EQUIVALENT NORMS ON DIRICHLET SPACES OF POLYHARMONIC FUNCTIONS ON THE BALL IN $\mathbb{R}^N$

OLIVERA DJORDJEVIĆ AND MIROSLAV PAVLOVIĆ

Abstract. We define $D^p_{α,k}(B)$, where $B$ is the unit ball in $\mathbb{R}^N$, to be the class of those polyharmonic functions $f$ of order $k$ on $B$ for which

$$|f(0)| + \left( \int_B |\nabla f(x)|^p (1 - |x|^2)^α \,dV(x) \right)^{1/p} < \infty,$$

and we present four equivalent norms on $D^p_{α,k}(B)$. We also consider some equivalent norms on Bloch type spaces.

1. Introduction

Let $\mathbb{R}^N (N \geq 2)$ denote the $N$-dimensional Euclidean space. A (real-valued) function $f$ defined in a domain $G \subset \mathbb{R}^N$ is said to be polyharmonic of order $k \geq 1$ if $\Delta^k f \equiv 0$ in $G$, where $\Delta$ denotes the ordinary Laplacian. The class of all such functions is denoted by $H_k(G)$. In this note we consider the case $G = B$, where $B = B_N$ is the unit ball centered at the origin. By the Almansi representation theorem [1], if $f \in H_k(B)$, then there exist unique harmonic functions $A_m f$ such that

$$f(x) = \sum_{m=0}^{k-1} (1 - |x|^2)^m A_m f(x), \quad x \in B. \quad (1.1)$$

Conversely, if $f_m \in H(B) := H_1(B)$, then the function $f(x) = \sum_{m=0}^{k-1} (1 - |x|^2)^m f_m(x)$ is polyharmonic of order $k$.

For $0 < p < \infty$, and $α > -1$, we define the Dirichlet type space $D^p_{α,k}(B)$ to be the class of $f \in H_k(B)$ for which

$$\|f\|_{D^p_{α,k}} := |f(0)| + \left( \int_B |\nabla f(x)|^p (1 - |x|^2)^α \,dV(x) \right)^{1/p} < \infty,$$

where $dV$ is the normalized Lebesgue measure on $B$, and $\nabla f$ is the gradient of $f$,

$$\nabla f = (D_1 f, \ldots, D_N f),$$

$$|x|^2 = x_1^2 + \ldots + x_N^2,$$

and

$$D_i f(x) = \frac{\partial f}{\partial x_i}.$$
In this note we present some equivalent norms on $D_p^\alpha$. For a function $f \in C^1(B)$ we denote by $Rf$ the radial derivative of $f$,

$$Rf(x) = \sum_{i=1}^N x_i D_i f(x), \quad x = (x_1, \ldots, x_N),$$

and

$$R_s f = sf + Rf, \quad s \geq 0.$$ 

Thus $R_0 = R$.

The main results of our paper are the following theorems.

**Theorem (1.2).** For $0 < p < \infty$, $\alpha > -1$, and $k \geq 2$, the following quantities are equivalent norms on $D_p^\alpha_k(B)$:

$$Q_1(f) = |f(0)| + \left( \int_B |Rf(x)|^p (1 - |x|^2)^\alpha dV(x) \right)^{1/p},$$

$$Q_2(f) = \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{i,j=1}^N \left( \int_B |T_{ij} f(x)|^p (1 - |x|^2)^\alpha dV(x) \right)^{1/p}.$$ 

Our second result concerns two equivalent norms which involve the “Almansi coordinates”.

**Theorem (1.3).** For $0 < p < \infty$, $\alpha > -1$, and $k \geq 2$, the following quantities are equivalent norms on $D_p^\alpha_k(B)$:

$$Q_3(f) = \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{m=0}^{k-1} \left( \int_B |\nabla A_m f(x)|^p (1 - |x|^2)^\alpha + m^p dV(x) \right)^{1/p},$$

$$Q_4(f) = \|A_0 f\|_{D_p^\alpha} + \sum_{m=1}^{k-1} \left( \int_B |A_m f(x)|^p (1 - |x|^2)^\alpha + (m-1)^p dV(x) \right)^{1/p}.$$ 

