MATEMATIKA 1– fizička hemija

NEODREDJENI INTEGRAL

Kažemo da je funkcija $F : X \to \mathbb{R}$, $X \subset \mathbb{R}$ primitivna funkcija funkcije $f : X \to \mathbb{R}$ ako je $F'(x) = f(x)$, $x \in X$, i pišemo $\int f(x)dx = F(x) + C$, $C = const.$

1. Osnovna svojstva

   (a) $d\left(\int f(x)\,dx\right) = f(x)\,dx$

   (b) $\int df(x) = f(x) + C$

   (c) $\int \lambda f(x)\,dx = \lambda \int f(x)\,dx,\ \lambda \in \mathbb{R}/\{0\}$

   (d) $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

2. Tablica osnovnih integrala

   (a) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$

   (b) $\int \frac{dx}{x} = \ln|x| + C$

   (c) $\int \frac{dx}{1+x^2} = \arctan x + C$

   (d) $\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

   (e) $\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C \\ -\arccos x + C \end{cases}$

   (f) $\int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln\left|x + \sqrt{x^2 \pm 1}\right| + C$

   (g) $\int a^x\,dx = \frac{a^x}{\ln a} + C,\ a > 0, a \neq 1$

   (h) $\int e^x\,dx = e^x + C$

   (i) $\int \sin x\,dx = -\cos x + C$

   (j) $\int \cos x\,dx = \sin x + C$

   (k) $\int \frac{dx}{\sin^2 x} = -\cot x + C$

   (l) $\int \frac{dx}{\cos^2 x} = \tan x + C$

   Primeri:

   1) $\int(\sqrt{x}+1)(x-\sqrt{x}+1)\,dx = \int\left(\sqrt{x^3}-1\right)\,dx = \int x^{3/2}\,dx + \int\,dx = \frac{2}{5}x^{5/2} + x + C$

   2) $\int(6x^2+8x+3)\,dx = 6\int x^2\,dx + 8\int x\,dx + 3\int\,dx = 2x^3 + 4x^2 + 3x + C$

3. Integracija prethodnim svodjenjem na oblik diferencijala

   Ako je $\int f(x)\,dx = F(x) + C$, $x \in X$ i $x = \varphi(t)$, $\varphi : Y \to \mathbb{R}$, φ - neprekidna i diferencijabilna, tada je $\int f(\varphi(t)) \cdot \varphi'(t)\,dt = F(\varphi(t)) + C$. Specijalno, $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$.

   Primeri:

   1) $\int \frac{dx}{x-a} = \ln|x-a| + C$

   2) $\int \frac{dx}{(x-a)^n} = \frac{1}{1-n}(x-a)^{1-n}$

   3) $\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \arcsin\frac{x}{a} + C$

   4) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \int \frac{d\left(\frac{x}{a}\right)}{\sqrt{\left(\frac{x}{a}\right)^2 \pm 1}} = \ln\left|x + \sqrt{\left(\frac{x}{a}\right) \pm 1}\right| + C_0 = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C,\ C = C_0 - \ln|a|$

   5) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\int \frac{d\left(\frac{x}{a}\right)}{1+\left(\frac{x}{a}\right)^2} = \frac{1}{a}\arctan\frac{x}{a} + C$

   6) $\int \frac{dx}{x^2-a^2} = \frac{1}{a}\int \frac{d\left(\frac{x}{a}\right)}{\left(\frac{x}{a}\right)^2 - 1} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

4. Parcijalna integracija

$u,\ v$ - diferencijabilne funkcije: $\int u\,dv = uv - \int v\,du$

Primeri:

1) $\int x\ln x\,dx = \left\{\begin{array}{c} u=\ln x \ \Rightarrow\ du=\frac{1}{x}\,dx \\ dv = x\,dx \ \Rightarrow\ v=\frac{x^2}{2}\end{array}\right\} = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$

