ON THE SOLID HULL OF THE HARDY-LORENTZ SPACE

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Abstract. The solid hulls of the Hardy-Lorentz spaces $H^{p,q}$, $0 < p < 1$, $0 < q \leq \infty$ and $H^{p,\infty}_0$, $0 < p < 1$, as well as of the mixed norm space $H^{p,\infty,\alpha}_0$, $0 < p \leq 1$, $0 < \alpha < \infty$, are determined.

Introduction

In [JP1], we determined the solid hull of the Hardy space $H^p$, $0 < p < 1$, i.e., we found the best possible function $\phi(t_1, t_2, \ldots)$, $t_k \geq 0$ ($k \geq 1$) such that $\|f\|_{H^p} \leq c \phi(|\hat{f}(0)|, |\hat{f}(1)|, |\hat{f}(2)|, \ldots)$, where $c$ is a positive constant. In this note we determine the solid hulls of the Hardy-Lorentz spaces $H^{p,q}$, $0 < p < 1$, $0 < q \leq \infty$ and $H^{p,\infty}_0$, $0 < p < 1$, as well as of the mixed norm space $H^{p,\infty,\alpha}_0$, $0 < p \leq 1$, $0 < \alpha < \infty$. Since $H^{p,p} = H^p$ our results generalize [JP1, Theorem 1].

Recall, the Hardy space $H^p$, $0 < p \leq \infty$, is the space of all functions $f$ holomorphic in the unit disk $U$ (f $\in H(U)$) for which

$$\|f\|_p = \lim_{r \to 1} M_p(r, f) < \infty,$$

where, as usual,

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p}, \quad 0 < p < \infty,$$

and

$$M_\infty(r, f) = \sup_{0 \leq t < 2\pi} |f(re^{it})|.$$

Now we introduce a generalization and refinement of the spaces $H^p$, the Hardy-Lorentz spaces $H^{p,q}$ $0 < p < \infty$, $0 < q \leq \infty$.

Let $\sigma$ denotes normalized Lebesgue measure on $T = \partial U$ and let $L^0(\sigma)$ be the space of complex -valued Lebesgue measurable functions on $T$. For $f \in L^0(\sigma)$ and $s \geq 0$ we write

$$\lambda_f(s) = \sigma(\{x \in T : |f(x)| > s\})$$

Date: 20 August, 2006.
2000 Mathematics Subject Classification. 30D55, 42A45.

The research of the authors was supported by a grant from MNZS ON144010, Serbia.
for the distribution function and 
\[ f^*(s) = \inf \{ t \geq 0 : \lambda_f(t) \leq s \} \]
for the decreasing rearrangement of \(|f|\) each taken with respect to \(\sigma\).

The Lorentz functional \(||| \cdot |||_{p,q}\) is defined at \(f \in L^0(\sigma)\) by
\[ \|f\|_{p,q} = \left( \int_0^1 (f^*(s)s^{1/p})^q \frac{ds}{s} \right)^{1/q} \quad \text{for} 0 < q < \infty, \]
and
\[ ||f||_{p,\infty} = \sup \{ f^*(s)s^{1/p} : s \geq 0 \}. \]
The corresponding Lorentz space is
\[ L^{p,q}(\sigma) = \{ f \in L^0(\sigma) : ||f||_{p,q} < \infty \}. \]
The space \(L^{p,q}(\sigma)\) is separable if and only if \(q \neq \infty\). The class of functions \(f \in L^0(\sigma)\) satisfying \(\lim_{s \to 0}(f^*(s)s^{1/p}) = 0\) is a separable closed subspace of \(L^{p,\infty}(\sigma)\) which is denoted by \(L_0^{p,\infty}(\sigma)\).

The Nevanlinna class \(N\) is the subclass of functions \(f \in H(U)\) for which
\[ \sup_{0 < r < 1} \int_T \log^+ |f(r\xi)| d\sigma(\xi) < \infty. \]
Functions in \(N\) are known to have non-tangential limits \(\sigma - a.e.\) on \(T\). Consequently every \(f \in N\) determines a boundary value function which we also denote by \(f\). Thus
\[ f(\xi) = \lim_{r \to 1} f(r\xi) \quad \sigma - a.e. \quad \xi \in T. \]
The Smirnov class \(N^+\) is the subclass of \(N\) consisting of those functions \(f\) for which
\[ \lim_{r \to 1} \int_T \log^+ |f(r\xi)| d\sigma(\xi) = \int_T \log^+ |f(\xi)| d\sigma(\xi). \]