Note that $Q_3$ can be written as

$$Q_3(f) = \sum_{m=0}^{k-1} \|A_m f\|_{D_p^\alpha + m^p},$$

while

$$Q_4(f) = \|A_0 f\|_{D_p^\alpha} + \sum_{m=0}^{k-1} \|A_m f\|_{L_p^{\alpha + (m-1)p}},$$

where

$$\|f\|_{L_p^{\alpha}} = \left( \int_B |f(x)|^p (1 - |x|^2)^\alpha dV(x) \right)^{1/p}.$$ 

(We will use the expression (1.4) even if $f$ is a vector-valued function.) Thus the equivalence $\|f\|_{D_p^\alpha} \asymp Q_3(f)$ means that the space $D_p^\alpha_{a,k}$ is the direct sum of the harmonic spaces $D_p^\alpha + m^p$, $m = 0, \ldots, k-1$, while the equivalence $\|f\|_{D_p^\alpha} \asymp Q_4(f)$ means that $D_p^\alpha_{a,k}$ is the direct sum of the harmonic space $D_p^\alpha$ and the harmonic Bergman spaces $\mathcal{H}(B) \cap L_p^{\alpha + (m-1)p}$, $m = 1, \ldots k - 1$. 

The equivalence $\|f\|_{D^p_\alpha} \asymp Q_1(f)$ can be viewed as a generalization of a theorem of Hardy and Littlewood on harmonic conjugates [3, Theorem 5], which we state as follows:

**Theorem (A).** Let $F$ be a function holomorphic in the unit disk $\mathbb{D} := B_2 \subset \mathbb{C} = \mathbb{R}^2$. If the function $\left( \partial / \partial r \right)(\text{Re} F)(re^{i\theta})$ belongs to $L^p_\alpha(\mathbb{D})$, where $p > 0$, $\alpha > -1$, then so does the derivative $F'$.

Note that in this situation, $R(\text{Re} f) = r \left( \partial / \partial r \right)(\text{Re} F)$ and $|F'| = |\nabla (\text{Re} f)|$, and so Theorem (A) coincides with the equivalence $\|f\|_{D^p_\alpha} \asymp Q_1(f)$ for $N = 2$, $k = 1$.

### 2. Some formulas

The tangential derivatives $T_{ij}f$ ($1 \leq i, j \leq N$) are defined by

$$T_{ij}f(x) = x_i D_j f(x) - x_j D_i f(x).$$

If $f \in C^2(B)$, then there hold the formulas:

(2.1) $$D_1 Rf = D_1 f + R D_1 f = R_1 D_1 f;$$

(2.2) $$D_i T_{ij} f = D_j f + T_{ij} D_i f,$$

$$D_j T_{ij} f = -D_i f + T_{ij} D_j f,$$

$$D_k T_{ij} f = T_{ij} D_k f \quad (k \neq i, k \neq j);$$

(2.3) $$RRf = Rf + \sum_{i,j=1}^{N} x_i x_j D_i D_j f;$$

$$T_{ij} T_{ij} f = x_i \delta_{ij} D_i f + x_i^2 D_j D_i f - x_i D_i f - x_i x_j D_i D_j f - x_j D_i f - x_i x_j D_j D_i f + x_j \delta_{ij} D_j f + x_j^2 D_i D_j f;$$

where $\delta_{ij}$ is the Kronecker delta. From this and (2.3) we get

(2.4) $$RRf + \frac{1}{2} \sum_{i,j=1}^{N} T_{ij} T_{ij} f = (2 - N)Rf + |x|^2 \Delta f,$$

where $\Delta$ is the ordinary Laplacian,

$$\Delta f = \sum_{i=1}^{N} D_i D_i f.$$

By successive application of (2.1) and (2.2) we get

$$\Delta^k Rf = 2k \Delta^k f + R \Delta^k f,$$

$$\Delta T_{ij} f = T_{ij} \Delta f.$$

As a consequence we get the well known facts:

**Proposition (A).** If $f$ is in $\mathcal{H}_k(B)$, then so are $Rf$ and $T_{ij} f$ ($1 \leq i, j \leq N$).

Using the fact that $T_{ij}$ annihilates radial functions, we get:
PROPOSITION (B). If $f$ is given by (1.1), then

$$T_{ij}f(x) = \sum_{m=0}^{k-1} (1 - |x|^2)^m T_{ij}A_m f(x), \quad x \in B.$$  \hspace{1cm} (2.5)

This provides another proof that $T_{ij}$ preserves $\mathcal{H}_q(B)$.

(We write $A(s) \asymp B(s)$ to denote that $A(s)/B(s)$ lies between two positive constants independent of $s$.)