2) $\int x\sin x dx = \left\{\begin{array}{c} u=x \ \Rightarrow\ du=dx \\ dv=\sin x\,dx \ \Rightarrow\ v=-\cos x\end{array}\right\} = -x\cos x + \int \cos x dx = -x\cos x + \sin x + C$

3) $I = \int e^x\cos x\,dx = \left\{\begin{array}{c} u=\cos x \ \Rightarrow\ du=-\sin x\,dx \\ dv=e^x\,dx \ \Rightarrow\ v=e^x\end{array}\right\} = e^x\cos x + \int e^x \sin x\,dx =$

$= \left\{\begin{array}{c} u=\sin x \ \Rightarrow\ du=\cos x\,dx \\ dv=e^x\,dx \ \Rightarrow\ v=e^x\end{array}\right\} = e^x\cos x + e^x\sin x - \int e^x\cos x\,dx = e^x(\cos x+\sin x) - I$

$\Rightarrow\ 2I = e^x(\cos x+\sin x)\ \Rightarrow\ I = \frac{e^x}{2}(\cos x+\sin x) + C$

4) $I_n = \int \frac{dx}{(x^2+a^2)^n} = \left\{\begin{array}{c} u=\frac{1}{(x^2+a^2)^n} \ \Rightarrow\ du=\frac{-2nx}{(x^2+a^2)^{n+1}}\,dx \\ dv=dx \ \Rightarrow\ v=x\end{array}\right\} = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}dx =$

$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{dx}{(x^2+a^2)^n} - 2na^2\int \frac{dx}{(x^2+a^2)^{n+1}} = \frac{x}{(x^2+a^2)^n} + 2nI_n - 2na^2 I_{n+1}$

$\Rightarrow\ I_{n+1} = \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2na^2}I_n,\ n\geq 1,\ \ I_1 = \frac{1}{a}\arctan\frac{x}{a} + C$

5. Smena promenljive

a) $x=\varphi(t),\ t$ - nova promenljiva, φ - ima neprekidan izvod po t i $\varphi'(t)\neq 0$
$\int f(x)\,dx = \int f(\varphi(t))\varphi'(t)\,dt$

Primeri (trigonometrijske smene):
1) $\int\sqrt{a^2-x^2}\,dx$ smena: $x = a\cos t$
2) $\int\sqrt{x^2-a^2}\,dx$ smena: $x=\frac{a}{\cos t}$
3) $\int\sqrt{a^2+x^2}\,dx$ smena: $x = a\tan t$

b) $u=\psi(x),\ f(x)\,dx = g(u)\,du$
$\int g(u)du = F(u)+C\ \Rightarrow \int f(x)\,dx = F(\psi(x))+C$

Primer:
$\int\frac{dx}{\sqrt{5x-2}} = \{smena:\ u=5x-2\} = \int\frac{\frac{1}{5}\,du}{\sqrt{u}} = \frac{2}{5}\sqrt{u}+C = \frac{2}{5}\sqrt{5x-2}+C$