We define the Hardy-Lorentz space \(H^{p,q}, 0 < p < \infty, 0 < q \leq \infty\), to be the space of functions \(f \in N^+\) with boundary value function in \(L^{p,q}(\sigma)\) and we put \(||f||_{H^{p,q}} = ||f||_{p,q}\). The functions in \(H^{p,\infty}\) with boundary value function in \(L_0^{p,\infty}(\sigma)\) form a closed subspace of \(H^{p,\infty}\) which is denoted by \(H_0^{p,\infty}\). The cases of major interest are of course \(p = q = \infty\); indeed \(H^{p,q}\) is nothing but \(H^p\), and \(H^{p,\infty}\) is the weak-\(H^p\).

The mixed norm space \(H^{p,q,\alpha}, 0 < p \leq \infty, 0 < q, \alpha \leq \infty\), consists of all \(f \in H(U)\) for which
\[ ||f||_{H^{p,q,\alpha}} = ||f||_{p,q,\alpha} = \left( \int_0^1 (1 - r)^{q\alpha - 1} M_p(r, f) q dr \right)^{1/q} < \infty. \]
$H^{p,q,\alpha}$ can also be defined when $q = \infty$, in which case it is sometimes known as the weighted Hardy space $H^{p,\infty,\alpha}$, and consists of all $f \in H(U)$ for which
\[
||f||_{p,\infty,\alpha} = \sup_{0 < r < 1} (1 - r)^\alpha M_p(r, f) < \infty.
\]
The functions in $H^{p,\infty,\alpha}$ form a closed subspace which is denoted by $H^p_{\infty,\alpha}$.

Throughout this paper, we identify holomorphic function $f$ with its sequence of Taylor coefficients $\{\hat{f}(k)\}_{k=0}^\infty$.

If $f(z) = \sum_{k=0}^\infty \hat{f}(k) z^k$ belongs to $H^{p,q}$, then
\[
(1) \quad \hat{f}(k) = O((k + 1)^{(1/p) - 1}), \text{ if } 0 < p < 1 \text{ and } 0 < q \leq \infty.
\]
(See [Al] and [Co].)

In this paper we find the strongest condition that the moduli of an $H^{p,q}$, $0 < p < 1$, $0 < q \leq \infty$, satisfy. Our result shows that the estimate (1) is optimal only if $q = \infty$.

To state our results in the form of theorems we need to introduce some more notations.

A sequence space $X$ is solid if $\{b_n\} \subseteq X$ whenever $\{a_n\} \subseteq X$ and $|b_n| \leq |a_n|$. More generally, we define $S(X)$, the solid hull of $X$. Explicitly,
\[
S(X) = \{\{\lambda_n\} : \text{there exists } \{a_n\} \subseteq X \text{ such that } |\lambda_n| \leq |a_n|\}.
\]

A complex sequence $\{a_n\}$ is of class $l(p,q)$, $0 < p, q \leq \infty$, if
\[
||\{a_n\}||_{p,q} = ||\{a_n\}||_{l(p,q)}^q = \sum_{n=0}^\infty (\sum_{k \in I_n} |a_k|^p)^{q/p} < \infty,
\]
where $I_0 = \{0\}$, $I_n = \{k \in N : 2^{n-1} \leq k < 2^n\}$, $n = 1, 2, \ldots$. In the case where $p$ or $q$ is infinite, replace the corresponding sum by a supremum.

Note that $l(p,p) = l^p$.

For $t \in R$ we write $D^t$ for the sequence $\{(n + 1)^t\}$, for all $n \geq 0$. If $\lambda = \{\lambda_n\}$ is a sequence and $X$ a sequence space, we write $\lambda X = \{\{\lambda_n x_n\} : \{x_n\} \subseteq X\}$; thus, for example, $\{a_n\} \subseteq D^t l^\infty$ if and only if $|a_n| = O(n^t)$.

We are now ready to state our first result.