3. Subharmonic behavior

The class $QNS(B)$. Let $QNS(B)$ denote the class of non-negative measurable functions $u$ on $B$ for which there exists a constant $Q = Q(u)$ such that

$$u(a) \leq Q\varepsilon^{-N} \int_{B(a, \varepsilon)} u \, dV$$  \hspace{1cm} (3.1)

whenever

$$B(a, \varepsilon) := \{x: |x - a| < \varepsilon\} \subset B.$$ Members of $QNS(B)$ are called quasi-nearly subharmonic functions [9, 10]. The class $QNS$ contains non-negative subharmonic functions. Observe that (3.1) implies

$$\sup_{B(a, \varepsilon)} u \leq C \varepsilon^{-N} \int_{B(a, 2\varepsilon)} u \, dV, \quad B(a, 2\varepsilon) \subset B.$$  \hspace{1cm} (3.2)

THEOREM (B). [5, 10] Let $p > 0$. If $u \in QNS(B)$, then $u^p \in QNS(B)$, and $Q(u^p) \leq C_p Q(u)$.

The class $HC^1(G)$. This class consists of all locally Lipschitz functions $f$ on $B$ for which there exists a constant $Q' = Q'(f)$ such that

$$|\nabla f(a)| \leq \frac{Q'}{\varepsilon} \sup_{B(a, \varepsilon)} |f|, \quad \text{whenever } B(a, \varepsilon) \subset B.$$  \hspace{1cm} (3.3)

Note that a locally Lipschitz function is differentiable almost everywhere and in particular the gradient is defined almost everywhere. If $\nabla f(a)$ does not exist, then we interpret $|\nabla f(a)|$ as

$$|\nabla f(a)| = \limsup_{x \to a} \frac{|f(x) - f(a)|}{|x - a|}.$$ $\nabla f(a)$ is a constant.

The class $OC^1(B)$. This is the subclass of $HC^1(B)$ consisting of those $f$ for which

$$|\nabla f(a)| \leq \frac{Q''}{\varepsilon} \sup_{B(a, \varepsilon)} \{|f(x) - f(a)| : x \in B(a, \varepsilon)\},$$

where $Q'' = Q''(f)$ is a constant.

THEOREM (C). [5] (a) If $f \in HC^1(B)$, then $|f| \in QNS(B)$, and $Q(|f|) \leq C_N Q(f)$.

(b) If $f \in OC^1(B)$, then both $|f|$ and $|\nabla f|$ belong to $QNS(B)$, and $Q(|f|) \leq C_N Q''(u)$.

THEOREM (D). [6, 7] If $f$ is a function polyharmonic in $B$, then $f \in OC^1(B)$. Moreover if $f \in \mathcal{H}_q(B)$, then $Q''(u) \leq C_{k,N}$ (where $C_{k,N}$ depends only on $k$ and $N$).
As a consequence we have the following generalization of a theorem of Hardy and Littlewood [3] \((N = 2)\) and Fefferman and Stein [2] \((N \geq 3)\).

**Theorem (E).** If \(u = |f|^p, u = |Rf|^p, u = |\nabla f|^p,\) or \(u(x) = |T_{i,j}f(x)|^p,\) where \(f \in \mathcal{H}_k(B),\) and \(p > 0,\) then \(u\) satisfies (3.2) with \(C\) depending only on \(p, k\) and \(N.\)

### 4. \(L^p\)-inequalities for QNS-functions

The following theorem was proved in [7] in the case of polyharmonic functions. However the proof was based only on the condition \(f \in \mathcal{H}^1(B).\) Therefore we omit the proof.

**Theorem (4.1).** If \(f \in \mathcal{H}^1(B), p > 0\) and \(\alpha \in (-\infty, \infty),\) then

\[
\left\| |\nabla f| \right\|_{L^p_\alpha} \leq C \left\| f \right\|_{L^p_\alpha}.
\]

In order to state a maximal theorem we let

\[
u^+(\rho y) = \sup_{0 \leq r \leq \rho} |u(ry)| = \sup_{0 \leq t \leq \rho} |u(ty)|, \quad 0 \leq \rho < 1, \ y \in \partial B.
\]

**Theorem (4.3).** If \(u \in \text{QNS}(B), \alpha > -1\) and \(p > 0,\) then

\[
\left\| u^+ \right\|_{L^p_\alpha} \leq C \left\| u \right\|_{L^p_\alpha},
\]

where \(C\) depends only on \(p, \alpha\) and \(Q(u).\)

