\begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \lambda_1 & \cdots & \lambda_m & 1 & \cdots & 1 \\ \lambda_1^2 & \cdots & \lambda_m^2 & 2\lambda_1 & \cdots & 2\lambda_m \\ \lambda_1^3 & \cdots & \lambda_m^3 & 3\lambda_1^2 & \cdots & 3\lambda_m^2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{2m-1} & \cdots & \lambda_m^{2m-1} & (2m-1)\lambda_1^{2m-2} & \cdots & (2m-1)\lambda_m^{2m-2} \end{pmatrix}\begin{pmatrix} U_1 \\ \vdots \\ U_m \\ W_1 \\ \vdots \\ W_m \end{pmatrix} = 0.$$

Since the $(m+a)$-th column is the derivative of the a-th column, our system has the determinant Δ derived from the Vandermonde determinant as

$$\Delta = \left.\frac{\partial^m V(\lambda_1,\dots,\lambda_m,\xi_1,\dots,\xi_m)}{\partial\xi_1\partial\xi_2\cdots\partial\xi_m}\right|_{\xi_1=\lambda_1,\dots,\xi_m=\lambda_m}.$$

On the other hand

$$V(\lambda_1,\dots,\lambda_m,\xi_1,\dots,\xi_m) = \prod_a(\xi_a-\lambda_a)\prod_{a<b}(\lambda_b-\lambda_a)\prod_{a<b}(\xi_b-\xi_a)\prod_{a<b}(\xi_b-\lambda_a)\prod_{a<b}(\xi_a-\lambda_b),$$

when after $\frac{\partial}{\partial\xi_a}\big|_{\xi_a=\lambda_a}$ survives just the term with differentiation of $(\xi_a-\lambda_a)$. Thus

$$\Delta = \prod_{a<b}(\lambda_b-\lambda_a)\prod_{a<b}(\lambda_b-\lambda_a)\prod_{a<b}(\lambda_b-\lambda_a)\prod_{a<b}(\lambda_a-\lambda_b)\neq 0,$$

and the system has the unique trivial solution $U_a = W_a = 0$ for all $1\le a\le m$. This leads to the following theorem given by Andrejić in 2006 [3] (see also [15]), which is a generalisation of the result given by Rakić[9] in 1998 [102] (see also [103]) that covers the definite case.

[9]Zoran Rakić (1964), Serbian mathematician

Theorem 8.8. *Let R be an Osserman algebraic curvature tensor and let $(E_1, \dots, E_n)$ be an orthonormal eigenbasis related to $\mathcal{J}_{E_x}$. Then for any $\lambda \in \mathbb{R}$ and $y \neq x$ holds*

$$\sum_{\kappa_{xi}=\lambda} \sum_{\kappa_{xj}=\lambda} \varepsilon_i \varepsilon_j (R_{ixyj} + R_{iyxj})^2 = 0.$$

8.5 Lorentzian zwei-stein

Let R be a Lorentzian timelike Osserman algebraic curvature tensor of dimension $n \geq 3$. Consider a unit timelike $T \in \mathcal{V}$. Since $T^\perp \leq \mathcal{V}$ is a positive definite subspace, by Lemma A.21 it has an orthonormal eigenbasis $(V_1, \dots, V_{n-1})$ related to $\widetilde{\mathcal{J}}_T$. Then, $\mathcal{J}_T V_i = \lambda_i V_i$ holds for $1 \leq i \leq n-1$ where $0, \lambda_1, \dots, \lambda_{n-1}$ are roots of the characteristic polynomial $\det(\lambda \mathbb{1} - \mathcal{J}_T)$ and therefore they are independent of T. Hence, for an arbitrary $X = \sum_{i=1}^{n-1} \alpha_i V_i \in T^\perp$ we obtain

$$\kappa(T, X) = \frac{g(\mathcal{J}_T(X), X)}{\varepsilon_T \varepsilon_X} = -\frac{\sum_{i=1}^{n-1} \alpha_i^2 \lambda_i}{\sum_{i=1}^{n-1} \alpha_i^2},$$

which implies the bounds

$$-\max_{1 \leq i \leq n-1} \lambda_i \leq \kappa(T, X) \leq -\min_{1 \leq i \leq n-1} \lambda_i.$$

The sectional curvature of indefinite planes is bounded, so Theorem 6.10 implies that κ is constant. García-Río, Kupeli, and Vázquez-Abal used this argument in 1997 [60] for the following theorem.

Theorem 8.9. *A Lorentzian timelike Osserman algebraic curvature tensor has constant sectional curvature.*

However, for Theorem 8.9 to hold, it is enough to assume that R is zwei-stein instead of Osserman. Let R be a zwei-stein algebraic curvature tensor on $(\mathcal{V}, g)$ of dimension n. For an arbitrary orthonormal basis and $1 \leq x \neq y \leq n$, using (8.12) and (8.10) from Theorem 8.6 we have

$$2 \sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j R_{ixxj} R_{iyyj} + \sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j (R_{ixyj} + R_{iyxj})^2 = \varepsilon_x \varepsilon_y \sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j ((R_{ixxj})^2 + (R_{iyyj})^2).$$

Hence

$$\sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j \left((R_{ixyj} + R_{iyxj})^2 - \varepsilon_x \varepsilon_y (\varepsilon_x R_{ixxj} - \varepsilon_y R_{iyyj})^2 \right) = 0,$$

which for $\varepsilon_x \varepsilon_y < 0$ becomes

$$\sum_{1 \leq i,j \leq n} \varepsilon_i \varepsilon_j \left((R_{ixyj} + R_{iyxj})^2 + (R_{ixxj} + R_{iyyj})^2 \right) = 0.$$

Let us assume additionally that $(\mathcal{V}, g)$ is Lorentzian ($\operatorname{Ind} g = 1$), where $\varepsilon_x \varepsilon_y < 0$ implies that $\varepsilon_i = 1$ holds for all $i \notin \{x, y\}$. We can split the previous sum into four parts,

$$\begin{aligned}
&\sum_{i,j \notin \{x,y\}} \varepsilon_i \varepsilon_j \left((R_{ixyj} + R_{iyxj})^2 + (R_{ixxj} + R_{iyyj})^2 \right) \\
&+ \sum_{j \notin \{x,y\}} \left(\varepsilon_x \varepsilon_j ((R_{xyxj})^2 + (R_{xyyj})^2) + \varepsilon_y \varepsilon_j ((R_{yxyj})^2 + (R_{yxxj})^2) \right) \\
&+ \sum_{i \notin \{x,y\}} \left(\varepsilon_i \varepsilon_x ((R_{ixyx})^2 + (R_{iyyx})^2) + \varepsilon_i \varepsilon_y ((R_{iyxy})^2 + (R_{ixxy})^2) \right) \\
&+ \left(\varepsilon_x \varepsilon_x (R_{xyyx})^2 + \varepsilon_x \varepsilon_y (R_{xyxy})^2 + \varepsilon_y \varepsilon_x (R_{yxyx})^2 + \varepsilon_y \varepsilon_y (R_{yxxy})^2 \right) = 0,
\end{aligned}$$

where all parts except the first one disappearing. Therefore

$$\sum_{i,j\notin\{x,y\}} \Big((R_{ixyj}+R_{iyxj})^2+(R_{ixxj}+R_{iyyj})^2\Big)=0,$$

so $R_{ixyj}+R_{iyxj}=0$ and $R_{ixxj}+R_{iyyj}=0$ hold for all $i,j\notin\{x,y\}$. Especially, for $i=j$ we have the same sectional curvatures

$$\kappa(E_i,E_x)=\kappa_{ix}=\varepsilon_i\varepsilon_x R_{ixxi}=\varepsilon_i\varepsilon_y R_{iyyi}=\kappa_{iy}=\kappa(E_i,E_y),$$

whenever i,x,y are different with $\varepsilon_x\varepsilon_y=-1$, for example $\varepsilon_x=-1$.

In order to complete the proof we use the following argument given by Andrejić in 2018 [9], that significantly shortens the proof, which was missed by the previous authors. Since, zwei-stein is Einstein by definition, for fixed x and $i\neq x$, the equality (8.8) gives

$$C_1=\kappa_{ix}+\sum_{y\neq i,x}\kappa_{iy}=\kappa_{ix}+\sum_{y\neq i,x}\kappa_{ix}=(n-1)\kappa_{ix},$$

and therefore $\kappa_{ix}=\kappa_{iy}=C_1/(n-1)$ for any orthonormal basis, so R has constant sectional curvature. In this way, we got the following theorem originally given by Blažić[10], Bokan[11], and Gilkey in 1997 [22].

Theorem 8.10. *Any Lorentzian zwei-stein algebraic curvature tensor has constant sectional curvature.*

8.6 Zwei-stein submanifolds

There is a vast literature dedicated to isometric immersions of Einstein manifolds into space forms. If an immersion has a large codimension, then great flexibility causes that there is no close link between the intrinsic and extrinsic geometry of the manifold. However, with a small enough codimension of an embedding, the links between the extrinsic geometry of the manifold and its curvature are strengthened.

The case of codimension one is well known from Fialkow[12] [46, Theorem 7.1], who gave the classification of Einstein hypersurfaces of space forms (see [41, Exercise 8.4]). It is interesting to investigate submanifolds of higher codimension in a space form, where it was considered that the merely Einstein condition is not good enough for some meaningful classification.

Many authors have set strong additional conditions to achieve some results. For example, minimal isometric immersions with codimension two of Einstein manifolds into space forms were classified by Matsuyama[13] [82] (see [41]). It is known that a minimal isometric immersion of Einstein manifolds into Euclidean space has homothetic Gauss map (its third fundamental form is a constant multiple of the metric) [50, Proposition 1], while in the case of codimension two it must be totally geodesic [50, Theorem 4].

One natural condition considers a flat normal bundle. We should mention that any proper isometric immersion with flat normal bundle of an Einstein manifold into a space form is locally holonomic [40, Theorem 1]. According to [36, Proposition 2], it is known that an Einstein submanifold of codimension two and dimension n in a space of constant curvature c has flat normal bundle in some special cases, namely for $\mathrm{Sc}\geq n(n-1)c$ unless $\mathrm{Sc}=n(n-1)c$ and all sectional curvatures are equal to c [41, Exercise 3.18].

[10] Novica Blažić (1959–2005), Serbian mathematician
[11] Neda Bokan (1947), Serbian mathematician
[12] Aaron Fialkow (1911-1999), American mathematician
[13] Yoshio Matsuyama, Japanese mathematician

Our main motivation comes from the work of Euh[14], Kim[15], Nikolayevsky, and Park[16] in 2022 [45]. They have proved that a connected zwei-stein submanifold with flat normal bundle in a space form has constant sectional curvature (Theorem 8.11). Additionally, they manage to prove that a connected zwei-stein submanifold of codimension two in a space form has flat normal bundle and therefore has constant sectional curvature (Theorem 8.12).

Let M be an Einstein submanifold in a space of constant curvature c. Let A^σ be the shape operators relative to an orthonormal basis ξ^σ in the normal space N_pM at a point $p \in M$. Because of $\mathrm{II}(X,Y) = \sum_\sigma g(\mathrm{II}(X,Y),\xi^\sigma)\xi^\sigma = \sum_\sigma g(A^\sigma X, Y)\xi^\sigma$, from the Gauss equation (7.9) we have

$$\mathcal{J}_X Y = c\varepsilon_X Y - cg(X,Y)X + \sum_\sigma g(A^\sigma X, X)A^\sigma Y - \sum_\sigma g(A^\sigma X, Y)A^\sigma X,$$

and therefore

$$\mathcal{J}_X = c(\varepsilon_X \mathbb{1} - X \otimes X^\flat) + \sum_\sigma (g(A^\sigma X, X)A^\sigma - (A^\sigma X) \otimes (A^\sigma X)^\flat).$$

Consider the new curvature tensor $R' = R - cR^1$, where R^1 is the curvature tensor of constant curvature 1. It just shifts the eigenvalues of the corresponding reduced Jacobi operator, so if R is k-stein for some positive integer k, then R' is also k-stein, where

$$\mathcal{J}'_X = \sum_\sigma (g(A^\sigma X, X)A^\sigma - (A^\sigma X) \otimes (A^\sigma X)^\flat).$$

Taking the trace, we obtain

$$\operatorname{Tr}\mathcal{J}'_X = \sum_\sigma (g(A^\sigma X, X)\operatorname{Tr}A^\sigma - (A^\sigma X)^\flat(A^\sigma X)) = \sum_\sigma (g(A^\sigma X, X)H^\sigma - g(A^\sigma X, A^\sigma X)),$$

where $H^\sigma = \operatorname{Tr}A^\sigma$, and since M is Einstein, there exists a constant C_1 such that

$$\operatorname{Tr}\mathcal{J}'_X = C_1\varepsilon_X = g\left(\sum_\sigma (H^\sigma A^\sigma - (A^\sigma)^2)X, X\right).$$

Since $\sum_\sigma (H^\sigma A^\sigma - (A^\sigma)^2) - C_1\mathbb{1}$ is self-adjoint, after the polarization, the nondegeneracy of scalar product implies

$$C_1\mathbb{1} = \sum_\sigma (H^\sigma A^\sigma - (A^\sigma)^2). \tag{8.16}$$

On the other hand, if we denote $T^{\sigma\tau} = \operatorname{Tr}(A^\sigma A^\tau)$, then

$$\operatorname{Tr}(\mathcal{J}'_X)^2 = \operatorname{Tr}\sum_\sigma (g(A^\sigma X, X)A^\sigma - (A^\sigma X) \otimes (A^\sigma X)^\flat)\sum_\tau (g(A^\tau X, X)A^\tau - (A^\tau X) \otimes (A^\tau X)^\flat)$$

implies

$$\begin{aligned}\operatorname{Tr}(\mathcal{J}'_X)^2 = \sum_{\sigma,\tau}\Big(&g(A^\sigma X, X)g(A^\tau X, X)T^{\sigma\tau} + g(A^\sigma X, A^\tau X)^2\\ &- g(A^\sigma X, X)g(A^\tau A^\sigma A^\tau X, X) - g(A^\tau X, X)g(A^\sigma A^\tau A^\sigma X, X)\Big).\end{aligned} \tag{8.17}$$

Theorem 8.11. *A connected Riemannian zwei-stein submanifold with flat normal bundle in a space form has constant curvature.*

[14] Yunhee Euh, South Korean mathematician
[15] Jihun Kim, South Korean mathematician
[16] JeongHyeong Park, South Korean mathematician

Proof. According to the Ricci equation (7.10) we have $R^{\perp}(X, Y, \xi^{\sigma}, \xi^{\tau}) = g([A^{\sigma}, A^{\tau}]X, Y)$, which means that the normal curvature operator $\mathcal{R}^{\perp}$ vanishes if all shape operators commutes. Hence, if the submanifold has flat normal connection then all the shape operators at any point are simultaneously diagonalisable relative to some orthonormal basis $(e_1, \dots e_n)$, which gives $A^{\sigma}e_i = \lambda_i^{\sigma}e_i$ for some $\lambda_i^{\sigma} \in \mathbb{R}$. Since M is zwei-stein, there exists a constant C_2 such that $\mathrm{Tr}(\mathcal{J}_X')^2 = C_2(\varepsilon_X)^2$.

If we apply the equality (8.16) on e_i, then we obtain

$$C_1 = \sum_{\sigma}(H^{\sigma}\lambda_i^{\sigma} - (\lambda_i^{\sigma})^2), \tag{8.18}$$

while the same substitution into the equality (8.17) gives

$$C_2 = \sum_{\sigma,\tau}(T^{\sigma\tau}\lambda_i^{\sigma}\lambda_i^{\tau} - (\lambda_i^{\sigma}\lambda_i^{\tau})^2). \tag{8.19}$$

On the other hand, if we substitute $X = e_i + e_j$ into (8.17), we have

$$\begin{aligned} 4C_2 = \sum_{\sigma,\tau} \Big(& T^{\sigma\tau}(\lambda_i^{\sigma} + \lambda_j^{\sigma})(\lambda_i^{\tau} + \lambda_j^{\tau}) + (\lambda_i^{\sigma}\lambda_i^{\tau} + \lambda_j^{\sigma}\lambda_j^{\tau})^2 \\ & - (\lambda_i^{\sigma} + \lambda_j^{\sigma})(\lambda_i^{\tau}\lambda_i^{\sigma}\lambda_i^{\tau} + \lambda_j^{\tau}\lambda_j^{\sigma}\lambda_j^{\tau}) - (\lambda_i^{\tau} + \lambda_j^{\tau})(\lambda_i^{\sigma}\lambda_i^{\tau}\lambda_i^{\sigma} + \lambda_j^{\sigma}\lambda_j^{\tau}\lambda_j^{\sigma})\Big), \end{aligned}$$

so if we subtract the equality (8.19) as well as the same equality for $i = j$, we obtain

$$2C_2 = \sum_{\sigma,\tau} \Big(T^{\sigma\tau}(\lambda_i^{\sigma}\lambda_j^{\tau} + \lambda_j^{\sigma}\lambda_i^{\tau}) + 2\lambda_i^{\sigma}\lambda_i^{\tau}\lambda_j^{\sigma}\lambda_j^{\tau} - (\lambda_i^{\sigma}\lambda_j^{\tau}\lambda_j^{\sigma}\lambda_j^{\tau} + \lambda_j^{\sigma}\lambda_i^{\tau}\lambda_i^{\sigma}\lambda_i^{\tau}) - (\lambda_i^{\tau}\lambda_j^{\sigma}\lambda_j^{\tau}\lambda_j^{\sigma} + \lambda_j^{\tau}\lambda_i^{\sigma}\lambda_i^{\tau}\lambda_i^{\sigma})\Big),$$

while the symmetry by σ and τ (because of $T^{\sigma\tau} = T^{\tau\sigma}$) gives

$$C_2 = \sum_{\sigma,\tau} \Big(T^{\sigma\tau}\lambda_i^{\sigma}\lambda_j^{\tau} + \lambda_i^{\sigma}\lambda_j^{\sigma}\lambda_i^{\tau}\lambda_j^{\tau} - \lambda_i^{\sigma}\lambda_j^{\sigma}(\lambda_j^{\tau})^2 - \lambda_j^{\sigma}\lambda_i^{\sigma}(\lambda_i^{\tau})^2 \Big).$$

Hence, since $\sum_j \lambda_j^{\sigma} = \mathrm{Tr}\, A^{\sigma} = H^{\sigma}$ and $\sum_j \lambda_j^{\sigma}\lambda_j^{\tau} = \mathrm{Tr}\, A^{\sigma}A^{\tau} = T^{\sigma\tau}$, using (8.18) we have

$$\sum_{\sigma,\tau,j} \lambda_i^{\sigma}\lambda_j^{\sigma}(\lambda_j^{\tau})^2 = \sum_{\sigma,j} \lambda_i^{\sigma}\lambda_j^{\sigma} \sum_{\tau}(\lambda_j^{\tau})^2 = \sum_{\sigma,j,\tau} H^{\tau}\lambda_i^{\sigma}\lambda_j^{\sigma}\lambda_j^{\tau} - C_1 \sum_{\sigma,j} \lambda_i^{\sigma}\lambda_j^{\sigma} = \sum_{\sigma,\tau} H^{\tau}T^{\sigma\tau}\lambda_i^{\sigma} - C_1 \sum_{\sigma} H^{\sigma}\lambda_i^{\sigma},$$

$$\sum_{\sigma,\tau,j} \lambda_i^{\sigma}\lambda_j^{\sigma}(\lambda_i^{\tau})^2 = \sum_{\sigma,j} \lambda_i^{\sigma}\lambda_j^{\sigma} \sum_{\tau}(\lambda_i^{\tau})^2 = \sum_{\sigma,j,\tau} H^{\tau}\lambda_i^{\sigma}\lambda_j^{\sigma}\lambda_i^{\tau} - C_1 \sum_{\sigma,j} \lambda_i^{\sigma}\lambda_j^{\sigma} = \sum_{\sigma,\tau} H^{\sigma}H^{\tau}\lambda_i^{\sigma}\lambda_i^{\tau} - C_1 \sum_{\sigma} H^{\sigma}\lambda_i^{\sigma},$$

and therefore

$$\sum_j C_2 = \sum_{\sigma,\tau} \Big(H^{\tau}T^{\sigma\tau}\lambda_i^{\sigma} + T^{\sigma\tau}\lambda_i^{\sigma}\lambda_i^{\tau} - H^{\tau}T^{\sigma\tau}\lambda_i^{\sigma} - H^{\sigma}H^{\tau}\lambda_i^{\sigma}\lambda_i^{\tau} \Big) + 2C_1 \sum_{\sigma} H^{\sigma}\lambda_i^{\sigma},$$

which yields

$$nC_2 = \sum_{\sigma,\tau}(T^{\sigma\tau} - H^{\sigma}H^{\tau})\lambda_i^{\sigma}\lambda_i^{\tau} + 2C_1 \sum_{\sigma} H^{\sigma}\lambda_i^{\sigma}.$$

Subtracting the equality (8.19) and using (8.18) we obtain

$$\begin{aligned} (n-1)C_2 &= 2C_1 \sum_{\sigma} H^{\sigma}\lambda_i^{\sigma} - \sum_{\sigma,\tau}(H^{\sigma}H^{\tau}\lambda_i^{\sigma}\lambda_i^{\tau} - (\lambda_i^{\sigma}\lambda_i^{\tau})^2) \\ &= 2C_1(C_1 + \sum_{\sigma}(\lambda_i^{\sigma})^2) - (C_1 + \sum_{\sigma}(\lambda_i^{\sigma})^2)(C_1 + \sum_{\tau}(\lambda_i^{\tau})^2) + \sum_{\sigma,\tau}(\lambda_i^{\sigma}\lambda_i^{\tau})^2 = (C_1)^2. \end{aligned}$$

However, if $0, \lambda_1^X, \dots, \lambda_{n-1}^X$ are (real) eigenvalues of $\mathcal{J}'_X$, then the arithmetic-quadratic mean inequality gives

$$(C_1 \varepsilon_X)^2 = (\mathrm{Tr}\, \mathcal{J}'_X)^2 = \Big(\sum_{i=1}^{n-1} \lambda_i^X \Big)^2 \leq (n-1) \sum_{i=1}^{n-1} (\lambda_i^X)^2 = (n-1)\, \mathrm{Tr}(\mathcal{J}'_X)^2 = (n-1) C_2 (\varepsilon_X)^2,$$

and consequently $\lambda_1^X = \dots = \lambda_{n-1}^X$, which means that R' has constant curvature $C_1/(n-1)$. Hence,

$$\mathcal{J}_X = \left(c + \frac{C_1}{n-1} \right) (\varepsilon_X \mathbb{1} - X \otimes X^\flat),$$

and consequently R has constant curvature. $\square$

Suppose that M has dimension n and codimension 2. Let $(e_1^1, \dots, e_n^1)$ and $(e_1^2, \dots, e_n^2)$ be the orthonormal bases that diagonalize the self-adjoint operators A^1 and A^2. In other words, there are $\lambda_i^1, \lambda_i^2 \in \mathbb{R}$ such that $A^1 e_i^1 = \lambda_i^1 e_i^1$ and $A^2 e_i^2 = \lambda_i^2 e_i^2$ for each $1 \leq i \leq n$. For an arbitrary unit vector $X = \sum_i \alpha_i^1 e_i^1 = \sum_i \alpha_i^2 e_i^2$, the Einstein condition (8.16) gives

$$C_1 = \sum_i (\alpha_i^1)^2 \lambda_i^1 (H^1 - \lambda_i^1) + \sum_i (\alpha_i^2)^2 \lambda_i^2 (H^2 - \lambda_i^2),$$

where $1 = \sum_i (\alpha_i^1)^2 = \sum_i (\alpha_i^2)^2$.

Starting with $m^1 = \min_i \lambda_i^1 (H^1 - \lambda_i^1)$ and $m^2 = \max_i \lambda_i^2 (H^2 - \lambda_i^2)$ we can easily conclude $C_1 = m^1 + m^2$. However, $\lambda_i^\sigma (H^\sigma - \lambda_i^\sigma) = \lambda_j^\sigma (H^\sigma - \lambda_j^\sigma)$ holds for $\lambda_i^\sigma \neq \lambda_j^\sigma$ if and only if $\lambda_i^\sigma + \lambda_j^\sigma = H^\sigma$. If we denote

$$\mathcal{M}_\lambda^\sigma = \mathrm{Ker}(A^\sigma - \lambda \mathbb{1}),$$

then it is easy to see that for $\lambda_x^1 (H^1 - \lambda_x^1) = m^1$ and $\lambda_y^2 (H^2 - \lambda_y^2) = m^2$ we have

$$\mathcal{M}^1_{\lambda_x^1} + \mathcal{M}^1_{H^1 - \lambda_x^1} = \mathrm{Ker}((A^\sigma)^2 - H^\sigma A^\sigma + m^\sigma \mathbb{1}) = \mathcal{M}^2_{\lambda_y^2} + \mathcal{M}^2_{H^2 - \lambda_y^2},$$

and we can repeat the process for the spectral decomposition $T_p M = \bigoplus_\lambda \mathcal{M}_\lambda^1 = \bigoplus_\lambda \mathcal{M}_\lambda^2$.

Hence A^1 and A^2 are simultaneously block-diagonal. Moreover, we can order the bases such that all the blocks from the diagonal being of sizes either 1×1 or 2×2. For example, in any large block we can find a unit $v \in \mathcal{M}^1_{\lambda_x^1}$ such that

$$w = A^2 v - g(A^2 v, v) v \in \mathcal{M}^1_{H^1 - \lambda_x^1},$$

where $A^2 A^2 v = H^2 A^2 v - m^2 v$ implies that A^2 is invariant on $\mathrm{Span}\{v, w\}$, so we can extract either 1×1 (if $w = 0$) or 2×2 (if $w \neq 0$) block, and repeat the process.

If the shape operators A^1 and A^2 are not simultaneously diagonalisable, then there exists a 2×2 block, so without loss of generality we have $A^\sigma e_\tau^\sigma = \lambda_\tau^\sigma e_\tau^\sigma$ for $1 \leq \sigma, \tau \leq 2$, where $\mathrm{Span}\{e_1^1, e_2^1\} = \mathrm{Span}\{e_1^2, e_2^2\}$ and $\lambda_1^\sigma + \lambda_2^\sigma = H^\sigma$. If we include the transition map between bases we get the restrictions of the shape operators related to the basis (e_1^1, e_2^1) as blocks

$$\begin{pmatrix} \lambda_1^1 & 0 \\ 0 & \lambda_2^1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c^2 \lambda_1^2 + s^2 \lambda_2^2 & sc(\lambda_2^2 - \lambda_1^2) \\ sc(\lambda_2^2 - \lambda_1^2) & s^2 \lambda_1^2 + c^2 \lambda_2^2 \end{pmatrix},$$

where $s = \sin t, c = \cos t$ for some $t \in \mathbb{R}$. If we denote

$$\alpha = \frac{\lambda_1^1 - \lambda_2^1}{2}, \quad \beta = (c^2 - s^2) \frac{\lambda_1^2 - \lambda_2^2}{2}, \quad \gamma = sc(\lambda_2^2 - \lambda_1^2),$$

then our blocks are equivalent to

$$\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} + \frac{1}{2}H^1 \mathbb{1} \quad \text{and} \quad \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix} + \frac{1}{2}H^2 \mathbb{1}\,.$$

Without loss of generality, we can suppose $H^2 = 0$ by rotating the orthonormal basis (ξ^1, ξ^2) in the normal space. Hence, if we denote $2h = H^1$ then our blocks become

$$\begin{pmatrix} h+\alpha & 0 \\ 0 & h-\alpha \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix},$$

where $\alpha\gamma \neq 0$ and $C_1 = h^2 - \alpha^2 - \beta^2 - \gamma^2$. The previous result is in fact equivalent to the Lemma from [45].

It is not hard to calculate $T^{11} + T^{22}$, where any 2×2 block yields $2(h^2+\alpha^2)+2(\beta^2+\gamma^2) = 4h^2 - 2C_1$, while any 1×1 block, with $A^1 e_i = \mu e_i$ and $A^2 e_i = \nu e_i$, yields $\mu^2 + \nu^2 = 2h\mu - C_1$, so the total amount is equal to $2h \operatorname{Tr} A^1 - nC_1$, and therefore

$$T^{11} + T^{22} = 4h^2 - nC_1. \tag{8.20}$$

Let us start with the equality (8.17). If we set $X = xe_1^1 + ye_2^1$, then

$$\begin{aligned} \operatorname{Tr}(\mathcal{J}_X')^2 =& ((h+\alpha)x^2 + (h-\alpha)y^2)^2 T^{11} + ((h+\alpha)^2x^2 + (h-\alpha)^2y^2)^2 \\ & - 2((h+\alpha)x^2 + (h-\alpha)y^2)((h+\alpha)^3x^2 + (h-\alpha)^3y^2) \\ & + 2((h+\alpha)x^2 + (h-\alpha)y^2)(\beta x^2 + 2\gamma xy - \beta y^2)T^{12} \\ & + 2((h+\alpha)\beta x^2 + 2h\gamma xy - (h-\alpha)\beta y^2)^2 \\ & - 2((h+\alpha)x^2 + (h-\alpha)y^2)(((h+\alpha)\beta^2 + (h-\alpha)\gamma^2)x^2 \\ & + 4\alpha\beta\gamma xy + ((h+\alpha)\gamma^2 + (h-\alpha)\beta^2)y^2) \\ & - 2(\beta x^2 + 2\gamma xy - \beta y^2)((h+\alpha)^2\beta x^2 + 2(h^2-\alpha^2)\gamma xy - (h-\alpha)^2\beta y^2) \\ & + (\beta x^2 + 2\gamma xy - \beta y^2)^2 T^{22} + (\beta^2+\gamma^2)^2(x^2+y^2)^2 \\ & - 2(\beta x^2 + 2\gamma xy - \beta y^2)(\beta^2+\gamma^2)(\beta x^2 + 2\gamma xy - \beta y^2) \end{aligned}$$

holds for all $x, y \in \mathbb{R}$. The trace of square is of the form $\operatorname{Tr}(\mathcal{J}_X')^2 = Ax^4 + Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4$, and we can calculate the coefficients $A, B, C, D, E \in \mathbb{R}$.

The coefficients of x^3y and xy^3 are

$$\begin{aligned} B =& 4(h+\alpha)\gamma T^{12} + 8(h+\alpha)h\beta\gamma - 8(h+\alpha)\alpha\beta\gamma \\ & - 4(h^2-\alpha^2)\beta\gamma - 4(h+\alpha)^2\beta\gamma + 4\beta\gamma T^{22} - 8(\beta^2+\gamma^2)\beta\gamma, \\ D =& 4(h-\alpha)\gamma T^{12} - 8(h-\alpha)h\beta\gamma - 8(h-\alpha)\alpha\beta\gamma \\ & + 4(h^2-\alpha^2)\beta\gamma + 4(h-\alpha)^2\beta\gamma - 4\beta\gamma T^{22} + 8(\beta^2+\gamma^2)\beta\gamma, \end{aligned}$$

and therefore

$$\begin{aligned} B/4 &= (h+\alpha)\gamma T^{12} + \beta\gamma T^{22} - 2h\alpha\beta\gamma - 2(\beta^2+\gamma^2+\alpha^2)\beta\gamma, \\ D/4 &= (h-\alpha)\gamma T^{12} - \beta\gamma T^{22} - 2h\alpha\beta\gamma + 2(\beta^2+\gamma^2+\alpha^2)\beta\gamma, \end{aligned}$$

which yields

$$\frac{B+D}{8\gamma} = h(T^{12} - 2\alpha\beta), \tag{8.21}$$

$$\frac{B-D}{8\gamma} = \alpha(T^{12} - 2\alpha\beta) + \beta(T^{22} - 2(\beta^2+\gamma^2)). \tag{8.22}$$

The coefficients of x^4 and y^4 are

$$\begin{aligned}A =&(h+\alpha)^2T^{11} + (h+\alpha)^4 - 2(h+\alpha)^4\\&+2(h+\alpha)\beta T^{12} + 2(h+\alpha)^2\beta^2 - 2(h+\alpha)((h+\alpha)\beta^2 + (h-\alpha)\gamma^2) - 2(h+\alpha)^2\beta^2\\&+\beta^2T^{22} + (\beta^2+\gamma^2)^2 - 2(\beta^2+\gamma^2)\beta^2,\\E =&(h-\alpha)^2T^{11} + (h-\alpha)^4 - 2(h-\alpha)^4\\&-2(h-\alpha)\beta T^{12} + 2(h-\alpha)^2\beta^2 - 2(h-\alpha)((h+\alpha)\gamma^2 + (h-\alpha)\beta^2) - 2(h-\alpha)^2\beta^2\\&+\beta^2T^{22} + (\beta^2+\gamma^2)^2 - 2(\beta^2+\gamma^2)\beta^2,\end{aligned}$$

and therefore

$$\begin{aligned}A =&(h+\alpha)^2T^{11} + 2(h+\alpha)\beta T^{12} + \beta^2T^{22} - (h+\alpha)^4 - 2(h^2-\alpha^2)\gamma^2 - 2(h+\alpha)^2\beta^2 + \gamma^4 - \beta^4,\\E =&(h-\alpha)^2T^{11} - 2(h-\alpha)\beta T^{12} + \beta^2T^{22} - (h-\alpha)^4 - 2(h^2-\alpha^2)\gamma^2 - 2(h-\alpha)^2\beta^2 + \gamma^4 - \beta^4,\end{aligned}$$

which yields

$$\frac{A+E}{2} = C_1^2 + (h^2+\alpha^2)(T^{11} - 2(h^2+\alpha^2)) + 2\alpha\beta(T^{12} - 2\alpha\beta) + \beta^2(T^{22} - 2(\beta^2+\gamma^2)). \quad (8.23)$$

$$\frac{A-E}{2} = 2h\alpha(T^{11} - 2(h^2+\alpha^2)) + 2h\beta(T^{12} - 2\alpha\beta). \quad (8.24)$$

Finally, the coefficient of x^2y^2 is

$$\begin{aligned}C =&2(h+\alpha)(h-\alpha)T^{11} + 2(h+\alpha)^2(h-\alpha)^2 - 2(h+\alpha)(h-\alpha)^3 - 2(h-\alpha)(h+\alpha)^3\\&+(-2(h+\alpha)\beta + 2(h-\alpha)\beta)T^{12} - 4(h+\alpha)(h-\alpha)\beta^2 + 8h^2\gamma^2\\&-2(h+\alpha)((h+\alpha)\gamma^2 + (h-\alpha)\beta^2) - 2(h-\alpha)((h+\alpha)\beta^2 + (h-\alpha)\gamma^2) + 2(h-\alpha)^2\beta^2\\&+2(h+\alpha)^2\beta^2 - 8(h^2-\alpha^2)\gamma^2 + (-2\beta^2+4\gamma^2)T^{22} + 2(\beta^2+\gamma^2)^2 - 2(\beta^2+\gamma^2)(-2\beta^2+4\gamma^2),\end{aligned}$$

and therefore

$$\frac{C}{2} = (h^2-\alpha^2)T^{11} - 2\alpha\beta T^{12} + (2\gamma^2-\beta^2)T^{22} + C_1^2 - 2h^4 + 2\alpha^4 - 4\gamma^4 + \beta^2(4\alpha^2 + 2\beta^2 - 2\gamma^2).$$

Together with (8.23) it gives $(A+E+C)/4 = h^2T^{11} + \gamma^2T^{22} + C_1^2 - 2h^2(h^2+\alpha^2) - 2\gamma^2(\beta^2+\gamma^2)$ which can be written as

$$\frac{A+E+C}{4} = C_1^2 + 2h^2\alpha^2 + h^2(T^{11} - 2(h^2+\alpha^2)) + \gamma^2(T^{22} - 2(\beta^2+\gamma^2)). \quad (8.25)$$

If M is zwei-stein, then there exists a constant C_2 such that $\mathrm{Tr}(\mathcal{J}_X')^2 = C_2(x^2+y^2)^2$, which gives $A = E = C_2$, $C = 2C_2$, and $B = D = 0$. In this case we introduce the short notation $K^{11} = T^{11} - 2(h^2+\alpha^2)$, $K^{22} = T^{22} - 2(\beta^2+\gamma^2)$, and $K^{12} = T^{12} - 2\alpha\beta$, where the equalities (8.21), (8.22), (8.23), (8.24), and (8.25) yield

$$\begin{aligned}hK^{12} &= 0,\\\alpha K^{12} + \beta K^{22} &= 0,\\(h^2+\alpha^2)K^{11} + 2\alpha\beta K^{12} + \beta^2K^{22} &= C_2 - C_1^2,\\2h\alpha K^{11} + 2h\beta K^{12} &= 0,\\h^2K^{11} + \gamma^2K^{22} &= C_2 - C_1^2,\end{aligned}$$

while (8.20) additionally gives $K^{11} + K^{22} = (2-n)C_1$.

The case $h \neq 0$ is easy, because first we have $K^{12} = 0$, from where $K^{11} = 0$ and $\beta K^{22} = 0$, which give $C_2 = C_1^2$, and therefore $K^{22} = 0$. However, then $(2-n)C_1 = 0$, and due to $n > 2$ we have $C_1 = 0$, as well as $C_2 = 0$ (the sum of squares of eigenvalues of the Jacobi operator is equal to zero), which implies $R' = 0$.

It remains the case $h = 0$ where the previous equalities yield

$$K^{11} = \frac{\beta^2+\gamma^2}{\alpha^2\gamma^2}(C_2 - C_1^2), \quad K^{22} = \frac{1}{\gamma^2}(C_2 - C_1^2), \quad K^{12} = -\frac{\beta}{\alpha\gamma^2}(C_2 - C_1^2),$$

and therefore

$$(2-n)C_1 = K^{11} + K^{22} = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha^2\gamma^2}(C_2 - C_1^2) = \frac{-C_1}{\alpha^2\gamma^2}(C_2 - C_1^2),$$

so since $C_1 \neq 0$ (otherwise $\alpha = \beta = \gamma = 0$ holds), we get $C_2 - C_1^2 = (n-2)\alpha^2\gamma^2$.

Since $H^1 = H^2 = 0$ we can rotate the orthonormal basis (ξ^1, ξ^2) in the normal space to get $T^{12} = 0$, that is, $K^{12} = -2\alpha\beta$, which gaves $\beta(n-4) = 0$, with additional $K^{22} = (n-2)\alpha^2$ and $K^{11} = (n-2)(\beta^2+\gamma^2)$.

The case $n = 4$ is Osserman, while in the case $\beta = 0$ we can discuss other blocks. In the case that we have a 1×1 block, there exists a vector e_3 such that $A^1 e_3 = \mu e_3$ and $A^2 e_3 = \nu e_3$. Then from (8.17) we calculate

$$\begin{aligned}
\mathrm{Tr}(\mathcal{J}'_{xe_1+ye_2+ze_3})^2 =& (\alpha x^2 - \alpha y^2 + \mu z^2)^2 T^{11} + (\alpha^2 x^2 + \alpha^2 y^2 + \mu^2 z^2)^2 \\
&- 2(\alpha x^2 - \alpha y^2 + \mu z^2)(\alpha^3 x^2 - \alpha^3 y^2 + \mu^3 z^2) \\
&+ 2(\mu\nu z^2)^2 - 2(\alpha x^2 - \alpha y^2 + \mu z^2)(-\alpha\gamma^2 x^2 + \alpha\gamma^2 y^2 + \mu\nu^2 z^2) \\
&- 2(2\gamma xy + \nu z^2)(-2\alpha^2\gamma xy + \mu^2\nu z^2) \\
&+ (2\gamma xy + \nu z^2)^2 T^{22} + (\gamma^2 x^2 + \gamma^2 y^2 + \nu^2 z^2)^2 - 2(2\gamma xy + \nu z^2)(2\gamma^3 xy + \nu^3 z^2).
\end{aligned}$$

The coefficient of xyz^2 is $4\gamma\nu(T^{22} - \mu^2 + \alpha^2 - \nu^2 - \gamma^2) = 0$, but because of a contradiction $T^{22} = \mu^2 + \nu^2 + \gamma^2 - \alpha^2 = 2\gamma^2 < T^{22}$ we obtain $\nu = 0$. Now, equating the coefficients of x^2z^2 and y^2z^2 we have $4\alpha\mu(T^{11} - \mu^2 - \alpha^2 - \gamma^2) = 0$, but because of $\mu \neq 0$ we obtain $T^{11} = 2\alpha^2 < T^{11}$, a contradiction.

Finally, the case of another 2×2 block includes vectors e_3 and e_4 such that $A^1 e_3 = \alpha' e_3$, $A^1 e_4 = -\alpha' e_4$, $A^2 e_3 = \gamma' e_4$, $A^2 e_4 = \gamma' e_3$. The coefficient of $xyzw$ in $\mathrm{Tr}(\mathcal{J}'_{xe_1+ye_2+ze_3+we_4})^2$ is $T^{22} + \alpha^2 - \gamma^2 + \alpha'^2 - \gamma'^2 = 0$, and therefore $T^{22} = \gamma^2 - \alpha^2 + \gamma'^2 - \alpha'^2 < T^{22}$, a contradiction.

Summarizing, we got the following theorem from [45]. Let us remark that our proof differs from the original, thanks to additional things we noticed.

Theorem 8.12. *A connected zwei-stein submanifold of codimension two in a space form has flat normal bundle and therefore has constant curvature.*

Chapter 9

DUALITY PRINCIPLE

9.1 Duality principle

Let R be an Osserman algebraic curvature tensor on a positive definite scalar product space $(\mathcal{V}, g)$ of dimension n. Then for unit vectors $X, Y \in \mathcal{V}$ the implication

$$\mathcal{J}_X Y = \lambda Y \quad \Longrightarrow \quad \mathcal{J}_Y X = \lambda X \tag{9.1}$$

appeared naturally, and it can significantly simplify some calculations. For a unit $X \in \mathcal{V}$, Lemma A.21 gives an orthonormal eigenbasis $(E_1, \dots, E_{n-1})$ in $X^\perp$ such that $\mathcal{J}_X E_i = \lambda_i E_i$ holds for $1 \leq i \leq n-1$. An arbitrary $Y = \sum_{i=1}^{n-1} \alpha_i E_i \in X^\perp$ satisfies $g(\mathcal{J}_X Y, Y) = \sum_{i=1}^{n-1} \alpha_i^2 \lambda_i$, so all sectional curvatures lie between $\min_i \lambda_i$ and $\max_i \lambda_i$ and they attain these extremal values precisely when X and Y are eigenvectors relative to each other. Thus, the statement (9.1) holds when λ is an extremal (minimum or maximum) eigenvalue of the Jacobi operator, which was used by Chi in 1988 [34, Lemma 3]. Moreover, Rakić used the implication (9.1) to formulate and prove the duality principle in 1998 [102, 103].

However, in an indefinite setting the implication (9.1) is inaccurate when X and Y belong to different unit pseudo-spheres. This was corrected, according to Theorem 8.2, with the following implication given by Andrejić in 2006 [3] (see also [15]),

$$\mathcal{J}_X(Y) = \varepsilon_X \lambda Y \quad \Longrightarrow \quad \mathcal{J}_Y(X) = \varepsilon_Y \lambda X. \tag{9.2}$$

Because of the compatibility $g(\mathcal{J}_Y X, X) = g(\mathcal{J}_X Y, Y)$, whenever X and Y are eigenvectors relative to each other, the corresponding eigenvalues are the same and we can exclude λ from (9.2) to introduce an equivalent implication,

$$Y \text{ is an eigenvector of } \mathcal{J}_X \quad \Longrightarrow \quad X \text{ is an eigenvector of } \mathcal{J}_Y. \tag{9.3}$$

It is important (especially if we deal with the converse problem from Section 9.5) to examine an optimal extension for our (X, Y) domain from (9.3). We use three kinds of duality depending on that domain. If (9.3) holds for mutually orthogonal units X and Y, just like in the original definition (see Rakić [102, 103]), we say that R is ***weak Jacobi-dual***. If (9.3) holds for all $X, Y \in \mathcal{V}$ with $\varepsilon_X \neq 0$, we say that R is ***Jacobi-dual*** or that R ***satisfies the duality principle*** (see Andrejić and Rakić [16]). Finally, if (9.3) holds with the only restriction $X \neq 0$ (a concept where the implication (9.2) has no restriction on X and Y), we say that R is ***totally Jacobi-dual***.

