Remarks on $L^p$-oscillation of the modulus of a holomorphic function

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Abstract

We extend Dyakonov’s theorem on the moduli of holomorphic functions to the case of $L^p$-norms.

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1. Introduction

A continuous, increasing function $\omega$ on the interval $[0, 2]$ is called a majorant if $\omega(0) = 0$ and the function $\omega(t)/t$ is nonincreasing. Given a majorant $\omega$ we define $\Lambda_\omega(\mathbb{D})$, where $\mathbb{D}$ is the unit disk of the complex plane, to be the class of those complex-valued functions $f$ for which there exists a constant $C$ such that

$$|f(w) - f(z)| \leq C\omega(|w - z|) \quad (z, w \in \mathbb{D}).$$

Our starting point here is the following result of Dyakonov [5].

Theorem A. Let $\omega$ be a majorant satisfying the following two conditions:

$$\int_0^x \frac{\omega(t)}{t} \, dt \leq C\omega(x) \quad (0 < x < 2),$$

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\[
\int_0^x \frac{\omega(t)}{t^2} \, dt \leq C \frac{\omega(x)}{x} \quad (0 < x < 2),
\]
where \( C \) is a positive constant. Then, a function \( f \) holomorphic in \( \mathbb{D} \) belongs to \( \Lambda_\omega(\mathbb{D}) \) if and only if so does its modulus \( |f| \).

Following Dyakonov [6], we call a majorant \( \omega \) fast (respectively, slow) if there exists a constant \( C \) such that there holds (1) (respectively, (2)). If \( \omega \) is both fast and slow, then it is said to be regular.

That condition (2) is perhaps superfluous can be seen by considering the majorant \( \omega(t) = t \). Namely, a function \( f \in H(\mathbb{D}) \) (\( = \) the class of all functions holomorphic in \( \mathbb{D} \)) satisfies the condition \(|f(z) - f(w)| \leq |z - w| \) in \( \mathbb{D} \) iff \(|f'| \leq 1 \) in \( \mathbb{D} \). On the other hand, the corresponding Lipschitz condition for \(|f|\) is satisfied iff \(|\nabla |f|| \leq 1\). Since \(|\nabla |f|| = |f'|\), we conclude that there holds the relation

\[
f \in A_1(\mathbb{D}) \iff |f| \in A_1(\mathbb{D}), \quad f \in H(\mathbb{D}).
\]

That condition (2) is actually superfluous can be deduced from [13], where a simple proof of Theorem A is given based on the following consequence of the Schwarz lemma.

**Lemma 1.** [13] If \( f \in H(\mathbb{D}) \), then there holds the inequality

\[
|f'(z)| \leq \frac{2}{\varepsilon} \sup\{w \in D_\varepsilon(z) : |f(w)| - |f(z)|\} \quad (0 < \varepsilon < 1 - |z|),
\]

where

\[
D_\varepsilon(z) = \{w : |w - z| < \varepsilon\}.
\]

This lemma is used in [15] to prove the following extension of Theorem A.

**Theorem 1.** Suppose \( \omega \) satisfies the (Dini) condition

\[
\psi(x) := \int_0^x \frac{\omega(t)}{t} \, dt < \infty \quad (0 < x < 2).
\]

If \(|f| \in A_\omega(\mathbb{D}) \cap H(\mathbb{D})\), then \( f \in A_\psi(\mathbb{D}) \).

In fact this theorem follows immediately from Lemma 1 and inequality (10) below (case \( p = \infty \)).

**Corollary 1.** Let \( A(\mathbb{D}) \) denote the disk-algebra, i.e., the subclass of \( H(\mathbb{D}) \) consisting of functions that are uniformly continuous on \( \mathbb{D} \). If \( f \in H(\mathbb{D}) \) and \(|f| \in A_\omega(\mathbb{D})\), where \( \omega \) satisfies the Dini condition, then \( f \in A(\mathbb{D}) \).

**Remark 1.** Concerning this corollary, it should be noted that there exists a function \( f \in H(\mathbb{D}) \setminus A(\mathbb{D}) \) such that the function \(|f|\) is uniformly continuous on \( \mathbb{D} \). Indeed, it is known that there exists a bounded holomorphic function \( u + iv \) such that \( u \) is continuous on \( \overline{\mathbb{D}} \), while \( v \) has no continuous extension to \( \overline{\mathbb{D}} \). Then there is a point \( \eta \in \mathbb{T} \) and two sequences \( \{z_n\} \subset \mathbb{D} \) and \( \{w_n\} \subset \mathbb{D} \) tending to \( \eta \) and two points \( a, b \in \mathbb{C} \) (\( a \neq b \)) such that \( v(z_n) \to a \) and \( v(w_n) \to b \). We can assume
that $e^{ia} \neq e^{ib}$ since otherwise we can consider the function $(u + iv)/t$ for a suitable $t > 0$. Then the desired function is $f = \exp(u + iv)$.

The validity of Theorem 1 can also be seen from a recent paper of Dyakonov [6] (see the proof of Theorem 1 [6]). In that paper Dyakonov uses Lemma 1, in conjunction with other techniques, to extend Theorem A in various directions; in particular, he considers Lipschitz spaces over arbitrary subdomains of the complex plane.