6. Integracija racionalnih funkcija

$R(x) = \frac{P(x)}{Q(x)} = T(x) + \frac{r(x)}{Q(x)},\quad P,Q,T,r$ - polinomi i $\deg r < \deg Q$
$\deg Q = n\ \Rightarrow\ Q$ - ima tačno n- nula (prostih ili višestrukih, realnih ili kompleksnih)
$Q(x) = \lambda_0(x-a_1)^{k_1}(x-a_2)^{k_2}\ldots(x-a_p)^{k_p}(x^2+b_1x+c_1)^{l_1}(x^2+b_2x+c_2)^{l_2}\ldots(x^2+b_qx+c_q)^{l_q}$
$k_1+k_2+\ldots+k_p+2(l_1+l_2+\ldots+l_q)$
$R(x)=\frac{A}{(x-a)^k}\ \wedge\ R(x) = \frac{Bx+C}{(x^2+bx+c)^l}$ - proste racionalne funkcije
$\frac{r(x)}{Q(x)} = \left(\frac{A_{11}}{x-a_1}+\frac{A_{12}}{(x-a_1)^2}+\ldots+\frac{A_{1k_1}}{(x-a_1)^{k_1}}\right)+\ldots+\left(\frac{A_{p1}}{x-a_p}+\frac{A_{p2}}{(x-a_p)^2}+\ldots+\frac{A_{pk_p}}{(x-a_p)^{k_p}}\right)+$
$+\left(\frac{B_{11}x+C_{11}}{x^2+b_1x+c_1}+\ldots+\frac{B_{1l_1}x+C_{1l_1}}{(x^2+b_1x+c_1)^{l_1}}\right)+\ldots+\left(\frac{B_{q1}x+C_{q1}}{x^2+b_qx+c_q}+\ldots+\frac{B_{ql_q}x+C_{ql_q}}{(x^2+b_qx+c_q)^{l_q}}\right)$

Primeri:

1) $\int\frac{x^3+1}{x^3-5x^2+6x}\,dx = \int\left(1+\frac{5x^2-6x+1}{x^3-5x^2+6x}\right)\,dx$
$\frac{5x^2-6x+1}{x^3-5x^2+6x} = \frac{5x^2-6x+1}{x(x-2)(x-3)} = \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x-3} = \frac{A(x^2-5x+6)+B(x^2-3x))+C(x^2-2x)}{x(x-2)(x-3)} = \frac{(A+B+C)x^2+(-5A-3B-2C)x+6A}{x^3-5x^2+6x}$

$$\begin{aligned} A + \quad B + C &= 5 \\ 5A + 3B + 2C &= 6 \quad \Rightarrow\ A = \tfrac{1}{6},\ B = -\tfrac{9}{2},\ C = \tfrac{28}{3} \\ 6A \qquad\qquad &= 1 \end{aligned}$$

$\int \frac{x^3+1}{x^3-5x^2+6x} dx = \int \left(1 + \frac{1}{6x} - \frac{9}{2(x-2)} + \frac{28}{3(x-3)}\right) dx = x + \frac{1}{6}\ln|x| - \frac{9}{2}\ln|x-2| + \frac{28}{3}\ln|x-3| + C$

2) $\int \frac{x}{x^3-3x+2}\, dx$

$\frac{x}{x^3-3x+2} = \frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+2} = \frac{(B+C)x^2+(A+B-2C)x+(2A-@B+C)}{x^3-3x+2}$

$$\begin{aligned} B + \quad C &= 0 \\ A + \quad B - 2C &= 1 \quad \Rightarrow\ A = \tfrac{1}{3},\ B = \tfrac{2}{9},\ C = -\tfrac{2}{9} \\ 2A + 2B + \quad C &= 0 \end{aligned}$$

$\int \frac{x^3+1}{x^3-5x^2+6x}\, dx = \frac{1}{3}\int \frac{dx}{(x-1)^2} + \frac{2}{9}\int \frac{dx}{x-1} - \frac{2}{9}\int \frac{dx}{x+2} = -\frac{1}{3(x-1)} + \frac{2}{9}\ln|x-1| - \frac{2}{9}\ln|x+2| + C$

3) $\int \frac{dx}{x^3+1}$

$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{(A+B)x^2+(-A+B+C)x+(A+C)}{x^3+1} \ \Rightarrow\ A = \frac{1}{3},\ B = -\frac{1}{3},\ C = \frac{2}{3}$

$\int \frac{dx}{x^3+1} = \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\, dx = \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{x-\frac{1}{2}}{x^2-x+1}\, dx + \frac{1}{2}\int \frac{dx}{x^2-x+1} =$

$= \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{6}\int \frac{d(x^2-x+1)}{x^2-x+1} + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left|x^2-x+1\right| + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C$