Let \(P(x, y)\) denote the Poisson kernel,

\[
P(x, y) = \frac{1 - |x|^2}{|x - y|^N}.
\]

Since \(\int_{\partial B} P(x, y) \, d\sigma(y) = 1,\) where \(d\sigma\) is the normalized surface measure on \(\partial B,\) we see that Theorem (4.3) is obtained from the following lemma, by using integration in polar coordinates, i.e., the formula

\[
\int_B \phi(x) \, dV(x) = N \int_0^1 r^{N-1} dr \int_{\partial B} \phi(ry) \, d\sigma(y).
\]

**Lemma (4.5).** If \(u \in \text{QNS}(B), \alpha > -1\) and \(p > 0,\) then

\[
\int_0^1 r^{N-1} u^+(ry)^p(1 - r^2)^\alpha \, dr \leq C \int_B u(x)^p(1 - |x|^2)^\alpha P(x, y) \, dV(x), \quad y \in \partial B.
\]

For the proof we need the following elementary lemma.

**Lemma (A).** [4] Let \(\beta > 0,\) and \(\{A_j\}_{j=0}^\infty\) a sequence of real numbers. Then

\[
\sum_{j=0}^\infty 2^{-j\beta} |A_{j+1}|^p \leq C|A_0|^p + C \sum_{j=0}^\infty 2^{-j\beta} |A_{j+1} - A_j|^p,
\]

where \(C\) depends only on \(\beta.\)
Proof of Lemma (4.5). We can assume that \( p = 1 \) because \( u \in \text{QNS}(B) \) implies \( u^p \in \text{QNS}(B) \), by Theorem (B). Let \( r_j = 1 - 2^{-j} \) for \( j \geq 0 \). Then, by Lemma (A),

\[
\int_0^1 r^{N-1} (1 - r^2) \bar{u}^+(ry) \, dr \leq C \sum_{j=0}^{\infty} 2^{-j(\alpha+1)} u^+(r_{j+1}y)
\]

\[
\leq Cu(0) + C \sum_{j=0}^{\infty} 2^{-j(\alpha+1)} \left( u^+(r_{j+1}y) - u^+(r_jy) \right)
\]

\[
\leq C \sum_{j=0}^{\infty} 2^{-j(\alpha+1)} \sup_{r_j \leq r \leq r_{j+1}} u(ry).
\]

By (3.2) with \( a = a_j := (r_j + r_{j+1})y/2 \) and \( c = (r_{j+1} - r_j)/2 = 2^{-j-2} \),

\[
2^{-j(\alpha+1)} \sup_{r_j \leq r \leq r_{j+1}} u(ry) \leq C 2^{-j(\alpha+1)2^N} \int_{B(a_j, 2^{-j-1})} u(x) \, dV(x).
\]

(4.6) \( 2^{-j(\alpha+1)} \sup_{r_j \leq r \leq r_{j+1}} u(ry) \leq C 2^{-j(\alpha+1)2^N} \int_{B(a_j, 2^{-j-1})} u(x) \, dV(x). \)

On the other hand, simple calculation shows that \(|x - a_j| \leq 2^{-j-1}\) implies

\[
2^{-j-2} \leq 1 - |x|, \quad |x - y| \leq 2^{-j+1}
\]

Hence

\[
2^{-j} 2^{jN} \leq 2^{N-2} P(x, y), \quad \text{for } x \in B(a_j, 2^{-j-1}).
\]

From this and (4.6) we get

\[
2^{-j(\alpha+1)} \sup_{r_j \leq r \leq r_{j+1}} u(ry) \leq C 2^{-ja} \int_{r_{j-1} \leq |x| \leq r_{j+2}} P(x, y) u(x) \, dV(x)
\]

\[
\leq C \int_{r_{j-1} \leq |x| \leq r_{j+2}} (1 - |x|)^{\alpha} P(x, y) u(x) \, dV(x)
\]

\((r_{-1} = 0)\) where we have used the inclusion

\[
\{x: |x - a_j| \leq 2^{-j-1}\} \subset \{x: r_{j-1} \leq |x| \leq r_{j+2}\}.
\]

Now the desired conclusion is easily obtained by summation from \( j = 0 \) to \( \infty \).

\[\Box\]

THEOREM (4.7). For a Borel measurable function \( u \) on \( B \) and \( s > 0 \), let

\[
I_s u(x) = \int_0^1 t^{s-1} u(tx) \, dt, \quad x \in B.
\]