By definition, totally Jacobi-dual implies Jacobi-dual, while Jacobi-dual implies weak Jacobi-dual. Let us consider other relations between our dual properties. Since

$$\mathcal{J}_{X/\sqrt{|\varepsilon_X|}}(Y/\sqrt{|\varepsilon_Y|}) = \varepsilon_{X/\sqrt{|\varepsilon_X|}} \lambda (Y/\sqrt{|\varepsilon_Y|})$$

is equivalent to $\mathcal{J}_X(Y) = \varepsilon_X \lambda Y$ (multiplication by $|\varepsilon_X|\sqrt{|\varepsilon_Y|} \neq 0$), we have a straightforward extension from unit to nonnull vectors. The mutually orthogonal condition can be easily removed using the following lemma, given by Andrejić in 2010 [5, Lemma 2].

Lemma 9.1. *If the implication* (9.2) *holds for all mutually orthogonal X and Y with $\varepsilon_X \neq 0$, then it holds with the only restriction $\varepsilon_X \neq 0$.*

Proof. Let us suppose $\mathcal{J}_X(Y) = \varepsilon_X \lambda Y$ for $\varepsilon_X \neq 0$ with $g(X,Y) \neq 0$. The orthogonal decomposition $Y = \alpha X + Z$ with $g(Z,X) = 0$ implies $\alpha \neq 0$. Since

$$0 = g(\mathcal{J}_X X, \alpha X + Z) = g(\mathcal{J}_X(\alpha X + Z), X) = g(\varepsilon_X \lambda(\alpha X + Z), X) = g(\varepsilon_X \lambda \alpha X, X) = (\varepsilon_X)^2 \alpha \lambda,$$

the conditions $\varepsilon_X \neq 0$ and $\alpha \neq 0$ imply $\lambda = 0$. Hence $\mathcal{J}_X Z = \mathcal{J}_X(\alpha X + Z) = \varepsilon_X \lambda(\alpha X + Z) = 0$, and because of $g(Z,X) = 0$ and $\varepsilon_X \neq 0$, the lemma assumptions give $\mathcal{J}_Z X = 0$. Thus arises

$$\mathcal{J}_Y X = \mathcal{J}_{\alpha X + Z}(X) = \alpha \mathcal{R}(X,Z)X + \mathcal{R}(X,Z)Z = -\alpha \mathcal{J}_X Z + \mathcal{J}_Z X = 0,$$

which proves $\mathcal{J}_Y X = 0 = \varepsilon_Y \lambda X$. □

In the case that R is Jacobi-diagonalisable, our domain can be equivalently extended to all $X, Y \in \mathcal{V}$ with $\varepsilon_X \neq 0$, which is given by Andrejić in 2006 [3, Theorem 3.1] (see also [15, Theorem 3.2] and [5, Theorem 2]).

Theorem 9.2. *If R is a Jacobi-diagonalisable algebraic curvature tensor, then R is Jacobi-dual if and only if it is weak Jacobi-dual.*

Proof. Let us suppose $\mathcal{J}_X Y = \varepsilon_X \lambda Y$, $g(X,Y) = 0$, $\varepsilon_X \neq 0$, and $\varepsilon_Y = 0$. Since R is Jacobi-diagonalisable, then $\operatorname{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X \lambda \mathbb{1}) \ni Y$ is an (generalised) eigenspace. According to Lemma A.22 it is nondegenerate, so Lemma 4.10 allows a decomposition $Y = S + T$, such that $S, T \in \operatorname{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X \lambda \mathbb{1})$ and $\varepsilon_S = -\varepsilon_T = 1$. For $\theta^2 \neq 1$, $S + \theta T$ is nonnull because of $\varepsilon_{S+\theta T} = \varepsilon_S + \theta^2 \varepsilon_T = (1 - \theta^2) \neq 0$. The vectors $S + \theta T$, S, and T are nonnull, orthogonal to X, eigenvectors of $\mathcal{J}_X$ corresponding to the eigenvalue $\varepsilon_X \lambda$, so the weak Jacobi-duality gives

$$\mathcal{J}_{S+\theta T} X = \varepsilon_{S+\theta T} \lambda X, \quad \mathcal{J}_S X = \varepsilon_S \lambda X, \quad \mathcal{J}_T X = \varepsilon_T \lambda X.$$

Hence

$$\varepsilon_{S+\theta T} \lambda X = \mathcal{J}_{S+\theta T} X = \mathcal{J}_S X + \theta^2 \mathcal{J}_T X + 2\theta \mathcal{J}(S,T)X = \varepsilon_S \lambda X + \theta^2 \varepsilon_T \lambda X + 2\theta \mathcal{J}(S,T)X,$$

which, for any $\theta \notin \{-1, 0, 1\}$, implies $\mathcal{J}(S,T)X = 0$. Then

$$\mathcal{J}_Y X = \mathcal{J}_{S+T} X = \mathcal{J}_S X + \mathcal{J}_T X + 2\mathcal{J}(S,T)X = \varepsilon_S \lambda X + \varepsilon_T \lambda X = 0 = \varepsilon_Y \lambda X,$$

which proves (9.2) for $X \perp Y$ and $\varepsilon_X \neq 0$. Finally, Lemma 9.1 removes the condition $X \perp Y$. □

Let R be a Jacobi-diagonalisable Osserman algebraic curvature tensor on a scalar product space $(\mathcal{V}, g)$ of dimension n. For any fixed unit $X = E_x \in \mathcal{V}$ there exists an orthonormal eigenbasis $(E_1, \ldots, E_n)$ in $\mathcal{V}$ related to $\mathcal{J}_X$. For any root $\lambda \in \mathbb{R}$ of $\det(\varepsilon_X \lambda \mathbb{1} - \mathcal{J}_X)$, we can set $\Lambda = \{1 \leq j \leq n : \varepsilon_x \varepsilon_j R_{xjjx} = \lambda\}$ and apply Theorem 8.8 to get

$$\sum_{i,j \in \Lambda} \varepsilon_i \varepsilon_j (R_{ixyj} + R_{iyxj})^2 = 0,$$

for all $1 \leq y \neq x \leq n$.

If we assume that $\mathcal{J}_X$ has no null eigenvectors, then any eigenspace consists of vectors of the same type, $\varepsilon_i \varepsilon_j = 1$ holds for $i, j \in \Lambda$, which implies $\sum_{i,j \in \Lambda} (R_{ixyj} + R_{iyxj})^2 = 0$. Thus arises $R_{ixyj} + R_{iyxj} = 0$ for all $i, j \in \Lambda$, and especially for $i = j$ it yields $R_{ixyi} = 0$ for all $i \in \Lambda$. Since any z belongs to some Λ we have $R_{xzzy} = 0$ for all $y \neq x$, and therefore the condition $\mathcal{J}_{E_x} E_z = \varepsilon_x \lambda E_z$ implies $\mathcal{J}_{E_z} E_x = \sum_p \varepsilon_p R_{xzzp} E_p = \varepsilon_x R_{xzzx} E_x = \varepsilon_z \lambda E_x$, which proves that R is Jacobi-dual.

Theorem 9.3. *Let R be a Jacobi-diagonalisable Osserman algebraic curvature tensor such that $\mathcal{J}_X$ has no null eigenvectors for all unit X. Then R is Jacobi-dual.*

Theorem 9.3 is established by Andrejić in 2006 [3, Theorem 3.4] (see also [15, Corollary 3.5]) and it has significant consequences. For example, an Osserman R with all different eigenvalues of Jacobi operator is necessarily Jacobi-diagonalisable with one-dimensional eigenspaces, and therefore it is Jacobi-dual. Let us remark that the previous statement can be seen as a consequence of Theorem 8.7, rather than Theorem 8.8.

Anyway, the most valuable consequence comes in the definite case, where null vectors do not exist and any R is Jacobi-diagonalisable. Therefore, any Riemannian Osserman R is Jacobi-dual, moreover it is totally Jacobi-dual. Let us notice that this valuable statement (in fact that a Riemannian Osserman R is weak Jacobi-dual) was first proven by Rakić in 1998 [102, Theorem 2.5] [103, Theorem 1.1], which is later reproved by Gilkey in 2001 [54, Theorem 3.1.2].

Theorem 9.4. *A Riemannian Osserman algebraic curvature tensor is totally Jacobi-dual.*

9.2 Quasi-Clifford curvature tensors

Let us recall first examples of algebraic curvature tensors on a scalar product space $(\mathcal{V}, g)$. These are R^1 from (6.14) and R^J from (6.15), where J is a skew-adjoint endomorphism on $\mathcal{V}$. Example 6.6 shows that, a linear combination

$$R = \mu_0 R^1 + \sum_{i=1}^{m} \mu_i R^{J_i} \tag{9.4}$$

is an algebraic curvature tensor for skew-adjoint endomorphisms $J_1, \dots, J_m$ on $\mathcal{V}$, where $\mu_j \in \mathbb{R}$ for $0 \le j \le m$.

A ***Clifford family*** of rank m is an anti-commuting family of skew-adjoint complex structures J_i for $1 \le i \le m$. Algebraic curvature tensors of form (9.4) associated with a Clifford family were introduced by Gilkey in 1994 [53] (see also [58]). However, we consider a generalisation, an anti-commuting family of J_i such that $J_i^2 = c_i \mathbb{1}$ for some $c_i \in \mathbb{R}$, which means that the Hurwitz-like relations,

$$J_i J_j + J_j J_i = 2 c_i \delta_{ij} \mathbb{1}, \tag{9.5}$$

hold for $1 \le i, j \le m$. We say that an algebraic curvature tensor R is ***quasi-Clifford*** if it has a form (9.4) such that the relations (9.5) hold, introduced by Andrejić and Lukić[1] in 2019 [12]. Especially, R is ***Clifford*** if it is quasi-Clifford with $c_i = -1$ for all $1 \le i \le m$.

It is well known that a Clifford algebraic curvature tensor is Osserman. However, according to Nikolayevsky [92, Section 2], in the definite setting the converse is true (any Riemannian Osserman R is Clifford) in all dimensions except $n = 16$, and also in many cases when $n = 16$.

Let us start with a quasi-Clifford R and an arbitrary vector $X \in \mathcal{V}$. Each J_i is skew-adjoint which implies $g(J_i X, X) = 0$ and simplifies the calculation of the Jacobi operator,

$$\mathcal{J}_X Y = \mathcal{R}(Y, X)X = \mu_0 (g(X,X)Y - g(Y,X)X) + \sum_{i=1}^{m} 3\mu_i g(J_i Y, X) J_i X,$$

and therefore

$$\mathcal{J}_X Y = \mu_0 (\varepsilon_X Y - g(Y,X)X) - 3 \sum_{i=1}^{m} \mu_i g(Y, J_i X) J_i X. \tag{9.6}$$

[1] Katarina Lukić (1994), Serbian mathematician

Additionally, the equality (9.5) implies $g(J_iX, J_jX) = 0$ and $\varepsilon_{J_iX} = -c_i\varepsilon_X$ for $1 \leq i \neq j \leq m$. If we denote

$$\mathcal{F}_t = \{X, J_1X, \ldots, J_tX\}$$

for $1 \leq t \leq m$, then we obtain

$$\mathcal{J}_X(J_iX) = \varepsilon_X(\mu_0 + 3c_i\mu_i)J_iX, \quad \mathcal{J}_XZ = \varepsilon_X\mu_0Z,$$

for all $1 \leq i \leq m$ and $Z \in \mathcal{F}_m^\perp$.

It is important to distinguish the case $c_i \neq 0, 1 \leq i \leq k$ from the case $c_i = 0, k < i \leq m$. For a nonnull X, the set $\mathcal{F}_k$ consists of mutually orthogonal nonnull eigenvectors, so Span $\mathcal{F}_k$ and $\mathcal{F}_k^\perp$ are nondegenerate, while $\operatorname{Span}(\mathcal{F}_m \setminus \mathcal{F}_k)$ is a totally isotropic subspace. Then, from $\mathcal{F}_k^\perp \cap \operatorname{Span}(\mathcal{F}_m \setminus \mathcal{F}_k)^\perp \leq \operatorname{Ker}((\mathcal{J}_X\restriction_{\mathcal{F}_k^\perp} - \varepsilon_X\mu_0\,\mathbb{1})^n)$ follows

$$\mathcal{F}_k^\perp \cap \operatorname{Ker}((\mathcal{J}_X\restriction_{\mathcal{F}_k^\perp} - \varepsilon_X\mu_0\,\mathbb{1})^n)^\perp \leq \operatorname{Span}\{J_{k+1}X, \ldots, J_mX\},$$

but a generalised eigenspace is nondegenerate (Lemma A.22), as well as its orthogonal, which cannot be a subspace of a totally isotropic subspace, unless it is trivial, which yields $\operatorname{Ker}((\mathcal{J}_X\restriction_{\mathcal{F}_k^\perp} - \varepsilon_X\mu_0\,\mathbb{1})^n) = \mathcal{F}_k^\perp$. Hence,

$$\det\left(\lambda\,\mathbb{1} - \frac{1}{\varepsilon_X}\mathcal{J}_X\right) = \lambda(\lambda - \mu_0)^{n-k-1}\prod_{i=1}^{k}(\lambda - (\mu_0 + 3c_i\mu_i))$$

is constant and according to Theorem 8.2, R is Osserman (see Andrejić and Lukić 2019 [12, Theorem 2.1]).

Theorem 9.5. *Any quasi-Clifford algebraic curvature tensor is Osserman.*

Additionally, the Jordan normal form of $\mathcal{J}_X$ has the critical part on $\mathcal{F}_k^\perp$, where we have

$$\operatorname{Im}(\mathcal{J}_X\restriction_{\mathcal{F}_k^\perp} - \varepsilon_X\mu_0\,\mathbb{1}) \subseteq \operatorname{Span}\{J_{k+1}X, \ldots, J_mX\} \subseteq \operatorname{Ker}(\mathcal{J}_X\restriction_{\mathcal{F}_k^\perp} - \varepsilon_X\mu_0\,\mathbb{1}),$$

and therefore we have two-step nilpotency, $(\mathcal{J}_X\restriction_{\mathcal{F}_k^\perp} - \varepsilon_X\mu_0\,\mathbb{1})^2 = 0$. Thus, a quasi-Clifford R do not allows Jordan blocks of size greater than 2. However, Example 8.6 shows a Jordan-Osserman curvature tensor such that the Jordan normal form has a Jordan block of size 3, which means that the converse question fails in the signature $(2, 2)$, where an Osserman R is not necessarily quasi-Clifford.

We follow the arguments given by Andrejić and Lukić in 2019 [12] (which is a generalisation of Andrejić and Rakić 2015 [16]), to investigate whether a quasi-Clifford $R = \mu_0R^1 + \sum_{i=1}^m \mu_iR^{J_i}$ is Jacobi-dual. Let $X \in \mathcal{V}$ be nonnull and suppose that Y is an eigenvector of $\mathcal{J}_X$, which means that $\mathcal{J}_XY = \varepsilon_X\lambda Y$ holds for some $\lambda \in \mathbb{R}$. Then (9.6) implies

$$\varepsilon_X(\lambda - \mu_0)Y = -\mu_0g(Y, X)X - 3\sum_{i=1}^{m}\mu_ig(Y, J_iX)J_iX, \tag{9.7}$$

while by interchanging the roles of X and Y in (9.6) we have

$$\mathcal{J}_YX = \mu_0(\varepsilon_YX - g(X, Y)Y) - 3\sum_{i=1}^{m}\mu_ig(X, J_iY)J_iY. \tag{9.8}$$

In the case $\lambda \neq \mu_0$, we can express Y from (9.7) and get

$$Y = \frac{-\mu_0g(Y, X)}{\varepsilon_X(\lambda - \mu_0)}X - 3\sum_{i=1}^{m}\frac{\mu_ig(Y, J_iX)}{\varepsilon_X(\lambda - \mu_0)}J_iX. \tag{9.9}$$

After the substitution in (9.8),

$$\begin{aligned}\mathcal{J}_Y X = \ & \mu_0 \left(\varepsilon_Y X - g(X,Y) \left(\frac{-\mu_0 g(Y,X)}{\varepsilon_X(\lambda-\mu_0)} X - 3\sum_{j=1}^m \frac{\mu_j g(Y,J_jX)}{\varepsilon_X(\lambda-\mu_0)} J_jX \right)\right) \\ & - 3\sum_{i=1}^m \mu_i g(X,J_iY)J_i \left(\frac{-\mu_0 g(Y,X)}{\varepsilon_X(\lambda-\mu_0)} X - 3\sum_{j=1}^m \frac{\mu_j g(Y,J_jX)}{\varepsilon_X(\lambda-\mu_0)} J_jX \right),\end{aligned}$$

which implies

$$\begin{aligned}\mathcal{J}_Y X = \ & \mu_0 \left(\varepsilon_Y + \frac{\mu_0 (g(X,Y))^2}{\varepsilon_X(\lambda-\mu_0)} \right) X \\ & + \frac{3\mu_0 g(X,Y)}{\varepsilon_X(\lambda-\mu_0)} \sum_{i=1}^m \mu_i \left(g(Y,J_iX) + g(X,J_iY) \right) J_iX \\ & + \frac{9}{\varepsilon_X(\lambda-\mu_0)} \sum_{i=1}^m \sum_{j=1}^m \mu_i\mu_j g(X,J_iY) g(Y,J_jX) J_iJ_jX.\end{aligned}$$

Since any J_i is skew-adjoint, the middle term on the right-hand side vanishes. Additionally, since $\mu_i\mu_j g(X,J_iY)g(Y,J_jX)J_iJ_jX + \mu_j\mu_i g(X,J_jY)g(Y,J_iX)J_jJ_iX = 0$ holds for $1 \le i \ne j \le m$, it reduces the last term on the right-hand side. Hence, we obtain

$$\mathcal{J}_Y X = \left(\mu_0\varepsilon_Y + \frac{\mu_0^2 (g(X,Y))^2}{\varepsilon_X(\lambda-\mu_0)} - \frac{9}{\varepsilon_X(\lambda-\mu_0)} \sum_{i=1}^m \mu_i^2 (g(Y,J_iX))^2 c_i \right) X,$$

and therefore X is an eigenvector of $\mathcal{J}_Y$.

Otherwise, we have the case $\lambda = \mu_0$ and the equality (9.7) becomes

$$\mu_0 g(Y,X)X + 3\sum_{i=1}^m \mu_i g(Y,J_iX)J_iX = 0. \tag{9.10}$$

If we separate $c_i \ne 0$ for $1 \le i \le k$ and $c_i = 0$ for $k < i \le m$, then $\operatorname{Span}\{X, J_1X, \dots, J_kX\}$ is a nondegenerate subspace of $\mathcal{V}$ of dimension $k+1$, which is orthogonal to $\{J_{k+1}X, \dots, J_mX\}$, for any nonnull X. For a nonnull X, from (9.10) we have $\mu_i g(X,J_iY) = \mu_i g(Y,J_iX) = 0$ for $1 \le i \le k$ with $\mu_0 g(Y,X) = 0$, so it remains $\sum_{i=k+1}^m \mu_i g(Y,J_iX)J_iX = 0$, while from (9.8) we obtain $\mathcal{J}_Y X = \varepsilon_Y \mu_0 X - 3\sum_{i=k+1}^m \mu_i g(X,J_iY)J_iY$.

The case $m = k+1$ gives $\mu_{k+1} g(Y,J_{k+1}X)J_{k+1}X = 0$, and therefore $\mu_{k+1} g(Y,J_{k+1}X) = 0$ holds anyway, and we get the duality, $\mathcal{J}_Y X = \varepsilon_Y \mu_0 X$. Hence, we have proven the following theorem (see [12, Theorem 3.1]).

Theorem 9.6. *Any quasi-Clifford algebraic curvature tensor with at most one $c_i = 0$ is Jacobi-dual.*

However, this statement is no longer true if there are at least two such J_i with $c_i = 0$. Let $(T_1, \dots T_p, S_1, \dots, S_q)$ be an orthonormal basis in a scalar product space $(\mathcal{V}, g)$ of signature (p,q). Let us define an endomorphism J on $\mathcal{V}$ by

$$\begin{aligned} JT_1 = T_2 + S_2 = -JS_1, \quad -JT_2 = T_1 + S_1 = JS_2 \\ JT_3 = T_4 + S_4 = -JS_3, \quad -JT_4 = T_3 + S_3 = JS_4 \\ JT_5 = \dots = JT_p = JS_5 = JS_6 = \dots = JS_q = 0. \end{aligned}$$

It is easy to check that J is skew-adjoint with $J^2 = 0$, so we make the algebraic curvature tensor by $R = R^J$. Its Jacobi operator has $\mathcal{J}_X Y = -3g(Y,JX)JX$ by (9.6), from which it can be seen that its characteristic polynomial is equal to λ^{p+q}. If $JX \neq 0$, then the Jordan normal form of $\mathcal{J}_X$ has exactly one block of size 2, while the case $JX = 0$ implies $\mathcal{J}_X = 0$ and there are no such blocks. If we set $X = \sum_{i=1}^{p} \alpha_i T_i + \sum_{i=1}^{q} \beta_j S_j$, we see that $JX = 0$ holds if and only if $\alpha_i = \beta_i$ holds for each $1 \leq i \leq 4$, which never happens for definite $X \in \mathrm{Span}\{T_1, \dots, T_4, S_1, \dots, S_4\}$, and always for arbitrary $X \in \mathrm{Span}\{T_5, \dots, T_p, S_5, \dots, S_q\}$. In the case $4 = p < q$, an Osserman algebraic curvature tensor $R = R^J$ is timelike Jordan-Osserman, but it is not spacelike Jordan-Osserman, which is similar to Gilkey and Ivanova [56, Theorem 3] and Gilkey [54, Section 3.2].

Theorem 9.7. *There exist timelike Jordan-Osserman algebraic curvature tensors that are not spacelike Jordan-Osserman.*

To make an example of quasi-Clifford R which is not Jacobi-dual, in addition to the already constructed skew-adjoint endomorphism J with $J^2 = 0$, according to Theorem 9.6, we need at least one more skew-adjoint endomorphism K with $K^2 = 0$, where it is necessary to ensure that there is a nonnull X such that JX and KX are linearly dependent. However, it is important to ensure the anti-commutativity $JK = -KJ$. Let us introduce an additional endomorphism K on $\mathcal{V}$ by

$$\begin{gathered}
KT_1 = T_2 + S_2 = -KS_1, \quad -KT_2 = T_1 + S_1 = KS_2 \\
KT_5 = T_6 + S_6 = -KS_5, \quad -KT_6 = T_5 + S_5 = KS_6 \\
KT_3 = KT_4 = KT_7 = \cdots = KT_p = KS_3 = KS_4 = KS_7 = \cdots = KS_q = 0.
\end{gathered}$$

A new endomorphism is modelled on J, where the vectors T_3, T_4, S_3, S_4 change roles with T_5, T_6, S_5, S_6, respectively. That is why K is also skew-adjoint with $K^2 = 0$, where we additionally provided $JK = KJ = 0$, which solves the problem of anti-commutativity. If we set

$$R = R^J - R^K,$$

we get a quasi-Clifford algebraic curvature tensor, which is Osserman according to Theorem 9.5.

From (9.6) follows $\mathcal{J}_X Y = 3(g(Y, KX)KX - g(Y, JX)JX)$. Since $JT_1 = KT_1 = T_2 + S_2$ we have $\mathcal{J}_{T_1} Y = 0$ for any $Y \in \mathcal{V}$. On the other hand, for $Y = T_2 + \sqrt{2}S_4$ we have

$$g(Y, JT_1) = g(Y, KT_1) = g(T_2 + \sqrt{2}S_4, T_2 + S_2) = -1,$$

and therefore $\mathcal{J}_Y T_1 = 3(g(Y, JT_1)JY - g(Y, KT_1)KY) = -3\sqrt{2}(T_3 + S_3)$. Thus, we obtain

$$\mathcal{J}_{T_1}(T_2 + \sqrt{2}S_4) = 0, \quad \mathcal{J}_{T_2+\sqrt{2}S_4}(T_1) = -3\sqrt{2}(T_3 + S_3),$$

which shows that R is not Jacobi-dual. Moreover, our counterexample contains mutually orthogonal unit vectors $X = T_1$ and $Y = T_2 + \sqrt{2}S_4$, such that the duality principle does not work.

Theorem 9.8. *There exist quasi-Clifford (and therefore Osserman) algebraic curvature tensors which are not Jacobi-dual.*

The first attempt to construct an example of Osserman algebraic curvature tensor that is not Jacobi-dual was published in 2019 [12, Theorem 3.2], where K is constructed on the model of J so that only S_4 and S_5 change roles. Unfortunately, this original example does not provide anti-commutativity for J and K, and the obtained curvature tensor is neither quasi-Clifford nor Osserman. However, the idea is good and our calculations after the introduction of K are literally identical to the original ones.

9.3 Total duality

Let R be a quasi-Clifford algebraic curvature tensor from the previous section. Skew-adjoint endomorphisms with $J_i^2 = 0$ change the Jordan normal form of $\mathcal{J}_X$ and therefore they are inadequate for the duality principle, which we have already seen in Theorem 9.8. Let us exclude them with $m = k$, which leaves only such J_i that are automorphisms. Without loss of generality, using the rescaled $(1/\sqrt{|c_i|})J_i$, we can suppose $c_i \in \{-1, 1\}$, and we say that such R is ***semi-Clifford***. It is easy to see that a semi-Clifford R is Jacobi-diagonalisable and consequently Jordan-Osserman.

Thus, we consider a family $\{J_1, \dots, J_k\}$ of anti-commuting skew-symmetric orthogonal and anti-orthogonal operators on $\mathcal{V}$. In fact, these are complex structures ($c_i = -1$) and product structures ($c_i = 1$). It is worth noting that a product structures J_i change the signature because of $\varepsilon_{J_iX} = -\varepsilon_X$, and therefore any semi-Clifford R that is not Kleinian is Clifford.

We have already seen that a semi-Clifford R is Jacobi-dual, and the next step is to investigate whether R is totally Jacobi-dual. Andrejić and Rakić in 2015 [16, Proposition 5.2] gave only a sufficient condition for the totally Jacobi-duality.

Theorem 9.9. *If semi-Clifford algebraic curvature tensor R is generated by a semi-Clifford family $\{J_1, \dots, J_m\}$, such that for every null X the set $\{X, J_1X, \dots, J_mX\}$ is linearly independent, then R is totally Jacobi-dual.*

Proof. By theorem 9.6, every semi-Clifford R is Jacobi-dual, which gives the implication (9.2) for a nonnull X. In the case of a null X, the assumption that Y is an eigenvector of $\mathcal{J}_X$, by Theorem 8.4 gives $\mathcal{J}_XY = 0$. Then the equality (9.6) implies (9.10), where the linear independence of $\mathcal{F}_m$, as before, yields $\mu_i g(Y, J_iX) = 0$ for $1 \le i \le m$ with $\mu_0 g(Y, X) = 0$. Hence $\mathcal{J}_YX = \varepsilon_Y\mu_0X$, so X is an eigenvector of $\mathcal{J}_Y$. □

Andrejić and Lukić in 2019 [12] considered the question of linear independence of the set $\mathcal{F}_m = \{X, J_1X, \dots, J_mX\}$ for a null vector X. The discussion begins with the following important theorem (see [12, Theorem 4.1]).

Theorem 9.10. *If $\theta_0X + \theta_1J_1X + \dots + \theta_mJ_mX = 0$ holds for a quasi-Clifford family $\{J_1, \dots, J_m\}$ with $J_i^2 = c_i\mathbb{1}$, $1 \le i \le m$, and $\theta_0, \dots, \theta_m \in \mathbb{R}$, $X \neq 0$, then $\theta_0^2 - c_1\theta_1^2 - \dots - c_m\theta_m^2 = 0$.*

Proof. For every $\alpha = (\alpha_1, \dots, \alpha_m) \in \{0, 1\}^m$ we define the endomorphism $J^\alpha = J_m^{\alpha_m} \dots J_1^{\alpha_1}$ and $(-1)^\alpha = (-1)^{\alpha_1 + \dots + \alpha_m}$. Applying $(-1)^\alpha J^\alpha$ on the equality

$$\theta_0X + \theta_1J_1X + \dots + \theta_mJ_mX = 0,$$

we obtain $\sum_{i=0}^m (-1)^\alpha \theta_i J^\alpha J^{e_i} X = 0$, where we introduce $e_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{im})$ with additional $e_0 = (0, \dots, 0)$, so $J^{e_i} = J_i$ and $J^{e_0} = \mathbb{1}$. Since $(-1)^\alpha J^\alpha J^{e_i} = (-1)^{\alpha_i + \dots + \alpha_m}(c_i)^{\alpha_i} J^{\alpha \pm e_i}$ holds for $1 \le i \le m$, where $\alpha \pm e_i$ and α differ only in the i-th slot, for every $\alpha \in \{0, 1\}^m$ we get the equality

$$\sum_{i=0}^m (-1)^{\alpha_i + \dots + \alpha_m}(c_i)^{\alpha_i}\theta_i J^{\alpha \pm e_i} X = 0.$$

In this way, we obtain a homogeneous system of 2^m linear equations with 2^m unknowns, $\sum_\beta M_{\alpha\beta} J^\beta X = 0$, where

$$M_{\alpha\alpha} = (-1)^{\alpha_1 + \dots + \alpha_m}\theta_0,$$
$$M_{\alpha(\alpha\pm e_i)} = (-1)^{\alpha_i + \dots + \alpha_m}(c_i)^{\alpha_i}\theta_i,$$

for $1 \le i \le m$, and $M_{\alpha\beta} = 0$ otherwise. Consider the matrix M^2, and calculate its entries,

$$(M^2)_{\alpha\alpha} = M_{\alpha\alpha}M_{\alpha\alpha} + M_{\alpha(\alpha\pm e_1)}M_{(\alpha\pm e_1)\alpha} + \dots + M_{\alpha(\alpha\pm e_m)}M_{(\alpha\pm e_m)\alpha} = \theta_0^2 - c_1\theta_1^2 - \dots - c_m\theta_m^2,$$
$$(M^2)_{\alpha(\alpha\pm e_i)} = M_{\alpha\alpha}M_{\alpha(\alpha\pm e_i)} + M_{\alpha(\alpha\pm e_i)}M_{(\alpha\pm e_i)(\alpha\pm e_i)} = 0,$$
$$(M^2)_{\alpha(\alpha\pm e_i\pm e_j)} = M_{\alpha(\alpha\pm e_i)}M_{(\alpha\pm e_i)(\alpha\pm e_i\pm e_j)} + M_{\alpha(\alpha\pm e_j)}M_{(\alpha\pm e_j)(\alpha\pm e_i\pm e_j)} = 0,$$

and $(M^2)_{\alpha\beta} = 0$ otherwise. Thus, M^2 is a diagonal matrix with

$$(\det M)^2 = \det(M^2) = (\theta_0^2 - c_1\theta_1^2 - \cdots - c_m\theta_m^2)^{2^m}.$$

For $\det M \neq 0$ the system has the unique zero solution $X = J_1X = \cdots = J_mX = 0$, but since $X \neq 0$, we have $\det M = 0$, which proves the theorem. □

In the case of a Clifford R, we have $c_i = -1$ for all $1 \leq i \leq m$, so Theorem 9.10 from the condition $\theta_0X + \theta_1J_1X + \cdots + \theta_mJ_mX = 0$ implies $\theta_0^2 + \theta_1^2 + \cdots + \theta_m^2 = 0$, and therefore $\theta_i = 0$ holds for all $0 \leq i \leq m$. Thus, $\mathcal{F}_m$ is a linearly independent set and according to Theorem 9.9, R is totally Jacobi-dual (see [12, Theorem 4.2]).

Theorem 9.11. *Any Clifford algebraic curvature tensor is totally Jacobi-dual.*

However, there are some problems in the case when we have $c_i = 1$ for some i. Every semi-Clifford R is Jacobi-dual according to Theorem 9.6, and if we want to check whether R is totally Jacobi-dual, it is enough to check the implication (9.2) for a null $X \in \mathcal{V}$. The initial condition for a null X is $\mathcal{J}_XY = 0$, from where (9.6) gives (9.10), where we get $\theta_0X + \theta_1J_1X + \cdots + \theta_mJ_mX = 0$ for concrete $\theta_0 = \mu_0 g(Y,X)$ and $\theta_i = 3\mu_i g(Y,J_iX)$ for $1 \leq i \leq m$. The question of whether R is totally Jacobi-dual is equivalent to whether the initial condition always give $\mathcal{J}_YX = \varepsilon_Y\mu_0X - \theta_0Y + \theta_1J_1Y + \cdots + \theta_mJ_mY$ which is proportional to X, that is obvious when $\theta_i = 0$ holds for all $0 \leq i \leq m$.

From the initial conditions $\theta_0 = \mu_0 g(Y,X)$ and $\theta_i = 3\mu_i g(Y,J_iX)$, we can calculate

$$\theta_0^2 = \mu_0 g(Y,\theta_0X) = -\mu_0\sum_{i=1}^m \theta_i g(Y,J_iX) = -\mu_0\sum_{i=1}^m \frac{\theta_i^2}{3\mu_i},$$

while from Theorem 9.10 follows

$$\theta_0^2 = \sum_{i=1}^m c_i\theta_i^2 = -\mu_0\sum_{i=1}^m \frac{\theta_i^2}{3\mu_i}, \tag{9.11}$$

and therefore

$$\sum_{i=1}^m (c_i + \frac{\mu_0}{3\mu_i})\theta_i^2 = 0.$$

If all the numbers $(3c_i\mu_i + \mu_0)/\mu_i$ are of the same sign for $1 \leq i \leq m$, then the only solution of the previous equation is $\theta_1 = \cdots = \theta_m = 0$, while (9.11) gives $\theta_0 = 0$, which implies the following theorem (see [12, Theorem 4.3]).

Theorem 9.12. *If* $(3c_i\mu_i + \mu_0)\mu_i > 0$ *for all* $1 \leq i \leq m$ *or* $(3c_i\mu_i + \mu_0)\mu_i < 0$ *for all* $1 \leq i \leq m$, *then the associated semi-Clifford R is totally Jacobi-dual.*

In the case that $c_i = 1$ holds for all $1 \leq i \leq m$ we say that R is ***anti-Clifford***. With this assumption, Theorem 9.10 for the hypothetical $\theta_0 = 0$ gives $\theta_1^2 + \cdots + \theta_m^2 = 0$ which shows that the set $\{J_1X,\ldots,J_mX\}$ is linearly independent, but $\mathcal{F}_m$ can be linearly dependent.

Since $J_1X,\ldots,J_mX$ form a basis of a totally isotropic subspace of $\mathcal{V}$, by Theorem 4.12 there exists a basis $(P_1,\ldots,P_m)$ of an isotropic supplement, such that $g(J_iX,P_j) = \delta_{ij}$ and $g(P_i,P_j) = 0$ hold for $1 \leq i,j \leq m$. Then

$$Z = \sum_{i=1}^m \frac{\theta_i}{3\mu_i}P_i$$

has the properties $\theta_i = 3\mu_i g(Z,J_iX)$, and consequently by (9.11), $\theta_0 = \mu_0 g(Z,X)$. It is easy to see that $Z + W$ has the same properties for any $W \in \{J_1X,\ldots,J_mX\}^\perp$.

From (9.8), we need such Y that $-\theta_0 Y + \sum_{i=1}^m \theta_i J_i Y$ is not proportional to X. Therefore, we search for $Y = Z+W$ such that $\mathcal{K}(Z+W)$ is not proportional to X where $\mathcal{K} = -\theta_0 \mathbb{1} + \sum_{i=1}^m \theta_i J_i$. For any nonnull D, the set $\{D, J_1 D, \dots, J_m D\}$ is linearly independent (mutually orthogonal nonnull vectors), and therefore $\mathcal{K}(D) = 0$ implies $\theta_0 = \cdots = \theta_m = 0$, which is impossible. Thus, we have $\mathcal{K}(D) \neq 0$ for all nonnull D.

The assumption $n > 2m$ enables a nonnull vector H from $\{J_1 X, \dots, J_m X, P_1, \dots, P_m\}^\perp$. We already have $\mathcal{K}(X) = -2\theta_0 X$. If R is totally Jacobi-dual, we have additional $\mathcal{K}(Z) = \zeta X$ and $\mathcal{K}(Z+H) = \xi X$. Then $\mathcal{K}(H) = (\xi - \zeta)X \neq 0$, so $\mathcal{K}(Z - (\zeta/(\xi-\zeta))H) = 0$, which implies that $Z - (\zeta/(\xi-\zeta))H$ is not nonnull, and therefore $\zeta = 0$. Similarly, $\mathcal{K}(X + (2\theta_0/\xi)H) = 0$, which implies that $X + (2\theta_0/\xi)H$ is not nonnull, and therefore $\theta_0 = 0$, that is a contradiction. Hence, we have the following theorem (see [12, Theorem 5.2]).

Theorem 9.13. *If there exist $\theta_0, \dots, \theta_m \in \mathbb{R}$ (where not all are equal to zero), such that $\theta_0 X + \theta_1 J_1 X + \cdots + \theta_m J_m X = 0$ holds for some null X, with the condition* (9.11)*, then the associated anti-Clifford algebraic curvature tensor of dimension $n > 2m$ is not totally Jacobi-dual.*

Let us show some concrete examples of anti-Clifford algebraic curvature tensors which are not totally Jacobi-dual.

Example 9.1. For $m = 1$, Theorem 9.12 gives the necessary condition $\mu_0 + 3\mu_1 = 0$. A skew-adjoint product structure given by $JT_i = S_i$ and $JS_i = T_i$ for $1 \leq i \leq t$, $n = 2t \geq 4$, where $(T_1, \dots, T_t, S_1, \dots, S_t)$ is an orthonormal basis in a scalar product space $(\mathcal{V}, g)$ of neutral signature, provides an anti-Clifford $R = 3\mu R^1 - \mu R^J$ for $\mu \neq 0$. We take $X = S_1 + T_1$, because of the linear dependence $X = JX$, and apply Theorem 9.13. Therefore $R = 3R^1 - R^J$ is anti-Clifford which is not totally Jacobi-dual. △

Example 9.2. For $m = 2$, Theorem 9.12 gives the condition $(\mu_0 + 3\mu_1)(\mu_0 + 3\mu_2)\mu_1\mu_2 \leq 0$ which is necessary. Consider skew-adjoint product structures J and K that are given by $JT_{2i-1} = S_{2i-1}$, $JT_{2i} = S_{2i}$, $KT_{2i-1} = S_{2i}$, and $KT_{2i} = -S_{2i-1}$, for all $1 \leq i \leq t$, $n = 4t \geq 8$, where $(T_1, \dots, T_{2t}, S_1, \dots, S_{2t})$ is an orthonormal basis in a scalar product space $(\mathcal{V}, g)$ of neutral signature. We can take $X = \cos\alpha\, T_1 + \sin\alpha\, T_2 + \cos\beta\, S_1 + \sin\beta\, S_2$ for some $\alpha, \beta \in \mathbb{R}$ to see that $X = \cos(\beta - \alpha) JX + \sin(\beta - \alpha) KX$. The condition (9.11) gives

$$\tan^2(\beta - \alpha) = \frac{\sin^2(\beta-\alpha)}{\cos^2(\beta-\alpha)} = \left(\frac{\theta_2}{\theta_1}\right)^2 = -\frac{\mu_0 + 3\mu_1}{\mu_0 + 3\mu_2} \cdot \frac{\mu_2}{\mu_1},$$

so we can take $\alpha = 0$,

$$\beta = \arctan \sqrt{-\frac{\mu_0 + 3\mu_1}{\mu_0 + 3\mu_2} \cdot \frac{\mu_2}{\mu_1}}$$

and apply Theorem 9.13 to get an anti-Clifford $R = \mu_0 R^1 + \mu_1 R^J + \mu_2 R^K$ which is not totally Jacobi-dual. △

9.4 Four-dimensional zwei-stein

Let R be a four-dimensional zwei-stein algebraic curvature tensor. Theorem 8.6 gives some useful information. From (8.8) for $1 \leq x \leq 4$ we have four equalities

$$\begin{aligned}
\varepsilon_1\varepsilon_2 R_{2112} + \varepsilon_1\varepsilon_3 R_{3113} + \varepsilon_1\varepsilon_4 R_{4114} &= C_1, \\
\varepsilon_2\varepsilon_1 R_{1221} + \varepsilon_2\varepsilon_3 R_{3223} + \varepsilon_2\varepsilon_4 R_{4224} &= C_1, \\
\varepsilon_3\varepsilon_1 R_{1331} + \varepsilon_3\varepsilon_2 R_{2332} + \varepsilon_3\varepsilon_4 R_{4334} &= C_1, \\
\varepsilon_4\varepsilon_1 R_{1441} + \varepsilon_4\varepsilon_2 R_{2442} + \varepsilon_4\varepsilon_3 R_{3443} &= C_1,
\end{aligned}$$

and after we solve the system of equations we get

$$\begin{aligned}\varepsilon_2\varepsilon_3 R_{3223} &= \varepsilon_1\varepsilon_4 R_{4114},\\ \varepsilon_2\varepsilon_4 R_{4224} &= \varepsilon_1\varepsilon_3 R_{3113},\\ \varepsilon_3\varepsilon_4 R_{4334} &= \varepsilon_1\varepsilon_2 R_{2112}.\end{aligned}\tag{9.12}$$

From (8.9) for $1 \leq x \neq y \leq 4$ we get six equalities

$$\begin{aligned}R_{2443} &= -\varepsilon_1\varepsilon_4 R_{2113}, & R_{1442} &= -\varepsilon_3\varepsilon_4 R_{1332},\\ R_{2334} &= -\varepsilon_1\varepsilon_3 R_{2114}, & R_{1443} &= -\varepsilon_2\varepsilon_4 R_{1223},\\ R_{3224} &= -\varepsilon_1\varepsilon_2 R_{3114}, & R_{1334} &= -\varepsilon_2\varepsilon_3 R_{1224}.\end{aligned}\tag{9.13}$$

From (8.10) for $1 \leq x \leq 4$ we have four equalities

$$\begin{aligned}R_{2112}^2 + R_{3113}^2 + R_{4114}^2 + 2\varepsilon_2\varepsilon_3 R_{2113}^2 + 2\varepsilon_2\varepsilon_4 R_{2114}^2 + 2\varepsilon_3\varepsilon_4 R_{3114}^2 &= C_2,\\ R_{1221}^2 + R_{3223}^2 + R_{4224}^2 + 2\varepsilon_1\varepsilon_3 R_{1223}^2 + 2\varepsilon_1\varepsilon_4 R_{1224}^2 + 2\varepsilon_3\varepsilon_4 R_{3224}^2 &= C_2,\\ R_{1331}^2 + R_{2332}^2 + R_{4334}^2 + 2\varepsilon_1\varepsilon_2 R_{1332}^2 + 2\varepsilon_1\varepsilon_4 R_{1334}^2 + 2\varepsilon_2\varepsilon_4 R_{2334}^2 &= C_2,\\ R_{1441}^2 + R_{2442}^2 + R_{3443}^2 + 2\varepsilon_1\varepsilon_2 R_{1442}^2 + 2\varepsilon_1\varepsilon_3 R_{1443}^2 + 2\varepsilon_2\varepsilon_3 R_{2443}^2 &= C_2,\end{aligned}$$

which after use (9.12) and (9.13) yield

$$\begin{aligned}\varepsilon_2\varepsilon_3 R_{2113}^2 + \varepsilon_2\varepsilon_4 R_{2114}^2 + \varepsilon_3\varepsilon_4 R_{3114}^2 &= \varepsilon_1\varepsilon_3 R_{1223}^2 + \varepsilon_1\varepsilon_4 R_{1224}^2 + \varepsilon_3\varepsilon_4 R_{3114}^2\\ &= \varepsilon_1\varepsilon_2 R_{1332}^2 + \varepsilon_1\varepsilon_4 R_{1224}^2 + \varepsilon_2\varepsilon_4 R_{2114}^2\\ &= \varepsilon_1\varepsilon_2 R_{1332}^2 + \varepsilon_1\varepsilon_3 R_{1223}^2 + \varepsilon_2\varepsilon_3 R_{2113}^2.\end{aligned}$$

The previous system of equations has the following solution,

$$\begin{aligned}\varepsilon_2\varepsilon_3 R_{2113}^2 &= \varepsilon_1\varepsilon_4 R_{1224}^2,\\ \varepsilon_2\varepsilon_4 R_{2114}^2 &= \varepsilon_1\varepsilon_3 R_{1223}^2,\\ \varepsilon_3\varepsilon_4 R_{3114}^2 &= \varepsilon_1\varepsilon_2 R_{1332}^2.\end{aligned}\tag{9.14}$$

The derived equations are enough to deal with the weak Jacobi-duality. The following theorem is originally proved by Andrejić in 2006 [3] (see also [15, Theorem 4.1]), but we present the simpler proof from [5, Theorem 3].