**Theorem 1.** If $0 < p < 1$ and $0 < q \leq \infty$, then $S(H^{p,q}) = D^{(1/p) - 1} l(\infty, q)$.

In particular, $S(H^p) = D^{(1/p) - 1} l(\infty, p)$, $0 < p < 1$. This was proved in [JP1]. Also, $S(H^{p,\infty}) = D^{(1/p) - 1} l^\infty$ means that the estimate (1) valid for the Taylor coefficients of an $H^{p,\infty}$ function, $0 < p < 1$, is sharp.
Our second result is as follows:

**Theorem 2.** If $0 < p < 1$, then $S(H_0^{p,\infty}) = D^{(1/p)-1}c_0$, where $c_0$ is the space of all null sequences.

Our method of proving Theorem 1 and Theorem 2 depend upon nested embedding [Le, Theorem 4.1] for Hardy-Lorentz spaces. Thus, the strategy is to trap $H^{p,q}$ between a pair of mixed norm spaces and then deduce the results for $H^{p,q}$ from corresponding results for mixed norm spaces. Our Theorem 1 will follow from the following two theorems:

**Theorem L [Le].** Let $0 < p_0 < p < s \leq \infty$, $0 < q \leq t \leq \infty$ and $\beta > (1/p_0) - (1/p)$. Then

(2) $D^{-\beta}H^{p_0,q,\beta+(1/p)-1/p_0} \subset H^{p,q} \subset H^{s,q,1/s -(1/s)}$,

and

(3) $D^{-\beta}H_0^{p_0,\infty,\beta+(1/p)-1/p_0} \subset H_0^{p,\infty} \subset H_0^{s,\infty,1/s -(1/s)}$.

**Theorem JP [JP1].** If $0 < p \leq 1$, $0 < q \leq \infty$ and $0 < \alpha < \infty$, then $S(H^{p,q,\alpha}) = D^{\alpha+(1/p)-1}(\infty,q)$.

To prove Theorem 2 we first determine the solid hull of the space $H_0^{p,\infty,\alpha}$, $0 < p \leq 1$, $0 < \alpha < \infty$. More precisely, we prove

**Theorem 3.** If $0 < p \leq 1$ and $0 < \alpha < \infty$, then $S(H_0^{p,\infty,\alpha}) = D^{\alpha+(1/p)-1}c_0$.

Given two vector spaces $X,Y$ of sequences we denote by $(X,Y)$ the space of multipliers from $X$ to $Y$. More precisely,

$$(X,Y) = \{ \lambda = \{ \lambda_n \} : \{ \lambda_n a_n \} \in Y, \text{for every} \{ a_n \} \in X \}.$$ 

As an application of our results we calculate multipliers $(H^{p,q}, l(u,v))$, $0 < p < 1$, $0 < q \leq \infty$, $(H_0^{p,\infty}, l(u,v))$, $0 < p < 1$, and $(H_0^{p,\infty}, X)$, $0 < p < 1$, where $X$ is a solid space. These results extend some of the results obtained by M. Lengfield [Le, Section 5].

1. **The solid hull of the Hardy-Lorentz space $H^{p,q}$**, $0 < p < 1$, $0 < q \leq \infty$

**Proof of Theorem 1.**

*Proof.* Let $0 < p < 1$. Chose $p_0$ and $s$ so that $p_0 < p < s \leq 1$ and a real number $\beta$ so that $\beta + (1/p) - (1/p_0) > 0$. As an easy consequence of Theorem JP we have

$S(D^{-\beta}H^{p_0,q,\beta+(1/p)-1/p_0}) = D^{(1/p)-1}(\infty,q)$. 
Also, by Theorem JP,
\[ S(H^{s,q,(1/p)-(1/s)}) = D^{(1/p)-1}l(\infty,q), \]
and consequently
\[ S(H^p,q) = D^{(1/p)-1}l(\infty,q), \]
by Theorem L. □

2. The solid hull of mixed norm space \( H_0^{p,\infty,\alpha} \), \( 0 < p \leq 1 \), \( 0 < \alpha < \infty \)

If \( f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k \) and \( g(z) = \sum_{k=0}^{\infty} \hat{g}(k)z^k \) are holomorphic functions in \( U \), then the function \( f \ast g \) is defined by \( (f \ast g)(z) = \sum_{k=0}^{\infty} \hat{f}(k)\hat{g}(k)z^k \).

The main tool for proving Theorem 3 are polynomials \( W_n \), \( n \geq 0 \), constructed in [JP1] and [JP3]. Recall the construction and some of their properties.