If \( u \in \text{QNS}(B) \), \( p > 0 \) and \( \alpha > -1 \), then

\[
\|I_s u\|_{L^p} \leq C \|u\|_{L^p_{\alpha+1}}.
\]

Proof. Write \( I_s u \) as

\[
I_s u(\rho y) = \frac{1}{\rho^s} \int_0^\rho t^{s-1} u(ty) \, dt, \quad y \in \partial B, \ 0 < \rho < 1.
\]

Hence

\[
|I_s u(\rho y)| \leq \frac{1}{s} \sup_{B(0, 1/2)} u + 2^{s+1} \int_0^\rho u(ty) \, dt, \quad y \in \partial B, \ 0 < \rho < 1.
\]
Since, by (3.2),
\[ \sup_{B(0,1/2)} u \leq C\|u\|_{L_{\alpha,p}^p}, \]
it suffices to prove that
\[ \|Ju\|_{L_{\alpha}^p} \leq C\|u\|_{L_{\alpha,p}^p}, \]
where
\[ Ju(\rho y) = \int_0^\rho u(ry) \, dr. \]

To show this we proceed in a similar way as in the proof of Lemma (4.5). Namely,
\[ \|f\|_{L_{\alpha}^p} = \sum_{j=0}^{\infty} \int_{r_j}^{r_{j+1}} \int_0^{\rho R_{N-1}^j} (Ju)(\rho y)^p (1 - r^2)^{\alpha-1} \, d\sigma(y). \]

On the other hand, it is easy to show that
\[ \sum_{j=0}^{\infty} \int_{r_j}^{r_{j+1}} N R_{N-1}^j (Ju)(\rho y)^p (1 - r^2)^{\alpha-1} \, dr \leq C \sum_{j=0}^{\infty} \left( \int_0^{r_{j+1}} u(\rho y) \, dr \right)^p 2^{-ja}. \]

Now we use Lemma (A) to show that the last quantity is equivalent to
\[ A := \sum_{j=0}^{\infty} \left( \int_{r_j}^{r_{j+1}} u(\rho y) \, dr \right)^p 2^{-ja}. \]

Since
\[ A \leq \sum_{j=0}^{\infty} 2^{-jp} 2^{-ja} \sup_{r_j < r < r_{j+1}} u(\rho y)^p, \]
we can proceed as in the proof of Lemma 1 to get
\[ \int_0^{1} N R_{N-1}^j (Ju)(\rho y)^p (1 - r^2)^{\alpha-1} \, dr \leq C \int_B u(x)^p (1 - |x|^2)^{\alpha+p} P(x, y) \, dV(x), \quad y \in \partial B. \]

Now integration in polar coordinates gives
\[ \|Ju\|_{L_{\alpha}^p} \leq C\|u\|_{L_{\alpha,p}^p}, \]
which completes the proof of the theorem.  

5. Inequalities for polyharmonic functions

In [7], the following decomposition theorem for polyharmonic Bergman spaces is proved.

**Theorem (F).** If \( f \) is given by (1.1), \( 0 < p < \infty \), and \( \alpha > -1 \), then
\[ \|f\|_{L_{\alpha}^p} \cong \sum_{m=0}^{k-1} \|A_m f\|_{L_{\alpha,m}^p}, \quad f \in \mathcal{H}_k(B). \]

Here we prove:

**Theorem (5.1).** Let \( \alpha > -1 \) and \( p > 0 \). Then
\[ \|f - f(0)\|_{L_{\alpha}^p} \cong \|Rf\|_{L_{\alpha,p}^p} \cong \|\nabla f\|_{L_{\alpha,p}^p}, \quad f \in \mathcal{H}_k(B). \]
Proof. We have
\[ f(x) - f(0) = \int_0^1 \frac{Rf(tx)}{t} \, dt. \]
Hence
\[ |f(x) - f(0)| \leq \sup_{|x| < 1/4} |\nabla f(x)| + 4 \int_0^1 |Rf(tx)| \, dt. \]
On the other hand,
\[ \nabla f(x) = \int_0^1 (\nabla Rf)(tx) \, dt, \]
whence
\[ \sup_{|x| < 1/4} |\nabla f(x)| \leq \sup_{|x| < 1/4} |\nabla Rf(x)|. \]
But
\[ \sup_{|x| < 1/4} |\nabla Rf(tx)| \leq C \sup_{|x| < 1/2} |Rf(x)| \]
because \( Rf \) is polyharmonic and therefore belongs to \( HC^1 \). Also since \( Rf \in QNS(B) \), we have
\[ \sup_{|x| < 1/2} |Rf(x)| \leq C\|Rf\|_{L^p_{r,p}}. \]
Combining the above inequalities we get
\[ |f(x) - f(0)| \leq C\|Rf\|_{L^p_{r,p}} + 4 \int_0^1 |Rf(tx)| \, dt. \]
Now the inequality
\[ \|f - f(0)\|_{L^p_r} \leq C\|Rf\|_{L^p_{r,p}} \]
follows from Theorem (4.7) \( (s = 1) \). Since \( |Rf(x)| \leq |x| |\nabla f(x)| \) and \( \|\nabla f\|_{L^p_{r,p}} \leq C\|f\|_{L^p_r} \), by Theorem (4.1), we see that the proof is finished.