Theorem 9.14. *Any four-dimensional zwei-stein algebraic curvature tensor is weak Jacobi-dual.*

Proof. Let $\mathcal{J}_X Y = \varepsilon_X \lambda Y$ holds for unit mutually orthogonal $X, Y \in \mathcal{V}$. Let us set $E_1 = X$, $E_2 = Y$ and extend them to an orthonormal basis (E_1, E_2, E_3, E_4) in $\mathcal{V}$. The assumption $\mathcal{J}_{E_1} E_2 = \varepsilon_1 \lambda E_2$ gives $R_{2112} = \varepsilon_1\varepsilon_2\lambda$, $R_{2113} = 0$, and $R_{2114} = 0$. Thus, by (9.14) we have $R_{1224} = 0$ and $R_{1223} = 0$. Finally, $\mathcal{J}_{E_2} E_1 = \varepsilon_1 R_{1221} E_1 + \varepsilon_3 R_{1223} E_3 + \varepsilon_4 R_{1224} E_4 = \varepsilon_2 \lambda E_1$, therefore $\mathcal{J}_Y X = \varepsilon_Y \lambda X$, so R is weak Jacobi-dual. □

Under the same initial conditions we can solve the Jacobi-duality problem. It is time to clarify the signature. If R is definite ($\operatorname{Ind} g = 0$ or $\operatorname{Ind} g = 4$) then it is Jacobi-diagonalisable, where Theorem 9.2 and Theorem 9.14 imply that R is Jacobi-dual. If R is Lorentzian ($\operatorname{Ind} g = 1$ or $\operatorname{Ind} g = 3$), any zwei-stein R is necessarily of constant sectional curvature (Theorem 8.10), so it is Clifford, and therefore Jacobi-dual (Theorem 9.6).

This is why we should check only the neutral signature $(2, 2)$. Without loss of generality we can set $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -\varepsilon_4$. Applying (8.11) for $x = 2, y = 3$ and for $x = 3, y = 2$, we get

$$\begin{aligned}R_{1221}R_{1231} - R_{1224}R_{1234} - R_{3221}R_{3231} + R_{3224}R_{3234} - R_{4221}R_{4231} + R_{4224}R_{4234} &= 0,\\ R_{1331}R_{1321} - R_{1334}R_{1324} + R_{2331}R_{2321} - R_{2334}R_{2324} - R_{4331}R_{4321} + R_{4334}R_{4324} &= 0,\end{aligned}$$

and after some arrangement and use of (9.12) and (9.13),

$$\begin{aligned}
&R_{2112}R_{2113} - R_{1224}R_{1234} + R_{1223}R_{1332} + R_{3114}R_{2114} - R_{1224}R_{1324} + R_{3113}R_{2113} = 0,\\
&R_{3113}R_{2113} - R_{1224}R_{1324} - R_{1332}R_{1223} - R_{2114}R_{3114} - R_{1224}R_{1234} + R_{2112}R_{2113} = 0.
\end{aligned}$$

The sum and the difference of the previous equations are

$$R_{2112}R_{2113} + R_{3113}R_{2113} - R_{1224}R_{1234} - R_{1224}R_{1324} = 0, \tag{9.15}$$

$$R_{1223}R_{1332} + R_{2114}R_{3114} = 0. \tag{9.16}$$

From (8.12) for $x = 2, y = 3$ we get

$$\begin{aligned}
&2(R_{1221}R_{1331} - 2R_{1224}R_{1334} + R_{4224}R_{4334})\\
&+ 2(R^2_{2112} + R^2_{3113} + R^2_{4114} - 2R^2_{2113} - 2R^2_{2114} + 2R^2_{3114}) + (R_{1231} + R_{1321})^2\\
&+ R^2_{1232} - R^2_{1323} - (R_{1234} + R_{1324})^2 + R^2_{2321} - R^2_{2323} - R^2_{2324} - R^2_{3231} - R^2_{3232}\\
&+ R^2_{3234} - (R_{4231} + R_{4321})^2 - R^2_{4232} + R^2_{4323} + (R_{4234} + R_{4324})^2 = 0,
\end{aligned}$$

which after use of (9.12), (9.13), and (9.14) becomes

$$\begin{aligned}
&2R_{2112}R_{3113} - 4R^2_{2113} + 2R_{3113}R_{2112}\\
&+ 2R^2_{2112} + 2R^2_{3113} + 2R^2_{4114} - 4R^2_{2113} - 4R^2_{2114} + 4R^2_{3114} + 4R^2_{2113}\\
&+ R^2_{2114} - R^2_{3114} - (R_{1234} + R_{1324})^2 + R^2_{2114} - R^2_{4114} - R^2_{3114} - R^2_{3114} - R^2_{4114}\\
&+ R^2_{2114} - (R_{1324} + R_{1234})^2 - R^2_{3114} + R^2_{2114} + 4R^2_{2113} = 0.
\end{aligned}$$

Thus arises

$$4R_{2112}R_{3113} + 2R^2_{2112} + 2R^2_{3113} - 2(R_{1234} + R_{1324})^2 = 0,$$

and finally

$$(R_{2112} + R_{3113})^2 = (R_{1234} + R_{1324})^2. \tag{9.17}$$

The derived formulas are sufficient to prove the following theorem given by Andrejić in 2010 [5, Theorem 4].

Theorem 9.15. *Any four-dimensional zwei-stein algebraic curvature tensor is Jacobi-dual.*

Proof. From Theorem 9.14, the weak Jacobi-duality holds, so by Lemma 9.1 it is enough to prove the implication (9.2) for $X \perp Y$ with $(\varepsilon_X)^2 = 1$ and $\varepsilon_Y = 0$. Let us set $E_1 = X$. Lemma 4.10 decomposes $Y = E_2 + E_3$ with mutually orthogonal unit E_2 and E_3, which are orthogonal to X, such that $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3$. Let E_4 be a vector which extends them to an orthonormal basis (E_1, E_2, E_3, E_4). Because of the signature $(2, 2)$, we have $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -\varepsilon_4$, which enables use of the previous equations. Our aim is to prove the following formula,

$$\mathcal{J}_{E_1}(E_2 + E_3) = \varepsilon_1\lambda(E_2 + E_3) \Longrightarrow \mathcal{J}_{E_2+E_3}E_1 = 0.$$

The assumption is

$$\begin{aligned}
\varepsilon_1\lambda(E_2 + E_3) &= \mathcal{J}_{E_1}(E_2 + E_3)\\
&= \varepsilon_2(R_{2112} + R_{3112})E_2 + \varepsilon_3(R_{2113} + R_{3113})E_3 + \varepsilon_4(R_{2114} + R_{3114})E_4,
\end{aligned}$$

which implies

$$\varepsilon_2(R_{2112} + R_{3112}) = \varepsilon_1\lambda = \varepsilon_3(R_{2113} + R_{3113}), \quad \varepsilon_4(R_{2114} + R_{3114}) = 0,$$

and finally

$$R_{2112} + R_{3113} + 2R_{2113} = 0, \tag{9.18}$$

$$R_{2114} + R_{3114} = 0. \tag{9.19}$$

The substitution (9.19) into (9.16) gives

$$R_{1223}R_{1332} = -R_{2114}R_{3114} = R_{2114}^2 = R_{1223}^2,$$

and therefore $R_{1223} = 0$ or $R_{1332} = R_{1223}$. However, from (9.14) and (9.19) we have

$$R_{1223} = 0 \Rightarrow R_{2114} = 0 \Rightarrow R_{3114} = 0 \Rightarrow R_{1332} = 0$$

and therefore the equation

$$R_{1332} = R_{1223}, \tag{9.20}$$

holds anyway. By the substitution (9.18) into (9.15) we get

$$(R_{1234} + R_{1324})R_{1224} = (R_{2112} + R_{3113})R_{2113} = -2R_{2113}^2 = -2R_{1224}^2.$$

Thus $R_{1224} = 0$ or $R_{1234} + R_{1324} = -2R_{1224}$. Similarly we use (9.14), (9.18), and (9.17) for

$$R_{1224} = 0 \Rightarrow R_{2113} = 0 \Rightarrow R_{2112} + R_{3113} = 0 \Rightarrow R_{1234} + R_{1324} = 0,$$

and undoubtedly

$$R_{1234} + R_{1324} = -2R_{1224} = -(R_{1224} + R_{1334}). \tag{9.21}$$

The final computation shows

$$\begin{aligned}
\mathcal{J}_{E_2+E_3}E_1 &= \sum_{p=1}^{4} \varepsilon_p (R_{122p} + R_{123p} + R_{132p} + R_{133p})E_p \\
&= \varepsilon_1(R_{2112} + R_{3113} + 2R_{2113})E_1 + \varepsilon_2(R_{1332} - R_{1223})E_2 \\
&\quad + \varepsilon_3(R_{1223} - R_{1332})E_3 + \varepsilon_4(R_{1224} + R_{1334} + R_{1234} + R_{1324})E_4.
\end{aligned}$$

The equations (9.18), (9.20), and (9.21) finally give $\mathcal{J}_{E_2+E_3}E_1 = 0$, which completes the proof. □

Of course, an Osserman R is zwei-stein and therefore any four-dimensional Osserman algebraic curvature tensor is Jacobi-dual.

9.5 Converse problem

In the previous sections we proved that Osserman algebraic curvature tensors are Jacobi-dual under assumptions of small index or low dimension. In the Riemannian setting ($\operatorname{Ind} g = 0$), any Osserman R is totally Jacobi-dual (Theorem 9.4). In the Lorentzian setting ($\operatorname{Ind} g = 1$), any Osserman R has a constant sectional curvature (Theorem 8.10) and therefore it is totally Jacobi-dual (Theorem 9.11).

In the case of dimension $n = 4$, any Osserman R is Jacobi-dual (Theorem 9.15). However, a four-dimensional Osserman R is not necessarily totally Jacobi-dual (Example 9.1), so the main converse question should be whether a Jacobi-dual R necessarily implies that R is Osserman.

Let us start with an elementary and elegant proof that every Riemannian Jacobi-dual R is at least Einstein (Andrejić and Lukić 2024 [14]).

Lemma 9.16. *Any Riemannian Jacobi-dual algebraic curvature tensor is Einstein.*

Proof. For an arbitrary unit $E_1 \in \mathcal{V}$ there exists an orthonormal eigenbasis $(E_1, \dots, E_n)$ in $\mathcal{V}$ related to $\mathcal{J}_{E_1}$, so there are $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ that depend on E_1, such that $\mathcal{J}_{E_1}E_i = \lambda_i E_i$ holds for $1 \leq i \leq n$. If R is Jacobi-dual then $\mathcal{J}_{E_i}E_1 = \lambda_i E_1$ and considering the sharp of the Ricci tensor as a linear operator we have $\mathrm{Ric}^\sharp = \mathcal{J}_{E_1} + \cdots + \mathcal{J}_{E_n}$, and therefore $\mathrm{Ric}^\sharp(E_1) = (\lambda_1 + \cdots + \lambda_n)E_1$. Hence, any unit $E_1 \in \mathcal{V}$ is an eigenvector of $\mathrm{Ric}^\sharp$, so all eigenvalues of $\mathrm{Ric}^\sharp$ are the same, which gives that $\lambda_1 + \cdots + \lambda_n$ is constant, and proves that R is Einstein. □

In solving the converse problem, the following universal lemma given by Andrejić in 2009 [4, Lemma 1] is very useful.

Lemma 9.17. *If $\mathcal{J}_X(Y) = \varepsilon_X \lambda Y$ and $\mathcal{J}_Y(X) = \varepsilon_Y \lambda X$ for $X \perp Y$, then for all $\alpha, \beta \in \mathbb{R}$ holds*

$$\mathcal{J}_{\alpha X+\beta Y}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y) = \varepsilon_{\alpha X+\beta Y}\lambda(\varepsilon_Y \beta X - \varepsilon_X \alpha Y).$$

Proof. This is a consequence of straightforward calculations,

$$\begin{aligned}\mathcal{J}_{\alpha X+\beta Y}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y) &= \mathcal{R}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y, \alpha X + \beta Y)(\alpha X + \beta Y)\\ &= (\varepsilon_X \alpha^2 + \varepsilon_Y \beta^2)\mathcal{R}(X, Y)(\alpha X + \beta Y)\\ &= \varepsilon_{\alpha X+\beta Y}(\beta \mathcal{J}_Y(X) - \alpha \mathcal{J}_X(Y)) = \varepsilon_{\alpha X+\beta Y}\lambda(\varepsilon_Y \beta X - \varepsilon_X \alpha Y).\end{aligned}$$

□

We consider the converse problem in low dimensions. Let us start with three-dimensional Jacobi-dual algebraic curvature tensor R. Since three-dimensional Einstein R has constant sectional curvature (Example 6.13), we expect the following theorem given by Andrejić in 2009 [4, Theorem 1].

Theorem 9.18. *Three-dimensional Jacobi-dual algebraic curvature tensor has constant sectional curvature.*

Proof. Let (E_1, E_2, E_3) be an arbitrary orthonormal basis in $\mathcal{V}$, such that $\varepsilon_2 = \varepsilon_3$. Since the Jacobi operator $\mathcal{J}_{E_1}$ has the matrix

$$\mathcal{J}_{E_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varepsilon_2 R_{2112} & \varepsilon_2 R_{3112} \\ 0 & \varepsilon_3 R_{2113} & \varepsilon_3 R_{3113} \end{pmatrix},$$

the characteristic polynomial of its reduced Jacobi operator $\widetilde{\mathcal{J}}_{E_1}$ is

$$\widetilde{\omega}_{E_1}(\lambda) = \lambda^2 - (\varepsilon_2 R_{2112} + \varepsilon_3 R_{3113})\lambda + \varepsilon_2\varepsilon_3 R_{2112}R_{3113} - \varepsilon_2\varepsilon_3 R_{3112}R_{2113}.$$

The discriminant D of the quadratic equation $\widetilde{\omega}_{E_1}(\lambda) = 0$ is

$$\begin{aligned}D &= (\varepsilon_2 R_{2112} + \varepsilon_3 R_{3113})^2 - 4\varepsilon_2\varepsilon_3 R_{2112}R_{3113} + 4\varepsilon_2\varepsilon_3 R_{3112}R_{2113}\\ &= (\varepsilon_2 R_{2112} + \varepsilon_3 R_{3113})^2 + 4(R_{2113})^2 \geq 0.\end{aligned}$$

For $D > 0$ our quadratic equation has two distinct real roots, which give two distinct eigenvalues of $\widetilde{\mathcal{J}}_{E_1}$, so there are two one-dimensional eigenspaces. Otherwise, $D = 0$ gives $\varepsilon_2 R_{2112} = \varepsilon_3 R_{3113}$ and $R_{2113} = 0$, and therefore $\widetilde{\mathcal{J}}_{E_1}$ is diagonal with a double root, so we have a two-dimensional eigenspace. Thus, there exists an orthonormal eigenbasis (E_1, E_2, E_3) related to $\mathcal{J}_{E_1}$.

Since R is Jacobi-dual, E_1 is an eigenvector of $\mathcal{J}_{E_2}$ which fulfils the requirements of Lemma 9.17, and hence $\varepsilon_2\beta E_1 - \varepsilon_1\alpha E_2$ is an eigenvector of $\mathcal{J}_{\alpha E_1+\beta E_2}$. Moreover, for $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2\varepsilon_1 + \beta^2\varepsilon_2 \neq 0$ holds, both $\alpha E_1 + \beta E_2$ and $\varepsilon_2\beta E_1 - \varepsilon_1\alpha E_2$ are nonnull and mutually orthogonal. Since $\mathcal{J}_{\alpha E_1+\beta E_2}$ is self-adjoint, $\mathcal{J}_{\alpha E_1+\beta E_2}E_3$ is orthogonal to both $\varepsilon_2\beta E_1 - \varepsilon_1\alpha E_2$ and $\alpha E_1 + \beta E_2$, and therefore E_3 is an eigenvector of $\mathcal{J}_{\alpha E_1+\beta E_2}$. By the Jacobi-duality, $\alpha E_1 + \beta E_2$ is an eigenvector of $\mathcal{J}_{E_3}$, which is possible only if $\mathrm{Span}\{E_1, E_2\}$ is a two-dimensional eigenspace of $\widetilde{\mathcal{J}}_{E_3}$. Similarly, one can prove that $\mathrm{Span}\{E_1, E_3\}$ is an eigenspace of $\widetilde{\mathcal{J}}_{E_2}$. Thus arises $\varepsilon_1\varepsilon_2 R_{1221} = \varepsilon_2\varepsilon_3 R_{2332} = \varepsilon_1\varepsilon_3 R_{1331} = \kappa$ and $R_{2113} = R_{1223} = R_{1332} = 0$. This equations completely determine R, and therefore R has constant sectional curvature κ. □

Let R be a four-dimensional Jacobi-dual algebraic curvature tensor on $(\mathcal{V}, g)$. Suppose that there exists some nonnull X such that $\mathcal{J}_X$ is diagonalisable. Then, there exists an orthonormal eigenbasis (E_1, E_2, E_3, E_4) in $\mathcal{V}$ such that $\mathcal{J}_{E_1}E_i = \varepsilon_1\lambda_{1i}E_i$ holds for $i = 2, 3, 4$. Without loss of generality we assume $\varepsilon_3 = \varepsilon_4$.

Since the Jacobi-duality gives $\mathcal{J}_{E_2}E_1 = \varepsilon_2\lambda_{12}E_1$, we see that $\mathcal{J}_{E_2}$ is invariant on the definite subspace $\mathrm{Span}\{E_3, E_4\}$. Hence, $\mathcal{J}_{E_2}$ is diagonalisable with two mutually orthogonal eigenvectors $\alpha_3E_3 + \alpha_4E_4$ and $\varepsilon_4\alpha_4E_3 - \varepsilon_3\alpha_3E_4$ for some $\alpha_3, \alpha_4 \in \mathbb{R}$ such that $\alpha_3^2\varepsilon_3 + \alpha_4^2\varepsilon_4 \neq 0$. Thus, we obtain the proportionalities

$$\begin{aligned}
&\alpha_3^2\mathcal{J}_{E_3}(E_2) + \alpha_4^2\mathcal{J}_{E_4}(E_2) + 2\alpha_3\alpha_4\mathcal{J}(E_3, E_4)(E_2) \propto E_2,\\
&\alpha_4^2\mathcal{J}_{E_3}(E_2) + \alpha_3^2\mathcal{J}_{E_4}(E_2) - 2\varepsilon_3\varepsilon_4\alpha_3\alpha_4\mathcal{J}(E_3, E_4)(E_2) \propto E_2,
\end{aligned}$$

and therefore

$$(\varepsilon_3\varepsilon_4\alpha_3^2 + \alpha_4^2)\mathcal{J}_{E_3}(E_2) + (\varepsilon_3\varepsilon_4\alpha_4^2 + \alpha_3^2)\mathcal{J}_{E_4}(E_2) \propto E_2.$$

Thus, $\mathcal{J}_{E_3}E_2 \perp E_4$ with $\mathcal{J}_{E_3}(\mathrm{Span}\{E_2, E_4\}) \subseteq \mathrm{Span}\{E_2, E_4\}$ (from $\mathcal{J}_{E_3}E_1 = \varepsilon_3\lambda_{13}E_1$) implies $\mathcal{J}_{E_3}E_2 \propto E_2$ and consequently $\mathcal{J}_{E_3}E_4 \propto E_4$. Hence $\mathcal{J}_{E_2}E_3 \propto E_3$ and $\mathcal{J}_{E_2}E_4 \propto E_4$, which means that our basis diagonalizes any of operators $\mathcal{J}_{E_1}, \mathcal{J}_{E_2}, \mathcal{J}_{E_3}, \mathcal{J}_{E_4}$. In this way, we obtain $\mathcal{J}_{E_i}(E_j) = \varepsilon_i\lambda_{ij}E_j$ where $\lambda_{ij} = \lambda_{ji}$ for all $1 \leq i, j \leq 4$.

This conclusion allows simple calculations,

$$\mathcal{J}(E_1, E_2)(E_3) = \sum_i \varepsilon_i\frac{1}{2}(R_{312i} + R_{321i})E_i = \frac{1}{2}\varepsilon_4(R_{3124} + R_{3214})E_4,$$

which holds in all permutations of indices from $\{1, 2, 3, 4\}$. Since,

$$\begin{aligned}
\mathcal{J}_{\alpha_1E_1+\alpha_2E_2}(E_3) &= (\alpha_1^2\varepsilon_1\lambda_{13} + \alpha_2^2\varepsilon_2\lambda_{23})E_3 + \alpha_1\alpha_2\varepsilon_4(R_{3124} + R_{3214})E_4,\\
\mathcal{J}_{\alpha_1E_1+\alpha_2E_2}(E_4) &= (\alpha_1^2\varepsilon_1\lambda_{14} + \alpha_2^2\varepsilon_2\lambda_{24})E_4 + \alpha_1\alpha_2\varepsilon_3(R_{4123} + R_{4213})E_3,
\end{aligned}$$

after the substitution $Q_{12} = R_{3124} + R_{3214} = R_{4123} + R_{4213}$, we have

$$\begin{aligned}
\mathcal{J}_{\alpha_1E_1+\alpha_2E_2}(\alpha_3E_3 + \alpha_4E_4) = {}&(\alpha_3(\alpha_1^2\varepsilon_1\lambda_{13} + \alpha_2^2\varepsilon_2\lambda_{23}) + \alpha_4\alpha_1\alpha_2\varepsilon_3Q_{12})E_3\\
&+ (\alpha_4(\alpha_1^2\varepsilon_1\lambda_{14} + \alpha_2^2\varepsilon_2\lambda_{24}) + \alpha_3\alpha_1\alpha_2\varepsilon_4Q_{12})E_4.
\end{aligned}$$

We should discuss two cases $Q_{12} \neq 0$ and $Q_{12} = 0$.

First, let us suppose that $Q_{12} \neq 0$. For any $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1^2\varepsilon_1 + \alpha_2^2\varepsilon_2 \neq 0$, by Lemma 9.17 we have $\mathcal{J}_{\alpha_1E_1+\alpha_2E_2}(\varepsilon_2\alpha_2E_1 - \varepsilon_1\alpha_1E_2) = \varepsilon_{\alpha_1E_1+\alpha_2E_2}\lambda_{12}(\varepsilon_2\alpha_2E_1 - \varepsilon_1\alpha_1E_2)$, so there exist $\alpha_3, \alpha_4 \in \mathbb{R}$ such that $\alpha_3^2\varepsilon_3 + \alpha_4^2\varepsilon_4 \neq 0$ and $\mathcal{J}_{\alpha_1E_1+\alpha_2E_2}(\alpha_3E_3 + \alpha_4E_4) \propto \alpha_3E_3 + \alpha_4E_4$. Thus

$$\alpha_4(\alpha_3(\alpha_1^2\varepsilon_1\lambda_{13} + \alpha_2^2\varepsilon_2\lambda_{23}) + \alpha_4\alpha_1\alpha_2\varepsilon_3Q_{12}) = \alpha_3(\alpha_4(\alpha_1^2\varepsilon_1\lambda_{14} + \alpha_2^2\varepsilon_2\lambda_{24}) + \alpha_3\alpha_1\alpha_2\varepsilon_4Q_{12}),$$

which implies

$$\alpha_3\alpha_4(\alpha_1^2\varepsilon_1(\lambda_{13} - \lambda_{14}) + \alpha_2^2\varepsilon_2(\lambda_{23} - \lambda_{24})) = \alpha_1\alpha_2Q_{12}(\alpha_3^2\varepsilon_4 - \alpha_4^2\varepsilon_3).$$

However, the Jacobi-duality implies that $\alpha_1E_1 + \alpha_2E_2$ is an eigenvector of $\mathcal{J}_{\alpha_3E_3+\alpha_4E_4}$, and because of $Q_{34} = R_{2341} + R_{2431} = R_{1342} + R_{1432} = Q_{12}$, we can use the symmetric formula to get

$$\alpha_1\alpha_2(\alpha_3^2\varepsilon_3(\lambda_{31} - \lambda_{32}) + \alpha_4^2\varepsilon_4(\lambda_{41} - \lambda_{42})) = \alpha_3\alpha_4Q_{12}(\alpha_1^2\varepsilon_2 - \alpha_2^2\varepsilon_1).$$

Let us rescale this with $\alpha_1 = x, \alpha_2 = 1$ and $\alpha_3 = y, \alpha_4 = 1$, which means that for any x (with $x^2 \neq -\varepsilon_1\varepsilon_2$) there exists y (with $y^2 \neq -\varepsilon_3\varepsilon_4$) such that

$$\begin{aligned}
(\varepsilon_4Q_{12}x)y^2 - (\varepsilon_1x^2(\lambda_{13} - \lambda_{14}) + \varepsilon_2(\lambda_{23} - \lambda_{24}))y - \varepsilon_3Q_{12}x = 0,\\
(\varepsilon_3(\lambda_{13} - \lambda_{23})x)y^2 - (Q_{12}(\varepsilon_2x^2 - \varepsilon_1))y + \varepsilon_4(\lambda_{14} - \lambda_{24})x = 0.
\end{aligned}$$

Since the solutions (of quadratic equation by y) goes in pairs (mutually orthogonal eigenvectors), the two previous equations give the same solutions, which implies

$$\begin{aligned} \varepsilon_4 Q_{12} x &= K\varepsilon_3(\lambda_{13}-\lambda_{23})x, \\ \varepsilon_1 x^2(\lambda_{13}-\lambda_{14})+\varepsilon_2(\lambda_{23}-\lambda_{24}) &= KQ_{12}(\varepsilon_2 x^2-\varepsilon_1), \\ -\varepsilon_3 Q_{12} x &= K\varepsilon_4(\lambda_{14}-\lambda_{24})x, \end{aligned}$$

for infinitely many x and some $K = K(x) \in \mathbb{R}$. Immediately, $\lambda_{13}-\lambda_{23} = -(\lambda_{14}-\lambda_{24})$, which gives

$$\lambda_{13}+\lambda_{14} = \lambda_{23}+\lambda_{24}.$$

Additionally, we have

$$\varepsilon_3(\lambda_{13}-\lambda_{23})(\varepsilon_1 x^2(\lambda_{13}-\lambda_{14})+\varepsilon_2(\lambda_{23}-\lambda_{24})) = \varepsilon_4 Q_{12}^2(\varepsilon_2 x^2-\varepsilon_1),$$

and the polynomial equation

$$(\varepsilon_1\varepsilon_3(\lambda_{13}-\lambda_{23})(\lambda_{13}-\lambda_{14})-\varepsilon_2\varepsilon_4 Q_{12}^2)x^2+\varepsilon_2\varepsilon_3(\lambda_{13}-\lambda_{23})(\lambda_{23}-\lambda_{24})+\varepsilon_1\varepsilon_4 Q_{12}^2 = 0$$

has infinitely many solutions, so we obtain

$$\begin{aligned} (\lambda_{13}-\lambda_{23})(\lambda_{13}-\lambda_{14}) &= \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 Q_{12}^2, \\ (\lambda_{13}-\lambda_{23})(\lambda_{23}-\lambda_{24}) &= -\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 Q_{12}^2. \end{aligned}$$

Hence $\lambda_{13}-\lambda_{14} = \lambda_{24}-\lambda_{23}$, which together with $\lambda_{13}+\lambda_{14} = \lambda_{24}+\lambda_{23}$ implies

$$\lambda_{13} = \lambda_{24} \quad \text{and} \quad \lambda_{14} = \lambda_{23}.$$

Moreover, we have

$$(\lambda_{13}-\lambda_{14})^2 = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 Q_{12}^2, \tag{9.22}$$

that immediately implies $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1$, which means that $Q_{12} \neq 0$ is impossible in the Lorentzian setting.

We turn now to the case $Q_{12} = 0$, where $\mathcal{J}_{\alpha_1 E_1+\alpha_2 E_2}(E_3) = (\alpha_1^2\varepsilon_1\lambda_{13}+\alpha_2^2\varepsilon_2\lambda_{23})E_3$ holds, and the Jacobi-duality gives

$$\alpha_1\varepsilon_3\lambda_{13}E_1+\alpha_2\varepsilon_3\lambda_{23}E_2 = \mathcal{J}_{E_3}(\alpha_1 E_1+\alpha_2 E_2) = \varepsilon_3\frac{\alpha_1^2\varepsilon_1\lambda_{13}+\alpha_2^2\varepsilon_2\lambda_{23}}{\alpha_1^2\varepsilon_1+\alpha_2^2\varepsilon_2}(\alpha_1 E_1+\alpha_2 E_2),$$

that implies $\lambda_{13} = \lambda_{23}$. Similarly, $\mathcal{J}_{\alpha_1 E_1+\alpha_2 E_2}(E_4) = (\alpha_1^2\varepsilon_1\lambda_{14}+\alpha_2^2\varepsilon_2\lambda_{24})E_4$ gives $\lambda_{14} = \lambda_{24}$.

The conclusion of the discussion is that $Q_{12} \neq 0$ gives $\lambda_{13} = \lambda_{24}$ and $\lambda_{14} = \lambda_{23}$, while otherwise $Q_{12} = 0$ yields $\lambda_{13} = \lambda_{23}$ and $\lambda_{14} = \lambda_{24}$. We obtain similar conclusions using the symmetries for $Q_{13} = R_{2134}+R_{2314}$ and $Q_{14} = R_{3142}+R_{3412}$ relative to $Q_{12} = R_{3124}+R_{3214}$. The case $Q_{13} \neq 0$ gives $\lambda_{12} = \lambda_{34}$ and $\lambda_{14} = \lambda_{23}$, while $Q_{13} = 0$ yields $\lambda_{12} = \lambda_{23}$ and $\lambda_{14} = \lambda_{34}$, as well as $Q_{14} \neq 0$ gives $\lambda_{13} = \lambda_{24}$ and $\lambda_{12} = \lambda_{34}$, while $Q_{14} = 0$ yields $\lambda_{13} = \lambda_{34}$ and $\lambda_{12} = \lambda_{24}$.

The case $Q_{12} = Q_{13} = Q_{14} = 0$ immediately gives $\lambda_{12} = \lambda_{13} = \lambda_{14} = \lambda_{23} = \lambda_{24} = \lambda_{34}$, while $0 = R_{1234}+R_{2314}+R_{3124} = R_{1234}+(Q_{13}-R_{2134})+(R_{3412}-Q_{14}) = 3R_{1234}$ follows from the first Bianchi identity, so $R_{1234} = -R_{1324} = 0$. Hence $R = \lambda_{12}R^1$, and therefore R has constant sectional curvature. Notice that this is the only possible case in the Lorentzian setting, because if $Q_{1i} \neq 0$ holds for some $2 \leq i \leq 4$, then immediately, due to (9.22), $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1$ holds.

Since the first Bianchi identity gives $Q_{12}+Q_{13}+Q_{14} = 0$, the other cases have at least two of Q_{12}, Q_{13}, and Q_{14} which are nonzero. Thus, it is obvious that in any case (including the one with $Q_{12} = Q_{13} = Q_{14} = 0$) we have

$$\lambda_{12} = \lambda_{34}, \quad \lambda_{13} = \lambda_{24}, \quad \lambda_{14} = \lambda_{23}.$$

Thanks to the formula (9.22) which holds for $Q_{12} \neq 0$, as well as its symmetric formulas, we additionally have $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = 1$ and the equations $(\lambda_{14} - \lambda_{13})^2 = Q_{12}^2$, $(\lambda_{12} - \lambda_{14})^2 = Q_{13}^2$, $(\lambda_{13} - \lambda_{12})^2 = Q_{14}^2$, because for a possible $Q_{1i} = 0$ we have $\lambda_{1j} = \lambda_{ik} = \lambda_{ij} = \lambda_{1k}$, where $\{i, j, k\} = \{1, 2, 3\}$. Changing the sign of one vector in our basis, if necessary, we can set

$$\lambda_{14} - \lambda_{13} = Q_{12}, \quad \lambda_{12} - \lambda_{14} = Q_{13}, \quad \lambda_{13} - \lambda_{12} = Q_{14}.$$

For example, the case $Q_{12} \neq 0$ implies $\lambda_{14} - \lambda_{13} = -Q_{12}$, which after we change E_4 by $-E_4$, gives $\lambda_{14} - \lambda_{13} = Q_{12}$. Hence, if $\lambda_{12} - \lambda_{14} \neq Q_{13}$ then we have $\lambda_{12} - \lambda_{14} = -Q_{13}$, and we obtain $\pm(\lambda_{13} - \lambda_{12}) = Q_{14} = -Q_{12} - Q_{13} = \lambda_{12} - 2\lambda_{14} + \lambda_{13}$. Since $\lambda_{14} \neq \lambda_{13}$ it implies $\lambda_{14} = \lambda_{12}$, so $Q_{13} = 0 = \lambda_{12} - \lambda_{14}$. Thus

$$\begin{aligned} \lambda_{14} - \lambda_{13} &= Q_{12} = -R_{1324} + R_{1432} = R_{1234} - 2R_{1324}, \\ \lambda_{12} - \lambda_{14} &= Q_{13} = -R_{1234} + R_{1423} = -2R_{1234} + R_{1324}, \end{aligned}$$

and therefore

$$R_{1234} = -\frac{2}{3}\lambda_{12} + \frac{1}{3}\lambda_{13} + \frac{1}{3}\lambda_{14}, \quad R_{1324} = -\frac{1}{3}\lambda_{12} + \frac{2}{3}\lambda_{13} - \frac{1}{3}\lambda_{14}.$$

In this way, we calculated the last two of the 20 independent coordinate curvature tensor components (see Theorem 6.4), which completely determines R. The calculations were made depending on three parameters $\lambda_{12}, \lambda_{13}, \lambda_{14} \in \mathbb{R}$, while previously we have 6 independent terms of the form $R_{ijji} = \lambda_{ij} = \lambda_{kl}$ and another 12 of the form $R_{jiik} = 0$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Every semi-Clifford algebraic curvature tensor is both Osserman (Theorem 9.5) and Jacobi-dual (Theorem 9.6), so the basic idea is to construct a semi-Clifford algebraic curvature tensor that will have components equal to those of our R, and from the uniqueness, we see that R is Osserman.

If the signature is definite, without loss of generality we suppose $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$, and introduce a quaternionic Clifford family $\{J_1, J_2, J_3\}$ given by

$$\begin{aligned} &J_1E_1 = E_2, && J_1E_2 = -E_1, && J_1E_3 = E_4, && J_1E_4 = -E_3, \\ &J_2E_1 = E_3, && J_2E_2 = -E_4, && J_2E_3 = -E_1, && J_2E_4 = E_2, \\ &J_3E_1 = E_4, && J_3E_2 = E_3, && J_3E_3 = -E_2, && J_3E_4 = -E_1, \end{aligned}$$

to see that our R is induced by Clifford

$$R = -\frac{\lambda_{12}}{3}R^{J_1} - \frac{\lambda_{13}}{3}R^{J_2} - \frac{\lambda_{14}}{3}R^{J_3}.$$

If the signature is neutral, we can suppose $\varepsilon_1 = \varepsilon_2 = -1$, $\varepsilon_3 = \varepsilon_4 = 1$, and introduce paraquaternionic semi-Clifford family $\{J_1, J_2, J_3\}$ given by

$$\begin{aligned} &J_1E_1 = -E_2, && J_1E_2 = E_1, && J_1E_3 = E_4, && J_1E_4 = -E_3, \\ &J_2E_1 = E_3, && J_2E_2 = E_4, && J_2E_3 = E_1, && J_2E_4 = E_2, \\ &J_3E_1 = E_4, && J_3E_2 = -E_3, && J_3E_3 = -E_2, && J_3E_4 = E_1, \end{aligned}$$

to see that our R is induced by semi-Clifford

$$R = -\frac{\lambda_{12}}{3}R^{J_1} + \frac{\lambda_{13}}{3}R^{J_2} + \frac{\lambda_{14}}{3}R^{J_3}.$$

The previous considerations prove the following theorem given by Andrejić in 2018 [9, Theorem 4.2].

Theorem 9.19. *Any four-dimensional Jacobi-dual algebraic curvature tensor such that $\mathcal{J}_X$ is diagonalisable for some nonnull X is Osserman.*

Consequently, any Riemannian four-dimensional Jacobi-dual R is Osserman, which is a variant of this theorem originally proved by Brozos-Vázquez[2] and Merino[3] in 2012 [27]. Additionally, we notice that any Lorentzian four-dimensional Jacobi-dual R has constant sectional curvature.

According to Theorem 8.10 a Lorentzian Osserman curvature tensor has a constant sectional curvature, so it is natural to have the following theorem given by Andrejić and Rakić in 2015 [16, Theorem 4.1].

Theorem 9.20. *A Lorentzian totally Jacobi-dual algebraic curvature tensor has constant sectional curvature.*

Proof. Let T be a unit timelike vector in a Lorentzian scalar product space $(\mathcal{V}, g)$. Then $T^\perp$ is positive definite, so $\widetilde{\mathcal{J}}_T$ is diagonalisable. Let $(S_1, \dots, S_{n-1})$ be an orthonormal eigenbasis of $T^\perp$ related to $\widetilde{\mathcal{J}}_T$. Then $\mathcal{J}_T S_i = \varepsilon_T \lambda_i S_i$ and the Jacobi-duality gives $\mathcal{J}_{S_i} T = \varepsilon_{S_i} \lambda_i T$ for all $1 \leq i \leq n-1$. Because of

$$\mathcal{J}_{T \pm S_i} T = \mathcal{R}(T, T \pm S_i)(T \pm S_i) = \mp \mathcal{J}_T S_i + \mathcal{J}_{S_i} T = \lambda_i (T \pm S_i),$$
$$\mathcal{J}_{T \pm S_i} S_i = \mathcal{R}(S_i, T \pm S_i)(T \pm S_i) = \mathcal{J}_T S_i \mp \mathcal{J}_{S_i} T = \mp \lambda_i (T \pm S_i),$$

it follows that $\mathcal{J}_{T \pm S_i}$ is invariant on $\mathcal{U}_i = \operatorname{Span}\{T, S_i\}$, and since it is self-adjoint, it is also invariant on $\mathcal{U}_i^\perp = \operatorname{Span} \bigcup_{j \neq i} \{S_j\}$. The restriction $\mathcal{J}_{T \pm S_i} \restriction_{\mathcal{U}_i^\perp}$ is diagonalisable as a self-adjoint operator on a definite space. If M is a nonnull eigenvector of $\mathcal{J}_{T \pm S_i} \restriction_{\mathcal{U}_i^\perp}$, then $\mathcal{J}_{T \pm S_i} M = \mu M$ together with the totally Jacobi-dual condition gives $\mathcal{J}_M (T \pm S_i) = \nu (T \pm S_i)$. However,

$$\mu \varepsilon_M = g(\mathcal{J}_{T \pm S_i} M, M) = g(\mathcal{J}_M (T \pm S_i), T \pm S_i) = \nu \varepsilon_{T \pm S_i} = 0,$$

so $\mu = 0$, and therefore $\mathcal{J}_{T \pm S_i} M = 0$. Consequently, $\mathcal{J}_{T \pm S_i} = 0$ on $\mathcal{U}_i^\perp$, and therefore $\mathcal{J}_{T \pm S_i} S_j = 0$ holds, for all $1 \leq i \neq j \leq n-1$. Then the relation $\mathcal{J}_{T+S_i} S_j = 0$ and the total Jacobi-duality implies that $T + S_i$ is an eigenvector of $\mathcal{J}_{S_j}$. Since $g(T, T + S_i) = -1 \neq 0$, eigenvectors T and $T + S_i$ of $\mathcal{J}_{S_j}$ are not orthogonal, so they have the same eigenvalues. Thus, $\mathcal{J}_{S_j} S_i = \lambda_j S_i$, which implies $\lambda_i = \lambda_j$ and R has constant sectional curvature.

Alternatively, from $\mathcal{J}_{T \pm S_i} = \mathcal{J}_T + \mathcal{J}_{S_i} \pm 2\mathcal{J}(T, S_i)$ we get $\mathcal{J}_{T+S_i} + \mathcal{J}_{T-S_i} = 2(\mathcal{J}_T + \mathcal{J}_{S_i})$. Thus $\mathcal{J}_{T \pm S_i} S_j = 0$ implies $\mathcal{J}_{S_i} S_j = -\mathcal{J}_T S_j = \lambda_j S_j$. Comparing this equation after (i, j)-symmetry $\mathcal{J}_{S_j} S_i = \lambda_i S_i$ and after the Jacobi-dual property $\mathcal{J}_{S_j} S_i = \lambda_j S_i$, we easily conclude that $\lambda_i = \lambda_j$ for $1 \leq i \neq j \leq n-1$, which proves that R has constant sectional curvature. $\square$

9.6 Osserman perturbations

Let R be a Jacobi-diagonalisable algebraic curvature tensor on a scalar product space $(\mathcal{V}, g)$. Consider an analytic curve $\gamma \colon I \to \mathcal{V}$ for some open interval $I \subseteq \mathbb{R}$. Form the one-parameter family of self-adjoint endomorphisms $A(t) = \mathcal{J}_{\gamma(t)}$ for $t \in I$. According to Lemma 6.5, the entries of the matrix $\mathcal{J}_{\gamma(t)}$ are homogeneous polynomials of degree 2 in the components of $\gamma(t)$, and therefore analytic on I. On the other hand, since R is Jacobi-diagonalisable, each $\mathcal{J}_{\gamma(t)}$ has a hyperbolic characteristic polynomial.

The conditions of Theorem A.26 are satisfied, which yields an analytic eigenvalue function $\lambda \colon I \to \mathbb{R}$ and an analytic eigenvector curve $V \colon I \to \mathcal{V}$ such that $\mathcal{J}_{\gamma(t)} V(t) = \lambda(t) V(t)$ holds for $t \in I$. Hence,

$$R(V(t), \gamma(t), \gamma(t), V(t)) = g(\mathcal{J}_{\gamma(t)} V(t), V(t)) = \lambda(t) g(V(t), V(t)) = \lambda(t) \varepsilon_{V(t)}.$$

[2] Miguel Brozos-Vázquez, Spanish mathematician
[3] Eugenio Merino, Spanish mathematician

Taking the derivative with respect to t, we obtain

$$2g(\mathcal{J}_{\gamma(t)}V(t), V'(t)) + 2g(\mathcal{J}_{V(t)}\gamma(t), \gamma'(t)) = \lambda'(t)g(V(t), V(t)) + 2\lambda(t)g(V(t), V'(t)),$$

which leads to the identity

$$2g(\mathcal{J}_{V(t)}\gamma(t), \gamma'(t)) = \varepsilon_{V(t)}\lambda'(t). \tag{9.23}$$

Suppose that R is Osserman, which gives $\lambda(t) = \varepsilon_{\gamma(t)}C$ for some constant $C \in \mathbb{R}$, and implies $\lambda'(t) = 2g(\gamma(t), \gamma'(t))C$. Using (9.23), we obtain $g(\mathcal{J}_{V(t)}\gamma(t), \gamma'(t)) = \varepsilon_{V(t)}g(\gamma(t), \gamma'(t))C$, which means that the vector $\mathcal{J}_{V(t)}\gamma(t) - \varepsilon_{V(t)}C\gamma(t)$ is orthogonal to $\gamma'(t)$. Now consider a nonnull vector $X \in \mathcal{V}$ such that $\mathcal{J}_X Y = \varepsilon_X \mu Y$ for some $\mu \in \mathbb{R}$ and nonzero vector $Y \in \mathcal{V}$. Take an arbitrary vector $Z \in \mathcal{V}$ and define the curve $\gamma\colon I \to \mathcal{V}$ by $\gamma(t) = X + tZ$ near $0 \in I$. Applying our result at $t = 0$ we conclude that $\mathcal{J}_{V(0)}X - \varepsilon_{V(0)}CX$ is orthogonal to each Z, so by nondegeneracy of the scalar product, we have $\mathcal{J}_{V(0)}X = \varepsilon_{V(0)}CX$. Theorem A.26 additionally ensures that $V(0) = Y$ and $\lambda(0) = \mu = C$, and therefore $\mathcal{J}_Y X = \varepsilon_Y \mu X$, which proves that R is Jacobi-dual.