7. Integrali oblika $I_{mn} = \int \sin^m x\ \cos^n x\, dx,\ \ m, n \in \mathbb{N}$

1) $m = 2k+1,\ k \geq 0{:}\ \ I_{mn} = -\int \sin^{2k} x\ \cos^n x\, d(\cos x) = -\int (1-\cos^2 x)^k \cos^n x\, d(\cos x)$
Analogno za $n = 2k+1$

Primer:
$\int \sin^{10} x\ \cos^3 x\, dx = \int \sin^{10} x(1-\sin^2 x) d(\sin x) = \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C$

2) $m = 2k,\ n = 2l$
Transformacije: $\sin^2 x = \frac{1}{2}(1-\cos 2x),\ \cos^2 x = \frac{1}{2}(1+\cos 2x),\ \sin x\cos x = \frac{1}{2}\sin 2x$

Primer:
$\int \sin^4 x\ \cos^2 x\, dx = \int (\cos x \sin x)^2 \sin^2 x\, dx = \int \frac{1}{4}\sin^2 2x \cdot \frac{1}{2}(1-\cos 2x)\, dx =$
$= \frac{1}{8}\int (\sin^2 2x - \sin^2 2x\cos 2x)\, dx = \frac{1}{8}\int \left(\frac{1-\cos 4x}{2} - \sin^2 2x\cos 2x\right) =$
$= \frac{1}{16}\int\ dx - \frac{1}{16}\int \cos 4x\, dx - \frac{1}{16}\int \sin^2 2x\, d(\sin 2x) = \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C$

8. Integrali oblika $\int R(\sin x,\ \cos x)\, dx$,
$R = R(u, v)$ - racionalna funkcija, $u = \sin x,\ v = \cos x$

1) $R(-u, v) = -R(u, v)$ - smena: $t = \cos x$

Primer:
$\int \frac{dx}{\sin x} = \int \frac{\sin x}{1-\cos^2 x}\, dx = \int \frac{-dt}{1-t^2} = \ln\left|\frac{t-1}{t+1}\right| + C = \ln\left|\frac{\cos x-1}{\cos x+1}\right| + C$

2) $R(u, -v) = -R(u, v)$ - smena: $t = \sin x$
Primer:
$\int \frac{\cos x}{\sin^4 x}\, dx = \int \frac{dt}{t^4} = -\frac{1}{3}t^{-3} + C = -\frac{1}{3\sin^3 x} + C$

3) $R(-u, -v) = R(u, v)$ - smena: $t = \tan x,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\sin x = \frac{t}{\sqrt{1+t^2}},\ \cos x = \frac{1}{\sqrt{1+t^2}}$

Primer:
$\int \frac{dx}{\sin^4 x\ \cos^2 x} = \int \frac{\frac{1}{1+t^2}\, dt}{\frac{t^4}{(1+t^2)^2}\cdot\frac{1}{1+t^2}} = \int \frac{(1+t^2)^2}{t^4}\, dt = \int \left(t^{-4} + 2t^{-2} + 1\right)\, dt = -\frac{1}{3}t^{-3} - 2t^{-1} + t + C =$
$= -\frac{1}{3\tan^3 x} - \frac{2}{\tan x} + \tan x + C$

4) opšta smena: $t = \tan\frac{x}{2},\ x \in (-\pi, \pi),\quad dx = \frac{2\,dt}{1+t^2},\ \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2}$

Primer:
$\int \frac{dx}{1+\sin x+\cos x} = \int \frac{\frac{2}{1+t^2}\,dt}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} = \int \frac{dt}{t+1} = \ln|t+1| + C = \ln\left|\tan\frac{x}{2}+1\right| + C$

9. Integrali binomnih diferencijala $I_{mnp} = \int x^m\,(a+bx^n)^p\,dx,\ \ m,n,p \in \mathbb{Q}$

1) $p \in \mathbb{Z}$ - smena: $x = t^\lambda,\ \lambda$ - najmanji zajednički sadržalac imenilaca brojeva m, n