6. Proof of the main results

**Theorem (6.1).** For \( 0 < p < \infty, \alpha > -1 \), and \( k \geq 2 \), we have \( Q_1(f) \asymp \|f\|_{D^\alpha_p}. \)

**Proof.** The inequality \( Q_1(f) \leq C\|f\|_{D^\alpha_p} \) is obvious. On the other hand, we have
\[ \|D_i Rf\|_{L^p_{r,p}} \leq C\|Rf\|_{L^p_r}, \quad 1 \leq i \leq N, \]
by Theorem (4.1). But since \( D_i Rf = R_1 D_i f \), we can use the formula
\[ I/R_i u = R_i I_0 u = u \]
(\( u = D_i f \)) together with Theorem (4.7) to get
\[ \|D_i f\|_{L^p_r} \leq C\|Rf\|_{L^p_r}, \]
which concludes the proof.

**Lemma (6.3).** If \( f \) is a harmonic function on \( B \), \( p > 0 \) and \( \alpha > 0 \), then
\[ Q_2(f) \asymp \|f\|_{D^\alpha_p}. \]
Proof. The inequality $Q_2(f) \leq C\|f\|_{D^p_n}$ is obvious. To prove the reverse inequality, observe that (2.4) implies
\begin{equation}
\frac{1}{2} \sum_{i,j=1}^{N} T_{ij} T_{ij} f = -R_{N-2} Rf.
\end{equation}
Since
\[ |A_m f(0)| + \sum_{i,j=1}^{N} \|T_{ij} f\|_{L^p_n} \]
we see that (6.4) and Theorem (4.1) imply
\[ \|R_{N-2} Rf\|_{L^p_n} \leq CQ_2(f). \]
Now we use the formula (6.2) with $u = Rf$, $s = N - 2$ together with Theorem (4.7) to conclude that
\[ \|Rf\|_{L^p_n} \leq C\|R_{N-2} Rf\|_{L^p_n} \leq CQ_2(f), \]
for $N \geq 3$. In the case $N = 2$ we can use Theorem (5.1) to get
\[ \|Rf\|_{L^p_n} \leq C\|Rf\|_{L^p_n} \leq CQ_2(f), \]
Now the result follows from Theorem (6.1). \hfill \Box

Theorem (6.5). Under the hypotheses of Theorem (1.2) we have $Q_2(f) \asymp \|f\|_{D^p_n}$.

Proof. First we prove that $\|f\|_{D^p_n} \leq CQ_2(f)$. By Proposition (B) and Theorem (F) we have
\[ Q_2(f) \asymp \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{m=0}^{k-1} \sum_{i,j=1}^{N} \|T_{ij} A_m f\|_{L^p_n}. \]
Hence, by Lemma (6.3),
\[ Q_2(f) \asymp \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{m=0}^{k-1} \|\nabla A_m f\|_{L^p_{n+mp}} \quad (= Q_3(f)). \]
Since $\|\nabla A_m f\|_{L^p_{n+mp}} \asymp \|A_m f - A_m f(0)\|_{L^p_{a+(m-1)p}}$ for $m \geq 1$, we have
\begin{align}
Q_2(f) &\asymp \sum_{m=0}^{k-1} |A_m f(0)| + \|\nabla A_0 f\|_{L^p_n} + \sum_{m=1}^{k-1} \|A_m f - A_m f(0)\|_{L^p_{a+(m-1)p}} \\
&\asymp \sum_{m=0}^{k-1} |A_m f(0)| + \|\nabla A_0 f\|_{L^p_n} + \sum_{m=1}^{k-1} \|A_m f\|_{L^p_{a+(m-1)p}} \\
&\asymp |A_0 f(0)| + \|\nabla A_0 f\|_{L^p_n} + \sum_{m=1}^{k-1} \|A_m f\|_{L^p_{a+(m-1)p}} \quad (= Q_4(f)).
\end{align}
On the other hand, since
\[ |\nabla f(x)| \leq |\nabla A_0 f(x)| + \sum_{m=1}^{m-1} (1 - |x|^2)^m |\nabla A_m f(x)| + 2m(1 - |x|^2)^{m-1} |A_m f(x)|, \]
we have, by Theorem (4.1),
\[ \| \nabla f \|_{L^p_B} \leq C \| A_0 f \|_{L^p_{saB}} + C \sum_{m=1}^{k-1} \| A_m f \|_{L^p_{saB_m-1p}}. \]
This inequality and (6.6) give the required inequality.