Conversely, suppose that R is Jacobi-dual, which implies $\mathcal{J}_{V(t)}\gamma(t) = (\varepsilon_{V(t)}/\varepsilon_{\gamma(t)})\lambda(t)\gamma(t)$ whenever $\gamma(t)$ is nonnull. In that case, using (9.23), we obtain $2\lambda(t)g(\gamma(t), \gamma'(t)) = \varepsilon_{\gamma(t)}\lambda'(t)$. Let $X, Z \in \mathcal{V}$ be such that $\varepsilon_X = \varepsilon_Z \neq 0$, and define the curve $\gamma\colon \mathbb{R} \to \mathcal{V}$ by $\gamma(t) = X\cos t + Z\sin t$. Then $\varepsilon_{\gamma(t)}$ is constant and nonzero, which implies $2g(\gamma(t), \gamma'(t)) = 0$, and therefore $\lambda'(t) = 0$. Hence, λ is constant, which means that $\mathcal{J}_X$ and $\mathcal{J}_Z$ has the same eigenvalues, and consequently R is Osserman.

Theorem 9.21. *A Jacobi-diagonalisable algebraic curvature tensor is Osserman if and only if it is Jacobi-dual.*

The previous theorem is a consequence of a more general result due to Nikolayevsky and Rakić, established in 2016 [94]. Specifically, any Jordan-Osserman algebraic curvature tensor is Jacobi-dual [94, Theorem 1], while a semisimple algebraic curvature tensor is Osserman if and only if it is Jacobi-dual [94, Theorem 2]. Let us also note that a Riemannian algebraic curvature tensor is Osserman if and only if it is Jacobi-dual, which is a special case of Theorem (9.21), previously proved by Nikolayevsky and Rakić in 2013 [93, Theorem 2.1].

CHAPTER 10

OSSERMAN TENSORS AND MANIFOLDS

10.1 Simple-root Osserman tensors

Let R be an algebraic curvature tensor on a scalar product space $(\mathcal{V}, g)$ of dimension n. We say that R is ***k*-root** if the reduced Jacobi operator $\widetilde{\mathcal{J}}_X$ has exactly k distinct eigenvalues (counting complex roots) for any nonnull $X \in \mathcal{V}$ (see Andrejić [10]). Accordingly, we say that a pseudo-Riemannian manifold (M, g) is ***k*-root** if the curvature tensor is k-root at each point, which means that the reduced Jacobi operator $\widetilde{\mathcal{J}}_X$ has exactly k distinct eigenvalues for any nonnull $X \in TM$.

Of course, any Osserman R is k-root for some fixed $k \in \mathbb{N}$. Consider a k-root Osserman R for which

$$\omega_X(\lambda) = \det(\lambda\,\mathbb{1} - \mathcal{J}_X) = \lambda(\lambda - \varepsilon_X\lambda_1)^{\nu_1} \cdots (\lambda - \varepsilon_X\lambda_k)^{\nu_k}$$

is the characteristic polynomial of the Jacobi operator, and $(\lambda - \varepsilon_X\lambda_1)^{\mu_1} \cdots (\lambda - \varepsilon_X\lambda_k)^{\mu_k}$ is the minimal polynomial of the corresponding reduced Jacobi operator. In general we know that $1 \le \mu_i \le \nu_i$ holds for $1 \le i \le k$, while for a Jacobi-diagonalisable R (for example, for a Riemannian R) we have $\mu_i = 1$.

Any of self-adjoint linear operators $\mathcal{K}_X^i = \mathcal{J}_X - \varepsilon_X\lambda_i\,\mathbb{1}$ for $1 \le i \le k$ share with $\mathcal{J}_X$ the same generalised eigenspaces,

$$\operatorname{Ker}(\mathcal{K}_X^i - \varepsilon_X(\lambda_j - \lambda_i)\,\mathbb{1})^{\mu_j} = \operatorname{Ker}(\mathcal{J}_X - \varepsilon_X\lambda_j\,\mathbb{1})^{\mu_j},$$

for $1 \le j \le k$. Every two such operators commute,

$$\mathcal{K}_X^i\mathcal{K}_X^j = \mathcal{K}_X^j\mathcal{K}_X^i = (\mathcal{J}_X)^2 - \varepsilon_X(\lambda_i + \lambda_j)\mathcal{J}_X + \varepsilon_X^2\lambda_i\lambda_j\,\mathbb{1},$$

and therefore their composition is also self-adjoint. This motivates us to define a self-adjoint linear operator $L(X) \colon \mathcal{V} \to \mathcal{V}$ for any nonnull $X \in \mathcal{V}$ by

$$L(X) = (\mathcal{K}_X^2)^{\mu_2}(\mathcal{K}_X^3)^{\mu_3} \cdots (\mathcal{K}_X^k)^{\mu_k}.$$

In a chosen orthonormal basis $(E_1, \dots, E_n)$ for $X = \sum_{i=1}^n x_iE_i$ we get

$$g(\mathcal{K}_X^i(E_a), E_b) = \sum_{p,q=1}^n (R_{apqb} - \varepsilon_a\varepsilon_p\delta_{ab}\delta_{pq}\lambda_i)x_px_q,$$

so the entries of the matrix $L(X)$ are homogeneous polynomials of degree $2(\mu_2 + \cdots + \mu_k)$. However, for a nonnull $X \in \mathcal{V}$, because of

$$\mathcal{V} = \operatorname{Span}\{X\} \oplus \bigoplus_{j=1}^k \operatorname{Ker}(\mathcal{K}_X^j)^{\mu_j},$$

it is easy to see that

$$\operatorname{Ker} L(X) \supseteq \bigoplus_{j=2}^{k} \operatorname{Ker}(\mathcal{K}_X^j)^{\mu_j}$$

holds with additional $L(X)X = (-\varepsilon_X)^{\mu_2+\cdots+\mu_k}\lambda_2^{\mu_2}\lambda_3^{\mu_3}\cdots\lambda_k^{\mu_k}X$.

In this section we examine the case $\nu_1 = 1$, which means a simple eigenvalue λ_1, where $L(X) = (\varepsilon_X)^{\mu_2+\cdots+\mu_k}(\lambda_1-\lambda_2)^{\mu_2}(\lambda_1-\lambda_3)^{\mu_3}\cdots(\lambda_1-\lambda_k)^{\mu_k}\,\mathbb{1}$ on $\operatorname{Ker}(\mathcal{K}_X^1)^{\mu_1}$. If $\lambda_2\lambda_3\cdots\lambda_k \neq 0$, then we consider a new algebraic curvature tensor defined by $R' = R - \lambda_k R^1$ instead of the original R, which is a common trick. The associated reduced Jacobi operator $\widetilde{\mathcal{J}}'_X$ has the same eigenspaces as those of $\widetilde{\mathcal{J}}_X$, but eigenvalues are shifted in such a way that $\lambda'_k = 0$, which allows $L(X)X = 0$.

In this way, for any nonnull $X \in \mathcal{V}$, we have the self-adjoint matrix $L(X)$ of rank one such that its entries L_{ij} are homogeneous polynomials of degree $2(\mu_2+\cdots+\mu_k)$ in n variables $x_1,\dots,x_n$. Any submatrix of order two in a rank one matrix is singular which gives

$$L_{ii}(X)L_{jj}(X) = L_{ij}(X)L_{ji}(X)$$

for all $1 \le i,j \le n$. However, $L(X)$ is self-adjoint which implies

$$\varepsilon_i L_{ij}(X) = g(L(X)E_j, E_i) = g(L(X)E_i, E_j) = \varepsilon_j L_{ji}(X),$$

and consequently

$$\varepsilon_i\varepsilon_j L_{ii}(X)L_{jj}(X) = L_{ij}(X)^2. \tag{10.1}$$

If we fix some monomial order (for example, the lexicographical order) then there is a unique monic (the coefficient of the largest monomial is 1) $G(X)$ which is the greatest common divisor of all $L_{ij}(X)$. Permuting the basis we can set

$$\varepsilon_i L_{ii}(X) = \sigma_i G(X) Q_i(X) P_i(X)^2,$$

where $P_i(X)$ and $Q_i(X)$ are some non-zero polynomials for $1 \le i \le m$, with additional $L_{ii}(X) = 0$ for $m < i \le n$, while $\sigma_i \in \{-1,1\}$. However, such decomposition is unique up to sign of $P_i(X)$ if we set that $Q_i(X)$ is monic square-free. Then

$$\sigma_i\sigma_j G(X)^2 Q_i(X)Q_j(X)P_i(X)^2P_j(X)^2 = L_{ij}(X)^2$$

implies $Q_i(X) = Q_j(X) = Q(X)$ and $\sigma_i = \sigma_j = \sigma$ for $1 \le i,j \le m$, and therefore we have

$$\varepsilon_i L_{ij}(X) = \varepsilon_j L_{ji}(X) = \sigma_{ij}\varepsilon_i\varepsilon_j G(X)Q(X)P_i(X)P_j(X),$$

where $\sigma_{ij} \in \{-1,1\}$. Additionally, by (10.1), $L_{ij} = 0$ holds whenever $m < i \le n$ or $m < j \le n$, which can be treated as $P_i(X) = 0$ for $m < i \le n$ and extend the indices to $m = n$. Since all the matrix entries are divisible by $G(X)Q(X)$ it must be $Q(X) = 1$.

Another submatrix of order two gives

$$L_{1i}(X)L_{ij}(X) = L_{1j}(X)L_{ii}(X),$$

which implies $\sigma_{1i}\sigma_{ij} = \sigma_{1j}\sigma_{ii}$, but $\sigma_{ii} = \sigma_i = \sigma$ yields $\sigma_{ij} = \sigma\sigma_{1i}\sigma_{1j}$, and therefore $\varepsilon_j L_{ij}(X) = \sigma G(X)\sigma_{1i}P_i(X)\sigma_{1j}P_j(X)$. Since the polynomials $P_i(X)$ are unique up to sign, we can use $\sigma_{1i}P_i(X)$ instead of $P_i(X)$ to obtain

$$\varepsilon_j L_{ij}(X) = \sigma G(X)P_i(X)P_j(X)$$

for all $1 \le i,j \le n$. The previous equality implies

$$\begin{aligned} L(X)\left(\sum_{i=1}^{n} P_i(X)E_i\right) &= \sum_{i=1}^{n} P_i(X)L(X)E_i = \sum_{i=1}^{n} P_i(X)\sum_{j=1}^{n} L_{ji}(X)E_j \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} P_j(X)(\sigma\varepsilon_j G(X)P_i(X)P_j(X))E_i \\ &= \sigma G(X)\sum_{j=1}^{n}\varepsilon_j P_j(X)^2\left(\sum_{i=1}^{n} P_i(X)E_i\right). \end{aligned}$$

Hence, $\sum_{i=1}^{n} P_i(X)E_i$ is an eigenvector of $L(X)$ associated to the simple eigenvalue

$$\sigma G(X)\sum_{j=1}^{n}\varepsilon_j P_j(X)^2 = \sum_{j=1}^{n} L_{jj}(X) = \operatorname{Tr} L(X).$$

Moreover, $\sum_{i=1}^{n} P_i(X)E_i$ is an eigenvector of $\widetilde{\mathcal{J}}'_X$ for $\varepsilon_X(\lambda_1-\lambda_k)$ and also an eigenvector of $\widetilde{\mathcal{J}}_X$ corresponding to the eigenvalue $\varepsilon_X\lambda_1$. However,

$$\sigma G(X)\sum_{i=1}^{n}\varepsilon_i P_i(X)^2 = \operatorname{Tr} L(X) = (\varepsilon_X)^{\mu_2+\cdots+\mu_k}(\lambda_1-\lambda_2)^{\mu_2}\cdots(\lambda_1-\lambda_k)^{\mu_k},$$

which gives $G(X) = (\varepsilon_X)^s = (\varepsilon_1 x_1^2 + \cdots + \varepsilon_n x_n^2)^s$ for some integer $s \ge 0$, and consequently

$$\sigma\sum_{i=1}^{n}\varepsilon_i P_i(X)^2 = (\varepsilon_X)^d(\lambda_1-\lambda_2)^{\mu_2}\cdots(\lambda_1-\lambda_k)^{\mu_k}, \tag{10.2}$$

where $d = \mu_2 + \cdots + \mu_k - s$.

Although we originally observed only nonnull X, P_i are homogeneous polynomials of order $d \le \mu_2 + \cdots + \mu_k$, and $P_i(X)$ exists for every $X \in \mathcal{V}$. This allows us to construct the map $P\colon \mathcal{V} \to \mathcal{V}$ by

$$P(X) = \frac{1}{\sqrt{|(\lambda_1-\lambda_2)^{\mu_2}\cdots(\lambda_1-\lambda_k)^{\mu_k}|}}\sum_{i=1}^{n} P_i(X)E_i.$$

If X is nonnull, then $P(X)$ is nonnull and orthogonal to X, such that

$$\mathcal{J}_X P(X) = \varepsilon_X\lambda_1 P(X), \tag{10.3}$$

while the equality (10.2) yields

$$\varepsilon_{P(X)} = \delta(\varepsilon_X)^d, \tag{10.4}$$

where $\delta = \sigma\operatorname{sgn}((\lambda_1-\lambda_2)^{\mu_2}\cdots(\lambda_1-\lambda_k)^{\mu_k})$ is the fixed sign.

Let R be Jordan-Osserman. Then it is Jacobi-dual (see comments after Theorem 9.21), so the equality (10.3) gives $\mathcal{J}_{P(X)}X = \varepsilon_{P(X)}\lambda_1 X$, and consequently by Lemma 9.17 for all $\alpha, \beta \in \mathbb{R}$ we have

$$\mathcal{J}_{\alpha X+\beta P(X)}(\varepsilon_{P(X)}\beta X - \varepsilon_X\alpha P(X)) = \varepsilon_{\alpha X+\beta P(X)}\lambda_1(\varepsilon_{P(X)}\beta X - \varepsilon_X\alpha P(X)).$$

Thus, for a nonnull $\alpha X + \beta P(X)$ we obtain $P(\alpha X+\beta P(X)) \propto \varepsilon_{P(X)}\beta X - \varepsilon_X\alpha P(X)$, which means that there exists the proportionality coefficient $0 \ne K_X(\alpha,\beta) \in \mathbb{R}$ such that

$$P(\alpha X+\beta P(X)) = K_X(\alpha,\beta)(\alpha P(X) - \beta(\varepsilon_{P(X)}/\varepsilon_X)X).$$

For a fixed $X = \sum_i x_i E_i$ we have fixed $P(X) = \sum_i p_i E_i$, so for each $1 \le i \le n$ holds

$$\frac{P_i(\ldots,\alpha x_j+\beta p_j,\ldots)}{\sqrt{|(\lambda_1-\lambda_2)^{\mu_2}\cdots(\lambda_1-\lambda_k)^{\mu_k}|}} = K_X(\alpha,\beta)(\alpha p_i - \beta\delta(\varepsilon_X)^{d-1}x_i),$$

from which it can be seen that $K_X(\alpha,\beta)$ is continuous, as a rational function of α and β, where $\alpha^2+\beta^2\delta(\varepsilon_X)^{d-1}\neq 0$, which is equivalent to the fact that $\alpha X+\beta P(X)$ is nonnull, as well as that $\alpha P(X)-\beta\delta(\varepsilon_X)^{d-1}X$ is nonnull.

Since $P_i(-X)=(-1)^dP_i(X)$ holds for $1\leq i\leq n$, we have $P(-X)=(-1)^dP(X)$ Therefore, in addition to the obvious $K_X(1,0)=1$, we also have $K_X(-1,0)=(-1)^{d-1}$. From (10.4) we get

$$(K_X(\alpha,\beta))^2=\frac{\varepsilon_{P(\alpha X+\beta P(X))}}{\varepsilon_{\alpha P(X)-\beta(\varepsilon_{P(X)}/\varepsilon_X)X}}=\frac{\delta(\alpha^2\varepsilon_X+\beta^2\varepsilon_{P(X)})^d}{\varepsilon_{P(X)}(\alpha^2+\beta^2\varepsilon_{P(X)}/\varepsilon_X)}=\left(\frac{\alpha^2\varepsilon_X+\beta^2\varepsilon_{P(X)}}{\varepsilon_X}\right)^{d-1},$$

and therefore for an even d we always have $\alpha^2+\beta^2\delta(\varepsilon_X)^{d-1}>0$, so the domain of K_X is the connected set $\mathbb{R}^2\setminus\{(0,0)\}$ whose image is not a connected set because of $K_X(1,0)>0$, $K_X(-1,0)<0$, and $K_X(\alpha,\beta)\neq 0$. Hence d is odd and $P(-X)=-P(X)$ holds.

Additionally, we have $K_X(\alpha,\beta)=S_X(\alpha,\beta)(\alpha^2+\beta^2\delta(\varepsilon_X)^{d-1})^{\frac{d-1}{2}}$, where $S_X(\alpha,\beta)\in\{-1,1\}$. Thus $S_X(\alpha,\beta)(\alpha^2+\beta^2\delta(\varepsilon_X)^{d-1})^{\frac{d-1}{2}}(\alpha p_i-\beta\delta(\varepsilon_X)^{d-1}x_i)$ is a homogeneous polynomial by α and β of degree d, and its coefficient by α^d is fixed and it is equal to $S_X(\alpha,\beta)p_i$, so choosing i such that $p_i\neq 0$ we obtain that S_X is constant. Hence $S_X(\alpha,\beta)=S_X(1,0)=1$, which leads us to the formula

$$P(\alpha X+\beta P(X))=(\alpha^2+\beta^2\delta(\varepsilon_X)^{d-1})^{\frac{d-1}{2}}\big(\alpha P(X)-\beta\delta(\varepsilon_X)^{d-1}X\big). \tag{10.5}$$

Let us remark that the formula (10.5) also holds when $\alpha X+\beta P(X)$ is null (the case $\delta=-1$), because from the continuity of polynomial map we have $P(\alpha X+\beta P(X))=0$. In particular, the formula (10.5) for $\alpha=0,\beta=1$ implies

$$P(P(X))=(\delta(\varepsilon_X)^{d-1})^{\frac{d-1}{2}}(-\delta(\varepsilon_X)^{d-1})X=-\delta^{\frac{d+1}{2}}(\varepsilon_X)^{\frac{d^2-1}{2}}X. \tag{10.6}$$

Lemma 10.1. *Let R be a k-root Jordan-Osserman algebraic curvature tensor on $(\mathcal{V},g)$ whose reduced Jacobi operator $\widetilde{\mathcal{J}}_X$ has a simple eigenvalue $\varepsilon_X\lambda_1$, while μ_i for $2\leq i\leq k$ are root multiplicities of the minimal polynomial. Then there exists a homogeneous polynomial map $P\colon\mathcal{V}\to\mathcal{V}$ of odd degree $d\leq\mu_2+\cdots+\mu_k$ such that for any nonnull $X\in\mathcal{V}$ holds* (10.3), (10.4), *and* (10.5).

The most often case of Lemma 10.1 considers a Jacobi-diagonalisable R where we have $\mu_i=1$ for $1\leq i\leq k$, while the most important case is a Riemannian R where additionally we have $\varepsilon_j=1$ for $1\leq j\leq n$, and consequently (10.4) gives $\delta=1$. Therefore as a corollary we have the following lemma similar to one given by Nikolayevsky in 2003 [88, Lemma 2.1].

Lemma 10.2. *Let R be a Riemannian k-root Osserman algebraic curvature tensor on $(\mathcal{V},g)$ whose reduced Jacobi operator $\widetilde{\mathcal{J}}_X$ has a simple eigenvalue $\varepsilon_X\lambda_1$. Then there exists a homogeneous polynomial map $P\colon\mathcal{V}\to\mathcal{V}$ of odd degree $d\leq k-1$ such that for any nonzero $X\in\mathcal{V}$ and all $\alpha,\beta\in\mathbb{R}$ we have*

$$\widetilde{\mathcal{J}}_XP(X)=\varepsilon_X\lambda_1P(X),\quad \varepsilon_{P(X)}=(\varepsilon_X)^d,$$
$$P(\alpha X+\beta P(X))=(\alpha^2+\beta^2(\varepsilon_X)^{d-1})^{(d-1)/2}\big(\alpha P(X)-\beta(\varepsilon_X)^{d-1}X\big).$$

10.2 Riemannian Osserman tensors

Let R be an algebraic curvature tensor on a positive definite scalar product space $(\mathcal{V},g)$ of dimension n. If $\lambda\in\mathbb{R}$ is an eigenvalue with a constant multiplicity r of the reduced Jacobi operators $\widetilde{\mathcal{J}}_X$ for all $X\in\mathbf{S}^{n-1}=\{X\in\mathcal{V}:\varepsilon_X=1\}\subset\mathcal{V}$, then

$$\widetilde{\mathcal{J}}_X-\lambda\,\mathbb{1}_{X^\perp}$$

is a tangent bundle homomorphism over $\mathbf{S}^{n-1}$ with the identification $T_X\mathbf{S}^{n-1} \cong X^\perp$. Since it has a constant rank $n-r$, according to Theorem 3.3, $\operatorname{Ker}(\widetilde{\mathcal{J}}_X - \lambda\,\mathbb{1}_{X^\perp})$ is a subbundle of $T\mathbf{S}^{n-1}$. Thus we have the following useful lemma.

Lemma 10.3. *Let R be a Riemannian algebraic curvature tensor of dimension n. If $\lambda \in \mathbb{R}$ is an eigenvalue with a constant multiplicity r of $\widetilde{\mathcal{J}}_X$ on $\mathbf{S}^{n-1}$, then $\operatorname{Ker}(\widetilde{\mathcal{J}}_X - \lambda\,\mathbb{1}_{X^\perp})$ is an r-dimensional distribution on $\mathbf{S}^{n-1}$.*

According to Theorem 3.5, $\mathbf{S}^{n-1}$ for $\rho(n) \leq k \leq n-1-\rho(n)$ does not admit a k-dimensional distribution, where ρ is the Hurwitz–Radon function given by the equation (3.3). Hence, Lemma 10.3 reduces the possibilities for the multiplicity of eigenvalues of $\mathcal{J}_X$ on $\mathbf{S}^{n-1}$.

Theorem 10.4. *A Riemannian Osserman algebraic curvature tensor of odd dimension has constant sectional curvature.*

Proof. The Osserman condition implies that any eigenvalue λ of the reduced Jacobi operator $\widetilde{\mathcal{J}}_X$ for $X \in \mathbf{S}^{n-1}$ has constant multiplicity r, which allows to apply Lemma 10.3. The fact that $\rho(n) = 1$ holds for odd n excludes multiplicities r with $1 \leq r \leq n-2$, which leaves us with $r = n-1$, and therefore $\operatorname{Ker}(\widetilde{\mathcal{J}}_X - \lambda\,\mathbb{1}_{X^\perp}) = T\mathbf{S}^{n-1}$. Hence, $\widetilde{\mathcal{J}}_X = \lambda\,\mathbb{1}$ holds for all $X \in \mathbf{S}^{n-1}$, so R has constant sectional curvature λ. □

Consider the next simplest case of twice an odd dimension $n = 4m+2$. Lemma 10.3 and the fact $\rho(4m+2) = 2$ left us with $r \in \{1, n-2, n-1\}$. There are no two eigenvalues λ and μ with multiplicities 1, because then $\operatorname{Ker}(\mathcal{J}_X - \lambda\,\mathbb{1}) \oplus \operatorname{Ker}(\mathcal{J}_X - \mu\,\mathbb{1})$ defines a two-dimensional distribution which is not possible. Thus, either we have a constant sectional curvature as before, or two different eigenvalues with multiplicities 1 and $n-2$.

Theorem 10.5. *A Riemannian Osserman algebraic curvature tensor of twice an odd dimension has constant sectional curvature or it is Clifford of rank* 1.

Proof. Let $\widetilde{\mathcal{J}}_X$ has exactly two eigenvalues, a simple eigenvalue $\varepsilon_X\lambda$ and $\varepsilon_X\mu$ with multiplicity $n-2$. Applying Lemma 10.2 we construct a homogeneous polynomial map $P\colon \mathcal{V} \to \mathcal{V}$ of degree $d = 1$ (it means that P is linear) such that $\widetilde{\mathcal{J}}_X P(X) = \varepsilon_X \lambda P(X)$ holds with $\varepsilon_{P(X)} = \varepsilon_X$. Moreover, P is a complex structure on $\mathcal{V}$, since the equality (10.6) gives $P^2 = -\mathbb{1}$. Additionally, if we set $Y = \alpha X + \beta P(X) + Z$ for $Z \in \operatorname{Span}\{X, P(X)\}^\perp = \operatorname{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X\mu\,\mathbb{1})$, then $X \in \operatorname{Ker}(\widetilde{\mathcal{J}}_Z - \varepsilon_Z\mu\,\mathbb{1}) = \operatorname{Span}\{Z, P(Z)\}^\perp$, which implies $X \perp P(Z)$. Therefore we have $g(P(X), Y) = \beta\varepsilon_{P(X)} = \beta\varepsilon_X = -g(X, P(\beta P(X))) = -g(X, P(Y))$ and P is skew-adjoint. Since our orthogonal skew-adjoint endomorphism P on $\mathcal{V}$ uniquely determines the Jacobi operators, it determines the curvature tensor. Thus, the only solution is $R = \mu R^1 + (\mu - \lambda)/3 R^P$, which is Clifford. □

We have already seen that thanks to Lemma 10.2 the previous theorem is very easy to prove. However, this case can be resolved by direct calculations, as Chi originally did in 1988 [34, Section 2]. Also interesting is the solution given by García-Río, Kupeli, and Vázquez-Lorenzo [62, Lemma 2.1.4]. In the following example we show some calculations based on their ideas.

Example 10.1. Let $\widetilde{\mathcal{J}}_X$ has exactly two eigenvalues, a simple eigenvalue $\varepsilon_X\lambda$ and $\varepsilon_X\mu$ with multiplicity $n-2$. A smooth one-dimensional distribution $\operatorname{Ker}(\widetilde{\mathcal{J}}_X - \lambda\,\mathbb{1})$ gives a vector field $J\colon \mathbf{S}^{n-1} \to T\mathbf{S}^{n-1}$ such that $X \mapsto JX \in X^\perp$. Let $\mathcal{U}_X = \operatorname{Span}\{X\} \oplus \operatorname{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X\lambda\,\mathbb{1}) = \operatorname{Span}\{X, JX\}$ and $\mathcal{U}_X^\perp = \operatorname{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X\mu\,\mathbb{1})$. By Theorem 9.4 we have $Y \in \mathcal{U}_X^\perp$ if and only if $X \in \mathcal{U}_Y^\perp$, while Lemma 9.17 additionally gives $Y \in \mathcal{U}_X$ if and only if $X \in \mathcal{U}_Y$. According to this, we have $J(X\cos\theta + JX\sin\theta) = \pm(JX\cos\theta - X\sin\theta)$ where the sign is determined using continuity and values at $\theta = 0$,

$$J(X\cos\theta + JX\sin\theta) = JX\cos\theta - X\sin\theta,$$

and consequently we obtain $J(-X) = -JX$ and $J^2 = -\mathbb{1}$.

We use the shortcuts $c = \cos\theta$, $s = \sin\theta$ to calculate $J(X\cos\theta + Y\sin\theta)$, where $Y \in \mathcal{U}_X^\perp$. From $Y \in \mathcal{U}_X^\perp = \mathcal{U}_{JX}^\perp$ we have $X, JX \in \mathcal{U}_Y^\perp$, so $JY \perp \{X, JX, Y\}$. For $Z \in \operatorname{Span}\{X, JX, Y, JY\}^\perp$, we have $Z \in \mathcal{U}_X^\perp \cap \mathcal{U}_Y^\perp$, hence $X, Y \in \mathcal{U}_Z^\perp$, and therefore $cX + sY \in \mathcal{U}_Z^\perp$, which finally gives $Z \in \mathcal{U}_{cX+sY}^\perp = \mathcal{U}_{J(cX+sY)}^\perp$. For every $Z \in \operatorname{Span}\{X, JX, Y, JY\}^\perp$ we have $Z \perp J(cX+sY)$, and therefore $J(cX+sY) \in \operatorname{Span}\{X, JX, Y, JY\}$. On the other hand, we have

$$\begin{aligned}\mathcal{J}_{cX+sY}(sX - cY) &= \mathcal{R}(sX - cY, cX + sY)(cX + sY) = \mathcal{R}(X, Y)(cX + sY)\\ &= s\mathcal{J}_Y X - c\mathcal{J}_X Y = \mu(sX - cY),\end{aligned}$$

which additionally gives $J(cX+sY) \perp \operatorname{Span}\{X, Y\}$ and yields $J(cX+sY) \in \operatorname{Span}\{JX, JY\}$.

Let us set $J(cX+sY) = \alpha JX + \beta JY$, where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ with $\alpha^2 + \beta^2 = 1$. From $\alpha JX + \beta JY \in \mathcal{U}_{cX+sY}$ follows $cX + sY \in \mathcal{U}_{\alpha JX+\beta JY}$, which gives $\mathcal{J}_{\alpha JX+\beta JY}(cX+sY) = \lambda(cX+sY)$. Hence,

$$\begin{aligned}\alpha^2\mathcal{J}_{JX}(cX+sY) + \beta^2\mathcal{J}_{JY}(cX+sY) + 2\alpha\beta\mathcal{J}(JX, JY)(cX+sY) &= \lambda(cX+sY),\\ \alpha^2(c\lambda X + s\mu Y) + \beta^2(c\mu X + s\lambda Y) + \alpha\beta(\mathcal{R}(cX+sY, JX)JY + \mathcal{R}(cX+sY, JY)JX) &= \lambda(cX+sY),\\ (\mu-\lambda)(\beta^2 cX + \alpha^2 sY) + \alpha\beta(c\mathcal{R}(X, JX)JY + s\mathcal{R}(Y, JX)JY + c\mathcal{R}(X, JY)JX + s\mathcal{R}(Y, JY)JX) &= 0.\end{aligned}$$

Since $X, JX, cX+sY \in \mathcal{U}_{JY}^\perp$ imply $JY \in \mathcal{U}_X^\perp \cap \mathcal{U}_{JX}^\perp \cap \mathcal{U}_{cX+sJX}^\perp$ we have $\mathcal{J}(X, JX)JY = 0$, that gives $\mathcal{R}(JY, X)JX + \mathcal{R}(JY, JX)X = 0$. If we include the first Bianchi identity, we get the equality $\mathcal{R}(X, JX)JY = 2\mathcal{R}(X, JY)JX$, and similarly $\mathcal{R}(Y, JY)JX = 2\mathcal{R}(Y, JX)JY$. Thus arises,

$$(\mu-\lambda)(\beta^2 cX + \alpha^2 sY) + 3\alpha\beta(c\mathcal{R}(X, JY)JX + s\mathcal{R}(Y, JX)JY) = 0.$$

Since $R(X, JY, JX, X) = 0 = R(Y, JX, JY, Y)$, taking the scalar product by X and Y we obtain

$$\begin{aligned}(\mu-\lambda)\beta^2 c + 3\alpha\beta s R(Y, JX, JY, X) &= 0\\ (\mu-\lambda)\alpha^2 s + 3\alpha\beta c R(X, JY, JX, Y) &= 0\end{aligned}$$

and therefore

$$(\mu-\lambda)\beta^2 c^2 = -3\alpha\beta sc R(X, JY, JX, Y) = (\mu-\lambda)\alpha^2 s^2.$$

Hence $\alpha^2 s^2 = \beta^2 c^2$ holds, which gives $\alpha^2 = \cos^2\theta$ and $\beta^2 = \sin^2\theta$. Again, by continuity we get

$$J(X\cos\theta + Y\sin\theta) = JX\cos\theta + JY\sin\theta,$$

which proves that J is linear. The extension $J(tX) = tJ(X)$ for $t \in \mathbb{R}$ defines an endomorphism J on $\mathcal{V}$, and so on... $\triangle$

10.3 Schur problems

In this section we consider pseudo-Riemannian manifolds and investigate results related to Schur type of problems, based on Theorem 6.8. We say that a pseudo-Riemannian manifold (M, g) is ***k*-stein** if its curvature tensor at each point of M is k-stein. In other words, (M, g) is k-stein if there exist smooth functions $C_j \in \mathfrak{F}(M)$ such that

$$\operatorname{Tr}(\mathcal{J}_X{}^j) = (\varepsilon_X)^j C_j(p)$$

holds for any $p \in M$, $X \in T_pM$, and $1 \leq j \leq k$. We want to study if the functions C_j are necessarily constant, and such problems are called Schur-like problems (see Gilkey [54, Section 1.13]).

The simplest case considers the function C_1, where the following theorem is just Theorem 6.12 written differently.

Theorem 10.6. *Any connected* 1*-stein manifold of dimension* $n \neq 2$ *has constant* C_1*.*

We say that a symmetric $A \in \mathfrak{T}_2^0(M)$ has the ***Schur property*** if, given $A = fg$ for some function $f \in \mathfrak{F}(M)$, then f is constant (see Carpenter 1980 [32, Chapter 2]). Hence, according to Theorem 10.6, the Ricci tensor has the Schur property, unless $\dim M = 2$.

Let us define an interesting symmetric covariant tensor $\Omega \in \mathfrak{T}_2^0(M)$ by setting in a neighbourhood of some point

$$\Omega(X,Y) = \sum_{1\le i,j,k\le n} \varepsilon_i\varepsilon_j\varepsilon_k R(X,E_i,E_j,E_k)R(Y,E_i,E_j,E_k),$$

where $(E_1,\dots,E_n)$ is an orthonormal local frame over that neighbourhood. If $(F_1,\dots,F_n)$ is some other orthonormal local frame, then the link with the old frame is $E_i = \sum_t m_{ti}F_t$ and $F_p = \sum_t m^{tp}E_t$, where $\sum_t m_{it}m^{tp} = \delta_{ip}$ (inverse matrix) and $\varepsilon_p m_{pi} = g(E_i,F_p) = \varepsilon_i m^{ip}$. Since $\sum_i \varepsilon_i m_{pi}m_{ui} = \sum_i \varepsilon_u m_{pi}m^{iu} = \varepsilon_u\delta_{pu}$, we have

$$\begin{aligned}
&\sum_{i,j,k} \varepsilon_i\varepsilon_j\varepsilon_k R(X,E_i,E_j,E_k)R(Y,E_i,E_j,E_k)\\
&=\sum_{i,j,k} \varepsilon_i\varepsilon_j\varepsilon_k \sum_{p,q,r} m_{pi}m_{qj}m_{rk}R(X,F_p,F_q,F_r)\sum_{u,v,w} m_{ui}m_{vj}m_{wk}R(Y,F_u,F_v,F_w)\\
&=\sum_{p,q,r,u,v,w} R(X,F_p,F_q,F_r)R(Y,F_u,F_v,F_w)\sum_i \varepsilon_i m_{pi}m_{ui}\sum_j \varepsilon_j m_{qj}m_{vj}\sum_k \varepsilon_k m_{rk}m_{wk}\\
&=\sum_{p,q,r} \varepsilon_p\varepsilon_q\varepsilon_r R(X,F_p,F_q,F_r)R(Y,F_p,F_q,F_r),
\end{aligned}$$

and Ω is well defined because it does not depend on the choice of an orthonormal frame.

This natural invariant is algebraically the simplest one after the scalar and Ricci curvatures, and it is mentioned by Besse in 1978 [18, pp.164–165]. If we introduce

$$\varepsilon_R = \|R\|^2 = \operatorname{tr}_g \Omega,$$

then we have the following lemma.

Lemma 10.7. *For an Einstein manifold holds* $\nabla\varepsilon_R = 4\operatorname{div}\Omega$*.*

Proof. In an orthonormal local frame we have,

$$\begin{aligned}
(\operatorname{div}\Omega)(E_k) &= \sum_{1\le i\le n} \varepsilon_i \nabla_{E_i}\Omega(E_k,E_i) = \sum_{1\le i,p,q,r\le n} \varepsilon_i\varepsilon_p\varepsilon_q\varepsilon_r \nabla_{E_i}(R_{kpqr}R_{ipqr})\\
&= \sum_{1\le i,p,q,r\le n} \varepsilon_i\varepsilon_p\varepsilon_q\varepsilon_r R_{ipqr}\nabla_i R_{kpqr} + \sum_{1\le i,p,q,r\le n} \varepsilon_i\varepsilon_p\varepsilon_q\varepsilon_r R_{kpqr}\nabla_i R_{ipqr}.
\end{aligned}$$

For an Einstein manifold $R_{ij} = \varepsilon_i\delta_{ij}C_1$ holds, so $\nabla R_{ij} = 0$. Thus (6.10) implies

$$\begin{aligned}
\sum_{1\le i\le n} \varepsilon_i\nabla_i R_{ipqr} &= \sum_{1\le i\le n} \varepsilon_i(-\nabla_r R_{ipiq} - \nabla_q R_{ipri}) = \sum_{1\le i,l\le n} g^{il}\nabla_r R_{ipql} - \sum_{1\le i,l\le n} g^{il}\nabla_q R_{iprl}\\
&= \nabla_{E_r}R_{pq} - \nabla_{E_q}R_{pr} = 0,
\end{aligned}$$

and therefore the second term vanishes, $\sum_{1\le p,q,r\le n}\varepsilon_p\varepsilon_q\varepsilon_r R_{kpqr}\sum_{1\le i\le n}\varepsilon_i\nabla_i R_{ipqr} = 0$. Then,

$$\begin{aligned}
(\operatorname{div}\Omega)(E_k) &= \sum_{1\le i,p,q,r\le n} \varepsilon_i\varepsilon_p\varepsilon_q\varepsilon_r R_{ipqr}\nabla_i R_{kpqr}\\
&= \frac{1}{2}\sum_{1\le i,p,q,r\le n} \varepsilon_i\varepsilon_p\varepsilon_q\varepsilon_r R_{ipqr}(-\nabla_k R_{piqr} - \nabla_p R_{ikqr} + \nabla_i R_{kpqr})\\
&= \frac{1}{2}\sum_{1\le i,p,q,r\le n} \varepsilon_i\varepsilon_p\varepsilon_q\varepsilon_r R_{ipqr}\nabla_k R_{ipqr} = \frac{1}{4}\sum_{1\le i,p,q,r\le n} \varepsilon_i\varepsilon_p\varepsilon_q\varepsilon_r \nabla_{E_k}(R_{ipqr})^2 = \frac{1}{4}\nabla_{E_k}\varepsilon_R.
\end{aligned}$$

□

According to Gray and Willmore [63], an Einstein manifold (M,g) of dimension $n>4$ is called ***super-Einstein*** if it satisfies $\Omega = fg$ for some $f \in \mathfrak{F}(M)$. In this case, taking the trace with respect to g, we have $\varepsilon_R = fn$, and therefore by (6.30),

$$\operatorname{div}\Omega = \operatorname{div} fg = \nabla f = \frac{1}{n}\nabla\varepsilon_R.$$

Using Lemma 10.7, we have

$$\frac{1}{4}\nabla\varepsilon_R = \frac{1}{n}\nabla\varepsilon_R,$$

which implies $\nabla\varepsilon_R = 0$, and therefore ε_R is constant. Thus, we see that in an Einstein manifold, Ω has the Schur property unless $n=4$. Additionally, a super-Einstein manifold of dimension $n=4$ is defined to be an Einstein manifold with constant ε_R.

Let us consider a 2-stein manifold (M,g) of dimension n. In an orthonormal local frame $(E_1,\dots,E_n)$, Theorem 8.6 for $1 \le x \ne y \le n$ gives the equality (8.12),

$$2\sum_{1\le i,j\le n}\varepsilon_i\varepsilon_j R_{ixxj}R_{iyyj} + \sum_{1\le i,j\le n}\varepsilon_i\varepsilon_j(R_{ixyj}+R_{iyxj})^2 = 2\varepsilon_x\varepsilon_y C_2.$$

Using the symmetry $\sum_{i,j}\varepsilon_i\varepsilon_j(R_{ixyj})^2 = \sum_{j,i}\varepsilon_j\varepsilon_i(R_{iyxj})^2$, it becomes

$$\sum_{1\le i,j\le n}\varepsilon_i\varepsilon_j\left(R_{ixxj}R_{iyyj} + R_{ixyj}R_{iyxj} + (R_{ixyj})^2\right) = \varepsilon_x\varepsilon_y C_2.$$

We multiply by ε_y and sum over the index y for all $y \ne x$. We add and subtract the case $y=x$ on the left hand side, while $\sum_{i,j}\varepsilon_i\varepsilon_j(R_{ixxj})^2 = C_2$ holds from (8.10), so

$$-3\varepsilon_x C_2 + \sum_{1\le y,i,j\le n}\varepsilon_y\varepsilon_i\varepsilon_j\left(R_{ixxj}R_{iyyj} + R_{ixyj}R_{iyxj} + (R_{ixyj})^2\right) = \sum_{1\le y\le n, y\ne x}\varepsilon_x C_2,$$

and therefore

$$\sum_{1\le y,i,j\le n}\varepsilon_y\varepsilon_i\varepsilon_j\left(R_{ixxj}R_{iyyj} + R_{ixyj}R_{iyxj} + (R_{ixyj})^2\right) = (n+2)\varepsilon_x C_2. \tag{10.7}$$

Let us discuss the terms on the left hand separately. We use the Einstein formulas (8.8) and (8.9) for the first term,

$$\begin{aligned}\sum_{1\le y,i,j\le n}\varepsilon_y\varepsilon_i\varepsilon_j R_{ixxj}R_{iyyj} &= \sum_{1\le i\ne j\le n}\varepsilon_i\varepsilon_j R_{ixxj}\sum_{1\le y\le n}\varepsilon_y R_{iyyj} + \sum_{1\le i\le n}R_{ixxi}\sum_{1\le y\le n}\varepsilon_y R_{iyyi}\\ &= \sum_{1\le i\le n}R_{ixxi}\cdot\varepsilon_i C_1 = \varepsilon_x C_1^2.\end{aligned}$$

For the second term in (10.7) we use some symmetries and the first Bianchi identity,

$$\begin{aligned}\sum_{1\le y,i,j\le n}\varepsilon_y\varepsilon_i\varepsilon_j R_{ixyj}R_{iyxj} &= \frac{1}{2}\sum_{1\le y,i,j\le n}\varepsilon_y\varepsilon_i\varepsilon_j(R_{xijy}R_{xjiy} + R_{xiyj}R_{xyij})\\ &= \frac{1}{2}\sum_{1\le y,i,j\le n}\varepsilon_y\varepsilon_i\varepsilon_j(R_{xijy}(-R_{xiyj} - R_{xyji}) + R_{xijy}R_{xyji})\\ &= \frac{1}{2}\sum_{1\le y,i,j\le n}\varepsilon_y\varepsilon_i\varepsilon_j R_{xijy}R_{xijy} = \frac{1}{2}\Omega(E_x,E_x).\end{aligned}$$

The last term is simple,

$$\sum_{1\le y,i,j\le n}\varepsilon_y\varepsilon_i\varepsilon_j(R_{ixyj})^2 = \Omega(E_x,E_x).$$

After the substitution in (10.7) we have

$$\varepsilon_X C_1^2 + \frac{1}{2}\Omega(E_X, E_X) + \Omega(E_X, E_X) = (n+2)\varepsilon_X C_2,$$

and therefore

$$\Omega(E_X, E_X) = \frac{2}{3}\varepsilon_X((n+2)C_2 - C_1^2).$$

After the polarization it yields

$$\Omega = \frac{2}{3}((n+2)C_2 - C_1^2)g,$$

and therefore

$$\varepsilon_R = \frac{2}{3}n((n+2)C_2 - C_1^2).$$

Our calculations show that a 2-stein manifold of dimension $n > 4$ is super-Einstein, which implies that ε_R is constant, and therefore C_2 is constant. Additionally, in dimension $n = 3$, according to Example 6.13, R has constant section curvature. However, there are problems in the remaining dimensions because C_1 may not be constant in $n = 2$ and ε_R may not be constant in $n = 4$.