Primer:
$\int \sqrt{x}(1+\sqrt[3]{x})^{-1}\,dx = \int x^{\frac{1}{2}}\left(1+x^{\frac{1}{3}}\right)^{-1}\,dx$
$m = \frac{1}{2},\ n = \frac{1}{3},\ p = -1 \in \mathbb{Z},\ \lambda = NZS(2,3) = 6\ \Rightarrow$ smena: $x = t^6$
$\int \sqrt{x}(1+\sqrt[3]{x})^{-1}\,dx = \int t^3(1+t^2)^{-1}\cdot 6t^5\,dt = 6\int \frac{t^8}{1+t^2}\,dt = 6\int\left(t^6-t^4+t^2+1+\frac{1}{1+t^2}\right)dt =$
$= \frac{6}{7}t^7 - \frac{6}{5}t^5 + 2t^3 + 6t + 6\arctan t + C = \frac{6}{7}x^{\frac{7}{6}} - \frac{6}{5}x^{\frac{5}{6}} + 2x^{\frac{1}{2}} + 6x^{\frac{1}{6}} + 6\arctan\sqrt[6]{x} + C$

2) $\frac{m+1}{n} \in \mathbb{Z}$ - smena: $x = \left(\frac{t^\nu - a}{b}\right)^{\frac{1}{n}},\ \nu$ - imenilac broja p

Primer:
$\int \frac{x}{\sqrt{1+\sqrt[3]{x^2}}}\,dx = \int x\left(1+x^{\frac{2}{3}}\right)^{-\frac{1}{2}}\,dx$
$m = 1,\ n = \frac{2}{3},\ p = -\frac{1}{2},\ \frac{m+1}{n} = 3 \in \mathbb{Z},\ \nu = 2\ \Rightarrow$ smena: $x = (t^2-1)^{\frac{3}{2}}$
$\int \frac{x}{\sqrt{1+\sqrt[3]{x^2}}}\,dx = \int \frac{(t^2-1)^{\frac{3}{2}}}{t}\cdot 3t(t^2-1)^{\frac{1}{2}}\,dt = 3\int (t^2-1)^2\,dt = \frac{3}{5}t^5 - 2t^3 + 3t + C =$
$= \frac{3}{5}(x^{\frac{2}{3}}+1)^{\frac{5}{2}} - 2(x^{\frac{2}{3}}+1)^{\frac{3}{2}} + 3(x^{\frac{2}{3}}+1)^{\frac{1}{2}} + C$

3) $\frac{m+1}{n} + p \in \mathbb{Z}$ - smena: $x = \left(\frac{a}{t^\nu - b}\right)^{\frac{1}{n}},\ \nu$ - imenilac broja p

Primer:
$\int \sqrt[3]{3x-x^3}\,dx = \int x^{\frac{1}{3}}(3-x^2)^{\frac{1}{3}}$
$m = \frac{1}{3},\ n = 2,\ p = \frac{1}{3},\ \frac{m+1}{n} = 1 \in \mathbb{Z},\ \nu = 3\ \Rightarrow$ smena: $x = \left(\frac{3}{t^3+1}\right)^{\frac{1}{2}}$
$\int \sqrt[3]{3x-x^3}\,dx = \int \left(\frac{3}{t^3+1}\right)^{\frac{1}{6}}\left(3-\frac{3}{t^3+1}\right)^{\frac{1}{3}}\cdot\frac{-3\sqrt{3}}{2}\frac{t^2}{(t^3+1)^{\frac{3}{2}}}\,dt = -\frac{9}{2}\int\frac{t^3}{(t^3+1)^2}\,dt = \dots$

10. Integrali oblika $\int R(x,\ \sqrt{ax^2+bx+c})\,dx$ – Ojlerove smene
$R = R(u,v),\ ax^2+bx+c = 0$ - nema dvostruko rešenje