In order to prove the reverse inequality it suffices to prove that
\[ (6.7) \sum_{m=0}^{k-1} |A_m f(0)| \leq C \| f \|_{D^p_B}. \]
Indeed, since $|\nabla f| \in QNS$, we have
\[ |f(0)| + \int_B |\nabla f(x)|^p(1-|x|^2)^\alpha dV(x) \geq c|f(0)| + \sup_{|x|<1/2} |\nabla f(x)|, \]
and hence
\[ \| f \|_{D^p_B} \geq c|f(0)| + \sup_{|x|<1/2} |f(x) - f(0)| \geq c \sup_{|x|<1/2} |f(x)|, \]
where $c$ is a positive constant. On the other hand,
\[ \int_{\partial B} f(ry) d\sigma(y) = \sum_{m=0}^{k-1} A_m f(0)(1-r^2)^m, \quad 0 < r < 1. \]
But the quantity $K(a_0, \ldots, a_{k-1}) = \sup_{0<r<1/2} |\sum_{m=0}^{k-1} a_m(1-r^2)^m|$ is a norm on $\mathbb{R}^k$, and this implies
\[ \sum_{m=0}^{k-1} |A_m f(0)| \leq C K(f) \leq C \sup_{0<r<1/2} \int_{\partial B} |f(ry)| d\sigma(y) \leq C \sup_{|x|<1/2} |f(x)|, \]
which completes the proof.

7. Bloch type spaces

We define $B_{a,k}$ ($\alpha > 0$, $k = 1, 2, \ldots$) to be the class of those $f \in H_B(B)$ for which
\[ (7.1) \| f \|_{B_{a,k}} := |f(0)| + \sup_{x \in B} (1-|x|^2)^\alpha |\nabla f(x)| < \infty. \]
The space $B_{1,1}$ is known as the harmonic Bloch space. The following theorem is proved in a similar way as Theorem (1.2); the proof is even simpler and therefore is omitted.

**Theorem (7.2).** For $\alpha > 0$, and $k \geq 2$, the following quantities are equivalent norms on $B_{a,k}(B)$:
\[ P_1(f) = |f(0)| + \sup_{x \in B} |Rf(x)|^p(1-|x|^2)^\alpha, \]
\[ P_2(f) = \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{i,j=1}^N \sup_{x \in B} |T_{ij} f(x)(1-|x|^2)^\alpha, \]
\[ P_3(f) = \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{m=0}^{k-1} \sup_{x \in B} |\nabla A_m f(x)(1-|x|^2)^{\alpha+m}|, \]
EQUIVALENT NORMS ON DIRICHLET SPACES OF POLYHARMONIC FUNCTIONS

\[ P_4(f) = \|A_0 f\|_{B_\alpha} + \sum_{m=1}^{k-1} \sup_{x \in B} |A_m f(x)|(1 - |x|^2)^{\alpha + (m-1)}. \]

In order to state another result, we define the \( L^q \)-oscillation (\( 0 < q \leq \infty \)) of \( f \) over the ball \( B(x, r) \subset B \) as

\[ \text{osc}_q(f, x, r) = \left( \frac{1}{r^N} \int_{B(x, r)} |f(z) - f(x)|^q dV(z) \right)^{1/q}. \]

Note that \( V(B(x, r)) = r^N \). In the case \( q = \infty \), equation (7.3) should be interpreted as

\[ \text{osc}(f, x, r) = \sup_{z \in B(x, r)} |f(z) - f(x)|. \]

**Theorem (7.4).** Let \( 0 < q \leq \infty \), \( \alpha > 0 \), and \( 0 < c < 1 \). Then the quantity

\[ P_5(f) = |f(0)| + \sup_{x \in B} (1 - |x|^2)^{\alpha - 1} \text{osc}_q(f, x, c(1 - |x|)) \]

is an equivalent norm on \( B_{\alpha, k} \).