Some variations of the following result, mostly restricted to the Riemannian setting, one can find in Besse [18, pp.164–165], Gilkey, Swann[1], Vanhecke[2] [58, Theorem 2.4], Gilkey [54, Section 1.13, pp.75–78], García-Río, Kupeli, Vázquez-Lorenzo [62, pp.10–15].

Theorem 10.8. *If (M, g) is a connected 2-stein manifold of dimension $n \notin \{2, 4\}$, then C_2 is constant.*

10.4 One-root manifolds

Let R be a k-root algebraic curvature tensor on a scalar product space $(\mathcal{V}, g)$. Consider the Jacobi operator $\mathcal{J}_X$ for a nonzero vector $X = \sum_{i=1}^n x_i E_i$. The entries of its (real) matrix related to some orthonormal basis $(E_1, \dots, E_n)$ are homogeneous polynomial functions of degree two in coefficients $x_1, \dots, x_n$,

$$(\mathcal{J}_X)_{ab} = \varepsilon_a g(\mathcal{J}_X(E_b), E_a) = \varepsilon_a \sum_{i,j=1}^n R_{bija} x_i x_j.$$

Since the k-root condition implies no crossing of eigenvalues, the multiplicities of eigenvalues do not change as X varies. According to Theorem A.28, this allows us to label the eigenvalues so that they depends analytically on the coordinates of $X(x_1, \dots, x_n) \neq 0$.

The simplest case $k = 1$ is associated with ***one-root manifolds*** which is considered by Andrejić in 2018 [9, Section 5]. Let R be an algebraic curvature tensor on $(\mathcal{V}, g)$ such that the reduced Jacobi operator $\widetilde{\mathcal{J}}_X$ has a single eigenvalue $\varepsilon_X \mu_X$ for any nonnull $X \in \mathcal{V}$. Let us remark that in this case, the single eigenvalue is necessarily real.

In the Riemannian setting we obtain $X^\perp = \mathrm{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X \mu_X \mathbb{1}_{X^\perp})$. Therefore, for mutually orthogonal nonnull $X, Y \in \mathcal{V}$ we have $\mathcal{J}_X(Y) = \varepsilon_X \mu_X Y$ and $\mathcal{J}_Y(X) = \varepsilon_Y \mu_Y X$, so

$$\varepsilon_X \varepsilon_Y \mu_X = g(\mathcal{J}_X(Y), Y) = \varepsilon_X \varepsilon_Y \kappa(X, Y) = g(\mathcal{J}_Y(X), X) = \varepsilon_X \varepsilon_Y \mu_Y,$$

which implies a constant sectional curvature $\mu_X = \mu_Y = \kappa(X, Y)$. The same proof remains valid for a Jacobi-diagonalisable R (see [62, p.102]). The final touch of the Schur's theorem (Theorem 6.8) implies the following theorem.

[1] Andrew Francis Swann, Danish mathematician
[2] Lieven Vanhecke, Belgian mathematician

Theorem 10.9. *A connected one-root Jacobi-diagonalisable manifold of dimension $n \geq 3$ is a space of constant sectional curvature. In particular, a connected Riemannian one-root manifold of dimension $n \geq 3$ is a space of constant sectional curvature.*

Let us consider the same problem in the Lorentzian setting. In this case we can use the fact that $\mathcal{J}_T$ is diagonalisable for any timelike T. For a spacelike X orthogonal to timelike T we can apply $\mathcal{J}_{X+tT} = \mathcal{J}_X + 2t\mathcal{J}(X,T) + t^2\mathcal{J}_T$ on an arbitrary $A \in \operatorname{Span}\{X,T\}^\perp$. Then $\mathcal{J}_{X+tT}(A) = \varepsilon_{X+tT}\mu_{X+tT}A$ for $\varepsilon_{X+tT} < 0$, while $\mathcal{J}_X(A) = \alpha A + B$, $\mathcal{J}(X,T)A = \beta A + C$, and $\mathcal{J}_T(A) = \varepsilon_T\mu_T A$, where $B, C \in A^\perp$. Thus, $B + 2tC = 0$ holds for all $|t| > \sqrt{-\varepsilon_X/\varepsilon_T}$ which implies $B = C = 0$, and hence A is an eigenvector of $\mathcal{J}_X$. Therefore $\mathcal{J}_X$ is diagonalisable for any spacelike X and like before we have a constant sectional curvature. This proves the following theorem given by Andrejić in 2018 [9, Theorem 5.1].

Theorem 10.10. *Any Lorentzian one-root algebraic curvature tensor has constant sectional curvature. A connected Lorentzian one-root manifold of dimension $n \geq 3$ is a space of constant sectional curvature.*

The problem of one-root algebraic curvature tensor in higher signatures are more complex. For example, in Section 8.2 we have seen some globally Osserman one-root manifolds which are not Jacobi-diagonalisable.

Additionally, let us suppose that a one-root manifold is pointwise Osserman, with the single eigenvalue $\varepsilon_X\mu(p) \in \mathbb{R}$ of $\widetilde{\mathcal{J}}_X$ for $X \in T_pM$. Since any Osserman R is Einstein, we have $\varepsilon_X C_1(p) = \operatorname{Tr}(\mathcal{J}_X) = \varepsilon_X(n-1)\mu(p)$. A connected 1-stein M of dimension $n \neq 2$ by Theorem 10.6 has constant C_1, therefore μ is constant on M and consequently M is globally Osserman.

Theorem 10.11. *A connected one-root pointwise Osserman manifold of dimension $n \geq 3$ is globally Osserman.*

10.5 Two-root Osserman manifolds

Consider the next simplest $k = 2$. Let (M, g) be a connected pointwise Osserman manifold of dimension n such that at each $p \in M$ for every $X \in T_pM$ the associated reduced Jacobi operator $\widetilde{\mathcal{J}}_X$ has exactly two distinct eigenvalues $\varepsilon_X\lambda(p)$ and $\varepsilon_X\mu(p)$ with multiplicities $\sigma(p)$ and $\tau(p)$ respectively.

According to Theorem 8.3, an Osserman algebraic curvature tensor for $n \geq 3$ is n-stein. This is why there exist smooth functions $C_1, \ldots, C_n \in \mathfrak{F}(M)$ such that

$$(\varepsilon_X)^j C_j(p) = \operatorname{Tr}(\mathcal{J}_X^j) = (\varepsilon_X)^j(\sigma(p)(\lambda(p))^j + \tau(p)(\mu(p))^j).$$

holds for all $1 \leq j \leq n$.

We already know that C_1 is constant for $n \neq 2$ (Theorem 10.6), and that C_2 is constant for $n \notin \{2, 4\}$ (Theorem 10.8). Hence, the condition $n \geq 5$ excludes the undesirable cases and we have constant both C_1 and C_2. Let us set $C_0 = n - 1$ and look at the system of equations for $j \in \{0, 1, 2\}$:

$$\begin{aligned}
\sigma(p) + \tau(p) &= C_0,\\
\sigma(p)\lambda(p) + \tau(p)\mu(p) &= C_1,\\
\sigma(p)\lambda(p)^2 + \tau(p)\mu(p)^2 &= C_2.
\end{aligned}$$

From the first two equations we have

$$\sigma = \frac{C_1 - C_0\mu}{\lambda - \mu}, \quad \tau = \frac{C_1 - C_0\lambda}{\mu - \lambda},$$

and after the substitution into the third equation it implies

$$C_1(\lambda+\mu) - C_0\lambda\mu = C_2.$$

Since $\tau \geq 1$, we have $C_1 - C_0\lambda \neq 0$, so the previous equation gives

$$\mu = \frac{C_2 - C_1\lambda}{C_1 - C_0\lambda}.$$

The other functions can be expressed in terms of λ in the following way

$$\sigma = \frac{C_0C_2 - C_1^2}{C_0\lambda^2 - 2C_1\lambda + C_2}, \quad \tau = \frac{(C_1 - C_0\lambda)^2}{C_0\lambda^2 - 2C_1\lambda + C_2}.$$

Hence, a concrete σ gives at most two values for λ, which are solutions of the related quadratic equation. Since values for σ are integers $1 \leq \sigma \leq n-2$, it follows that λ can get at most $2(n-2)$ concrete values. From $C_j(p) = \sigma(p)\lambda(p)^j + \tau(p)\mu(p)^j$, we can see that $C_j(p)$ for $j > 2$ can get at most $2(n-2)$ concrete values. The function $C_j(p)$ is smooth, so it is continuous, and therefore it can get exactly one value. Thus C_j is constant as is the case with all other functions (σ, τ, λ, and μ), which proves that M is globally Osserman.

The result is also valid in the case of complex eigenvalues. The existence of a complex eigenvalue $\alpha(p) + i\beta(p)$ of $\widetilde{\mathcal{J}}_X$ implies that its conjugate $\alpha(p) - i\beta(p)$ is also a root of characteristic polynomial with the same multiplicity $(n-1)/2$. Then we have $C_1 = (n-1)\alpha(p)$ and $C_2 = (n-1)(\alpha(p)^2 - \beta(p)^2)$, which imply that $\alpha(p)$ and $\beta(p)$ are constant, too. Constant α and β give constant eigenvalues and therefore we have a globally Osserman manifold.

Theorem 10.12. *Any connected two-root pointwise Osserman manifold of dimension $n \geq 5$ is globally Osserman.*

There is a variant of Theorem 10.12 that requires constant multiplicities $\sigma(p)$ and $\tau(p)$ (see García-Río, Kupeli, Vázquez-Lorenzo [62, pp.10–16]), but we have already seen that this requirement is not necessary since σ is an integer by definition as Andrejić showed in 2013 [7, Theorem 2.1].

10.6 Two-root Riemannian tensors

Let R be a two-root algebraic curvature tensor on a positive definite scalar product space $(\mathcal{V}, g)$ of dimension n. Then, the reduced Jacobi operator $\widetilde{\mathcal{J}}_X$ for a nonzero $X \in \mathcal{V}$ has exactly two eigenvalues $\varepsilon_X\mu_X$ and $\varepsilon_X\nu_X$ with constant multiplicities, so the characteristic polynomial of Jacobi operator is

$$\omega_X(\lambda) = \det(\lambda\mathbb{1} - \mathcal{J}_X) = \lambda(\lambda - \varepsilon_X\mu_X)^p(\lambda - \varepsilon_X\nu_X)^q$$

for fixed integers $p \geq q \geq 1$ with $p + q = n - 1$.

Let us focus on the unit sphere $\mathcal{S} = \mathbf{S}^{n-1} = \{X \in \mathcal{V} : \varepsilon_X = 1\} \subset \mathcal{V}$. Consider $\widetilde{\mathcal{J}}_X - \mu_X\mathbb{1}_{X^\perp}$ as a smooth tangent bundle homomorphism over $\mathcal{S}$ with the identification $T_X\mathcal{S} \cong X^\perp$. Since it has a constant rank q, $\operatorname{Ker}(\widetilde{\mathcal{J}}_X - \mu_X\mathbb{1}_{X^\perp})$ is a subbundle of $T\mathcal{S}$ (see Theorem 3.3), that is, a p-dimensional distribution on $\mathcal{S}$. Similarly, $\operatorname{Ker}(\widetilde{\mathcal{J}}_X - \nu_X\mathbb{1}_{X^\perp})$ is a q-dimensional distribution on $\mathcal{S}$. By Theorem 3.5, $\mathcal{S}$ for $\rho(n) \leq k \leq n - 1 - \rho(n)$ does not admit a k-dimensional distribution, which leaves us with

$$q < \rho(n), \tag{10.8}$$

where ρ is the Hurwitz–Radon function from (3.3).

The inequality (10.8) significantly reduces the possibilities for the multiplicities p and q. For example, it immediately removes an odd n because of $\rho(n) = 1$, which means that n must be even, noticed by Andrejić in 2023 [10, Theorem 3].

Theorem 10.13. *There is no odd-dimensional Riemannian two-root algebraic curvature tensor. In particular, there is no Riemannian two-root manifold of odd dimension.*

Without loss of generality we can suppose $\mu_X < \nu_X$, since otherwise we consider $-R$ as a new algebraic curvature tensor. For any nonzero $X \in \mathcal{V}$ we define the eigenspaces,

$$\mathcal{M}(X) = \mathrm{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X \mu_X \mathbb{1}_{X^\perp}), \quad \mathcal{N}(X) = \mathrm{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X \nu_X \mathbb{1}_{X^\perp}),$$

where $\dim \mathcal{M}(X) = p$ and $\dim \mathcal{N}(X) = q$, which allows an orthogonal decomposition

$$X^\perp = \mathcal{M}(X) \oplus \mathcal{N}(X).$$

For nonzero $X, Y \in \mathcal{V}$ that satisfy $Y \in \mathcal{M}(X)$, we can decompose $X = M + N$ such that $M \in \mathcal{M}(Y)$ and $N \in \mathcal{N}(Y)$. Because of

$$\begin{aligned}
&g(\mathcal{J}_X(Y), Y) = g(\varepsilon_X \mu_X Y, Y) = \varepsilon_X \varepsilon_Y \mu_X, \\
&g(\mathcal{J}_Y(X), X) = g(\varepsilon_Y \mu_Y M + \varepsilon_Y \nu_Y N, M + N) = \varepsilon_Y \varepsilon_M \mu_Y + \varepsilon_Y \varepsilon_N \nu_Y,
\end{aligned}$$

we have $\varepsilon_X \mu_X = \varepsilon_M \mu_Y + (\varepsilon_X - \varepsilon_M)\nu_Y$, and consequently

$$\varepsilon_M = \varepsilon_X \frac{\nu_Y - \mu_X}{\nu_Y - \mu_Y}, \tag{10.9}$$

which gives

$$0 \le \frac{\nu_Y - \mu_X}{\nu_Y - \mu_Y} \le 1,$$

and hence $\mu_Y \le \mu_X \le \nu_Y$. In a similar fashion, $Y \in \mathcal{N}(X)$ implies

$$\varepsilon_N = \varepsilon_X \frac{\mu_Y - \nu_X}{\mu_Y - \nu_Y} \quad \text{and} \quad 0 \le \frac{\nu_X - \mu_Y}{\nu_Y - \mu_Y} \le 1,$$

and therefore $\mu_Y \le \nu_X \le \nu_Y$. Hence

$$\begin{aligned}
0 \ne Y \in \mathcal{M}(X) \quad &\Longrightarrow \quad \mu_Y \le \mu_X \le \nu_Y, \\
0 \ne Y \in \mathcal{N}(X) \quad &\Longrightarrow \quad \mu_Y \le \nu_X \le \nu_Y.
\end{aligned} \tag{10.10}$$

The restrictions $\mu\restriction_{\mathcal{S}} \colon \mathcal{S} \to \mathbb{R}$ and $\nu\restriction_{\mathcal{S}} \colon \mathcal{S} \to \mathbb{R}$ are continuous functions on a compact, so their ranges are closed intervals. Because of $\mathcal{J}_{tX}/\varepsilon_{tX} = \mathcal{J}_X/\varepsilon_X$ we obtain $\mu(tX) = \mu(X)$ for all $X \ne 0$ and $t \in \mathbb{R}$. Hence, for a nonzero $X \in \mathcal{V}$ we reach $\mu_X \in [\mu_{\min}, \mu_{\max}]$ and $\nu_X \in [\nu_{\min}, \nu_{\max}]$, which allows us to define

$$\mathcal{U} = \mu^{-1}(\mu_{\min}) \cup \{0\}, \quad \mathcal{W} = \nu^{-1}(\nu_{\max}) \cup \{0\}.$$

If $0 \ne Y \in \mathcal{M}(X)$ holds for $0 \ne X \in \mathcal{U}$, then (10.10) gives $Y \in \mathcal{U}$, while (10.9) implies $\varepsilon_M = \varepsilon_X$, that is, $X = M \in \mathcal{M}(Y)$. Similarly it can be done for $Y \in \mathcal{N}(X)$ and $X \in \mathcal{W}$. In this way we get some kind of the Rakić duality principle when the eigenvalues are extremal,

$$\begin{aligned}
0 \ne Y \in \mathcal{M}(X) \wedge 0 \ne X \in \mathcal{U} \quad &\Longleftrightarrow \quad 0 \ne X \in \mathcal{M}(Y) \wedge 0 \ne Y \in \mathcal{U}, \\
0 \ne Y \in \mathcal{N}(X) \wedge 0 \ne X \in \mathcal{W} \quad &\Longleftrightarrow \quad 0 \ne X \in \mathcal{N}(Y) \wedge 0 \ne Y \in \mathcal{W}.
\end{aligned} \tag{10.11}$$

If we have both $\mathcal{J}_X(Y) = \varepsilon_X \lambda Y$ and $\mathcal{J}_Y(X) = \varepsilon_Y \lambda X$ for nonzero mutually orthogonal vectors $X, Y \in \mathcal{V}$ and $\lambda \in \mathbb{R}$, then for all $\alpha, \beta \in \mathbb{R}$, by Lemma 9.17 we have

$$\mathcal{J}_{\alpha X + \beta Y}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y) = \varepsilon_{\alpha X + \beta Y} \lambda (\varepsilon_Y \beta X - \varepsilon_X \alpha Y). \tag{10.12}$$

According to (10.11), $Y \in \mathcal{M}(X)$ with $X \in \mathcal{U}$ implies $X \in \mathcal{M}(Y)$ with $Y \in \mathcal{U}$, so (10.12) yields $\varepsilon_Y \beta X - \varepsilon_X \alpha Y \in \mathcal{M}(\alpha X + \beta Y)$ with $\alpha X + \beta Y \in \mathcal{U}$. Hence, $0 \neq U \in \mathcal{U}$ gives $\mathrm{Span}\{U\} \oplus \mathcal{M}(U) \subseteq \mathcal{U}$, or consequently $\mathcal{U}^\perp \subseteq \mathcal{N}(U)$, as well as its analogue for $\mathcal{W}$,

$$\begin{aligned} 0 \neq U \in \mathcal{U} \quad &\Longrightarrow \quad \mathrm{Span}\{U\} \oplus \mathcal{M}(U) \subseteq \mathcal{U}, \\ 0 \neq W \in \mathcal{W} \quad &\Longrightarrow \quad \mathrm{Span}\{W\} \oplus \mathcal{N}(W) \subseteq \mathcal{W}. \end{aligned} \tag{10.13}$$

Since $\dim(\mathrm{Span}\{U\} \oplus \mathcal{M}(U)) = p + 1$, $\dim(\mathrm{Span}\{W\} \oplus \mathcal{N}(W)) = q + 1$, while we have $\dim \mathcal{V} = p+q+1 > (p+1)+(q+1)$, the Grassmann formula gives a non-trivial intersection,

$$0 \neq (\mathrm{Span}\{U\} \oplus \mathcal{M}(U)) \cap (\mathrm{Span}\{W\} \oplus \mathcal{N}(W)) \subseteq \mathcal{U} \cap \mathcal{W}. \tag{10.14}$$

The formula (10.14) allows to take $0 \neq A \in \mathcal{U} \cap \mathcal{W}$, as an initial step, and exploit its nice properties $\mu_A = \mu_{\min}$ and $\nu_A = \nu_{\max}$.

Due to Theorem 10.13, n must be even, so we consider the next simplest case of twice an odd dimension n. In that case $\rho(n) = 2$, so the inequality (10.8) gives $q = 1$, which means a simple root.

Let us assume that one eigenvalue is simple, that is, $q = 1$. If we suppose $p > n/2$ (which excludes only $n = 4$), then according to the Grassmann formula any two $\mathcal{M}$ spaces have a non-trivial intersection. Thus, for nonzero $X, Y \in \mathcal{U}$ there exists $0 \neq S \in \mathcal{M}(X) \cap \mathcal{M}(Y)$, so (10.11) yields $X, Y \in \mathcal{M}(S)$ with $S \in \mathcal{U}$, and therefore by (10.13), $\mathrm{Span}\{X, Y\} \subset \mathcal{U}$, which proves that $\mathcal{U}$ is a subspace of $\mathcal{V}$.

We want to show that μ is constant, or equivalently $\mathcal{U} = \mathcal{V}$. Assuming the opposite, $\mathcal{U} \neq \mathcal{V}$, since $\mathcal{U}$ is a subspace, applying (10.13) we have $\dim \mathcal{U} = n - 1$, and therefore

$$0 \neq X \in \mathcal{U} \quad \Longrightarrow \quad \mathcal{U} = \mathrm{Span}\{X\} \oplus \mathcal{M}(X) \quad \wedge \quad \mathcal{U}^\perp = \mathcal{N}(X). \tag{10.15}$$

Let us start with $0 \neq A \in \mathcal{U} \cap \mathcal{W}$ from (10.14). For $0 \neq Z \in \mathcal{N}(A) = \mathcal{U}^\perp$, by (10.11) we have $A \in \mathcal{N}(Z)$ with $Z \in \mathcal{W}$, so $\mathcal{M}(A) = \mathcal{M}(Z) = \mathrm{Span}\{A, Z\}^\perp$. For $0 \neq B \in \mathcal{M}(A) = \mathcal{M}(Z)$ we have $B \in \mathcal{U}$, so by (10.15), $Z \in \mathcal{U}^\perp = \mathcal{N}(B)$. Then $g(\mathcal{J}_B(Z), Z) = g(\mathcal{J}_Z(B), B)$ gives $\nu_B = \mu_Z = c$. However, if B and Z are units, then by (10.12) holds

$$\mathcal{J}_{Z \cos t + B \sin t}(B \cos t - Z \sin t) = c(B \cos t - Z \sin t),$$

which implies $\overline{\mu}(t) = \mu(Z \cos t + B \sin t) = c$ or $\overline{\nu}(t) = \nu(Z \cos t + B \sin t) = c$, for any $t \in \mathbb{R}$. The functions $\overline{\mu}, \overline{\nu} \colon \mathbb{R} \to \mathbb{R}$ are continuous with $\overline{\mu} < \overline{\nu}$ and $\overline{\mu}(0) = \overline{\nu}(\pi/2) = c$, and hence $\mathbb{R} = \overline{\mu}^{-1}(c) \sqcup \overline{\nu}^{-1}(c)$ is a disjoint union of nonempty closed sets, which is not possible.

It proves that $\mu_X = \mu$ must be constant for $q = 1$, unless $n = 4$. This allows us to introduce a new algebraic curvature tensor $R' = R - \mu R^1$, where $R^1 \in \mathfrak{T}^0_4(\mathcal{V})$ is the tensor of constant sectional curvature one given by (6.14). This trick just shifts the eigenvalues, which means the characteristic polynomial of the new Jacobi operator becomes $\det(\lambda \mathbb{1} - \mathcal{J}'_X) = \lambda^{n-1}(\lambda - \varepsilon_X(\nu_X - \mu))$.

In order not to complicate things too much, we shall keep the previous notation and assume that $\mathcal{J}_X$ has a simple eigenvalue $\varepsilon_X \nu_X > 0$, while other eigenvalues are all zero. This essentially means that the original reduced Jacobi operator $\widetilde{\mathcal{J}}_X$ has a simple eigenvalue $\varepsilon_X(\nu_X + \mu)$, while the other root is $\varepsilon_X \mu$ with multiplicity $n - 2$.

A simple eigenvalue motivates us to modify the ideas from Section 10.1. Let us choose an arbitrary orthonormal basis $(E_1, \ldots, E_n)$ in $\mathcal{V}$. Then for any nonzero $X = \sum_{i=1}^n x_i E_i \in \mathcal{V}$, the Jacobi operator $\mathcal{J}_X$ is of rank one such that its matrix entries $\mathcal{J}_{ij}(X)$ are quadratic forms in n variables $x_1, \ldots, x_n$. Any submatrix of order two in a rank one symmetric matrix is singular which gives

$$\mathcal{J}_{ii}(X)\mathcal{J}_{jj}(X) = \mathcal{J}_{ij}(X)^2 \tag{10.16}$$

for all $1 \leq i, j \leq n$. If we fix some monomial order then there is a unique monic $G(X)$ which is the greatest common divisor of all $\mathcal{J}_{ij}(X)$. Permuting the basis we can set

$$\mathcal{J}_{ii}(X) = \sigma_i G(X) Q_i(X) P_i(X)^2,$$

where $P_i(X)$ and $Q_i(X)$ are some nonzero polynomials for $1 \leq i \leq m$, with additional $\mathcal{J}_{ii}(X) = 0$ for $m < i \leq n$, while $\sigma_i \in \{-1, 1\}$. However, such decomposition is unique up to sign of $P_i(X)$ if we set that $Q_i(X)$ is monic square-free. Then

$$\sigma_i \sigma_j G(X)^2 Q_i(X) Q_j(X) P_i(X)^2 P_j(X)^2 = \mathcal{J}_{ij}(X)^2$$

implies $Q_i(X) = Q_j(X) = Q(X) = 1$ and $\sigma_i = \sigma_j = \sigma$ for $1 \leq i \leq m$, and therefore we have $\mathcal{J}_{ij}(X) = \sigma_{ij} G(X) P_i(X) P_j(X)$, where $\sigma_{ij} \in \{-1, 1\}$. Additionally, by (10.16), $\mathcal{J}_{ij}(X) = 0$ holds whenever $m < i \leq n$ or $m < j \leq n$, which can be treated as $P_i(X) = 0$ for $m < i \leq n$ and extend the indices to $m = n$.

Another submatrix of order two gives $\mathcal{J}_{1i}(X)\mathcal{J}_{ij}(X) = \mathcal{J}_{1j}(X)\mathcal{J}_{ii}(X)$, so $\sigma_{1i}\sigma_{ij} = \sigma_{1j}\sigma_{ii}$. Because of $\sigma_{ii} = \sigma_i = \sigma$ we have $\sigma_{ij} = \sigma\sigma_{1i}\sigma_{1j}$, and therefore $\mathcal{J}_{ij}(X) = \sigma G(X) \sigma_{1i} P_i(X) \sigma_{1j} P_j(X)$. Since the polynomials $P_i(X)$ are unique up to sign, we can use $\sigma_{1i}P_i(X)$ instead of $P_i(X)$ to obtain $\mathcal{J}_{ij}(X) = \sigma G(X) P_i(X) P_j(X)$ for all $1 \leq i \leq n$.

Moreover, comparing the degrees in a polynomial $\mathcal{J}_{ij}(X)$ we conclude that all P_i have the same degree, zero or one. The degree zero yields constant polynomials P_i, so $\mathcal{J}_X = G(X)M$, for some constant matrix M. In that case, if $\mathcal{J}_X(Y) = \varepsilon_X \nu_X Y$, then $\mathcal{J}_X(Z) = 0$ for all $Z \in Y^\perp$, which gives $MZ = 0$. However, then $\mathcal{J}_Y(Z) = G(Y)MZ = 0$, which gives the contradiction $\mathcal{J}_Y = 0$. Therefore, all P_i have degree one, while $G(X)$ has degree zero and consequently $G(X) = 1$.

Summarizing the previous results, the equality

$$\mathcal{J}_{ij}(X) = \sigma P_i(X) P_j(X)$$

holds for all $1 \leq i, j \leq n$, where P_i are linear homogeneous polynomials. If we set

$$P(X) = \sum_{i=1}^{n} P_i(X) E_i,$$

then it implies

$$\mathcal{J}_X(P(X)) = \sum_{i=1}^{n} P_i(X) \sum_{j=1}^{n} \mathcal{J}_{ji}(X) E_j = \sigma \sum_{i=1}^{n} P_i(X)^2 (P(X)).$$

Thus, $P(X)$ is an eigenvector of $\widetilde{\mathcal{J}}_X$ associated to the simple eigenvalue

$$\sigma \varepsilon_{P(X)} = \sigma \sum_{i=1}^{n} P_i(X)^2 = \operatorname{Tr} \mathcal{J}_X = \varepsilon_X \nu_X,$$

but since we set $\nu_X > 0$, it must be $\sigma = 1$. In this way we construct a linear map $P\colon \mathcal{V} \to \mathcal{V}$ such that $\mathcal{N}(X) = \operatorname{Span}\{P(X)\}$ and

$$\nu_X = \frac{\varepsilon_{P(X)}}{\varepsilon_X}$$

with $\nu_{P(X)} \geq \nu_X$ (because of (10.10)) for any nonzero $X \in \mathcal{V}$.

Let us start with $0 \neq A \in \mathcal{W}$, when $P(A) \in \mathcal{N}(A)$, because of (10.11), implies $A \in \mathcal{N}(P(A))$ with $P(A) \in \mathcal{W}$. Hence, by (10.12),

$$\mathcal{J}_{\alpha A + \beta P(A)}(\varepsilon_{P(A)} \beta A - \varepsilon_A \alpha P(A)) = \varepsilon_{\alpha A + \beta P(A)} \nu_{\max} (\varepsilon_{P(A)} \beta A - \varepsilon_A \alpha P(A)),$$

which gives $\nu(\alpha A + \beta P(A)) = \nu_{\max}$ and $P(\alpha A + \beta P(A)) \propto \varepsilon_{P(A)} \beta A - \varepsilon_A \alpha P(A)$. Using the linearity of P and the fact that $P(A) \perp \operatorname{Span}\{A, P^2(A)\}$, we get the coefficient of proportionality equal to $-1/\varepsilon_A$, and consequently $P^2(A) = -\nu_A A$ with $\operatorname{Span}\{A, P(A)\} \subseteq \mathcal{W}$.

We can continue in a similar manner, using $A_1 = A$ and $\nu_1 = \nu_{\max}$ as the induction basis. Let us suppose that we already have mutually orthogonal nonzero vectors $A_1, P(A_1), \ldots, A_k, P(A_k)$ such that

$$\operatorname{Span}\{A_i, P(A_i)\} \setminus \{0\} \subseteq \nu^{-1}(\nu_i) \quad \text{and} \quad P^2(A_i) = -\nu_i A_i \tag{10.17}$$

hold for all $1 \leq i \leq k$ with $0 < \nu_k \leq \cdots \leq \nu_1$. We define

$$\nu_{k+1} = \max\{\nu_X : X \in \mathcal{S} \cap \mathcal{M}(A_1) \cdots \cap \mathcal{M}(A_k)\} \leq \nu_k$$

and take arbitrarily $0 \neq A_{k+1} \in \nu^{-1}(\nu_{k+1})$. It is fruitful to notice that, since μ is constant, the duality (10.11) always provides $Y \in \mathcal{M}(X) \iff X \in \mathcal{M}(Y)$. As a consequence of this, $A_{k+1} \in \mathcal{M}(A_i) = \mathcal{M}(P(A_i)) = \operatorname{Span}\{A_i, P(A_i)\}^{\perp}$ implies $A_i, P(A_i) \in \mathcal{M}(A_{k+1}) \perp \mathcal{N}(A_{k+1})$, and therefore $P(A_{k+1}) \in \operatorname{Span}\{A_i, P(A_i)\}^{\perp} = \mathcal{M}(A_i)$. Thus we have $\nu_{P(A_{k+1})} \leq \nu_{k+1}$, which by the formula (10.10) yields $\nu_{P(A_{k+1})} = \nu_{k+1}$, while (10.12) gives $\operatorname{Span}\{A_{k+1}, P(A_{k+1})\} \setminus \{0\} \subseteq \nu^{-1}(\nu_{k+1})$ and $P^2(A_{k+1}) = -\nu_{k+1}A_{k+1}$.

This procedure uses constants $0 < \nu_{n/2} \leq \cdots \leq \nu_1$ with the properties (10.17) to exhaust the space

$$\mathcal{V} = \bigoplus_{i=1}^{n/2} \operatorname{Span}\{A_i, P(A_i)\}. \tag{10.18}$$

Having that on mind, it is easy to conclude that P is skew-adjoint. Namely, if we introduce $X = \sum_{i=1}^{n/2}(x_iA_i + \bar{x}_iP(A_i))$ and $Y = \sum_{i=1}^{n/2}(y_iA_i + \bar{y}_iP(A_i))$, then

$$g(P(X), Y) = \sum_{i=1}^{n/2} g(x_iP(A_i) - \bar{x}_i\nu_iA_i, y_iA_i + \bar{y}_iP(A_i)) = \sum_{i=1}^{n/2} \nu_i\varepsilon_{A_i}(x_i\bar{y}_i - \bar{x}_iy_i) = -g(P(Y), X).$$

The key idea is that any skew-adjoint endomorphism P on $\mathcal{V}$, according to Example 8.3 generates an algebraic curvature tensor $R^P \in \mathfrak{T}_4^0(\mathcal{V})$, where we have

$$\mathcal{J}_X^P = \begin{cases} -3\varepsilon_X\nu_X \mathbb{1} & \text{on } \operatorname{Span}\{P(X)\} \\ 0 & \text{on } \operatorname{Span}\{P(X)\}^{\perp} \end{cases}.$$

Therefore, taking into account the shifting of eigenvalues for $\varepsilon_X\mu$, and the possible choice of $\nu_X < \mu_X$ from the beginning of discussion, the algebraic curvature tensor must be of form

$$R = \pm\left(-\frac{1}{3}R^P + \mu R^1\right). \tag{10.19}$$

This result is better expressed in an orthonormal basis $(E_1, F_1, \ldots, E_{n/2}, F_{n/2})$ obtained from (10.18) by rescaling $E_i = A_i/\sqrt{\varepsilon_{A_i}}$ and $F_i = P(A_i)/\sqrt{\varepsilon_{P(A_i)}}$. Conversely, it is easy to see that for any orthonormal basis $(E_1, F_1, \ldots, E_{n/2}, F_{n/2})$ in $\mathcal{V}$, constants $0 < \nu_{n/2} \leq \cdots \leq \nu_1$ define a skew-adjoint endomorphism P on $\mathcal{V}$ by

$$P(E_i) = \sqrt{\nu_i}F_i, \quad P(F_i) = -\sqrt{\nu_i}E_i, \tag{10.20}$$

for all $1 \leq i \leq n/2$. Hence we obtain the following theorem given by Andrejić in 2023 [10, Theorem 4].

Theorem 10.14. *Any Riemannian two-root algebraic curvature tensor of dimension $n > 4$ with a simple root is of the form* (10.19)*, for $\mu \in \mathbb{R}$ and some skew-adjoint endomorphism P defined by* (10.20) *using positive constants $\nu_1, \ldots, \nu_{n/2} \in \mathbb{R}$.*

10.7 Two-root Riemannian manifolds

Theorem 10.14 and the formula (10.19) characterize all possible two-root algebraic curvature tensors of twice an odd dimension. The next step is then based on the use of the second Bianchi identity with an idea to decide which of these algebraic curvature tensors may be realized as curvature tensors of a Riemannian manifold.

We shall study the Riemannian manifold M locally in a neighbourhood $U \subset M$ of some point. There we can set a local orthonormal frame and smoothly extend the elements of our construction. The smoothness of the curvature tensor $R \in \mathfrak{T}_4^0(U)$ gives the smoothness of $\mu \in \mathfrak{F}(U)$, while ν is smooth on the tangent bundle TU minus the zero section. Then, the way we constructed P brings the smoothness of $P_i(X) \in \mathfrak{F}(U)$, which yields a skew-adjoint operator $P \in \mathfrak{T}_1^1(U)$. Finally, $\nu \in \mathfrak{F}(TU \setminus (M \times \{0\}))$ implies $\nu_1, \dots, \nu_{n/2} \in \mathfrak{F}(U)$, and we can extend our orthonormal bases from the construction to a local orthonormal frame $(E_1, F_1, \dots, E_{n/2}, F_{n/2})$ in $\mathfrak{X}(U)$ that fits the formula (10.20). It is convenient to use this frame in the following proof.

Such extensions allow us to apply covariant derivatives to our tensors. It is important to notice that $\nabla_V P \in \mathfrak{T}_1^1(U)$ is also skew-adjoint, since $PX \perp X$ implies

$$0 = \nabla_V(g(PX,X)) = g(\nabla_V(PX),X) + g(PX,\nabla_V X) = g(\nabla_V(PX),X) - g(P\nabla_V X,X) = g((\nabla_V P)X,X),$$

which after the polarization gives

$$g((\nabla_V P)X,Y) = -g((\nabla_V P)Y,X),$$

for all $X, Y, V \in \mathfrak{X}(U)$.

Since $\nabla R^1 = 0$, the covariant derivative along a vector field $V \in \mathfrak{X}(U)$ of our curvature tensor R from the formula (10.19) can be expressed by

$$\nabla_V R = \mp\frac{1}{3}\nabla_V R^P \pm (\nabla_V \mu)R^1.$$

For all $X, Y, Z, W, V \in \mathfrak{X}(U)$ we can calculate

$$\begin{aligned}(\nabla_V R^P)(X,Y,Z,W) = \; & g\big(g(PX,Z)(\nabla_V P)Y - g(PY,Z)(\nabla_V P)X + 2g(PX,Y)(\nabla_V P)Z, W\big)\\ &+ g\big(g((\nabla_V P)X,Z)PY - g((\nabla_V P)Y,Z)PX + 2g((\nabla_V P)X,Y)PZ, W\big),\end{aligned} \tag{10.21}$$

and

$$\begin{aligned}&(\nabla_V R^P)(X,Y,Y,X) + (\nabla_X R^P)(Y,V,Y,X) + (\nabla_Y R^P)(V,X,Y,X)\\ &\quad= \; 3g(PX,Y)\big(2g((\nabla_V P)Y,X) - g((\nabla_X P)Y,V) + g((\nabla_Y P)X,V)\big)\\ &\qquad - 3g\big(g((\nabla_X P)X,Y)PY + g((\nabla_Y P)Y,X)PX, V\big).\end{aligned}$$

Thus, applying the second Bianchi identity yields

$$\begin{aligned}0 =& (\nabla_V R)(X,Y,Y,X) + (\nabla_X R)(Y,V,Y,X) + (\nabla_Y R)(V,X,Y,X)\\ =& \mp g(PX,Y)\big(2g((\nabla_V P)Y,X) - g((\nabla_X P)Y,V) + g((\nabla_Y P)X,V)\big)\\ &\pm g\big(g((\nabla_X P)X,Y)PY + g((\nabla_Y P)Y,X)PX, V\big)\\ &\pm (\nabla_V\mu)(\varepsilon_X\varepsilon_Y - g(X,Y)^2) \pm (\nabla_X\mu)(g(X,Y)g(Y,V) - \varepsilon_Y g(X,V))\\ &\pm (\nabla_Y\mu)(g(X,Y)g(X,V) - \varepsilon_X g(Y,V)).\end{aligned} \tag{10.22}$$

Assuming $Y \perp PX$ in (10.22) we get

$$\begin{aligned}0 = \; & g\big(g((\nabla_X P)X,Y)PY + g((\nabla_Y P)Y,X)PX, V\big)\\ &+ (\nabla_X\mu)g(g(X,Y)Y - \varepsilon_Y X, V) + (\nabla_Y\mu)g(g(X,Y)X - \varepsilon_X Y, V) + (\nabla_V\mu)(\varepsilon_X\varepsilon_Y - g(X,Y)^2),\end{aligned}$$

and thus

$$\begin{aligned}(\varepsilon_X\varepsilon_Y - g(X,Y)^2)(\nabla\mu)^\sharp =& - g((\nabla_X P)X,Y)PY - g((\nabla_Y P)Y,X)PX\\ &+ (X(\mu)\varepsilon_Y - Y(\mu)g(X,Y))X + (Y(\mu)\varepsilon_X - X(\mu)g(X,Y))Y.\end{aligned}$$

Therefore, for nowhere vanishing $X, Y \in \mathfrak{X}(U)$ such that $Y \in \operatorname{Span}\{X, PX\}^\perp$ we have that $(\nabla\mu)^\sharp \in \operatorname{Span}\{X, PX, Y, PY\}$ holds. However, using our frame with (10.20) we get

$$(\nabla\mu)^\sharp \in \bigcap_{1\le i<j\le n/2} \operatorname{Span}\{E_i, F_i, E_j, F_j\} = 0,$$

which gives $\nabla\mu = 0$. Therefore μ must be constant, while the formula (10.22) yields

$$\begin{aligned} &g(PX,Y)\big(2g((\nabla_V P)Y,X) - g((\nabla_X P)Y,V) + g((\nabla_Y P)X,V)\big) \\ &\quad - g\big(g((\nabla_X P)X,Y)PY + g((\nabla_Y P)Y,X)PX, V\big) = 0. \end{aligned} \tag{10.23}$$

Again, $Y \perp PX$ gives $g((\nabla_X P)X,Y)PY + g((\nabla_Y P)Y,X)PX = 0$, while the additional $Y \perp X$ provides linear independence for PX and PY (since X and Y are linearly independent as orthogonal), and therefore $g((\nabla_X P)X,Y) = 0$ for $Y \in \mathrm{Span}\{X,PX\}^\perp$. However, we know that $g((\nabla_X P)X,X) = 0$ holds, which implies $(\nabla_X P)X \propto PX$.

Let us define the map $\lambda_X\colon U \to \mathbb{R}$ for any $X \in \mathfrak{X}(U)$ by $\lambda_X = g((\nabla_X P)X,PX)/g(PX,PX)$ on $(\varepsilon_{PX})^{-1}(\mathbb{R}_+) = (\varepsilon_X)^{-1}(\mathbb{R}_+) \subseteq U$ and $\lambda_X = 0$ on $(\varepsilon_X)^{-1}(\{0\})$, where the previously proven proportionality yields

$$(\nabla_X P)X = \lambda_X PX. \tag{10.24}$$

It is easy to check that $\lambda_{fX} = f\lambda_X$ holds for $f \in \mathfrak{F}(U)$. On the other hand, from (10.24) for any $X,Y \in \mathfrak{X}(U)$ we have

$$(\nabla_X P)Y + (\nabla_Y P)X = (\lambda_{X+Y} - \lambda_X)PX + (\lambda_{X+Y} - \lambda_Y)PY,$$

which after taking the scalar product by X gives

$$(\lambda_{X+Y} - \lambda_Y)g(PY,X) = g((\nabla_X P)Y,X) = -g((\nabla_X P)X,Y) = -\lambda_X g(PX,Y),$$

and therefore $(\lambda_{X+Y} - \lambda_Y - \lambda_X)g(PX,Y) = 0$. Hence, the additivity

$$\lambda_{X+Y} = \lambda_X + \lambda_Y$$

holds whenever $g(Y,PX)$ is nowhere zero. For any $p \in U$, the condition $Y_p \perp PX_p$ can be excluded by continuity of $\lambda(p)\colon T_pU \to \mathbb{R}$ given by $\lambda(p)(X_p) = \lambda_X(p)$. Thus, the additivity holds for all $X,Y \in \mathfrak{X}(U)$, which means that λ is $\mathfrak{F}(U)$-linear. Consequently, since $\lambda_E \in \mathfrak{F}(U)$ for a unit $E \in \mathfrak{X}(U)$, we have $\lambda_X \in \mathfrak{F}(U)$ for any $X \in \mathfrak{X}(U)$, and finally $\lambda \in \mathfrak{T}^0_1(U) = \mathfrak{X}^*(U)$.