Let \( \omega : R \rightarrow R \) be a nonincreasing function of class \( C^\infty \) such that \( \omega(t) = 1 \), for \( t \leq 1 \), and \( \omega(t) = 0 \), for \( t \geq 2 \). We define polynomials \( W_n = W_n^\omega \), \( n \geq 0 \), in the following way:

\[ W_0(z) = \sum_{k=0}^{\infty} \omega(k)z^k \quad \text{and} \quad W_n(z) = \sum_{k=2^{n-1}}^{2^n-1} \varphi\left(\frac{k}{2^n-1}\right)z^k, \text{for} \ n \geq 1, \]

where \( \varphi(t) = \omega(t/2) - \omega(t), \ t \in R \).

The coefficients \( \hat{W}_n(k) \) of these polynomials have the following properties:

\[ \text{(4)} \quad \text{supp} \{ \hat{W}_n \} \subset [2^{n-1}, 2^{n+1}]; \]
\[ \text{(5)} \quad 0 \leq \hat{W}_n(k) \leq 1, \text{ for all } k, ; \]
\[ \text{(6)} \quad \sum_{n=0}^{\infty} \hat{W}_n(k) = 1, \text{ for all } k, \]
\[ \text{(7)} \quad \hat{W}_n(k) + \hat{W}_{n+1}(k) = 1, \text{ for } 2^n \leq k \leq 2^{n+1}, n \geq 0. \]

The property (5) implies that
\[ f(z) = \sum_{n=0}^{\infty} (W_n \ast f)(z), \quad f \in H(U), \]
the series being uniformly convergent on compact subsets of \( U \).
If $0 < p < 1$ then there exists a constant $C > 0$ depending only on $p$ such that
\begin{equation}
\|W_n\|_p^p \leq C p 2^{-n(1-p)} \quad n \geq 0.
\end{equation}

**Proof of Theorem 3.**

**Proof.** Let $f \in H^{p,\infty,\alpha}_0$, $0 < p < 1$, $0 < \alpha < \infty$. By using the familiar inequality
\[ M_p(r, f) \geq C (1 - r)^{(1/p) - 1} M_1(r^2, f), \quad 0 < p \leq 1, \]
(see [Du, Theorem 5. 9]), we obtain
\begin{align*}
\sup_{k \in \mathbb{I}} |\hat{f}(k)|^r 2^k \leq M_1(r^2, f) \leq CM_p(r, f) (1 - r)^{1-(1/p)}, 0 < r < 1.
\end{align*}

Now we take $r_n = 1 - 2^{-n}$ and let $n \to \infty$, to get \{\hat{f}(k)\} $\in D^{\alpha+(1/p)-1} c_0$. Thus $H^{p,\infty,\alpha}_0 \subset D^{\alpha+(1/p)-1} c_0$.

To show that $D^{\alpha+(1/p)-1} c_0$ is the solid hull of $H^{p,\infty,\alpha}_0$, it is enough to prove that if \{\{a_n\}\} $\in D^{\alpha+(1/p)-1} c_0$, then there exists \{\{b_n\}\} $\in H^{p,\infty,\alpha}_0$ such that $|b_n| \geq |a_n|$, for all $n$.

Let \{\{a_n\}\} $\in D^{\alpha+(1/p)-1} c_0$. Define
\[ g(z) = \sum_{j=0}^{\infty} B_j(W_j(z) + W_{j+1}(z)) = \sum_{k=0}^{\infty} c_k z^k, \]
where $B_j = \sup_{2^j \leq k < 2^{j+1}} |a_k|$. Using (4) and (8)we find that
\begin{align*}
M^p_p(r, g) &\leq \sum_{j=0}^{\infty} B^p_j \left( M^p_p(r, W_j) + M^p_p(r, W_{j+1}) \right) \\
&\leq C \left( B_0^p + \sum_{j=1}^{\infty} B^p_j r^{2^{j-1}2^{j(1-p)}} \right)
\end{align*}

Set $B^p_j 2^{-j(\alpha p+1-p)} = \lambda_j$. Then
\begin{align*}
M^p_p(r, g) &\leq C(\lambda_0 + \sum_{j=1}^{\infty} \lambda_j r^{2^{j-1}2^{j\alpha p}}),
\end{align*}

where $\lambda_j \to 0$, as $j \to \infty$. From this easily follows that $(1-r)^{\alpha p} M^p_p(r, g) \to 0$, as $r \to 1$. Thus $g \in H^{p,\infty,\alpha}_0$. 