**Proof.** We consider only the case \( q < \infty \). It is easy to check that a QNS-function \( u \) on \( B \) satisfies the condition

\[ \sup_{|z-x|<\varepsilon/2} u(z)^q \leq \frac{C}{\varepsilon^N} \int_{B(x, \varepsilon)} u(z)^q dV(z), \quad B(x, \varepsilon) \subset B. \]

Since a polyharmonic function \( f \) belongs to \( OC^1(B) \) (Theorem (D)), this implies

\[ |\nabla f(x)|^q \leq \frac{C}{\varepsilon^{N+q}} \int_{B(x, \varepsilon)} |f(z) - f(x)|^q dV(x) \]

\[ = \frac{C}{\varepsilon^q} \left( \text{osc}_q(f, x, \varepsilon) \right)^q. \]

where \( C \) is independent of \( f \). Now we take \( \varepsilon = c(1 - |x|) \) and use the hypothesis

\[ (1 - |x|^2)^{\alpha - 1} \text{osc}_q(f, x, c(1 - |x|)) \leq P_5(f) \]

to get

\[ |\nabla f(x)| \leq \frac{CP_5(f)}{c(1 - |x|)} (1 - |x|)^{(1-a)} = C P_5(f) \frac{(1 - |x|)^{-a}}{c}. \]

This proves part of the theorem.

In the other direction, assume that

\[ M(f) := \sup_{x \in B} (1 - |x|^2)^\alpha |\nabla f(x)| < \infty. \]

It is enough to prove that

\[ \sup_{x \in B} (1 - |x|^2)^{\alpha - 1} \text{osc}(f, x, c(1 - |x|)) \leq CM(f), \]

because \( \text{osc}_q(f, x, c(1 - |x|)) \leq \text{osc}(f, x, c(1 - |x|)) \). For we have, by Lagrange's theorem,

\[ \text{osc}(f, x, c(1 - |x|)) \leq c(1 - |x|) \sup_{z \in B(x, c(1 - |x|))} \|\nabla f(z)\|. \]
whence, by hypothesis,

\[ \text{osc}(f, x, c(1 - |x|)) \leq c(1 - |x|)M(f) \sup_{z \in B(x, c(1 - |x|))} (1 - |z|^2)^{-\alpha}. \]

Now the desired result follows from the inequality

\[ 1 - c < \frac{1 - |z|}{1 - |x|} < 1 + c, \quad z \in B(x, c(1 - |x|)), \]

which is easily deduced from the inequalities \(|x| - |z| < c(1 - |x|)\) and \(|z| - |x| < c(1 - |x|)\) valid for \(z \in B(x, c(1 - |x|))\). This concludes the proof of the theorem. \( \square \)

Finally we define the mean values of \(|f|\) on the ball \(B(x, r)\) by

\[ m_q(f, x, r) = \left( \frac{1}{r^N} \int_{B(x, r)} |f|^q \, dV \right)^{1/q}, \quad 0 < q \leq \infty. \]

Using Theorems (E) and (7.2) one can prove the following:

THEOREM (7.7). Let \(\alpha > 0\), \(0 < q \leq \infty\), and \(k \geq 2\), and \(0 < c < 1\), then the following quantities are equivalent norm on \(B_{\alpha, k}\):

\[ P_6(f) = |f(0)| + \sup_{x \in B} m_q(Rf, x, c(1 - |x|))(1 - |x|^2)^\alpha, \]

\[ P_7(f) = \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{i,j=1}^{N} \sup_{x \in B} m_q(T_{ij} f, x, c(1 - |x|))(1 - |x|^2)^\alpha. \]

Acknowledgment

We are very grateful to the referee who suggested to us to consider Bloch spaces as well as to help us to make some proofs, in particular of Lemma (4.5) and Theorem (4.7), more clear.

Received February 05, 2007

Final version received August 10, 2007
REFERENCES

5. M. Pavlović, Multipliers of \( H^p \) and \( \text{BMOA} \), Pacific J. Math. 146 (1990), 71–84.