With this in mind, the equality (10.23) becomes

$$g(PX,Y)\Big(2g((\nabla_V P)Y,X) - g\big((\nabla_X P)Y - (\nabla_Y P)X + \lambda_X PY - \lambda_Y PX, V\big)\Big) = 0.$$

Hence,

$$2g((\nabla_V P)Y,X) = g\big((\nabla_X P)Y - (\nabla_Y P)X + \lambda_X PY - \lambda_Y PX, V\big) \tag{10.25}$$

holds in the case that $g(PX,Y)$ is nowhere zero. However, since the right hand side is linear in Y and there is a frame consisting of vector fields that are not orthogonal to PX, the equality (10.25) holds for all $X,Y,V \in \mathfrak{X}(U)$. Applying (10.25) twice, we have

$$4g((\nabla_V P)Y,X) = g\big((\nabla_V P)Y - (\nabla_Y P)V + \lambda_V PY - \lambda_Y PV, X\big) + 2g\big(-(\nabla_Y P)X + \lambda_X PY - \lambda_Y PX, V\big),$$

and therefore

$$2\lambda_X g(PY,V) = g\big(3(\nabla_V P)Y - (\nabla_Y P)V - \lambda_V PY - \lambda_Y PV, X\big),$$

which implies

$$2g(PY,X)\lambda^\sharp = 3(\nabla_X P)Y - (\nabla_Y P)X - \lambda_X PY - \lambda_Y PX.$$

On the other hand, the definition of $\lambda \in \mathfrak{X}^*(U)$ gives $(\nabla_X P)Y + (\nabla_Y P)X = \lambda_Y PX + \lambda_X PY$, and therefore we obtain $g(X,PY)\lambda^\sharp = (\nabla_X P)Y - (\nabla_Y P)X$, which can be written as

$$2(\nabla_X P)Y = g(X,PY)\lambda^\sharp + \lambda_Y PX + \lambda_X PY. \tag{10.26}$$

Now that we know ∇P, it remains to calculate $\nabla^2 P$ and use the Ricci identity for the tensor field $P \in \mathfrak{T}^1_1(U)$. From (10.26) we have

$$\begin{aligned}
2(\nabla_X\nabla_Y P)Z =& 2\nabla_X((\nabla_Y P)Z) - 2(\nabla_Y P)(\nabla_X Z)\\
=& \nabla_X(g(Y,PZ)\lambda^\sharp + \lambda_Z PY + \lambda_Y PZ) - (g(Y,P(\nabla_X Z))\lambda^\sharp + \lambda_{\nabla_X Z}PY + \lambda_Y P(\nabla_X Z))\\
=& (g(\nabla_X Y,PZ) + g(Y,\nabla_X PZ))\lambda^\sharp + g(Y,PZ)\nabla_X\lambda^\sharp\\
&+ (g(\nabla_X\lambda^\sharp,Z) + g(\lambda^\sharp,\nabla_X Z))PY + \lambda_Z\nabla_X PY\\
&+ (g(\nabla_X\lambda^\sharp,Y) + g(\lambda^\sharp,\nabla_X Y))PZ + \lambda_Y\nabla_X PZ\\
&- g(Y,P(\nabla_X Z))\lambda^\sharp - \lambda_{\nabla_X Z}PY - \lambda_Y P(\nabla_X Z)\\
=& (g(\nabla_X Y,PZ) + g(Y,(\nabla_X P)Z))\lambda^\sharp + g(\nabla_X\lambda^\sharp,Z)PY + (g(\nabla_X\lambda^\sharp,Y) + g(\lambda^\sharp,\nabla_X Y))PZ\\
&+ g(Y,PZ)\nabla_X\lambda^\sharp + \lambda_Z\nabla_X PY + \lambda_Y(\nabla_X P)Z,
\end{aligned}$$

and therefore

$$\begin{aligned}
2(\nabla^2_{X,Y}P - \nabla^2_{Y,X}P)Z =& 2(\nabla_X\nabla_Y P - \nabla_Y\nabla_X P - \nabla_{\nabla_X Y - \nabla_Y X}P)Z\\
=& (g(\nabla_X Y,PZ) + g(Y,(\nabla_X P)Z))\lambda^\sharp + g(\nabla_X\lambda^\sharp,Z)PY + (g(\nabla_X\lambda^\sharp,Y) + g(\lambda^\sharp,\nabla_X Y))PZ\\
&+ g(Y,PZ)\nabla_X\lambda^\sharp + \lambda_Z\nabla_X PY + \lambda_Y(\nabla_X P)Z\\
&- (g(\nabla_Y X,PZ) + g(X,(\nabla_Y P)Z))\lambda^\sharp - g(\nabla_Y\lambda^\sharp,Z)PX - (g(\nabla_Y\lambda^\sharp,X) + g(\lambda^\sharp,\nabla_Y X))PZ\\
&- g(X,PZ)\nabla_Y\lambda^\sharp - \lambda_Z\nabla_Y PX - \lambda_X(\nabla_Y P)Z\\
&- g(\nabla_X Y - \nabla_Y X,PZ)\lambda^\sharp - \lambda_Z P(\nabla_X Y - \nabla_Y X) - \lambda_{\nabla_X Y - \nabla_Y X}PZ,\\
=& (g(Y,(\nabla_X P)Z) - g(X,(\nabla_Y P)Z))\lambda^\sharp - g(\nabla_Y\lambda^\sharp,Z)PX + g(\nabla_X\lambda^\sharp,Z)PY\\
&+ (g(\nabla_X\lambda^\sharp,Y) - g(\nabla_Y\lambda^\sharp,X))PZ + g(Y,PZ)\nabla_X\lambda^\sharp - g(X,PZ)\nabla_Y\lambda^\sharp\\
&+ \lambda_Z(\nabla_X P)Y - \lambda_Z(\nabla_Y P)X + \lambda_Y(\nabla_X P)Z - \lambda_X(\nabla_Y P)Z.
\end{aligned}$$

Applying (10.26) again we obtain

$$\begin{aligned}
&2(\nabla^2_{X,Y}P - \nabla^2_{Y,X}P)Z\\
&= \lambda_Z g(Y,PX)\lambda^\sharp - g(\nabla_Y\lambda^\sharp,Z)PX + g(\nabla_X\lambda^\sharp,Z)PY\\
&\quad + (g(\nabla_X\lambda^\sharp,Y) - g(\nabla_Y\lambda^\sharp,X))PZ + g(Y,PZ)\nabla_X\lambda^\sharp - g(X,PZ)\nabla_Y\lambda^\sharp\\
&\quad + \lambda_Z g(X,PY)\lambda^\sharp + \frac{1}{2}(\lambda_Y g(X,PZ) - \lambda_X g(Y,PZ))\lambda^\sharp + \frac{1}{2}\lambda_Y\lambda_Z PX - \frac{1}{2}\lambda_X\lambda_Z PY\\
&= \frac{1}{2}(\lambda_Y g(X,PZ) - \lambda_X g(Y,PZ))\lambda^\sharp + (\frac{1}{2}\lambda_Y\lambda_Z - g(\nabla_Y\lambda^\sharp,Z))PX - (\frac{1}{2}\lambda_X\lambda_Z - g(\nabla_X\lambda^\sharp,Z))PY\\
&\quad + (g(\nabla_X\lambda^\sharp,Y) - g(\nabla_Y\lambda^\sharp,X))PZ + g(Y,PZ)\nabla_X\lambda^\sharp - g(X,PZ)\nabla_Y\lambda^\sharp.
\end{aligned}$$

We introduce the operator $Q \in \mathfrak{T}^1_1(U)$ defined by $QX = \frac{1}{2}\lambda_X\lambda^\sharp - \nabla_X\lambda^\sharp$ to simplify the notation, so the previous equality becomes

$$\begin{aligned}
2(\nabla^2_{X,Y}P - \nabla^2_{Y,X}P)Z =& g(X,PZ)QY - g(Y,PZ)QX\\
&+ g(Z,QY)PX - g(Z,QX)PY + (g(QY,X) - g(QX,Y))PZ.
\end{aligned} \tag{10.27}$$

On the other hand, for the curvature operator $\mathcal{R}$ we have $\pm 3\mathcal{R} = -\mathcal{R}^P + 3\mu\mathcal{R}^1$ from

(10.19), and therefore

$$\begin{aligned}
&\pm 3(\mathcal{R}(X,Y)PZ - P(\mathcal{R}(X,Y)Z))\\
&\qquad = -\mathcal{R}^P(X,Y)PZ + 3\mu\mathcal{R}^1(X,Y)PZ + P(\mathcal{R}^P(X,Y)Z) - 3\mu P(\mathcal{R}^1(X,Y)Z)\\
&\qquad = -g(PX,PZ)PY + g(PY,PZ)PX - 2g(PX,Y)P^2Z + 3\mu(g(Y,PZ)X - g(X,PZ)Y)\\
&\qquad\quad + g(PX,Z)P^2Y - g(PY,Z)P^2X + 2g(PX,Y)P^2Z - 3\mu(g(Y,Z)PX - g(X,Z)PY)\\
&\qquad = -g(PX,PZ)PY + g(PY,PZ)PX + 3\mu g(Y,PZ)X - 3\mu g(X,PZ)Y\\
&\qquad\quad + g(PX,Z)P^2Y - g(PY,Z)P^2X - 3\mu g(Y,Z)PX + 3\mu g(X,Z)PY.
\end{aligned}$$

We introduce the self-adjoint operator $S \in \mathfrak{T}^1_1(U)$ defined by $SX = 3\mu X + P^2X$, so the previous equality becomes

$$\pm 3(\mathcal{R}(X,Y)PZ - P(\mathcal{R}(X,Y)Z)) = g(PX,Z)SY - g(PY,Z)SX + g(SX,Z)PY - g(SY,Z)PX.$$

The Ricci identity (6.36), $((\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})P)Z = \mathcal{R}(X,Y)PZ - P(\mathcal{R}(X,Y)Z)$, holds for all $X, Y, Z \in \mathfrak{X}(U)$, which using (10.27) yields

$$\begin{aligned}
&g(X,PZ)QY - g(Y,PZ)QX + g(Z,QY)PX - g(Z,QX)PY + (g(QY,X) - g(QX,Y))PZ\\
&= \pm\frac{2}{3}\Big(g(PX,Z)SY - g(PY,Z)SX + g(SX,Z)PY - g(SY,Z)PX\Big).
\end{aligned}$$

It is convenient to introduce another operator $K = Q \pm \frac{2}{3}S \in \mathfrak{T}^1_1(U)$, that is

$$KX = \frac{1}{2}\lambda_X\lambda^\sharp - \nabla_X\lambda^\sharp \pm \frac{2}{3}(3\mu X + P^2X),$$

for $X \in \mathfrak{X}(U)$, so the previous equality becomes

$$g(X,PZ)KY - g(Y,PZ)KX + g(Z,KY)PX - g(Z,KX)PY + (g(KY,X) - g(KX,Y))PZ = 0.$$

The special case $Z = Y$ implies

$$g(X,PY)KY + g(Y,KY)PX + (g(KY,X) - 2g(KX,Y))PY = 0, \tag{10.28}$$

which holds for all $X, Y \in \mathfrak{X}(U)$. For an arbitrary nowhere vanishing $Y \in \mathfrak{X}(U)$ we can take a nowhere vanishing $X \in \operatorname{Span}\{Y, PY\}^\perp$. In this case $X \perp PY$ gives

$$g(Y,KY)PX + (g(KY,X) - 2g(KX,Y))PY = 0,$$

but since X and Y are linearly independent as mutually orthogonal, PX and PY are linearly independent, which implies $g(Y,KY) = 0$.

Hence, $KY \perp Y$ holds for any $Y \in \mathfrak{X}(U)$, so the polarization gives $g(KX,Y) + g(KY,X) = 0$, and proves that K is also skew-adjoint. With this in mind, the equality (10.28) becomes

$$g(X,PY)KY + 3g(KY,X)PY = 0,$$

and holds for all $X, Y \in \mathfrak{X}(U)$. Substituting $X = PY$ for a nowhere vanishing $Y \in \mathfrak{X}(U)$, we obtain $KY \propto PY$, while taking the inner product by PY we get $4\varepsilon_{PY}g(KY,PY) = 0$, and therefore $K = 0$. Thus arises the important formula

$$\nabla_X\lambda^\sharp = \frac{1}{2}\lambda_X\lambda^\sharp \pm \frac{2}{3}(3\mu X + P^2X). \tag{10.29}$$

For any $X, Z \in \mathfrak{X}(U)$, we use (10.26) to calculate

$$\begin{aligned}
\nabla_X(\varepsilon_{PZ}) &= \nabla_X g(PZ,PZ) = 2g(\nabla_X(PZ),PZ) = 2g((\nabla_XP)Z + P\nabla_XZ, PZ)\\
&= g(X,PZ)\lambda_{PZ} + g(PX,PZ)\lambda_Z + g(PZ,PZ)\lambda_X - 2g(\nabla_XZ, P^2Z).
\end{aligned}$$

On the other hand, $\nabla_X(\varepsilon_{PZ}) = \nabla_X(\nu_Z\varepsilon_Z) = \nu_Z\nabla_X\varepsilon_Z + \varepsilon_Z\nabla_X\nu_Z$, which gives

$$\nu_Z\nabla_X\varepsilon_Z + \varepsilon_Z\nabla_X\nu_Z = g(X,PZ)\lambda_{PZ} - g(X,P^2Z)\lambda_Z + \varepsilon_{PZ}\lambda_X - 2g(\nabla_X Z, P^2Z).$$

Consider the eigenspaces of P^2, defined by

$$\mathcal{P}_j = \mathrm{Ker}(P^2 + \nu_j\,\mathbb{1}) = \bigoplus_{\nu_i=\nu_j} \mathrm{Span}\{E_i, F_i\}.$$

If $Z \in \mathcal{P}_j$ holds for some $1 \le j \le n/2$, then $-2g(\nabla_X Z, P^2Z) = 2\nu_j g(\nabla_X Z, Z) = \nu_j\nabla_X\varepsilon_Z$, which implies

$$\varepsilon_Z\nabla_X\nu_j = g(X,PZ)\lambda_{PZ} + \nu_j g(X,Z)\lambda_Z + \nu_j\varepsilon_Z\lambda_X.$$

Hence, we obtain

$$d(\ln\nu_j)(X) = \frac{\nabla_X\nu_j}{\nu_j} = \begin{cases} 2\lambda_X \text{ for } X \in \mathrm{Span}\{Z, PZ\} \\ \lambda_X \text{ for } X \in \mathrm{Span}\{Z, PZ\}^\perp \end{cases}, \tag{10.30}$$

whenever $Z \in \mathcal{P}_j$, and therefore

$$\lambda = \frac{2}{n+2}\sum_{i=1}^{n/2} d(\ln\nu_i) = \frac{2}{n+2}d(\ln(\nu_1\nu_2\cdots\nu_{n/2})). \tag{10.31}$$

The equation (10.31) shows that λ cannot be any covector field, but at least one that is the differential of a smooth function. Moreover, using (10.29) we have the necessary condition,

$$\nabla_X \operatorname{grad}(\ln(\nu_1\nu_2\cdots\nu_{n/2})) = \frac{1}{2}\lambda_X \operatorname{grad}(\ln(\nu_1\nu_2\cdots\nu_{n/2})) \pm \frac{n+2}{3}(3\mu X + P^2X),$$

that holds for any $X \in \mathfrak{X}(U)$. This finally proves the following theorem given by Andrejić in 2023 [10, Theorem 5].

Theorem 10.15. *A two-root Riemannian manifold of dimension $n \equiv 2 \pmod 4$ locally has the curvature tensor of the form* (10.19)*, for a constant μ and some skew-adjoint linear operator P defined by* (10.20) *using positive smooth functions $\nu_1, \dots, \nu_{n/2}$. In addition, the equations* (10.26), (10.29), (10.30), *and* (10.31) *hold.*

The most natural case has $\lambda = 0$, where the equation (10.26) implies $\nabla P = 0$, so (10.21) gives $\nabla R^P = 0$, and consequently $\nabla R = 0$, which means that M is locally symmetric. Moreover, the equation (10.29) for $\lambda = 0$ implies $P^2 = -3\mu\,\mathbb{1}$, which implies $\nu = 3\mu$, and consequently M is globally Osserman, where the reduced Jacobi operator $\widetilde{\mathcal{J}}_X$ has a simple eigenvalue $4\varepsilon_X\mu$, while the other eigenvalue (with multiplicity $n-2$) is four times smaller. Thus, a connected two-root Riemannian manifold of dimension $n \ge 3$ with $n \equiv 2 \pmod 4$ that has $\lambda = 0$ is globally Osserman, and hence is two-points homogeneous.

Let us remark, that if $\nabla\nu_j = 0$ holds for some $1 \le j \le n/2$, then $\lambda = 0$, and the previous conclusion holds. The question whether there are two-root Riemannian manifolds of twice an odd dimension that are not Osserman remains open and requires a construction of concrete manifolds with $\lambda \ne 0$. Let us remark that the first attempt could be $\lambda^\sharp = E_k$ for some $1 \le k \le n/2$, where (10.30) yields $d(\ln\nu_k) = 2\lambda = 2d(\ln\nu_i)$ for any $i \ne k$, and therefore there exist constants C_i such that $\nu_k = C_i\nu_i^2$.

10.8 Proportionality principle

Let R be a k-root Jacobi-diagonalisable algebraic curvature tensor on a scalar product space $(\mathcal{V}, g)$ such that

$$\widetilde{\omega}_X(\lambda) = \det(\lambda\,\mathbb{1} - \widetilde{\mathcal{J}}_X) = (\lambda - \varepsilon_X\lambda_1(X))^{\nu_1} \dots (\lambda - \varepsilon_X\lambda_k(X))^{\nu_k}$$

holds for all nonnull $X \in \mathcal{V}$ and some scalars $\nu_1, \dots, \nu_k \in \mathbb{N}$ and $\lambda_1(X), \dots, \lambda_k(X) \in \mathbb{R}$, which are ordered by $\lambda_1(X) < \dots < \lambda_k(X)$. In other words, $\widetilde{\mathcal{J}}_X$ has eigenvalues $\varepsilon_X\lambda_1(X), \dots, \varepsilon_X\lambda_k(X)$ with multiplicities $\nu_1, \dots, \nu_k$, respectively. Let us define ν_i-dimensional (generalised) eigenspaces

$$\mathcal{V}_i(X) = \mathrm{Ker}(\widetilde{\mathcal{J}}_X - \varepsilon_X\lambda_i(X)\,\mathbb{1})$$

of $\widetilde{\mathcal{J}}_X$ for any nonnull $X \in \mathcal{V}$ and $1 \leq i \leq k$. In this way, for any nonnull $X \in \mathcal{V}$ one has a direct orthogonal decomposition

$$\mathcal{V} = \bigoplus_{i=0}^{k} \mathcal{V}_i(X), \tag{10.32}$$

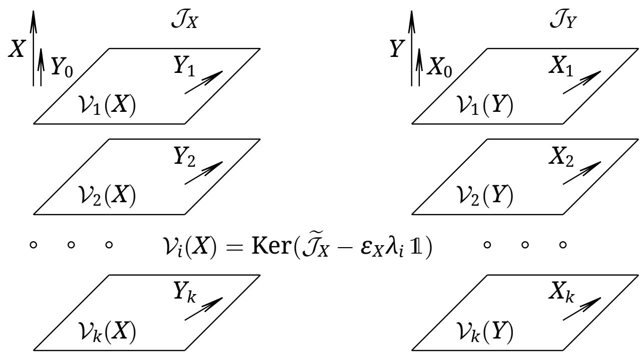
where $\mathcal{V}_0(X) = \mathrm{Span}\{X\}$.

We say that a k-root Jacobi-diagonalisable R is ***Jacobi-proportional*** if for any pair of nonnull vectors $X, Y \in \mathcal{V}$ we have the proportionality

$$\varepsilon_X(\varepsilon_{Y_0}, \varepsilon_{Y_1}, \dots, \varepsilon_{Y_k}) = \varepsilon_Y(\varepsilon_{X_0}, \varepsilon_{X_1}, \dots, \varepsilon_{X_k}),$$

where $X = \sum_{i=0}^{k} X_i$ and $Y = \sum_{j=0}^{k} Y_j$ are decomposed such that $X_i \in \mathcal{V}_i(Y)$ and $Y_j \in \mathcal{V}_j(X)$.

In the Riemannian setting, the important special case, where $Y \in \mathcal{V}_i(X)$ for some $1 \leq i \leq k$ implies $X \in \mathcal{V}_i(Y)$, is related to the Jacobi-dual implication (9.3). Hence, it is clear that a Riemannian Jacobi-proportional R is weak Jacobi-dual, by Theorem 9.2 it is Jacobi-dual, and according to Theorem 9.21, R is Osserman, with $\lambda_i(X) = \lambda_i$ for $1 \leq i \leq k$.



The Jacobi-proportionality for Riemannian Osserman tensors was invented by Andrejić in 2022 [11], while a generalisation to the pseudo-Riemannian case is presented in [13]. The main idea was to consider a Jacobi-proportional Osserman algebraic curvature tensor R, and use $\mathcal{J}_X V = \varepsilon_X\lambda_i V$ to define $\mathcal{K}_X V = \varepsilon_X\mu_i V$ for arbitrary $\mu_1, \dots, \mu_k \in \mathbb{R}$. Of course, we need to add $\mathcal{K}_X X = 0$, which means $\mu_0 = 0$. In this way, we obtain endomorphisms $\mathcal{K}_X$ for any nonnull $X \in \mathcal{V}$ by keeping the existing eigenspaces and replacing the eigenvalues.

Since (10.32) decomposes $Y = \sum_{i=0}^{k} Y_i$ and $Z = \sum_{i=0}^{k} Z_i$ with $Y_i, Z_i \in \mathcal{V}_i(X)$ for $0 \leq i \leq k$, we have

$$g(\mathcal{K}_X Y, Z) = g\left(\sum_{i=0}^{k} \mathcal{K}_X Y_i, \sum_{j=0}^{k} Z_j\right) = \varepsilon_X \sum_{i,j=0}^{k} \mu_i g(Y_i, Z_j) = \varepsilon_X \sum_{j=0}^{k} \mu_j g(Y_j, Z_j) = g(Y, \mathcal{K}_X Z),$$

which means that each $\mathcal{K}_X$ is self-adjoint.

Since R is Jacobi-proportional, if we decompose nonnull $X, Y \in \mathcal{V}$ by $X = \sum_{i=0}^{k} X_i$ and $Y = \sum_{j=0}^{k} Y_j$ such that $X_i \in \mathcal{V}_i(Y)$ and $Y_j \in \mathcal{V}_j(X)$ hold, then we have $\varepsilon_X \varepsilon_{Y_i} = \varepsilon_Y \varepsilon_{X_i}$ for all $0 \leq i \leq k$. Hence, it yields

$$g(\mathcal{K}_X Y, Y) = g\left(\sum_{i=0}^{k} \varepsilon_X \mu_i Y_i, Y\right) = \varepsilon_X \sum_{i=0}^{k} \mu_i g(Y_i, Y) = \varepsilon_X \sum_{i=0}^{k} \mu_i \varepsilon_{Y_i},$$
$$g(\mathcal{K}_Y X, X) = g\left(\sum_{i=0}^{k} \varepsilon_Y \mu_i X_i, X\right) = \varepsilon_Y \sum_{i=0}^{k} \mu_i g(X_i, X) = \varepsilon_Y \sum_{i=0}^{k} \mu_i \varepsilon_{X_i},$$

and therefore the compatibility $g(\mathcal{K}_Y X, X) = g(\mathcal{K}_X Y, Y)$ holds for nonnull $X, Y \in \mathcal{V}$. Thus, we fulfilled the conditions of Theorem 6.7, which gives the following theorem (see [13, Theorem 2] and [11, Theorem 2]).

Theorem 10.16. *If there exists a Jacobi-proportional Jacobi-diagonalisable k-root Osserman algebraic curvature tensor such that* $\det(\lambda \mathbb{1} - \widetilde{\mathcal{J}}_X) = \prod_{i=1}^{k} (\lambda - \varepsilon_X \lambda_i)^{\nu_i}$, *then for any* $\mu_1, \ldots, \mu_k \in \mathbb{R}$ *there is a new Osserman algebraic curvature tensor such that* $\det(\lambda \mathbb{1} - \widetilde{\mathcal{J}}_X) = \prod_{i=1}^{k} (\lambda - \varepsilon_X \mu_i)^{\nu_i}$.

Any Riemannian Jacobi-proportional R is Osserman, while it turns out that any known example of Riemannian Osserman R is Jacobi-proportional. Since $\varepsilon_Y X_0 = g(X, Y) Y$ and $\varepsilon_X Y_0 = g(Y, X) X$ always hold, we obtain

$$\varepsilon_Y \varepsilon_{X_0} = (g(X, Y))^2 = \varepsilon_X \varepsilon_{Y_0}. \tag{10.33}$$

Any two-point homogeneous manifold of nonconstant sectional curvature is two-root. In that case $k = 2$, because of $\varepsilon_X = \varepsilon_{X_0} + \varepsilon_{X_1} + \varepsilon_{X_2}$ and $\varepsilon_Y = \varepsilon_{Y_0} + \varepsilon_{Y_1} + \varepsilon_{Y_2}$, the equality (10.33) yields

$$\varepsilon_X \varepsilon_{Y_1} - \varepsilon_Y \varepsilon_{X_1} = \varepsilon_Y \varepsilon_{X_2} - \varepsilon_X \varepsilon_{Y_2}.$$

On the other hand, $g(\mathcal{J}_X Y, Y) = g(\mathcal{J}_Y X, X)$ gives $\varepsilon_X(\lambda_1 \varepsilon_{Y_1} + \lambda_2 \varepsilon_{Y_2}) = \varepsilon_Y(\lambda_1 \varepsilon_{X_1} + \lambda_2 \varepsilon_{X_2})$, and therefore

$$\lambda_1(\varepsilon_X \varepsilon_{Y_1} - \varepsilon_Y \varepsilon_{X_1}) = \lambda_2(\varepsilon_Y \varepsilon_{X_2} - \varepsilon_X \varepsilon_{Y_2}).$$

However, since $\lambda_1 \neq \lambda_2$, from the previous two equalities we obtain

$$\varepsilon_X \varepsilon_{Y_1} - \varepsilon_Y \varepsilon_{X_1} = 0 = \varepsilon_Y \varepsilon_{X_2} - \varepsilon_X \varepsilon_{Y_2},$$

which implies $\varepsilon_Y(\varepsilon_{X_0}, \varepsilon_{X_1}, \varepsilon_{X_2}) = \varepsilon_X(\varepsilon_{Y_0}, \varepsilon_{Y_1}, \varepsilon_{Y_2})$ and gives the following theorem.

Theorem 10.17. *Any two-root Jacobi-diagonalisable Osserman algebraic curvature tensor is Jacobi-proportional.*

The previous theorem is given by Andrejić in 2022 [11, Theorem 3], but it is essentially the weak duality lemma published by the same author in 2010 [6, Lemma 5.6] (see also [8, Lemma 9]), which can be considered a precursor to the proportionality principle for Osserman manifolds.

However, although every known Riemannian Osserman manifold is either one-root or two-root, this does not necessarily apply to Riemannian Osserman algebraic curvature tensors. Fortunately, every known example of a Riemannian k-root Osserman algebraic curvature for $k > 2$ is Clifford.

Let R be semi-Clifford, and consequently R is Jacobi-diagonalisable Osserman. If we split $Y = \alpha_0 X + \sum_{i=1}^{m} \alpha_i J_i X + P$ and $X = \beta_0 Y + \sum_{i=1}^{m} \beta_i J_i Y + Q$, where $P \in \operatorname{Span}\{X, J_1 X, \ldots, J_m X\}^{\perp}$ and $Q \in \operatorname{Span}\{Y, J_1 Y, \ldots, J_m Y\}^{\perp}$, then we obtain

$$-c_i \varepsilon_X \alpha_t = \varepsilon_{J_t X} \alpha_t = g(Y, J_t X) = -g(X, J_t Y) = -\varepsilon_{J_t Y} \beta_t = c_i \varepsilon_Y \beta_t$$

for $1 \le t \le m$, which gives $\varepsilon_X^2\alpha_t^2 = \varepsilon_Y^2\beta_t^2$, so $\varepsilon_Y\varepsilon_{\beta_tJ_tY} = -c_i\varepsilon_Y^2\beta_t^2 = -c_i\varepsilon_X^2\alpha_t^2 = \varepsilon_X\varepsilon_{\alpha_tJ_tX}$. We already have $\varepsilon_Y^2\beta_0^2 = \varepsilon_X^2\alpha_0^2$ from (10.33), so it remains

$$\varepsilon_Y\varepsilon_Q = \varepsilon_Y\varepsilon_X - \varepsilon_Y^2\beta_0^2 + \varepsilon_Y^2\sum_{i=1}^{m} c_i\beta_i^2 = \varepsilon_X\varepsilon_Y - \varepsilon_X^2\alpha_0^2 + \varepsilon_X^2\sum_{i=1}^{m} c_i\alpha_i^2 = \varepsilon_X\varepsilon_P,$$

which is enough to conclude that R is Jacobi-proportional (see [11, Theorem 4]).

Theorem 10.18. *Any semi-Clifford algebraic curvature tensor is Jacobi-proportional.*

Let R be a Riemannian k-root Osserman algebraic curvature tensor on a scalar product space $(\mathcal{V}, g)$ of dimension n, such that $\widetilde{\omega}_X(\lambda) = \prod_{i=1}^{k}(\lambda - \varepsilon_X\lambda_i)^{\nu_i}$. According to Nikolayevsky [88, 89, 90, 91], any such R is Clifford, except for $n = 16$, with $\nu_i = 7$ or $\nu_i = 8$ for some $1 \le i \le k$.

Let us suppose that R is Riemannian Jacobi-proportional k-root for $k \ge 3$ that is not Clifford, which implies $n = 16$. We can order the multiplicities by $\nu_1 \ge \nu_2 \ge \cdots \ge \nu_k$, to conclude that $\nu_1 = 8$ or $\nu_1 = 7$. According to Theorem 10.16, there is a new Osserman algebraic curvature tensor such that $\widetilde{\omega}_X(\lambda) = \prod_{i=1}^{k}(\lambda - \varepsilon_X\mu_i)^{\nu_i}$ for any scalars $\mu_1, \ldots, \mu_k \in \mathbb{R}$. Let us use $\mu_2 = \lambda_1$ with $\mu_i = \lambda_i$ for $i \ne 2$ to create R_1, and $\mu_2 = \lambda_2 - \lambda_1$ with $\mu_i = 0$ for $i \ne 2$ to create R_2. It is easy to conclude that $R = R_1 + R_2$. However, R_1 is $(k-1)$-root Osserman with $\nu_{\max} = \nu_1 + \nu_2$ and therefore Clifford, unless

$$\nu_1 = 7, \nu_2 = \cdots = \nu_9 = 1, \tag{10.34}$$

while R_2 is 2-root Osserman with $\nu_{\max} = n - 1 - \nu_2$ which is Clifford, except

$$\nu_1 = \nu_2 = 7, \nu_3 = 1. \tag{10.35}$$

Hence, if we exclude the cases (10.34) and (10.35), $R = R_1 + R_2$ is a sum of two Clifford tensors. However, R_1 and R_2 have compatible Jacobi operators, so if J_1 is a complex structure from Clifford family of rank $\nu_3 + \cdots + \nu_k$ in R_1, and J_2 is a complex structure from Clifford family of rank ν_2 in R_2, then $J_1X \perp J_2X$ holds for all $X \in \mathcal{V}$. Hence, we obtain $g(J_1J_2X, X) = 0$, which after the polarization gives $g(J_1J_2X, Y) + g(J_1J_2Y, X) = 0$ for all $X, Y \in \mathcal{V}$. Thus we have $g((J_1J_2 + J_2J_1)X, Y) = 0$, and consequently $J_1J_2 + J_2J_1 = 0$, which means that the union of our Clifford families is also a Clifford family, and therefore R is Clifford.

Moreover, the case (10.34) is also Clifford, because in a similar way as before, it decomposes into two Clifford tensors, R_1 given by $\mu_2 = \mu_3 = \lambda_1$ with $\mu_i = \lambda_i$ for $i \ne 2, 3$ and R_2 given by $\mu_2 = \lambda_2 - \lambda_1, \mu_3 = \lambda_3 - \lambda_1$ with $\mu_i = 0$ for $i \ne 2, 3$. Then, R_1 is 7-root Osserman with $\nu_{\max} = 9$, while R_2 is 3-root Osserman with $\nu_{\max} = 13$ and therefore both of them are Clifford.

On the other hand, the remaining case $\nu_1 = 8, \nu_2 = 7$ was mentioned by Nikolayevsky [91, Theorem 1.2] who announced the statement that such an Osserman manifold is locally isometric to $\mathbb{O}\mathbf{P}^2$ or $\mathbb{O}\mathbf{H}^2$ (but one should be careful since no proof is given). Let us remark that these are the only known Osserman manifolds which are not Clifford. Other potential counterexamples that are Jacobi-proportional should be sought in the case (10.35).

Theorem 10.19. *If R is a Riemannian Jacobi-proportional algebraic curvature tensor which is not Clifford then R is 2-root with multiplicities* 8 *and* 7*, or it is 3-root with multiplicities* 7*,* 7*, and* 1.

The previous theorem, given by Andrejić in 2022 [11, Theorem 5], can be useful in solving the Osserman conjecture. Therefore, the key question is whether we can prove that Osserman algebraic curvature tensors are Jacobi-proportional.

CHAPTER A

APPENDIX

A.1 Set theory

To avoid potential ambiguities, let us clarify the notation for the standard sets we use throughout this book. We denote the set of all natural numbers (without zero) by $\mathbb{N} = \{1, 2, \dots\}$, while $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$. The set of integers is $\mathbb{Z}$, the set of rational numbers is $\mathbb{Q}$, the set of real numbers is $\mathbb{R}$, and the set of complex numbers is $\mathbb{C}$.

A set is ***countable*** if there is an injective map from it into the set of natural numbers $\mathbb{N}$. In other words, each element of the set can be assigned a unique natural number, which means that either the set is finite or there is a bijective map between it and $\mathbb{N}$. If we want to emphasize that a countable set is not finite, we say that is is ***countably infinite***. Let us remark that there are authors who use the term at most countable instead of our countable, while they use the word countable only if the set is not finite. Anyway, we consider a finite set to be countable.

Let us state some basic facts about countable sets. The sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable, while $\mathbb{R}$ is not countable. Every subset of a countable set is countable. The product of two countable sets is countable. A countable union of countable sets is countable.

The following lemma tells us how a map between sets behaves with respect to set operations.

Lemma A.1. *The image of the union of subsets under an arbitrary map is equal to the union of their images. The image of the intersection of subsets under an injective map is equal to the intersection of their images.*

Proof. Let X and Y be arbitrary sets, $A_\alpha \subseteq X$ for each $\alpha \in \Lambda$, and $f\colon X \to Y$ be a map. From $A_\beta \subseteq \bigcup_\alpha A_\alpha$ it follows $f(A_\beta) \subseteq f(\bigcup_\alpha A_\alpha)$ for each $\beta \in \Lambda$, so $\bigcup_\alpha f(A_\alpha) \subseteq f(\bigcup_\alpha A_\alpha)$. On the other hand, for $x \in f(\bigcup_\alpha A_\alpha)$ there exists $y \in \bigcup_\alpha A_\alpha$ such that $f(y) = x$, so there is $\beta \in \Lambda$ such that $y \in A_\beta$, whence $x = f(y) \in f(A_\beta) \subseteq \bigcup_\alpha f(A_\alpha)$, which gives $f(\bigcup_\alpha A_\alpha) \subseteq \bigcup_\alpha f(A_\alpha)$ and proves

$$f\Big(\bigcup_\alpha A_\alpha\Big) = \bigcup_\alpha f(A_\alpha). \tag{A.1}$$

From $\bigcap_\alpha A_\alpha \subseteq A_\beta$ it follows $f(\bigcap_\alpha A_\alpha) \subseteq f(A_\beta)$ for each $\beta \in \Lambda$, so $f(\bigcap_\alpha A_\alpha) \subseteq \bigcap_\alpha f(A_\alpha)$. On the other hand, for $x \in \bigcap_\alpha f(A_\alpha)$ we have $x \in f(A_\beta)$ for each $\beta \in \Lambda$, and using the condition that f is injective we get $f^{-1}(x) \in A_\beta$ for each $\beta \in \Lambda$ and therefore $x \in f(\bigcap_\alpha A_\alpha)$, which gives $\bigcap_\alpha f(A_\alpha) \subseteq f(\bigcap_\alpha A_\alpha)$ and proves

$$f\Big(\bigcap_\alpha A_\alpha\Big) = \bigcap_\alpha f(A_\alpha). \tag{A.2}$$

Let us remark that the formula (A.2) does not have to hold if f is not injective, for example for constant f and disjoint A_α. □

A.2 Topology

This book is written for readers who already have a solid understanding of basic topology, while Lee [77] and Crainic[1] [37] can serve as a reminder. In this section we give the basic ideas from topology we have used throughout this book.

A ***topology*** on a set X is a family $\mathcal{T}$ of subsets of X such that $\emptyset, X \in \mathcal{T}$, while an arbitrary union of elements from $\mathcal{T}$, as well as a finite intersection of elements from $\mathcal{T}$ are also in $\mathcal{T}$. The elements of topology are ***open subsets***, while a subset $A \subseteq X$ is ***closed*** if its complement $X \setminus A$ is open. A ***topological space*** is an ordered pair $(X, \mathcal{T})$ consisting of a set X equipped with a topology $\mathcal{T}$ on X, but when topology is well known, it is common to say that X is a topological space.

For a given $x \in X$, a ***neighbourhood*** of x is any open subset $U \subseteq X$ such that $x \in U$. Many authors use the term neighbourhood more broadly to refer to any subset containing an open subset around x. We use the term ***generalised neighbourhood*** for such subsets that are not necessarily open. In this text, however, a neighbourhood is always considered to be an open set.

Example A.1. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. The ***subspace topology*** is the topology $\mathcal{T}_A$ on a set A given by $\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}$. $\triangle$

Example A.2. Let $(X, \mathcal{T})$ be a topological space, and $q\colon X \to Y$ be a surjective map. The ***quotient topology*** is the topology $\mathcal{T}_q = \{V \subseteq Y : q^{-1}(V) \in \mathcal{T}\}$ given on the set Y. The surjection q is often the natural projection that sends elements from X to their equivalence classes for some equivalence relation on X. $\triangle$

A map $f\colon X \to Y$ between topological spaces is ***continuous*** if the inverse image $f^{-1}(U) \subseteq X$ is open for every open $U \subseteq Y$. A bijective map $f\colon X \to Y$ such that both f and f^{-1} are continuous is called ***homeomorphism***. If there is a homeomorphism from X to Y, then we say that X and Y are ***homeomorphic***.

The union of all open subsets that are subset of the set A is called the ***interior*** of A and is denoted by $\operatorname{Int} A$. The intersection of all closed subsets that are a superset of the set A is called the ***closure*** of A and is denoted by $\overline{A}$. The ***boundary*** of the set A is the set $\partial A = \overline{A} \setminus \operatorname{Int} A$. A point $x \in A$ is said to be an ***isolated point*** of A if x has a neighbourhood U such that $U \cap A = \{x\}$.

For a sequence $(x_n)_{n\in\mathbb{N}}$ of points of a topological sapce X we say that ***converges*** to $x \in X$ if for every neighbourhood U of x there is $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$, then we write $\lim_{n\to\infty} x_n = x$.

A topological space X is said to be ***Hausdorff space*** if for every pair of distinct points $p, q \in X$ there exist open subsets $U, V \subseteq X$ with $p \in U$ and $q \in V$ such that $U \cap V = \emptyset$. In other words, a topological space is Hausdorff if every two distinct points can be separated by its disjoint neighbourhoods. A topological space X is a ***regular space*** if for every $x \in X$ and every closed subset $C \not\ni x$, the point x and the set C can be separated by disjoint neighbourhoods. The space is ***normal*** if any two disjoint closed subsets of X can be separated by disjoint neighbourhoods.

Let $(X, \mathcal{T})$ be a topological space. A ***topology basis*** is a family $\mathcal{B} \subseteq \mathcal{T}$ such that each $U \in \mathcal{T}$ is the union of some elements from $\mathcal{B}$. A ***local basis*** at $x \in X$ is a family $\mathcal{B}_x \subseteq \mathcal{T}$ of neighbourhoods of x such that each neighbourhood of x contains at least one element from $\mathcal{B}_x$. Some authors allow elements of a local basis not to be open sets, so a ***generalised local basis*** at $x \in X$ is a family of generalised neighbourhoods of x such that each neighbourhood of x contains at least one element from that family. We say that a topological space is ***first countable*** if every point in the space has a countable basis, and it is ***second countable*** if it has a countable topology basis.

[1] Marius Crainic (1974), Romanian-Dutch mathematician

Two most frequently studied topological properties throughout the history of topology are certainly connectedness and compactness. We say that a topological space is ***connected*** if it cannot be expressed as the union of two disjoint nonempty open sets. A topological space is ***path-connected*** if there is a path joining any two points (for any $x, y \in X$ there exists a continuous map $f\colon [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$). We say that a subset of a topological space is ***connected*** or ***path-connected*** if it has these properties with respect to the subspace topology. Connectedness is a continuous map invariant, and every path-connected topological space is connected.

Roughly speaking, we say that a topological property is satisfied locally if each small region of the space resembles a region of a space where that property holds. More concretely, if X is a topological space and P is an adjective which can apply to spaces, then X is ***locally*** P often means that for every $x \in X$ and every neighbourhood $U \ni x$ there exists a subset $A \subseteq U$ that satisfies the property P and contains a neighbourhood of x. In other words, a topological space is locally P if and only if every point has a generalised local basis consisting of sets with the property P.

A topological space is ***locally connected*** if every point has a local basis consisting of connected neighbourhoods. Since the union of local bases over all points forms a topology basis, and every topology basis includes a local basis for each point, a topological space is locally connected if and only if it has a basis consisting of connected sets. Similarly, a topological space is ***locally path-connected*** if it has a basis of path-connected sets.

A ***connected component*** of a topological space is its maximal connected subset. Each point of a topological space X belongs to some of its connected components, and the connected components are disjoint and closed in X. If a topological space is locally connected, then all connected components are open, from which it follows that it is the disjoint union space of its connected components, which is not true in the general case.

Example A.3. A connected space that is not locally connected nor path-connected can even be found as a subset of the Euclidean plane, and classic examples are the topologist's sine curve (the graph of $\sin(1/x)$ for $x \in (0,1]$, together with the segment $\{0\} \times [-1,1]$) and the infinite broom (the union of segments joining the origin to the point $(1, 1/n)$ for $n \in \mathbb{N}$, together with the segment $[1/2, 1] \times \{0\}$) $\triangle$

A ***cover*** of a set is a family of sets whose union contains this set as a subset, and if all elements of that family are open, then it is an ***open cover***. A ***subcover*** of some cover is its subset that is still a cover. A topological space or some its subset is called ***compact*** if each of its open covers has a finite subcover. We say that a subset is ***relatively compact*** if its closure is compact.

Every closed subset of a compact space is compact, while every compact subset of a Hausdorff space is closed. Compactness is an invariant of a continuous map, and the following lemma also holds.

Lemma A.2. *If $f\colon M \to N$ is a continuous map between a compact space M and Hausdorff space N, then f is closed.*

Proof. Every closed subset $X \subseteq M$ of a compact set is compact, so $f(X) \subseteq N$ is compact and therefore closed. $\square$

The structure of compact subsets of Euclidean space was well understood through the ***Heine[2]–Borel[3] theorem***.

Theorem A.3. *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

[2]Heinrich Eduard Heine (1821–1881), German mathematicain

[3]Félix Édouard Justin Émile Borel (1871–1956), French mathematician and politician

A topological space is ***locally compact*** if every point has a generalised local basis consisting of compact sets. If the space is Hausdorff (which is almost always the case), it can be shown that it is locally compact if and only if every point has a local basis consisting of relatively compact sets. However, it turns out that a Hausdorff space is locally compact if and only if every point has a compact generalised neighbourhood, which is the most commonly used definition. Of course, any compact Hausdorff space is locally compact.

It is interesting to consider the ***domain invariance theorem***, which is a deep result introduced by Brouwer[4] in 1911 [26], and we give it without proof.

Theorem A.4. *If $U \subseteq \mathbb{R}^n$ is open for some $n \in \mathbb{N}$ and $f\colon U \to \mathbb{R}^n$ is continuous and injective, then $f(U) \subseteq \mathbb{R}^n$ is open.*

In other words, any continuous injective map between Euclidean spaces of the same dimension is an open map. An important consequence of domain invariance is the ***dimension invariance theorem***.

Theorem A.5. *A nonempty subset of $\mathbb{R}^m$ is not homeomorphic to an open subset of $\mathbb{R}^n$ except for $m = n$.*

Proof. Let an open $U \subseteq \mathbb{R}^m$ be homeomorphic to $V \subseteq \mathbb{R}^n$ under a homeomorphism $f\colon U \to V$. Without loss of generality we can assume that $n \leq m$ (otherwise consider f^{-1}), so there exists a linear inclusion $\imath\colon \mathbb{R}^n \hookrightarrow \mathbb{R}^m$. The map $\imath \circ f\colon U \to \mathbb{R}^m$ is continuous and injective, and according to Theorem A.4 $\imath(f(U))$ is open. However, this is impossible for $n < m$ because the inclusion image lies in a hyperplane, while every neighbourhood of a hyperplane point contains points that are not in the hyperplane, which implies $m = n$. □

A.3 Eigen-structure of endomorphisms

In this section, we review some fundamental concepts from linear algebra and discuss both the algebraic and analytic aspects related to the eigenstructure of a vector space. A primary reference for this material is the book by Friedberg[5], Insel[6], and Spence[7] [51].