To prove that $|c_k| \geq |a_k|$, $k = 1, 2, \ldots$, choose $n$ so that $2^n \leq k < 2^{n+1}$. It follows from (7)

$$c_k = \sum_{j=0}^{\infty} B_j(\hat{W}_j(k) + \hat{W}_{j+1}(k)) \geq B_n(\hat{W}_n(k) + \hat{W}_{n+1}(k))$$

$$= B_n = \sup_{2^n \leq j < 2^{n+1}} |a_j| \geq |a_k|.$$ 

Now the function $h(z) = \sum_{n=0}^{\infty} b_n z^n$, where $b_0 = a_0$ and $b_n = c_n$, for $n \geq 1$, belongs to $H^{p,\infty,\alpha}_0$ and $|b_n| \geq |a_n|$ for all $n \geq 0$. This finishes the proof of Theorem 3. \hfill \Box

3. The solid hull of the space $H^{p,\infty}_0$, $0 < p < 1$

Proof of Theorem 2.

Proof. Let $0 < p < 1$. Chose $p_0$ and $s$ so that $p_0 < p < s \leq 1$ and $\beta \in \mathbb{R}$ so that $\beta + (1/p) - (1/p_0) > 0$. Then

$$S(D^{-\beta} H^{p_0,\infty,\beta+(1/p)-(1/p_0)}_0) = D^{(1/p)-1} c_0,$$

and

$$S(H^{p,\infty,(1/p)-(1/s)}_0) = D^{(1/p)-1} c_0,$$

by Theorem 3. By Theorem L we have

$$S(H^{p,\infty}_0) = D^{(1/p)-1} c_0.$$ \hfill \Box

4. Applications to multipliers

As it was noticed in Introduction another objective of this paper is to extend some of the results given in [Le, section 5].

The next lemma due to Kellog (see [K]) (who states it for exponents no smaller than 1, but it then follows for all exponents, since $\{\lambda_n\} \in (l(a, b), l(c, d))$ if and only if $\{\lambda_n^{(1/t)}\} \in (l(at, bt), l(ct, dt))$).

Lemma 1. If $0 < a, b, c, d \leq \infty$, then

$$(l(a, b), l(c, d)) = l(a \odot c, b \odot d),$$

where $a \odot c = \infty$ if $a \leq c$, $b \odot d = \infty$, if $b \leq d$, and

$$\frac{1}{a \odot c} = \frac{1}{c} - \frac{1}{a}, \text{ for } 0 < c < a,$$

$$\frac{1}{b \odot d} = \frac{1}{d} - \frac{1}{b}, \text{ for } 0 < d < b.$$
In particular, \((l^\infty, l(u,v)) = l(u,v)\). Also, it is known that \((c_0, l(u,v)) = l(u,v)\).

In [AS] it is proved that if \(X\) is any solid space and \(A\) any vector space of sequences then \((A, X) = (S(A), X)\).

Since \(l(u,v)\) are solid spaces, we have \((H^{p,q}, l(u,v)) = (S(H^{p,q}), l(u,v))\) and \((H_0^{p,\infty}, l(u,v)) = (S(H_0^{p,\infty}), l(u,v))\). Using this, Lemma 1, Theorem 1 and Theorem 2 we get

**Theorem 4.** Let \(0 < p < 1\) and \(0 < q \leq \infty\). Then
\[
(H^{p,q}, l(u,v)) = D^{1/(1/p)} l(u, q \odot v).
\]

**Theorem 5.** Let \(0 < p < 1\). Then
\[
(H_0^{p,\infty}, l(u,v)) = D^{1/(1/p)} l(u,v).
\]

In particular, \((H^{p,\infty}, l(u,v)) = D^{1/(1/p)} l(u,v)\). In fact more is true.

**Theorem 6.** Let \(0 < p < 1\) and let \(X\) be a solid space. Then
\[
(H^{p,\infty}, X) = D^{1/(1/p)} X.
\]

**Proof.** Since \(X\) is a solid space, we have \((l^\infty, X) = X\). Hence, using Theorem 1 we get
\[
(H^{p,\infty}, X) = (S(H^{p,\infty}), X) = (D^{1/(1/p)} l^\infty, X) = D^{1/(1/p)} (l^\infty, X) = D^{1/(1/p)} X.
\]

This theorem improves Theorem 5.2 of [Le], where it is assumed that \(X \subset H(U)\) is an \(F\)-space, while we do not require any topological condition on \(X\).

**References**


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