1) I Ojlerova smena: $a > 0$, smena: $\sqrt{ax^2+bx+c} = t \pm \sqrt{a}x$

Primer:
$\int \frac{dx}{x+\sqrt{x^2+x+1}} = \left\{smena:\ \sqrt{x^2+x+1} = t - x\right\} = \int \frac{-\frac{2(1-t+t^2)}{(1-2t)^2}}{t}\,dt =$
$= \int\left(-\frac{2}{t} - \frac{3}{(2t-1)^2} + \frac{3}{2t-1}\right)dt = -2\ln|t| + \frac{3}{2(2t-1)} + \frac{3}{2}\ln|2t-1| + C =$
$= -2\ln(x+\sqrt{x^2+x+1}) + \frac{3}{2(2x+2\sqrt{x^2+x+1}-1)} + \frac{3}{2}\ln(2x+2\sqrt{x^2+x+1}-1) + C$

2) II Ojlerova smena: $c > 0$, smena: $\sqrt{ax^2+bx+c} = xt \pm \sqrt{c}$

Primer:
$\int \frac{x\,dx}{1+\sqrt{1+x-x^2}} = \left\{smena:\ \sqrt{1+x-x^2} = tx - 1\right\} = \int 2\frac{1-t-t^2}{(1+t^2)^2}\frac{1}{t}\,dt =$
$= 2\int\frac{dt}{t} - \int\frac{dt}{1+t^2}\,dt - \int\frac{1+4t-t^2}{(1+t^2)^2}\,dt - \int\frac{2t}{1+t^2}\,dt = 2\ln t - \arctan t - \frac{t-2}{1+t^2} - \ln(1+t^2) + C =$
$= 1 - \sqrt{1+x-x^2} - \arctan\frac{1+\sqrt{1+x-x^2}}{x} + \ln(3-x+2\sqrt{1+x-x^2}) + C$

3) III Ojlerova smena: $ax^2+bx+c=a(x-\lambda)(x-\mu),\ \lambda\neq\mu,\ \lambda,\ \mu\in\mathbb{R}$,
smena: $\sqrt{ax^2+bx+c}=t(x-\lambda)\ \vee\ t(x-\mu)$

Primer:
$\int\sqrt{2x-x^2}\,dx=\left\{smena:\ \sqrt{2x-x^2}=tx\right\}=\int\frac{2t}{t^2-1}\frac{-4t}{(t^2-1)^3}\,dt=-8\int\frac{t^2}{(t^2-1)^4}\,dt=$
$=-8\int\frac{dt}{(t^2-1)^3}-8\int\frac{dt}{(t^2-1)^4}=-8\left(I_3+I_4\right)$

11. Integrali oblika $\int R\left(x,\ \left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)^{p_1/n_1},\ \ldots,\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)^{p_k/n_k}\right)\,dx$

Smena: $x=\frac{\delta t^n-\beta}{\alpha-\gamma t^n},\quad n=NZS(n_1,n_2,\ldots,n_k)$

Primer:
$\int\frac{dx}{\sqrt{2x-1}-\sqrt[4]{2x-1}}$
$\alpha=2,\ \beta=-1,\ \gamma=0,\ \delta=1,\ p_1=p_2=1,\ n_1=2,\ n_2=4\ \Rightarrow$ smena: $x=\frac{t^4+1}{2}$
$\int\frac{dx}{\sqrt{2x-1}-\sqrt[4]{2x-1}}=\int\frac{2t^3}{t^2-t}\,dt=2\int\frac{t^2}{t-1}\,dt=2\int\left(t+1+\frac{1}{t-1}\right)\,dt=t^2+2t+2\ln|t-1|+C=$
$=\sqrt{2x-1}+2\sqrt[4]{2x-1}+\ln\left|\sqrt[4]{2x-1}-1\right|+C$