Let $\mathcal{V}$ be a vector space of dimension $n \in \mathbb{N}$ over a field $\mathbb{F}$, where we primarily consider the cases $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$. An ***endomorphism*** of the vector space $\mathcal{V}$ is a linear map $A\colon \mathcal{V} \to \mathcal{V}$, and the set of all endomorphisms of $\mathcal{V}$ is denoted by $\mathrm{End}(\mathcal{V})$.

An ***eigenvector*** of $A \in \mathrm{End}(\mathcal{V})$ is a nonzero vector that does not change direction after applying A. In this case, the endomorphism A simply scales X by some $\lambda \in \mathbb{F}$ which is called the ***eigenvalue*** of A, which can be written as $AX = \lambda X$ with $X \neq 0$. The set of all eigenvectors of A corresponding to the value λ, together with the zero vector, is called the ***eigenspace*** and is denoted by $\mathcal{E}_\lambda = \mathrm{Ker}(A - \lambda\,\mathbb{1}) \leq \mathcal{V}$.

In the presence of a basis $\boldsymbol{e} = (E_1, \ldots, E_n)$ for $\mathcal{V}$, each $A \in \mathrm{End}(\mathcal{V})$ is identified with the matrix $[A]_{\boldsymbol{e}} \in \mathbb{F}^{n\times n}$ for which the relation $A(E_j) = \sum_{i=1}^n ([A]_{\boldsymbol{e}})_{ij} E_i$ holds for each $1 \leq j \leq n$. If X is an eigenvector of A for the eigenvalue λ, then $(\lambda\,\mathbb{1} - A)X = 0$, and the matrix $[\lambda\,\mathbb{1} - A]_{\boldsymbol{e}} = \lambda\,\mathbb{1} - [A]_{\boldsymbol{e}}$ cannot be invertible, as this would lead to a unique solution $X = 0$, so we must have $\det(\lambda\,\mathbb{1} - [A]_{\boldsymbol{e}}) = 0$. Conversely, if $\det(\lambda\,\mathbb{1} - [A]_{\boldsymbol{e}}) = 0$, the equation $(\lambda\,\mathbb{1} - A)X = 0$ has infinitely many solutions (since $X = 0$ is certainly a solution), so there exists an eigenvector for which λ is an eigenvalue.

The polynomial $\omega_A(x) = \det(x\,\mathbb{1} - A) = \det(x\,\mathbb{1} - [A]_{\boldsymbol{e}})$ does not depend on the choice of basis $\boldsymbol{e}$, and we call it the ***characteristic polynomial*** of A, while the eigenvalues of the endomorphism A are exactly the roots of the polynomial ω_A. The characteristic polynomial

[4] Luitzen Egbertus Jan Brouwer (1881–1966), Dutch mathematician and philosopher
[5] Stephen Howard Friedberg, American mathematician
[6] Arnold Joseph Insel (1940), American mathematician
[7] Lawrence Edward Spence (1946), American mathematician

is of degree n, so the number of distinct eigenvalues cannot exceed the dimension of the space.

Multiplicity (or ***algebraic multiplicity***) of an eigenvalue λ of the endomorphism $A \in \mathrm{End}(\mathcal{V})$ is the largest $r \in \mathbb{N}$ such that the polynomial $(x - \lambda)^r$ divides $\omega_A(x)$, while its ***geometric multiplicity*** is given by $s = \dim \mathcal{E}_\lambda$. If a basis for $\mathcal{E}_\lambda$ is extended to a basis $\boldsymbol{e}$ for $\mathcal{V}$, then $[A]_{\boldsymbol{e}}$ is an upper triangular block matrix. From this, we can see that the polynomial $(\lambda - x)^s$ divides $\omega_A(x)$, which implies that the geometric multiplicity of the eigenvalue is not greater than its algebraic multiplicity, and is certainly at least 1.

For an endomorphism $A \in \mathrm{End}(\mathcal{V})$, we say that A is ***diagonalisable*** if there exists a basis $\boldsymbol{e}$ for $\mathcal{V}$ such that the matrix $[A]_{\boldsymbol{e}}$ is diagonal. If some eigenvectors of A form a basis for $\mathcal{V}$, we call this an ***eigenbasis***, and A is diagonalisable if and only if $\mathcal{V}$ has an eigenbasis with respect to A. A polynomial from $\mathbb{F}[x]$ is ***split*** (over $\mathbb{F}$) if it can be factored into linear factors, that is, if all of its roots lie in $\mathbb{F}$. The characteristic polynomial of a diagonalisable endomorphism is obviously split.

Lemma A.6. *Eigenvectors of an endomorphism of a vector space corresponding to distinct eigenvalues are linearly independent.*

Proof. Let $r \in \mathbb{N}$ be a minimal number of linearly dependent vectors corresponding to distinct eigenvalues of $A \in \mathrm{End}(\mathcal{V})$, and let $\alpha_1 X_1 + \cdots + \alpha_r X_r = 0$ hold for some $X_i \in \mathcal{V} \setminus \{0\}$, $\alpha_i \in \mathbb{F}$ where $AX_i = \lambda_i X_i$ for $1 \leq i \leq r$, with distinct eigenvalues $\lambda_i \in \mathbb{F}$. Then, we have $\alpha_1 AX_1 + \cdots + \alpha_r AX_r = 0$, which leads to $\alpha_2(\lambda_2 - \lambda_1)X_2 + \cdots + \alpha_r(\lambda_r - \lambda_1)X_r = 0$. Since r is minimal, the vectors $X_2, \ldots, X_r$ are linearly independent, so $\alpha_2(\lambda_2 - \lambda_1) = \cdots = \alpha_r(\lambda_r - \lambda_1) = 0$. From this, we obtain $\alpha_2 = \cdots = \alpha_r = 0$, and then $\alpha_1 X_1 = 0$ implies $\alpha_1 = 0$, so $X_1, \ldots, X_r$ are linearly independent, which contradicts the assumption, thus proving the statement. $\square$

As a consequence of the previous lemma, it can be shown that $\mathcal{E}_\mu \cap \sum_{\lambda \neq \mu} \mathcal{E}_\lambda = \{0\}$, which means that the eigenspaces form a direct sum $\bigoplus_\lambda \mathcal{E}_\lambda \leq \mathcal{V}$. Note that the notation $\bigoplus_\lambda$ formally represents a direct sum over all scalars $\lambda \in \mathbb{F}$, but if λ is not an eigenvalue, then by definition $\mathcal{E}_\lambda = \{0\}$, so essentially we have a direct sum over the eigenvalues λ. The equality $\bigoplus_\lambda \mathcal{E}_\lambda = \mathcal{V}$ holds if and only if A is diagonalisable, and in the case of a diagonalisable A, the geometric multiplicity is equal to the algebraic multiplicity for each eigenvalue of A.

For $\mathcal{W} \leq \mathcal{V}$, we say that it is A-***invariant*** subspace of $\mathcal{V}$ for $A \in \mathrm{End}(\mathcal{V})$ if $A(\mathcal{W}) \subseteq \mathcal{W}$ holds, that is, if $A(X) \in \mathcal{W}$ for every $X \in \mathcal{W}$. In this case $A\restriction_{\mathcal{W}} \in \mathrm{End}(\mathcal{W})$, and if we extend a basis for $\mathcal{W}$ to a basis $\boldsymbol{e}$ for $\mathcal{V}$ then the matrix $[A]_{\boldsymbol{e}}$ is an upper triangular block matrix, which shows that the polynomial $\omega_{A\restriction_{\mathcal{W}}}(x)$ divides $\omega_A(x)$.

Lemma A.7. *The characteristic polynomial of the restriction of an endomorphism A on a finite-dimensional vector space to an A-invariant subspace divides the characteristic polynomial of A.*

A basic example of an A-invariant subspace of $\mathcal{V}$ for $A \in \mathrm{End}(\mathcal{V})$ is the A-***cyclic*** subspace $\mathcal{C}_X = \mathrm{Span} \bigcup_{r \in \mathbb{N}_0} \{A^r(X)\}$ generated by a nonzero vector $X \in \mathcal{V}$, which is the smallest A-invariant subspace of $\mathcal{V}$ that contains X. If $m \in \mathbb{N}$ is the largest integer such that the set $\bigcup_{r=0}^{m-1} \{A^r(X)\}$ is linearly independent ($m \leq n$), then $\mathrm{Span} \bigcup_{r=0}^{m-1} \{A^r(X)\}$ is A-invariant and equal to $\mathcal{C}_X$. In this case $\boldsymbol{e} = (X, A(X), \ldots, A^{m-1}(X))$ is a basis for $\mathcal{W} = \mathcal{C}_X$, and we have $\dim \mathcal{C}_X = m$. If for $\alpha_0, \ldots, \alpha_{m-1} \in \mathbb{F}$ it holds that

$$\alpha_0 X + \alpha_1 A(X) + \cdots + \alpha_{m-1} A^{m-1}(X) + A^m(X) = 0,$$

then the matrix of the restricted endomorphism A in our basis is

$$[A\restriction_{\mathcal{W}}]_{\boldsymbol{e}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{m-1} \end{pmatrix},$$

and we obtain the characteristic polynomial $\omega_{A\restriction_{\mathcal{W}}}(x) = a_0 + a_1 x + \cdots + a_{m-1}x^{m-1} + x^m$.

Hence $(\omega_{A\restriction_{\mathcal{W}}}(A))(X) = (a_0\,\mathbb{1} + a_1 A + \cdots + a_{m-1}A^{m-1} + A^m)(X) = 0$, and since by Lemma A.7 the polynomial $\omega_{A\restriction_{\mathcal{W}}}(x)$ divides $\omega_A(x)$, it follows that $(\omega_A(A))(X) = 0$ for every $X \neq 0$. Of course, $\omega_A(A)$ is linear, so $(\omega_A(A))(0) = 0$, which completes the equality $\omega_A(A) = 0$, known as the ***Cayley–Hamilton theorem***.

Theorem A.8. *An endomorphism of a finite-dimensional vector space annihilates its characteristic polynomial.*

Every endomorphism $A \in \operatorname{End}(\mathcal{V})$ annihilates its characteristic polynomial, so there exists a monic polynomial $\mu_A(x) \in \mathbb{F}[x]$ of the smallest possible degree that satisfies $\mu_A(A) = 0$. Such a polynomial μ_A is unique and is called the ***minimal polynomial***, and it is clear that it must divide the polynomial ω_A. If $p(x) = \sum_{j=0}^m a_j x^j \in \mathbb{F}[x]$ is such that $p(A) = 0$, then we have

$$p(x)\,\mathbb{1} = p(x)\,\mathbb{1} - p(A) = \sum_{j=0}^{m} a_j((x\,\mathbb{1})^j - A^j) = (x\,\mathbb{1} - A)\sum_{j=1}^{m} a_j \sum_{k=1}^{j} x^{j-k}A^{k-1},$$

from which it follows that $\omega_A(x) = \det(x\,\mathbb{1} - A)$ divides the polynomial $\det(p(x)\,\mathbb{1}) = (p(x))^n$, where $n = \dim\mathcal{V}$, and proves the following lemma, which especially holds for the minimal polynomial.

Lemma A.9. *If an endomorphism annihilates a polynomial, then this polynomial has as factors all the irreducible factors of the characteristic polynomial.*

The basic idea is to decompose $\mathcal{V}$ into a direct sum of A-invariant subspaces because the behaviour of the endomorphism A on $\mathcal{V}$ can be understood from its behaviour on the individual subspaces. If $\mathcal{V} = \bigoplus_{k=1}^m \mathcal{W}_k$, where $\mathcal{W}_k \leq \mathcal{V}$ are A-invariant for $1 \leq k \leq m$, then from the individual bases, we can construct a basis for $\mathcal{V}$ in which A had a block diagonal matrix, so the characteristic polynomial of A is equal to the product of the characteristic polynomials of the individual restrictions,

$$\omega_A(x) = \prod_{k=1}^{m} \omega_{A\restriction_{\mathcal{W}_k}}(x).$$

Let A be an endomorphism of an n-dimensional vector space $\mathcal{V}$ over a field $\mathbb{F}$. A nonzero vector $X \in \mathcal{V}$ for which there exists $r \in \mathbb{N}$ such that $(A - \lambda\,\mathbb{1})^r(X) = 0$ is called a ***generalised eigenvector*** of A corresponding to $\lambda \in \mathbb{F}$. The set of all eigenvectors of A for the value λ, together with the zero vector, is denoted by $\mathcal{V}_\lambda$ and is called the ***generalised eigenspace***. It is easy to verify that $\mathcal{V}_\lambda$ is a vector subspace of $\mathcal{V}$. For a more concise notation, we define $A_\lambda = A - \lambda\,\mathbb{1} \in \operatorname{End}(\mathcal{V})$ for $\lambda \in \mathbb{F}$.

If $d < n$ is the multiplicity of the eigenvalue λ, then there exist a polynomial $p(x) \in \mathbb{F}[x]$ and $0 \neq \mu \in \mathbb{F}$ such that $\omega_A(x) = (x - \lambda)^d(p(x)(x - \lambda) + \mu)$. By the Cayley–Hamilton theorem (Theorem A.8) we have $0 = \omega_A(A) = p(A)(A_\lambda)^{d+1} + \mu(A_\lambda)^d$. From this, it follows that $(A_\lambda)^d = -(1/\mu)p(A)(A_\lambda)^{d+1}$, so if $(A_\lambda)^r X = 0$ for $r > d$, then $(A_\lambda)^d X = 0$ as well, and this holds for $d = n$ by Theorem A.8. Therefore, we conclude that $\mathcal{V}_\lambda = \operatorname{Ker}(A_\lambda)^d$, and since $d \leq n$, we can always express this as $\mathcal{V}_\lambda = \operatorname{Ker}(A_\lambda)^n$.

For arbitrary polynomials $p(x), q(x) \in \mathbb{F}[x]$ it holds $p(A)q(A) = q(A)p(A)$. Applying this to $q(A) = (A_\lambda)^d$ yields $p(A)(\mathcal{V}_\lambda) \subseteq \mathcal{V}_\lambda$, which means that every generalised eigenspace $\mathcal{V}_\lambda$ is $p(A)$-invariant, and hence also A-invariant. Moreover, if a polynomial $p(x)$ is not divisible by $x - \lambda$, then the restriction of $p(A)$ to $\mathcal{V}_\lambda$ is injective. Assuming $X \in \mathcal{V}_\lambda \cap \operatorname{Ker} p(A)$, we obtain the minimal $0 \leq r \leq d$ such that $(A_\lambda)^r X = 0$, as well as $0 = p(A)(X) = q(A)A_\lambda X + \mu X$ for some $q(x) \in \mathbb{F}[x]$ and $0 \neq \mu \in \mathbb{F}$. For $r \geq 1$, this leads to $q(A)(A_\lambda)^r X + \mu(A_\lambda)^{r-1}X = 0$. Since r is minimal, this implies $(A_\lambda)^{r-1}X = 0$ and therefore $r = 0$, that is, $X = 0$. Thus $p(A)\restriction_{\mathcal{V}_\lambda}$ is an automorphism of the subspace $\mathcal{V}_\lambda$ whenever $p(x)$ is not divisible by $x - \lambda$. In particular, this holds for A_μ when $\mu \neq \lambda$.

Lemma A.10. *Let A be an endomorphism of a finite-dimensional vector space $\mathcal{V}$ over a field $\mathbb{F}$, let λ be its eigenvalue, and let $p(x) \in \mathbb{F}[x]$ be an arbitrary polynomial. The generalised eigenspace $\mathcal{V}_\lambda$ is $p(A)$-invariant, and if the polynomial $x - \lambda$ does not divide $p(x)$ then the restriction $p(A)\restriction_{\mathcal{V}_\lambda}$ is an automorphism of the subspace $\mathcal{V}_\lambda$.*

By the rank–nullity theorem applying to $(A_\lambda)^d \in \operatorname{End}(\mathcal{V})$, the A-invariant subspaces $\operatorname{Ker}(A_\lambda)^d = \mathcal{V}_\lambda$ and $\operatorname{Im}(A_\lambda)^d = (A_\lambda)^d(\mathcal{V})$ are complementary, meaning that $\mathcal{V} = \mathcal{V}_\lambda \oplus (A_\lambda)^d(\mathcal{V})$. For $X = (A_\lambda)^d(Y)$, the condtion $A_\lambda(X) = 0$ implies that $Y \in \mathcal{V}_\lambda$ which leads to $X = 0$. Thus, there is no $X \in (A_\lambda)^d(\mathcal{V})$ that is an eigenvector of A for λ, meaning that λ is not an eigenvalue of the restriction of A to $(A_\lambda)^d(\mathcal{V})$. Since $A\restriction_{\mathcal{V}_\lambda}$ annihilates the polynomial $(x - \lambda)^d$, its characteristic polynomial, by Lemma A.9, has no irreducible factors other than $x - \lambda$, which means it must be exactly $(x - \lambda)^d$.

Lemma A.11. *If d is the multiplicity of an eigenvalue λ for an endomorphism A of a finite-dimensional vector space $\mathcal{V}$, then* $\dim \mathcal{V}_\lambda = d$ *and the characteristic polynomial of the restriction $A\restriction_{\mathcal{V}_\lambda}$ is $(x - \lambda)^d$.*

If $X \in \mathcal{V}_\lambda \cap \sum_{\mu\neq\lambda} \mathcal{V}_\mu$, then $(A_\lambda)^n(X) = 0$, and since, by Lemma A.10, A_λ is an automorphism on $\mathcal{V}_\mu$ for $\mu \neq \lambda$, it follows that $(A_\lambda)^n$ is an automorphism on $\sum_{\mu\neq\lambda} \mathcal{V}_\mu \ni X$, from which we conclude that $X = 0$. Thus, the generalised eigenspaces form a direct sum $\bigoplus_\lambda \mathcal{V}_\lambda \leq \mathcal{V}$, generalising Lemma A.6. If the characteristic polynomial ω_A is split, then the sum of the multiplicities of the eigenvalues is equal to n, which gives $\mathcal{V} = \bigoplus_\lambda \mathcal{V}_\lambda$.

Theorem A.12. *If an endomorphism A of a finite-dimensional vector space $\mathcal{V}$ has a split characteristic polynomial, then $\mathcal{V}$ decomposes as a direct sum of the generalised eigenspaces with respect to A.*

Let us consider how an endomorphism A acts on the generalised eigenspace $\mathcal{V}_\lambda$. For a fixed eigenvalue λ, let $\mathcal{W} = \mathcal{V}_\lambda$ and $B = A_\lambda\restriction_{\mathcal{W}} \in \operatorname{End}(\mathcal{W})$. By Lemma A.11, we have $B^d = 0$ for $d = \dim \mathcal{W}$, so it follows that for every $0 \neq Y \in \mathcal{W}$ there exists $r(Y) \in \mathbb{N}$ such that $B^{r(Y)}Y = 0$. Thus, for $f(Y) = B^{r(Y)-1}Y \neq 0$ and $f(0) = 0$, we define the map $f\colon \mathcal{W} \to \mathcal{E}_\lambda = \operatorname{Ker} B$, which assigns to each generalised eigenvector for the eigenvalue λ an eigenvector for λ.

For every nonzero $X \in \mathcal{E}_\lambda$ we introduce the nonempty set $\mathcal{U}_X = \{Y \in \mathcal{W} : B^{r(Y)-1}Y = X\}$, where an arbitrary nonzero $Y \in \mathcal{U}_X$ for which $s = r(Y)$ holds, gives the B-cyclic subspace $\mathcal{C}_Y = \operatorname{Span} \bigcup_{k=0}^{s-1}\{B^k(Y)\} \leq \mathcal{W}$ with the property $f(\mathcal{C}_Y) = \operatorname{Span}\{X\}$. The cycle $(B^{s-1}Y, \ldots, B^2Y, BY, Y)$ is a canonical basis for $\mathcal{C}_Y$, and with respect to this basis, the matrix of the endomorphism $A\restriction_{\mathcal{C}_Y} = A_\lambda\restriction_{\mathcal{C}_Y} + \lambda\,\mathbb{1}_{\mathcal{C}_Y} = B\restriction_{\mathcal{C}_Y} + \lambda\,\mathbb{1}_{\mathcal{C}_Y}$ takes the form

$$\mathcal{J}_s(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \in \mathbb{F}^{s\times s}, \tag{A.3}$$

which is referred to as the (standard) ***Jordan block*** or ***Jordan matrix*** of order s for the value $\lambda \in \mathbb{F}$.

Let $(E_1, \ldots, E_m)$ be an arbitrary basis of the eigenspace $\mathcal{E}_\lambda$, and $Y_i \in \mathcal{U}_{E_i}$ for $1 \leq i \leq m$ an arbitrary choice of vectors. By mathematical induction on m we prove that $\sum_{k=1}^m \mathcal{C}_{Y_k} = \bigoplus_{k=1}^m \mathcal{C}_{Y_k}$, which is obvious for $m = 1$. For the induction step, we have $\sum_k \mathcal{C}_{Y_k} = \mathcal{C}_{Y_j} + \sum_{k\neq j} \mathcal{C}_{Y_k} = \mathcal{C}_{Y_j} + \bigoplus_{k\neq j} \mathcal{C}_{Y_k}$, and since $\mathcal{C}_{Y_j} \cap \bigoplus_{k\neq j} \mathcal{C}_{Y_k} = \{0\}$ holds because of $f(\mathcal{C}_{Y_j}) = \operatorname{Span}\{E_j\}$ and $f(\bigoplus_{k\neq j} \mathcal{C}_{Y_k}) \subseteq \operatorname{Span} \bigcup_{k\neq j}\{E_k\}$, we obtain the direct sum $\bigoplus_{k=1}^m \mathcal{C}_{Y_k} \leq \mathcal{W}$.

If we choose $Y_i \in \mathcal{U}_{E_i}$ for $1 \leq i \leq m$ so that $r(Y_i)$ is maximal for $Y_i \in \mathcal{U}_{E_i}$, we can prove that $\mathcal{W} = \bigoplus_{k=1}^m \mathcal{C}_{Y_k}$. We perform the proof by mathematical induction on $d = \dim \mathcal{W}$, which is obvious for $d = 1$. For the induction step, we consider the subspace $B(\mathcal{W}) \leq \mathcal{W}$ and the restriction $B\restriction_{B(\mathcal{W})}$, where we have $\operatorname{Ker} B = \mathcal{E}_\lambda$, and therefore $\dim B(\mathcal{W}) = d - m < d$. By the induction hypothesis, we have $B(\mathcal{W}) = \bigoplus_{k=1}^{m} \mathcal{C}_{BY_k}$, where by convention $C_0 = \{0\}$, which we have in the case $Y_k = E_k$. Since $\dim \bigoplus_{k=1}^{m} \mathcal{C}_{Y_k} = m + \dim \bigoplus_{k=1}^{m} \mathcal{C}_{BY_k} = d$, we obtain $\mathcal{W} = \bigoplus_{k=1}^{m} \mathcal{C}_{Y_k}$.

Lemma A.13. *A generalised eigenspace of an endomorphism A decomposes into a direct sum of A-cyclic subspaces.*

As a consequence of Theorem A.12, we obtain the following theorem.

Theorem A.14. *If an endomorphism A of a finite-dimensional vector space $\mathcal{V}$ has a split characteristic polynomial, then $\mathcal{V}$ decomposes into a direct sum of A-cyclic subspaces, and there exists a basis consisting of disjoint cycles of generalised eigenvectors in which the matrix of A is block-diagonal with Jordan matrices on the diagonal.*

Let us consider the most important case for us, where $\mathbb{F} = \mathbb{R}$, meaning $\mathcal{V}$ is a finite-dimensional real vector space. The characteristic polynomial ω_A of $A \in \operatorname{End}(\mathcal{V})$ has real coefficients, but its roots are not necessarily real. Thus, it is possible for an endomorphism A to have a non-split characteristic polynomial and, consequently, not be diagonalisable. The field of complex numbers is algebraically closed, and the ***fundamental theorem of algebra*** states that every non-constant polynomial in $\mathbb{C}[x]$ has at least one complex root, so in the case $\mathbb{F} = \mathbb{C}$, every endomorphism has a split characteristic polynomial.

That motivates us to naturally extend the real vector space $\mathcal{V}$ to a complex vector space $\mathcal{V}_\mathbb{C} = \{(X, Y) : X, Y \in \mathcal{V}\} = \mathcal{V} \times \mathcal{V} = \mathcal{V} \oplus i\mathcal{V} = \{X + iY : X, Y \in \mathcal{V}\}$ which we call the ***complexification*** of $\mathcal{V}$. Any basis $\mathcal{V}$ is also a basis of its complexification $\mathcal{V}_\mathbb{C}$. Of course, if $(E_1, \ldots, E_n)$ is a basis for $\mathcal{V}$, and we view $\mathcal{V}_\mathbb{C}$ as a real vector space, then its natural basis is given by $(E_1, iE_1, \ldots, E_n, iE_n)$.

For $A \in \operatorname{End}(\mathcal{V})$, the ***complexification*** of A is the endomorphism $A_\mathbb{C} \in \operatorname{End}(\mathcal{V}_\mathbb{C})$ defined by $A_\mathbb{C}(X + iY) = A(X) + iA(Y)$ for all $X, Y \in \mathcal{V}$. It is common to identify the eigenvalues and eigenvectors of A with those of $A_\mathbb{C}$, allowing us to say that the eigenvalues of A are precisely the roots of its characteristic polynomial ω_A. If $\lambda \notin \mathbb{R}$ is an eigenvalue of $A_\mathbb{C}$, we refer to $\lambda \in \mathbb{C}$ as a ***complex eigenvalue*** of A. Complex eigenvalues must appear in conjugate pairs, as demonstrated in the following lemma.

Lemma A.15. *Let A be an endomorphism of a real vector space $\mathcal{V}$. If $\lambda \in \mathbb{C}$ is an eigenvalue of $A_\mathbb{C}$ with eigenvector $Z \in \mathcal{V}_\mathbb{C}$, then $\overline{\lambda}$ is also an eigenvalue of $A_\mathbb{C}$ with eigenvector $\overline{Z} \in \mathcal{V}_\mathbb{C}$.*

Proof. If $\lambda = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$ and $Z = X + iY$ for $X, Y \in \mathcal{V}$, then by assumption,

$$A(X) + iA(Y) = A_\mathbb{C}(X + iY) = (\alpha + i\beta)(X + iY) = (\alpha X - \beta Y) + i(\alpha Y + \beta X).$$

From this, we obtain the real and imaginary parts, $A(X) = \alpha X - \beta Y$ and $A(Y) = \beta X + \alpha Y$, and therefore

$$A_\mathbb{C}(X - iY) = A(X) - iA(Y) = (\alpha X - \beta Y) - i(\alpha Y + \beta X) = (\alpha - i\beta)(X - iY),$$

which proves $A_\mathbb{C}(\overline{Z}) = \overline{\lambda}\overline{Z}$, while $\overline{Z} \neq 0$ because of $Z \neq 0$. □

If the characteristic polynomial of A is not split, we can use complexification to obtain $A_\mathbb{C} \in \operatorname{End}(\mathcal{V}_\mathbb{C})$ from $A \in \operatorname{End}(\mathcal{V})$. The characteristic polynomial of $A_\mathbb{C}$ is split, so by Theorem A.12, we get the decomposition $\mathcal{V}_\mathbb{C} = \bigoplus_\lambda (\mathcal{V}_\mathbb{C})_\lambda$. Since by Lemma A.15 complex eigenvalues appear in conjugate pairs, we define, in addition to usual $A_\lambda = A - \lambda \mathbb{1}$ for $\lambda \in \mathbb{R}$, the endomorphism

$$A_\lambda = (A - \lambda \mathbb{1})(A - \overline{\lambda} \mathbb{1}) = A^2 - 2\Re(\lambda)A + |\lambda|^2 \mathbb{1}$$

for complex eigenvalues $\lambda \in \mathbb{C} \setminus \mathbb{R}$, which gives additional ***generalised eigenspaces*** of the form

$$\mathcal{V}_\lambda = \mathrm{Ker}(A_\lambda)^n = \Re\left(\mathrm{Ker}\left((A_\mathbb{C} - \lambda\,\mathbb{1})(A_\mathbb{C} - \overline{\lambda}\,\mathbb{1})\right)^n\right) = \Re\left((\mathcal{V}_\mathbb{C})_\lambda \oplus (\mathcal{V}_\mathbb{C})_{\overline{\lambda}}\right)$$

and allows the decomposition

$$\mathcal{V} = \bigoplus_{\lambda, \Im(\lambda)\geq 0} \mathcal{V}_\lambda.$$

For $Z = X + iY \in \mathcal{V}_\mathbb{C}$ and $\lambda = \alpha + i\beta \in \mathbb{C}$, we can compute for any $r \in \mathbb{N}$:

$$\begin{aligned}(A_\mathbb{C} - (\alpha \pm i\beta)\,\mathbb{1})^r(X \pm iY) &= \sum_{k=0}^{r} \binom{r}{k}(-1)^k(\alpha \pm i\beta)^k (A_\mathbb{C})^{r-k}(X \pm iY)\\ &= \sum_{k=0}^{r} \binom{r}{k}(-1)^k \sum_{j=0}^{k} \binom{k}{j} \alpha^{k-j}\beta^j(\pm i)^j(A^{r-k}X \pm iA^{r-k}Y),\end{aligned}$$

where the vectors $(i)^j(A^{r-k}X + iA^{r-k}Y)$ and $(-i)^j(A^{r-k}X - iA^{r-k}Y)$ are complex conjugate of each other (which can be easily verified for both even and odd j), from which we obtain:

$$\overline{(A_\mathbb{C} - \lambda\,\mathbb{1})^r Z} = (A_\mathbb{C} - \overline{\lambda}\,\mathbb{1})^r \overline{Z}. \tag{A.4}$$

Therefore $(A_\mathbb{C} - \lambda\,\mathbb{1})^s Z = 0$ if and only if $(A_\mathbb{C} - \overline{\lambda}\,\mathbb{1})^s \overline{Z} = 0$, so every $(A_\mathbb{C} - \lambda\,\mathbb{1})$-cyclic subspace $\mathcal{C}_Z$ of $(\mathcal{V}_\mathbb{C})_\lambda$ generated by the vector Z and of dimension s has a corresponding conjugate $(A_\mathbb{C} - \overline{\lambda}\,\mathbb{1})$-cyclic subspace $\mathcal{C}_{\overline{Z}}$ of $(\mathcal{V}_\mathbb{C})_{\overline{\lambda}}$ generated by the vector $\overline{Z}$ and of dimension s. This allows us to combine the two subspaces and obtain the space $\mathcal{C}_Z \oplus \mathcal{C}_{\overline{Z}} \leq (\mathcal{V}_\mathbb{C})_\lambda \oplus (\mathcal{V}_\mathbb{C})_{\overline{\lambda}}$. If we introduce $E_j = (A_\mathbb{C} - \lambda\,\mathbb{1})^{s-j}Z$ and $F_j = (A_\mathbb{C} - \overline{\lambda}\,\mathbb{1})^{s-j}\overline{Z}$ for $1 \leq j \leq s$, then $e = (E_1, \ldots, E_s, F_1, \ldots, F_s)$ is a natural basis for $\mathcal{C}_Z \oplus \mathcal{C}_{\overline{Z}}$, while the endomorphism $A_\mathbb{C}$ in this basis has the form,

$$\left[A_\mathbb{C}\restriction_{\mathcal{C}_Z\oplus\mathcal{C}_{\overline{Z}}}\right]_e = \begin{pmatrix} \mathcal{J}_s(\lambda) & 0 \\ 0 & \mathcal{J}_s(\overline{\lambda})\end{pmatrix} = \begin{pmatrix} \lambda & 1 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 0 & \lambda & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda & 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & \overline{\lambda} & 1 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & \overline{\lambda} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \overline{\lambda}\end{pmatrix} \in \mathbb{C}^{2s\times 2s}.$$

Since E_j and F_j are conjugate to each other, by formula (A.4), we can rearrange the basis vectors as

$$G_j = \Re(E_j) = \frac{E_j + F_j}{2} \in \mathcal{V}, \quad H_j = \Im(E_j) = \frac{E_j - F_j}{2i} \in \mathcal{V},$$

and define a new basis $e' = (G_1, H_1, \ldots, G_s, H_s)$. Since for $2 \leq j \leq s$ we have $A_\mathbb{C}E_j = \lambda E_j + E_{j-1}$ and $A_\mathbb{C}F_j = \overline{\lambda}F_j + F_{j-1}$, for $\lambda = \alpha + i\beta$ we obtain

$$\begin{aligned} AG_j &= \Re(\lambda E_j) + \Re(E_{j-1}) = \alpha G_j - \beta H_j + G_{j-1},\\ AH_j &= \Im(\lambda E_j) + \Im(E_{j-1}) = \alpha H_j + \beta G_j + H_{j-1},\end{aligned}$$

with the additional relations $AG_1 = \alpha G_1 - \beta H_1$ and $AH_1 = \alpha H_1 + \beta G_1$. In the new basis, the restriction of $A_\mathbb{C}$ to $\mathcal{C}_Z \oplus \mathcal{C}_{\overline{Z}}$, and consequently the restriction of A to $\Re(\mathcal{C}_Z \oplus \mathcal{C}_{\overline{Z}}) \leq \mathcal{V}_\lambda$, takes the form

$$\mathcal{J}_s(\alpha,\beta) = \begin{pmatrix} \alpha & \beta & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & \alpha & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha & \beta & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\beta & \alpha & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\beta & \alpha \end{pmatrix} \in \mathbb{R}^{2s\times 2s},$$

which is a ***real Jordan block*** or a ***real Jordan matrix*** of order $2s$ corresponding to the pair of conjugate eigevalues $\alpha \pm i\beta \in \mathbb{C}\setminus\mathbb{R}$. A consequence of Theorem A.14 is the following theorem, which allows us, in the case $\mathbb{F}=\mathbb{R}$, to decompose $\mathcal{V}$ canonically as a direct sum where each summand corresponds to a Jordan block.

Theorem A.16. *If A is an endomorphism of a finite-dimensional real vector space $\mathcal{V}$, then there exists a basis for $\mathcal{V}$ in which the matrix of A is block-diagonal, with (standard or real) Jordan matrices on the diagonal.*

The unordered family of Jordan blocks from the preceding theorem is called the ***Jordan normal form*** of the endomorphism.

The coefficients of the characteristic polynomial ω_A of an endomorphism A are expressed polynomially in terms of the matrix of A. In particular, we recognise the trace and determinant as coefficients in the equation

$$\omega_A(x) = x^n - \mathrm{Tr}(A)x^{n-1} + \cdots + (-1)^n \det(A),$$

while additional invariants are given by the traces of powers, as shown in the following lemma.

Lemma A.17. *If A is an endomorphism of a vector space with characteristic polynomial $\omega_A(x) = x^n + \sigma_1 x^{n-1} + \cdots + \sigma_{n-1}x + \sigma_n$, then*

$$m\sigma_m + \sigma_{m-1}\mathrm{Tr}(A) + \sigma_{m-2}\mathrm{Tr}(A^2) + \cdots + \sigma_1 \mathrm{Tr}(A^{m-1}) + \mathrm{Tr}(A^m) = 0 \qquad \text{(A.5)}$$

holds for every $1 \le m \le n$.

Proof. Let $\omega(x) = x^n + \sigma_1 x^{n-1} + \cdots + \sigma_n = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$ (if complex eigenvalues exist, we may consider $A_{\mathbb{C}}$ instead of A). The coefficients σ_m for $1 \le m \le n$ are expressed using Viète's[8] formulas,

$$\sigma_m = (-1)^m \sum_{i_1<\cdots<i_m} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_m}.$$

For $1 \le j \le m-1$, we obtain

$$\sigma_{m-j}\mathrm{Tr}(A^j) = (-1)^{m-j}(\lambda_1^j + \lambda_2^j + \cdots + \lambda_n^j) \sum_{i_1<\cdots<i_{m-j}} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{m-j}},$$

which, after substituting

$$S_q^p = \sum_{i=1}^{n} \left(\lambda_i^p \cdot \sum_{j_1<\cdots<j_q,\, j_l\neq i} \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_q} \right),$$

[8] François Viète (1540-1603), French mathematician

becomes $\sigma_{m-j}\operatorname{Tr}(A^j) = (-1)^{m-j}(S^{j+1}_{m-j-1} + S^j_{m-j})$, from which it follows that

$$\sum_{j=1}^{m-1} \sigma_{m-j}\operatorname{Tr}(A^j) = \sum_{j=1}^{m-1}(-1)^{m-j}(S^{j+1}_{m-j-1} + S^j_{m-j}) = -S^m_0 + (-1)^{m-1}S^1_{m-1}.$$

Clearly, $S^m_0 = \sum_{i=1}^n \lambda_i^m = \operatorname{Tr}(A^m)$, while we also have

$$S^1_{m-1} = \sum_{i=1}^{n}\left(\lambda_i \cdot \sum_{j_1<\cdots<j_{m-1},\, j_l\neq i} \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_{m-1}}\right).$$

Each term in

$$(-1)^m\sigma_m = \sum_{i_1<\cdots<i_m} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_m},$$

for instance $\lambda_1\lambda_2\cdots\lambda_m$, appears exaclty m times in S^1_{m-1}, which gives $S^1_{m-1} = m(-1)^m\sigma_m$. Finally, we obtain

$$\sum_{j=1}^{m-1} \sigma_{m-j}\operatorname{Tr}(A^j) = -\operatorname{Tr}(A^m) - m\sigma_m,$$

which proves the equation (A.5). □

The characteristic and minimal polynomial share the same roots, so if $\lambda_1,\ldots,\lambda_k$ are distinct eigenavlues, then $\omega_A(x) = (x-\lambda_1)^{\alpha_1}(x-\lambda_2)^{\alpha_2}\cdots(x-\lambda_k)^{\alpha_k}$, where α_j for $1 \le j \le k$ are the (algebraic) multiplicities of the eigenvalues. The minimal polynomial must be of the form $\mu_A(x) = (x-\lambda_1)^{\beta_1}(x-\lambda_2)^{\beta_2}\cdots(x-\lambda_k)^{\beta_k}$, where its multiplicities satisfy $1 \le \beta_j \le \alpha_j$. Considering Theorem A.14 and Theorem A.16, we observe that β_j is the largest order of a (standard) Jordan block withing the subspace $\mathcal{V}_{\lambda_j}$.

A.4 Self-adjoint endomorphisms

This section is a natural continuation of the previous one, where we studied the eigenstructure of endomorphisms. If we introduce a scalar product that interacts well with endomorphism, we can fine-tune aspects related to the eigenstructure and achieve certain advantages. As a reference, we recommend Malcev [9] [81].

It is customary to consider a self-adjoint endomorphism on a scalar product space, but it is also useful to have an appropriate extension for a complex vector space.

Let $\mathcal{V}$ be a finite-dimensional vector space over the field $\mathbb{C}$. A ***complex scalar product*** g on $\mathcal{V}$ is usually a nondegenerate symmetric bilinear form $g\colon \mathcal{V}\times\mathcal{V} \to \mathbb{C}$, which generalises the scalar product. A ***complex scalar product space*** is a complex vector space equipped with a complex scalar product.

A ***sesquilinear form*** on $\mathcal{V}$ is a function $g\colon \mathcal{V}\times\mathcal{V} \to \mathbb{C}$ that is $\mathbb{C}$-linear in the first argument and conjugate-linear in the second argument, i.e., such that $g(\alpha X + \beta Y, Z) = \alpha g(X,Z) + \beta g(Y,Z)$ and $g(X, \alpha Y + \beta Z) = \overline{\alpha} g(X,Y) + \overline{\beta} g(X,Z)$ hold for all $X, Y \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{C}$. An ***Hermitian form*** on $\mathcal{V}$ is a sesquilinear form g on $\mathcal{V}$ such that for all $X, Y \in \mathcal{V}$, we have $g(Y,X) = \overline{g(X,Y)}$. An ***Hermitian scalar product*** is a nondegenerate Hermitian form, while an ***Hermitian scalar product space*** is a complex vector space equipped with an Hermitian scalar product.

[9] Anatoly Ivanovich Malcev, Russian mathematician

Example A.4. An Hermitian scalar product space that is positive definite is called a ***unitary space***. A standard example of a unitary space is $\mathbb{C}^n$ with an Hermitian form g defined by

$$g((X_1,\dots,X_n),(Y_1,\dots,Y_n)) = X_1\overline{Y_1} + X_2\overline{Y_2} + \cdots + X_n\overline{Y_n},$$

for all $X_1,\dots,X_n,Y_1,\dots,Y_n \in \mathbb{C}$. $\triangle$

Following the real case, the ***quadratic form*** associated with g is a function $\varepsilon\colon \mathcal{V} \to \mathbb{C}$ defined by $\varepsilon_X = g(X,X)$. The sesquilinear form g is certainly $\mathbb{R}$-bilinear, and for $X,Y \in \mathcal{V}$, we have the polarization identities

$$4g(X,Y) = \varepsilon_{X+Y} - \varepsilon_{X-Y} + i\varepsilon_{X+iY} - i\varepsilon_{X-iY}, \quad 2g(X,Y) = (1+i)\varepsilon_X + (1+i)\varepsilon_Y - \varepsilon_{X-Y} - i\varepsilon_{X-iY}.$$

For the Hermitian form g, for every $X,Y \in \mathcal{V}$ and $\alpha,\beta \in \mathbb{C}$, the following equality holds:

$$g(\alpha X + \beta Y, \alpha X + \beta Y) = |\alpha|^2\varepsilon_X + 2\Re(\alpha\overline{\beta}g(X,Y)) + |\beta|^2\varepsilon_Y,$$

which implies that $\varepsilon_X \in \mathbb{R}$ for every $X \in \mathcal{V}$.

Let $(\mathcal{V},g)$ be a (real) scalar product space. The complexification yields the complex vector space $\mathcal{V}_\mathbb{C}$, and we can perform complexification of the scalar product g in two natural ways. For any $X,Y,Z,W \in \mathcal{V}$, the formulas

$$\begin{aligned} g_\mathbb{C}(X+iY,Z+iW) &= (g(X,Z) - g(Y,W)) + i(g(Y,Z) + g(X,W)),\\ g_\mathbb{C}^H(X+iY,Z+iW) &= (g(X,Z) + g(Y,W)) + i(g(Y,Z) - g(X,W)), \end{aligned}$$

define the complex scalar product $g_\mathbb{C}\colon \mathcal{V}_\mathbb{C} \times \mathcal{V}_\mathbb{C} \to \mathbb{C}$ and the Hermitian scalar product $g_\mathbb{C}^H\colon \mathcal{V}_\mathbb{C} \times \mathcal{V}_\mathbb{C} \to \mathbb{C}$. If for every $Z \in \mathcal{V}$ we have $g_\mathbb{C}(X+iY,Z) = 0$ or $g_\mathbb{C}^H(X+iY,Z) = 0$, then we obtain $g(X,Z) = 0$ and $g(Y,Z) = 0$, so $X+iY = 0$, proving that both forms are nondegenerate. Moreover, in a Hermitian scalar product space, Sylvester's law of inertia holds, and the Hermitian extension preserves the signature of the original space. In any case $(\mathcal{V},g)$ induces both a complex scalar product space $(\mathcal{V}_\mathbb{C},g_\mathbb{C})$ and an Hermitian scalar product space $(\mathcal{V}_\mathbb{C},g_\mathbb{C}^H)$.

If A is a self-adjoint endomorphism on $\mathcal{V}$, then in the complex scalar product space $(\mathcal{V}_\mathbb{C},g_\mathbb{C})$ we have

$$\begin{aligned} g_\mathbb{C}(A_\mathbb{C}(X+iY),Z+iW) &= (g(AX,Z) - g(AY,W)) + i(g(AY,Z) + g(AX,W))\\ &= (g(X,AZ) - g(Y,AW)) + i(g(Y,AZ) + g(X,AW)) = g_\mathbb{C}(X+iY,A_\mathbb{C}(Z+iW)), \end{aligned}$$

while in the Hermitian scalar product space $(\mathcal{V}_\mathbb{C},g_\mathbb{C}^H)$ we have

$$\begin{aligned} g_\mathbb{C}^H(A_\mathbb{C}(X+iY),Z+iW) &= (g(AX,Z) + g(AY,W)) + i(g(AY,Z) - g(AX,W))\\ &= (g(X,AZ) + g(Y,AW)) + i(g(Y,AZ) - g(X,AW)) = g_\mathbb{C}^H(X+iY,A_\mathbb{C}(Z+iW)), \end{aligned}$$

which means that the complexification $A_\mathbb{C} \in \operatorname{End}(\mathcal{V}_\mathbb{C})$ is also self-adjoint in both cases.

It is important to note that if A is a self-adjoint endomorphism of a complex scalar product space, then $A - \lambda\mathbb{1}$ for $\lambda \in \mathbb{C}$, as well as $p(A)$ where $p(x) \in \mathbb{C}[x]$ is an arbitrary polynomial, are also self-adjoint. However, this does not hold in the Hermitian scalar product space, where we have $g((A-\lambda\mathbb{1})X,Y) = g(AX,Y) - \lambda g(X,Y) = g(X,(A-\overline{\lambda}\mathbb{1})Y)$.

From the proof of Theorem A.12, we know that the generalised eigenspaces have a trivial intersection and thus form a direct sum. In the case where A is a self-adjoint endomorphism, this direct sum is orthogonal.

Lemma A.18. *The generalised eigenspaces of a self-adjoint endomorphism of a real or complex scalar product space are orthogonal to each other.*

Proof. Let $A \in \mathrm{End}(\mathcal{V})$, and let $X \in \mathcal{V}_\lambda$ and $Y \in \mathcal{V}_\mu$ be arbitrary for $\lambda \neq \mu$. By Lemma A.10, the restriction of A_λ is an automorphism of $\mathcal{V}_\mu$, so there exists $Z \in \mathcal{V}_\mu$ such that $(A_\lambda)^n Z = Y$. Since A is self-adjoint, $(A_\lambda)^n$ is also self-adjoint, so we have $g(X,Y) = g(X,(A_\lambda)^n Z) = g((A_\lambda)^n X, Z) = 0$, which proves that $\mathcal{V}_\lambda \perp \mathcal{V}_\mu$. □

Lemma A.19. *The generalised eigenspaces of a self-adjoint endomorphism of an Hermitian scalar product space corresponding to non-conjugate eigenvalues are orthogonal to each other. In particular, the generalised eigenspace corresponding to eigenvalues that are not real are totally isotropic.*

Proof. Similarly to the proof of Lemma A.18, for $\overline{\lambda} \neq \mu$, there exists $Z \in \mathcal{V}_\mu$ such that $(A_{\overline{\lambda}})^n Z = Y$. Since A is self-adjoint, we get $g(X,Y) = g(X,(A_{\overline{\lambda}})^n Z) = g((A_\lambda)^n X, Z) = 0$. □

Additional advantages arise if the (real) scalar product is (positively) definite, as in this case all eigenvalues are real.

Lemma A.20. *The eigenvalues of a self-adjoint endomorphism on a definite scalar product space are real.*

Proof. Let $\lambda = \alpha + i\beta \in \mathbb{C}$ be a complex eigenvalue of a self-adjoint endomorphism A on a definite scalar product space $(\mathcal{V}, g)$. Then there exists a nonzero $Z = X + iY \in \mathcal{V}_\mathbb{C}$ such that $A_\mathbb{C} Z = \lambda Z$, which gives $AX = \alpha X - \beta Y$ and $AY = \beta X + \alpha Y$. Since A is self-adjoint, we have $g(\alpha X - \beta Y, Y) = g(X, \beta X + \alpha Y)$, which leads to $\beta(\varepsilon_X + \varepsilon_Y) = 0$. As $\mathcal{V}$ is definite, $\varepsilon_X + \varepsilon_Y \neq 0$, thus we conclude $\beta = 0$, which gives $\lambda \in \mathbb{R}$. □

For $0 \neq X \in \mathcal{V}$ there exists $r \in \mathbb{N}$ such that $(A_\lambda)^r X \neq 0$, from which the definite scalar product gives $g((A_\lambda)^{2r} X, X) = g((A_\lambda)^r X, (A_\lambda)^r X) \neq 0$, but $(A_\lambda)^{2r} X = 0$ for $2r > r$, which implies $r = 0$, and thus $A_\lambda X = 0$ so $\mathcal{V}_\lambda = \mathcal{E}_\lambda$. By Lemma A.20, the characteristic polynomial ω_A is split, and by Theorem A.12, $\mathcal{V}$ decomposes into a direct sum of eigenspaces $\mathcal{V}_\lambda$, which, due to Lemma A.18, is orthogonal. Thus, A is diagonalisable and admits a spectral decomposition $\mathcal{V} = \bigoplus_\lambda \mathcal{E}_\lambda$, which is the statement we call the ***spectral theorem***.

Lemma A.21. *A self-adjoint endomorphism A on a definite scalar product space $\mathcal{V}$ is diagonalisable and satisfies $\mathcal{V} = \bigoplus_\lambda \mathrm{Ker}(A - \lambda \mathbb{1})$.*

Spectral decomposition is not always possible when g is indefinite, primarily because ω_A may not be split. The characteristic polynomial of $A_\mathbb{C}$ is split, and from Theorem A.12 and Lemma A.18, we obtain $\mathcal{V}_\mathbb{C} = \bigoplus_\lambda (\mathcal{V}_\mathbb{C})_\lambda$, which subsequently allows the decomposition $\mathcal{V} = \bigoplus_{\lambda, \Im(\lambda) \geq 0} \mathcal{V}_\lambda$. Since the direct sum is orthogonal, the generalised eigenspaces $\mathcal{V}_\lambda$ are nondegenerate.

Lemma A.22. *The generalised eigenspaces of a self-adjoint endomorphism on a scalar product space are nondegenerate, and the space is equal to their orthogonal sum.*

The generalised eigenspace $\mathcal{V}_\lambda$ for $A \in \mathrm{End}(\mathcal{V})$ is, by Lemma A.13, decomposed into a direct sum of A-cyclic subspaces. Consider $\mathcal{C}_Y = \mathrm{Span} \bigcup_{k=0}^{s-1} \{B^k Y\}$, where $B = (A - \lambda \mathbb{1})\restriction_{\mathcal{V}_\lambda}$, and $s = r(Y)$ ($B^{s-1} Y$ is an eigenvector).

In the presence of a real or complex scalar product g, we can construct a sequence of scalars $\alpha_0, \alpha_1, \ldots, \alpha_{s-1} \in \mathbb{F}$ by recursively defining $\alpha_0 = 1$ and

$$\alpha_{s-1-m} = -\frac{1}{2} \sum_{j=0}^{s-2-m} \sum_{k=0}^{\min(s-1-m-j, s-2-m)} \alpha_j \alpha_k g(B^{m+j+k} Y, Y)$$

for $s-2 \geq m \geq 0$, in order to introduce $Z = \sum_{j=0}^{s-1} \alpha_j B^j Y \in \mathcal{C}_Y$ and obtain

$$
\begin{aligned}
g(B^m Z, Z) &= g\left(B^m \sum_{j=0}^{s-1} \alpha_j B^j Y, \sum_{k=0}^{s-1} \alpha_k B^k Y\right) = \sum_{j=0}^{s-1} \sum_{k=0}^{s-1} \alpha_j \alpha_k g(B^{m+j} Y, B^k Y) \\
&= \sum_{j=0}^{s-1-m} \sum_{k=0}^{s-1-m-j} \alpha_j \alpha_k g(B^{m+j+k} Y, Y) = 0.
\end{aligned}
$$

The sequence of vectors $E_j = B^{s-j} Z$ for $1 \leq j \leq s$ forms a basis $e = (E_1, \ldots, E_s)$ for $\mathcal{C}_Y = \mathcal{C}_Z$. With respect to this basis, the endomorphism has a Jordan matrix $[A\restriction_{\mathcal{C}_Y}]_e = \mathcal{J}_s(\lambda)$, while the Gram matrix is $[g\restriction_{\mathcal{C}_Y}]_e = \nu N_s$, where $\nu = g(B^{s-1} Z, Z)$ and

$$
N_s = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & \cdots & 1 & 0 & 0 \\
\vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix} \in \mathbb{F}^{s \times s}. \tag{A.6}
$$

If $\nu = 0$, we cannot do much, as $\mathcal{C}_Y$ is a totally isotropic subspace. However, if $\nu \neq 0$, in the construction, we can replace the vector Z with the vector μZ for any $0 \neq \mu \in \mathbb{F}$, so we obtain $g(B^{s-1} \mu Z, \mu Z) = \mu^2 g(B^{s-1} Z, Z)$, which allows us, in the case $\mathbb{F} = \mathbb{R}$ to choose $\mu = 1/\sqrt{|g(B^{s-1} Z, Z)|}$ and thus normalise $\nu = \pm 1$, while in the case $\mathbb{F} = \mathbb{C}$, the choice $\mu = (g(B^{s-1} Z, Z))^{-1/2}$ gives $\nu = 1$.

From the existing basis, we can construct a new basis with

$$
G_j = E_j + \operatorname{sgn}(2j - s - 1) E_{s+1-j},
$$

for $1 \leq j \leq s$, from which we obtain

$$
\begin{aligned}
g(G_j, G_k) &= g\left(E_j + \operatorname{sgn}(2j - s - 1) E_{s+1-j}, E_k + \operatorname{sgn}(2k - s - 1) E_{s+1-k}\right) \\
&= \nu \delta^j_{s+1-k} + \nu \operatorname{sgn}(2k - s - 1) \delta^j_k + \nu \operatorname{sgn}(2j - s - 1) \delta^j_k \\
&\quad + \nu \operatorname{sgn}((2j - s - 1)(2k - s - 1)) \delta^j_{s+1-k} \\
&= 2\nu \delta^j_k \operatorname{sgn}(2j - s - 1) + \nu \delta^j_{s+1-k} \left(1 - \operatorname{sgn}(2j - s - 1)^2\right) \\
&= \nu \delta^j_k \left(2 \operatorname{sgn}(2j - s - 1) + \delta^{2j}_{s+1}\right),
\end{aligned}
$$

which gives

$$
g(G_j, G_k) = \begin{cases}
2\nu \delta_{jk} \text{ за } 2j > s + 1 \\
\nu \delta_{jk} \text{ за } 2j = s + 1 \\
-2\nu \delta_{jk} \text{ за } 2j < s + 1
\end{cases}
$$

and proves that our basis is orthogonal.

The main idea is to decompose each $\mathcal{V}_\lambda$ into an orthogonal sum of nondegenerate subspaces of the form $\mathcal{C}_Y$. If we choose an arbitrary definite vector $D \in \mathcal{V}_\lambda$, we have $X = f(D) \in \mathcal{E}_\lambda$, and since $\mathcal{C}_Y \ni D$ is not totally isotropic, it follows $\nu \neq 0$. Now, the orthogonal basis we had can easily be transformed into an orthonormal one, and $\mathcal{C}_Y$ is a nondegenerate subspace of index $s/2$ for even s, or $(s - \operatorname{sgn} \nu)/2$ for odd s. This procedure can be continued on the nondegenerate subspace $\mathcal{V}_\lambda \cap (\mathcal{C}_Y)^\perp$, and thus in $\dim \mathcal{E}_\lambda$ steps, we decompose the entire $\mathcal{V}_\lambda$.

Theorem A.23. *If A is a self-adjoint endomorphism on a complex scalar product space $\mathcal{V}$, then $\mathcal{V}$ decomposes into an orthogonal sum of nondegenerate A-cyclic subspaces, and there exists a basis for $\mathcal{V}$ in which the matrix of A is block-diagonal with Jordan blocks, and the Gram matrix is block-diagonal with blocks of the form* (A.6).

It remains to consider $\mathcal{V}_\lambda$ for a complex eigenvalue λ in the case $\mathbb{F} = \mathbb{R}$. In the proof of Theorem A.16, we decomposed $\mathcal{V}_\lambda$ into a direct sum of subspaces of the form $\mathcal{C}_Z \oplus \mathcal{C}_{\overline{Z}} \leq (\mathcal{V}_\mathbb{C})_\lambda \oplus (\mathcal{V}_\mathbb{C})_{\overline{\lambda}}$. We had a canonical basis $(E_1, \ldots, E_s, F_1, \ldots, F_s)$ in which the endomorphism has a block-diagonal matrix with $\mathcal{J}_s(\lambda)$ and $\mathcal{J}_s(\overline{\lambda})$ on the diagonal. The new basis $(G_1, H_1, \ldots, G_s, H_s)$, where $G_j = \Re(E_j)$ and $H_j = \Im(E_j)$, led us to the matrix $\mathcal{J}_s(\alpha, \beta)$, where $\lambda = \alpha + i\beta$, and it remains to determine how the Gram matrix looks in this basis.

If we extend g to the complex scalar product $g_\mathbb{C}$, by Lemma A.18 we have $g_\mathbb{C}(E_j, F_k) = 0$, while $g_\mathbb{C}(F_j, F_k) = g_\mathbb{C}(\overline{E_j}, \overline{E_k}) = \overline{g_\mathbb{C}(E_j, E_k)}$. Now, we can compute:

$$
\begin{aligned}
g(G_j, G_k) &= g_\mathbb{C}(\Re(E_j), \Re(E_k)) = \frac{1}{2}\Re(g_\mathbb{C}(E_j, E_k)),\\
g(G_j, H_k) &= g_\mathbb{C}(\Re(E_j), \Im(E_k)) = 0,\\
g(H_j, H_k) &= g_\mathbb{C}(\Im(E_j), \Im(E_k)) = -\frac{1}{2}\Re(g_\mathbb{C}(E_j, E_k)),
\end{aligned}
$$

and since $g_\mathbb{C}(E_j, E_k) = \delta^j_{s+1-k}\nu$, if $\nu \neq 0$, by appropriately choosing the vector μZ instead of Z, we can normalise $\nu = 2$ to obtain $g(G_j, G_k) = 1$ and $g(H_j, H_k) = -1$ only for $j + k = s + 1$, while the other components are zero, Thus, we arrive at a basis in which the endomorphism has the matrix $\mathcal{J}_s(\alpha, \beta)$, while the Gram matrix takes the form

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 1 & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & -1\\
0 & 0 & \cdots & 1 & 0 & 0 & 0\\
0 & 0 & \cdots & 0 & -1 & 0 & 0\\
\vdots & \vdots & ⋰ & \vdots & \vdots & \vdots & \vdots\\
1 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & -1 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix} \in \mathbb{R}^{2s\times 2s}, \tag{A.7}
$$

which has index s and means the neutral signature.

As in the case $\lambda \in \mathbb{R}$, we choose an arbitrary definite vector $D \in (\mathcal{V}_\mathbb{C})_\lambda$, to ensure that $\mathcal{C}_Y \ni D$ is not totally isotropic (although by Lemma A.19, in the Hermitian case $(\mathcal{V}_\mathbb{C})_\lambda$ is totally isotropic, but our case is complex), and by exhaustion, we decompose the entire $\mathcal{V}_\lambda$ as before.

Theorem A.24. *If A is a self-adjoint endomorphism of a real scalar product space $\mathcal{V}$, then $\mathcal{V}$ decomposes into an orthogonal sum of nondegenerate subspaces, and there exists a basis for $\mathcal{V}$ in which the matrix of A is block-diagonal with (standard or real) Jordan matrices, and the Gram matrix is block-diagonal with blocks of the form* (A.6) *and* (A.7).

A.5 Perturbation theory

Understanding how the spectral decomposition of an endomorphism on a scalar product space depends on parameters is a natural and significant problem. It is well known that the n roots of a real or complex polynomial of degree n depend continuously on the polynomial's coefficients. Consequently, the eigenvalues of an endomorphism vary continuously with the entries of its matrix. It is important to emphasise that we consider the roots as an unordered n-tuple of scalars. Since there is no canonical way to order the roots, it is not always possible to label the eigenvalues so that they individually form continuous functions.

However, we are primarily interested in the question of the smoothness or analyticity of the eigenstructure. The history of perturbation theory began in 1937 with Rellich[10] and his first article in a series [104], which ultimately led to his book [105]. The theory culminates in the classic monograph by Kato[11] [71]. For a modern perspective, we recommend the survey by Parusiński[12] and Rainer[13] [99].

Let $P_A(x) = x^n + a_1x^{n-1} + \cdots + a_n$ denote a monic polynomial of degree n with real coefficients $A = (a_1, \ldots, a_n) \in \mathbb{R}^n$. A monic polynomial P_A is called ***hyperbolic*** if all its roots $\lambda_1, \ldots, \lambda_n$ (counted with multiplicities) are real.

For an open interval $I \subseteq \mathbb{R}$ containing 0, we consider a one-parameter family of hyperbolic monic polynomials $P_{A(t)}$ of degree n for $t \in I$, where $A = (a_1, \ldots, a_n) \colon I \to \mathbb{R}^n$ is analytic. Applying the ***Tschirnhaus transformation*** [14]

$$P_B(x) = P_A\left(x - \frac{a_1}{n}\right) = \sum_{j=0}^{n} a_j \sum_{k=0}^{n-j} \binom{n-j}{k} x^k \left(\frac{-a_1}{n}\right)^{n-j-k} = \sum_{k=0}^{n} b_{n-k}x^k,$$

we obtain $B = (b_1, \ldots, b_n) \colon I \to \mathbb{R}^n$, where $a_0 = 1$ and

$$b_{n-k} = \sum_{j=0}^{n-k} \binom{n-j}{k} \left(\frac{-1}{n}\right)^{n-j-k} a_j a_1^{n-j-k},$$

for $0 \le k \le n$. Thus, we obtain a family of polynomials $P_{B(t)}$ that remain monic ($b_0 = 1$) and have the coefficient of x^{n-1} equal to zero ($b_1 = 0$). Such a polynomial P_B is said to be in ***Tschirnhaus form***. Its roots are $\mu_j = \lambda_j + a_1/n$ for $1 \le j \le n$, so P_B is also hyperbolic, and $B \colon I \to \mathbb{R}^n$ remains analytic.

Since $\sum_j \mu_j^2 = \sum_{j,k} \mu_j\mu_k - 2\sum_{j<k} \mu_j\mu_k = (-b_1)^2 - 2b_2$, the condition $b_1 = 0$ implies $-2b_2 = \mu_1^2 + \cdots + \mu_n^2$, and consequently $b_2 \le 0$. The case $b_2 = 0$ is straightforward, since then $\mu_j = 0$ for $1 \le j \le n$, which implies that the original roots $\lambda_j = -a_1/n$ are analytic. Otherwise, we may shrink the interval so that $-b_2(t) = t^{2m}u(t)$, where $u(0) > 0$ for some $m \in \mathbb{N}_0$ (note that if $b_2(0) \neq 0$ then $m = 0$) and $t \in I'$, with $0 \in I' \subseteq I$.

Since the parameter space is one-dimensional, we can always choose $\theta(t) = \pm t^m u^{1/2}(t)$ as one of the two analytic square roots of $-b_2(t)$, and define

$$P_C(x) = \theta^{-n} P_B(\theta x) = x^n - x^{n-2} + \sum_{j=3}^{n} \theta^{-j} b_j x^{n-j}.$$

In this way, we obtain an analytic function $C = (c_1, \ldots, c_n) \colon I' \to \mathbb{R}^n$ such that $c_1 = 0$ and $c_2 = -1$. Hence, $\xi_1 + \cdots + \xi_n = 0$ and $\xi_1^2 + \cdots + \xi_n^2 = 2$, where $\xi_j = \theta^{-1}\mu_j$ for $1 \le j \le n$ are the roots of P_C. This allows us to factor $P_{C(0)} = P_D P_E$, where P_D and P_E are monic real polynomials of positive degree with no common root.

Consider the map $\psi \colon \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{p+q}$ defined by $\psi(Y, Z) = W$, where $P_W = P_Y P_Z$ holds. In other words, for $Y = (y_1, \ldots, y_p)$ and $Z = (z_1, \ldots, z_q)$, we obtain $W = (w_1, \ldots, w_{p+q})$ such that $w_k = \sum_{j=0}^{k} y_{k-j} z_j$ for all $1 \le k \le p + q$, where $y_0 = z_0 = 1$. The Jacobian matrix of this polynomial map ψ is exactly the Sylvester matrix associated with P_Y and P_Z. Furthermore,

[10] Franz Rellich (1906–1955), Austrian-German mathematician
[11] Tosio Kato (1917–1999), Japanese mathematician
[12] Adam Parusiński, Polish mathematician
[13] Armin Rainer, Austrian mathematician
[14] Ehrenfried Walther von Tschirnhaus (1651–1708), German mathematician, physicist, and philosopher

the resultant of P_Y and P_Z is, up to sign, equal to the determinant of the Sylvester matrix. Hence, P_Y and P_Z have no common (complex) roots if and only if ψ is invertible.

Since P_D and P_E have no common roots, ψ is invertible at (D,E), and by the Inverse function theorem (Theorem A.30), it remains invertible in a neighbourhood of (D,E). Thus, there exists a neighbourhood $\mathcal{U} \subseteq \mathbb{R}^n$ of $C(0) = \psi(D,E)$ such that, for any $W \in \mathcal{U}$, the corresponding polynomial splits uniquely as $P_W = P_Y P_Z$. Moreover, the inverse map ψ^{-1} is analytic on $\mathcal{U}$, so the compositions $\pi_1 \circ \psi^{-1}\colon W \mapsto Y$ and $\pi_2 \circ \psi^{-1}\colon W \mapsto Z$ are analytic as well, with $\pi_1 \circ \psi^{-1}(C(0)) = D$ and $\pi_2 \circ \psi^{-1}(C(0)) = E$.

In this way, we obtain the identity $P_{Y(t)}P_{Z(t)} = P_{C(t)}$ for all $t \in I''$, where $0 \in I'' \subseteq I' \subseteq I$, and the maps $Y\colon I'' \to \mathbb{R}^p$, $Z\colon I'' \to \mathbb{R}^q$ are analytic. Since $p = \deg D < n$ and $q = \deg E < n$, we may apply the induction hypothesis on $\deg A = n$ to obtain analytic functions $\lambda_j\colon I'' \to \mathbb{R}$ for $1 \le j \le n$, which parametrise the roots of $P_{A(t)}$.

This local result may be regarded as a reformulation of Rellich's theorem [104, Lemma 2] inspired by the approach developed in [99]. However, owing to the uniqueness of analytic continuation, we may, without loss of generality, assume that $I \subseteq \mathbb{R}$ is an arbitrary open interval and that $I'' = I$. Consequently, we obtain the following global version of the theorem.

Theorem A.25. *Let $P_{A(t)}(x) = x^n + a_1(t)x^{n-1} + \cdots + a_n(t)$ for $t \in I$ be a one-parameter family of hyperbolic monic polynomials of degree n with real analytic coefficients $a_j\colon I \to \mathbb{R}$ for $1 \le j \le n$, where $I \subseteq \mathbb{R}$ is an open interval. Then there exist analytic functions $\lambda_j\colon I \to \mathbb{R}$ for $1 \le j \le n$ such that $P_{A(t)}(x) = \prod_{j=1}^n (x - \lambda_j(t))$ for all $t \in I$.*

Let $I \subseteq \mathbb{R}$ be an open interval containing 0, and consider a one-parameter family of endomorphisms $A(t)$ on a vector space $\mathcal{V}$ of dimension $n \in \mathbb{N}$. Fix a basis $(E_1, \ldots, E_n)$ of $\mathcal{V}$. The matrix entries of $A(t)$ in this basis are given by $A(t)(E_j) = \sum_k A_{kj}(t)E_k$ for $1 \le j,k \le n$, and we assume that all functions $A_{jk}\colon I \to \mathbb{R}$ are analytic. It is important to note that the analyticity of the matrix entries A_{jk} is preserved under a change of basis, provided that the transition matrix depends analytically on t.

If the characteristic polynomial of $A(t)$ is hyperbolic for all $t \in I$, then we may apply Theorem A.25. This yields analytic eigenvalues $\lambda_j\colon I \to \mathbb{R}$ for $1 \le j \le n$, such that

$$\det(x\mathbb{1} - A(t)) = \prod_{j=1}^{n} (x - \lambda_j(t))$$

holds for all $t \in I$.

Let $V(t) = \sum_j v_j(t)E_j$ be an eigenvector corresponding to an eigenvalue $\lambda(t)$. Denote $M(t) = A(t) - \lambda(t)\mathbb{1}$. Then the eigenvalue equation $M(t)V(t) = 0$ yields the system $\sum_{k=1}^n M_{jk}(t)v_k(t) = 0$ for $1 \le j \le n$, where $M_{jk}(t) = A_{jk}(t) - \lambda(t)\delta_{jk}$ are analytic functions.

If $M(0) \neq 0$, then, since $\det M(0) = 0$, there exists $r \in \mathbb{N}$ such that all minors of order $r+1$ vanish, while there is a minor of order $r = |\mathcal{A}| = |\mathcal{B}|$ for which $\det((M_{jk}(0))_{j\in\mathcal{A},k\in\mathcal{B}} \neq 0$. Let $N_{jk}(t)$ denote the cofactor of $M_{jk}(t)$ in the $(r+1) \times (r+1)$ matrix $(M_{jk}(t))_{j\in\mathcal{A}',k\in\mathcal{B}'}$, where $\mathcal{A}' = \mathcal{A} \sqcup \{a\}$ and $\mathcal{B}' = \mathcal{B} \sqcup \{b\}$. Define $v_k(t) = N_{ak}(t)$ for $k \in \mathcal{B}'$ and $v_k(t) = 0$ for $k \notin \mathcal{B}'$, so that the vector $V(t) = \sum_k v_k(t)E_k$ satisfies

$$\sum_{k=1}^{n} M_{jk}(t)v_k(t) = \sum_{k\in\mathcal{B}'} M_{jk}(t)N_{ak}(t) = \begin{vmatrix} M_{p_1q_1}(t) & M_{p_1q_2}(t) & \cdots & M_{p_1q_r}(t) & M_{p_1b}(t) \\ M_{p_2q_1}(t) & M_{p_2q_2}(t) & \cdots & M_{p_2q_r}(t) & M_{p_2b}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{p_rq_1}(t) & M_{p_rq_2}(t) & \cdots & M_{p_rq_r}(t) & M_{p_rb}(t) \\ M_{aq_1}(t) & M_{aq_2}(t) & \cdots & M_{aq_r}(t) & M_{ab}(t) \end{vmatrix} = 0,$$

for all $1 \le j \le n$, where $\mathcal{A} = \{p_1, \ldots, p_r\}$ and $\mathcal{B} = \{q_1, \ldots, q_r\}$. By continuity, we may shrink the interval so that

$$D(t) = \det(((M_{jk}(t))_{j\in\mathcal{A},k\in\mathcal{B}}) \neq 0,$$

for all $t \in I'$, where $0 \in I' \subseteq I$. Hence, $v_b(t) = N_{ab}(t) = D(t) \neq 0$, so $V(t)$ is an eigenvector of $A(t)$ corresponding to the eigenvalue $\lambda(t)$ for all $t \in I'$.

If $M(0) = 0$, then the case $M(t) = 0$ for all $t \in I$ is trivial. Otherwise, since all functions $M_{jk}\colon I \to \mathbb{R}$ are analytic and satisfy $M_{jk}(0) = 0$, there exists $m \in \mathbb{N}$ such that $M(t) = t^m M^o(t)$, where $M^o(0) \neq 0$. In this case, we may solve the system $\sum_{k=1}^{n} M^o_{jk}(t)v_k(t) = 0$ as before and construct the eigenvector $V(t)$.

Suppose that $A(0)X = \mu X$ holds for some nonzero vector $X \in \mathcal{V}$ and $\mu \in \mathbb{R}$. There exists a basis $(E_1, \ldots, E_n)$ of $\mathcal{V}$ in which the matrix of $A(0)$ is block-diagonal with Jordan matrices on the diagonal such that $E_1 = X$ (see Theorem A.14). Using $\lambda(0) = \mu$, there exist indices $1 \leq a, b \leq n$ such that $M_{ak}(0) = 0$ and $M_{kb}(0) = 0$ hold for all $1 \leq k \leq n$ (with $b = 1$, of course), and therefore $a \notin \mathcal{A}$ and $b \notin \mathcal{B}$. Applying the previous results we obtain the eigenvector $V(t) = \sum_k v_k(t)E_k$ for $\lambda(t)$, where $v_k(0) = 0$ for all $k \neq b = 1$, which gives $V(0) = v_1(0)E_1$, and therefore $W(t) = V(t)/D(t)$ generates an analytic eigenvector corresponding to $\lambda(t)$ such that $W(0) = X$ and $\lambda(0) = \mu$.

Theorem A.26. *Let $A(t)$, for $t \in I$, be a one-parameter family of endomorphisms on a finite-dimensional vector space $\mathcal{V}$ with a hyperbolic characteristic polynomial, such that the entries of the corresponding matrices are real analytic functions on an open interval $I \subseteq \mathbb{R}$ containing 0. If $A(0)X = \mu X$ holds for some nonzero vector $X \in \mathcal{V}$ and $\mu \in \mathbb{R}$, then there exist an analytic function $\lambda\colon I' \to \mathbb{R}$ and an analytic map $V\colon I' \to \mathcal{V} \setminus \{0\}$ such that $A(t)V(t) = \lambda(t)V(t)$ holds for $t \in I'$, where $0 \in I' \subseteq I$, with $V(0) = X$ and $\lambda(0) = \mu$.*

If we have a scalar product space $(\mathcal{V}, g)$, then it is natural to assume that all $A(t)$ are self-adjoint. We also assume that each $A(t)$ is diagonalisable (with hyperbolic characteristic polynomial). This condition is geometrically natural and automatically satisfied when $\mathcal{V}$ is definite (see Lemma A.20 and Lemma A.21).

Let $(E_1, \ldots, E_n)$ be an orthonormal basis such that $A(0)E_i = \mu_i E_i$ for all $1 \leq i \leq n$. We have already shown (Theorem A.26) that for any μ_i, say μ_1, there exist an analytic eigenvalue function $\lambda\colon I' \to \mathbb{R}$ and a corresponding analytic map $V\colon I' \to \mathcal{V}$ such that $V(t)$ is an eigenvector of $A(t)$ with eigenvalue $\lambda(t)$ for all $t \in I'$, where $0 \in I' \subseteq I$, $\lambda(0) = \mu_1$, and $V(0) = E_1$. Furthermore, possibly after shrinking the interval, the map V can be normalised to obtain a unit analytic map.

Define the endomorphism $P(t)\colon \mathcal{V} \to \mathcal{V}$, called the ***projection operator***, by $P(t)U = g(V(t), U)V(t)$, where V is the analytic map constructed above. The matrix entries of $P(t)$ with respect to a fixed orthonormal basis are given by $p_{jk}(t) = \varepsilon_k v_j(t)v_k(t)$. These entries are analytic functions of t. We now consider the new family of self-adjoint endomorphisms defined by $B(t) = A(t) - P(t)$.

Any eigenvector of $A(0)$ corresponding to the eigenvalue $\lambda(0)$ that is orthogonal to $V(0)$ is also an eigenvector of $B(0)$ for the same eigenvalue. Moreover, consider an eigenvector map $W\colon I'' \to \mathbb{R}$ such that $A(t)W(t) - g(V(t), W(t))V(t) = B(t)W(t) = \lambda(t)W(t)$ for $t \in I''$, where $0 \in I'' \subseteq I'$. Since $A(t) - \lambda(t)\,\mathbb{1}$ is self-adjoint, we obtain

$$0 = g((A(t) - \lambda(t)\,\mathbb{1})V(t), W(t)) = g((A(t) - \lambda(t)\,\mathbb{1})W(t), V(t)) = g(V(t), W(t))g(V(t), V(t)),$$

which implies $V(t) \perp W(t)$ for all $t \in I''$.

This construction allows us to proceed by induction on the multiplicity ν of $\lambda(0)$, thereby obtaining ν orthonormal analytic eigenvector maps V_i associated with λ satisfying $AV_i(t) = \lambda(t)V_i(t)$, with $V_i(0) = E_i$ whenever $\mu_i = \lambda(0)$. Applying the same argument to each eigenvalue of $A(0)$ recovers the classical result of Rellich: the eigenvalues and corresponding eigenvectors of $A(t)$ can be chosen to depend analytically on t in a neighbourhood of 0 (see [104, Proposition 1], [105, Theorem 1], [99]). As before, this local construction implies a global statement on any open interval I, yielding $I = I'$ after analytic continuation.

Theorem A.27. *Let $A(t)$, for $t \in I$, be a one-parameter family of diagonalisable self-adjoint endomorphisms on a scalar product space $\mathcal{V}$ of dimension n, such that the entries of the corresponding matrices are real analytic functions on an open interval $I \subseteq \mathbb{R}$. Then, there exist analytic functions $\lambda_j \colon I \to \mathbb{R}$ and analytic maps $V_j \colon I \to \mathcal{V}$ for $1 \leq j \leq n$, such that $(V_1(t), \ldots, V_n(t))$ forms an orthonormal basis of $\mathcal{V}$ for all $t \in I$, and each $V_j(t)$ satisfies $A(t)V_j(t) = \lambda_j(t)V_j(t)$ for all $t \in I$ and $1 \leq j \leq n$.*

If the entries of the corresponding matrices are not analytic, but merely smooth, the eigenvectors may fail to admit even a continuous choice, as shown in the following example from [104].

Example A.5. Consider the one-parameter family of self-adjoint endomorphisms defined by the matrices

$$A(t) = e^{-\frac{1}{t^2}} \begin{pmatrix} \cos\frac{1}{t} & \sin\frac{1}{t} \\ \sin\frac{1}{t} & -\cos\frac{1}{t} \end{pmatrix}$$

for $t \neq 0$ and set $A(0) = 0$. This family is smooth for all $t \in \mathbb{R}$, including $t = 0$, but is not analytic at $t = 0$ (see Example A.7). Since $\operatorname{Tr} A(t) = 0$ and $\det A(t) = -e^{-2/t^2}$, the eigenvalues of $A(t)$ are $\lambda(t) = \pm e^{-1/t^2}$. For example, the vector $V(t) = (\cos(1/2t), \sin(1/2t))$ is an eigenvector corresponding to $\lambda(t) = e^{-1/t^2}$. However, this eigenvector cannot be extended to a non-vanishing continuous map at $t = 0$, since the map $t \mapsto V(t)$ oscillates wildly and has no limit as $t \to 0$. △

However, both Theorem A.25 and Theorem A.27 do not extend to the multi-parameter case, as demonstrated by the following example from [104].

Example A.6. Consider the two-parameter family of self-adjoint endomorphisms defined by the matrices

$$A(t,s) = \begin{pmatrix} 1+2t & t+s \\ t+s & 1+2s \end{pmatrix}$$

for $t, s \in \mathbb{R}$. The eigenvalues of $A(t,s)$ are given by $\lambda(t,s) = 1 + t + s \pm \sqrt{2(t^2+s^2)}$, which are not analytic in any neighbourhood of the origin. △

Some multi-parameter versions have been proposed by Kurdyka[15] and Paunescu[16] [75], but for our purposes, it is sufficient to consider only simpler cases. When dealing with multi-parameter families of self-adjoint endomorphisms, analytic dependence of eigenvalues and eigenvectors is generally lost, as demonstrated in Example A.6.

If we consider a multi-parameter family of hyperbolic monic polynomials $P_{A(T)}$ of degree n, where $T \in U \subseteq \mathbb{R}^m$ is an open set with $m > 1$, then the proof of Theorem A.25 cannot be directly extended, as we are no longer able to choose an analytic square root of $-b_2(T)$. For instance, taking $A(t,s) = (0, -t^2 - s^2)$ yields the family $P_{A(t,s)}(x) = x^2 - t^2 - s^2$, whose roots $\pm\sqrt{t^2+s^2}$ are not analytic at the origin.

However, if the eigenvalue multiplicities remain constant in a neighbourhood of a given point, then partial regularity can be recovered. When the multiplicities do not change as T varies, we have no crossing of eigenvalues, so the polynomial $P_{B(T)}$ in Tschirnhaus form splits simultaneously, which prevents the case of non-constant b_2 with $b_2(0) = 0$. Thus, the square roots of $-b_2 > 0$ are analytic, and we can proceed in the same manner as before. If the domain U is simply connected, the eigenvalues may be labelled so as to be analytic.

Theorem A.28. *Let $P_{A(T)}(x) = x^n + a_1(T)x^{n-1} + \cdots + a_n(T)$ for $T \in U$ be a multi-parameter family of hyperbolic monic polynomials of degree n with real analytic coefficients $a_j \colon U \to \mathbb{R}$ for $1 \leq j \leq n$, where $U \subseteq \mathbb{R}^m$ is a simply connected open set and $m > 1$. If the multiplicities of the roots remain constant throughout U, then there exist analytic functions $\lambda_j \colon U \to \mathbb{R}$ for $1 \leq j \leq n$ such that $P_{A(T)}(x) = \prod_{j=1}^n (x - \lambda_j(T))$ for all $T \in U$.*

[15] Krzysztof Kurdyka (1957), Polish mathematician

[16] Laurentiu Paunescu (1952), Romanian-Australian mathematician

A.6 Analysis

In this section, we have some results from the real analysis that we need in the book.

Example A.7. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}.$$

For $x < 0$ it is obviously $f^{(n)}(x) = 0$. For $x > 0$ we can use the induction over n to prove that the n-th derivative of the function f has the form

$$f^{(n)}(x) = \frac{P_{n-1}(x)}{x^{2n}} f(x), \tag{A.8}$$

where $P_{n-1}(x)$ is a polynomial of degree $n-1$. The induction basis $n = 1$ gives $f'(x) = f(x)/x^2$, which is realized for $P_0(x) = 1$. The induction step follows from

$$\begin{aligned} f^{(n+1)}(x) &= \left(\frac{P'_{n-1}(x)}{x^{2n}} - 2n \frac{P_{n-1}(x)}{x^{2n+1}} + \frac{P_{n-1}(x)}{x^{2n+2}} \right) f(x) \\ &= \frac{x^2 P'_{n-1}(x) - (2nx - 1) P_{n-1}(x)}{x^{2n+2}} f(x) = \frac{P_n(x)}{x^{2(n+1)}} f(x), \end{aligned}$$

where $P_n(x)$ is a polynomial of degree n given by $P_n(x) = x^2 P'_{n-1}(x) - (2nx - 1)P_{n-1}(x)$. Although it is sufficient to say that $P_n(x)$ is a polynomial of degree not greater than n, it is easy to see that the leading coefficient of $P_n(x)$ is obtained by multiplying the leading coefficient of $P_{n-1}(x)$ by $-(n+1)$, and it is equal to $(-1)^n (n+1)! \neq 0$.

It remains to show that the right-hand derivative of f at $x = 0$ is zero. The exponential dominates the powers of $x > 0$, for example for all $m \in \mathbb{N}_0$ we have

$$\frac{1}{x^m} = x \left(\frac{1}{x} \right)^{m+1} \leq (m+1)! x \sum_{i=0}^{\infty} \frac{1}{i!} \left(\frac{1}{x} \right)^i = (m+1)! x e^{\frac{1}{x}},$$

and therefore

$$\lim_{x \searrow 0} \frac{e^{-\frac{1}{x}}}{x^m} \leq (m+1)! \lim_{x \searrow 0} x = 0. \tag{A.9}$$

Using (A.9) for $m = 1$, we see that

$$f'(0) = \lim_{x \searrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \searrow 0} \frac{e^{-\frac{1}{x}}}{x} = 0.$$

Since (A.8) is established for $x > 0$, the limit (A.9) for $m = 2n + 1$, with the assumption that $f^{(n)}(0) = 0$, implies

$$f^{(n+1)}(0) = \lim_{x \searrow 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x \searrow 0} \frac{P_{n-1}(x)}{x^{2n+1}} e^{-\frac{1}{x}} = P_{n-1}(0) \lim_{x \searrow 0} \frac{e^{-\frac{1}{x}}}{x^{2n+1}} = 0,$$

which by induction over n proves that $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}$, and therefore f is smooth.

The function f is not analytic (at zero), since $f^{(n)}(0) = 0$ holds for all $n \in \mathbb{N}_0$, and therefore the Maclaurin[17] series of f converges everywhere to the zero function, which cannot be equal to $f(x)$ for $x > 0$. In this way we obtain an example of smooth function that is not analytic. △

[17] Colin Maclaurin (1698–1746), Scottish mathematician

Lemma A.29 (Hadamard's lemma). *If $U \ni a$ is an open convex neighbourhood in $\mathbb{R}^n$, then for $f \in \mathfrak{F}(U)$ there exist functions $l_i \in \mathfrak{F}(U)$ for $1 \leq i \leq n$ such that for each $x \in U$ we have*

$$f(x) = f(a) + \sum_{i=1}^{n} (\pi_i(x) - \pi_i(a)) l_i(x),$$

where $l_i(a) = (\partial f/\partial \pi_i)(a)$.

Proof. If we define $h\colon [0,1] \to \mathbb{R}$ by $h(t) = f(a + t(x - a))$, then since

$$h'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial \pi_i}(a + t(x - a))(\pi_i(x) - \pi_i(a)),$$

we have

$$f(x) - f(a) = h(1) - h(0) = \int_0^1 h'(t)\, dt = \sum_{i=1}^{n} (\pi_i(x) - \pi_i(a)) \int_0^1 \frac{\partial f}{\partial \pi_i}(a + t(x - a))\, dt,$$

where

$$l_i(x) = \int_0^1 \frac{\partial f}{\partial \pi_i}(a + t(x - a))\, dt,$$

so the function l_i is smooth and $l_i(a) = (\partial f/\partial \pi_i)(a)$ holds. □

If $f\colon U \to V$ is a smooth function, where $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are open subsets of Euclidean spaces, then the (total) ***derivative*** of f at $p \in U$ is the linear map $f'(p)\colon \mathbb{R}^m \to \mathbb{R}^n$ such that for $v \in \mathbb{R}^m$ we have

$$(f'(p))(v) = \left.\frac{d}{dt}\right|_{t=0} f(p + tv).$$

Theorem A.30 (Inverse function theorem). *Let $f\colon U \to \mathbb{R}^n$ be a smooth function for an open $U \subseteq \mathbb{R}^n$. If $f'(p)$ is invertible for $p \in U$, then there exists a neighbourhood $V \subseteq U$ of p such that $f{\restriction}_V\colon V \to f(V)$ is a diffeomorphism.*

Proof. For example, see Lee [78, Theorem C.34]. □

A.7 Algebra

Let a set $\mathcal{V}$ has two binary operations $+$ and $\times$, and let 0 be the neutral element for $+$. The ***Jacobi identity*** is the equality given by

$$x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0, \tag{A.10}$$

for all $x, y, z \in \mathcal{V}$. The standard cross product $x \times y$ satisfies the Jacobi identity, as well as the commutator operation $[x, y] = x \times y - y \times x$. In a general case, we have the following lemma.

Lemma A.31. *In a submodule of an algebra over a ring that is closed to the commutator, the Jacobi identity* (A.10) *holds for the commutator operation.*

Proof. The Jacobi identity $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ can be easily checked by computation, since the direct substitution resulting twelve terms on the left hand side cancel in pairs, $xyz - xzy - yzx + zyx + yzx - yxz - zxy + xzy + zxy - zyx - xyz + yxz = 0$. □

It is interesting to consider a bilinear map $f\colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$. In general, the image $\operatorname{Im} f \subseteq \mathbb{R}^n$ is not a vector subspace. However, if it is, then the following inequalities apply, as stated in the theorem given below due to Howard[18] in 1980 [69] (see also [98] for a simpler proof).

Theorem A.32. *If $f\colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ is bilinear and surjective, then $n \leq 2m - 1$. Additionally, if f is skew-symmetric, then $n \leq 2m - 3$.*

A.8 Number theory

Lemma A.33 (Dirichlet's approximation theorem). *For every $c \in \mathbb{R}$ and $N \in \mathbb{N}$, there exist $m, n \in \mathbb{N}_0$ with $n \leq N$ such that $|nc - m| \leq 1/N$.*

Proof. Consider the numbers $pc - \lfloor pc \rfloor \in [0, 1)$ for integers $0 \leq p \leq N$. If we split the interval $[0, 1)$ into N subintervals of equal lengths, according to the pigeonhole principle, two of our numbers $0 \leq p < q \leq N$ must be in the same subinterval from where we obtain $|(qc - \lfloor qc \rfloor) - (pc - \lfloor pc \rfloor)| < 1/N$. Finally, we put $n = q - p$ and $m = \lfloor qc \rfloor - \lfloor pc \rfloor$. □

[18] Ralph Elwood Howard (1950), American mathematician

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