2. $L^p$-oscillation of the modulus

As a consequence of (3) we have

$$\left| \nabla u(z) \right| \leq \frac{K}{\varepsilon} \text{osc}(u; z, \varepsilon) \quad (0 < \varepsilon < 1 - |z|) \quad (4)$$

($u = |f|$, $K = 2$), where $\text{osc}(u; z, \varepsilon)$ denotes the oscillation of $u$ over $D_\varepsilon(z)$,

$$\text{osc}(u; z, \varepsilon) = \sup \{|u(w) - u(z)| : w \in D_\varepsilon(z)\}.$$

The $L^p$-oscillation of $u$ over $D_\varepsilon(z)$ is defined by

$$\text{osc}_p(u; z, \varepsilon) = \left\{ \frac{1}{\varepsilon^2} \int_{D_\varepsilon(z)} |u(w) - u(z)|^p \, dm(w) \right\}^{1/p},$$

where $dm$ is the Lebesgue measure normalized so that $m(\mathbb{D}) = 1$.

**Lemma 2.** Let $p > 0$ and $0 < \varepsilon < 1 - |z|$, $z \in \mathbb{D}$. If $f$ is holomorphic in $\mathbb{D}$, then there holds the inequality

$$|f'(z)| \leq \frac{C}{\varepsilon} \text{osc}_p(|f|; z, \varepsilon), \quad (5)$$

where $C$ is a constant depending only on $p$.

**Proof.** Let $z \in \mathbb{D}$ and $0 < \varepsilon < 1 - |z|$. Let

$$u(w) = |f(w)| - |f(z)| \quad (w \in \mathbb{D}),$$

and

$$u^+(w) = \max\{u(w), 0\}.$$

The function $u$ is subharmonic and therefore so is $u^+$. Hence, by (3) and the maximum principle,

$$|f'(z)| \leq \frac{4}{\varepsilon} \sup \{u^+(w) : w \in D_\delta(z)\}, \quad (6)$$

where $\delta = \varepsilon/2$. If $p \geq 1$, then $(u^+)^p$ is subharmonic and therefore

$$\left( u^+(w) \right)^p \leq \frac{1}{\delta^2} \int_{D_\varepsilon(z)} (u^+(\eta))^p \, dm(\eta) \quad (w \in D_\delta(z)).$$

From this and (6) we obtain

$$|f'(z)|^p \leq \frac{16}{\varepsilon} \frac{1}{\varepsilon^2} \int_{D_\varepsilon(z)} (u^+)^p \, dm.$$
Now (5), in the case \( p \geq 1 \), follows from the inequality
\[
u^+(w) \leq \left\| f(w) - |f(z)| \right\|.
\]

In the case \( p < 1 \) the function \((u^+)^p\) need not be subharmonic. However then we can use the inequality
\[
U(z)^p \leq C_p \frac{1}{\delta^2} \int_{D_\delta(z)} U(\eta)^p \, dm(\eta),
\]

essentially due to Hardy and Littlewood, which is valid for any subharmonic function \( U \geq 0 \) (cf. [9]; see also [14] for a simple proof). \( \square \)

Applying (5) to the functions \( z \mapsto f(e^{i\theta}z) \) and then integrating with respect to \( \theta \) we get the following:

**Proposition 1.** If \( f \) is holomorphic in \( \mathbb{D} \), \( p > 0 \) and \( \varepsilon = (1 - r)/2 \), \( r = |z| \), then
\[
M_p^p(r, f')(1-r)^p \leq \frac{C_p}{(1-r)^2} \int_{D_\varepsilon(z)} \|f_w - f_z\|^p_p \, dm(w).
\]

Here, as usual,
\[
f_w(\zeta) = f(w\zeta), \quad \text{for } |\zeta| < 1/|w|,
\]

\[
\|g\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^p \, d\theta \right\}^{1/p},
\]

and
\[
M_p(r, g) = \|g_r\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta \right\}^{1/p}.
\]

**Corollary 2.** If \( f \) is holomorphic in \( \mathbb{D} \), \( p > 0 \), and
\[
\|f_w - f_z\|_p \leq C\omega(1 - |z|) \quad \text{whenever } |w - z| \leq (1 - |z|)/2,
\]

where \( \omega \) is an arbitrary majorant, then
\[
M_p(r, f') \leq C \frac{\omega(1-r)}{1-r}.
\]

For a Borel function \( g \) defined on \( \mathbb{D} \) we define the \( L^p \)-modulus of continuity over \( \mathbb{D} \):
\[
\Omega_p(g, \delta) = \sup\{ \|g_w - g_z\|_p : |z - w| < \delta, \, z, w \in \mathbb{D} \}.
\]

The following fact, an immediate consequence of Proposition 1, improves the well-known inequality
\[
M_p(r, f') \leq C_p (1 - r)^{-1} \|f\|_p, \quad f \in H^p,
\]
due to Hardy and Littlewood (cf. [4] for information and references). Here \( H^p \) denotes the standard \( p \)-Hardy space of holomorphic functions in \( \mathbb{D} \) such that \( \sup_{0<r<1} M_p(r, f) < \infty \).
Corollary 3. If \( f \) is holomorphic in \( \mathbb{D} \) and \( p > 0 \), then
\[
M_p(r, f') \leq C_p (1 - r)^{-1} \Omega_p (|f|, 1 - r) \quad (r < 1).
\]

A weaker improvement of (7) was proved by Storoženko in [16], namely:
\[
M_p(r, f') \leq C_p (1 - r)^{-1} \Omega_p (f, 1 - r) \quad (r < 1).
\]

2.1. Remarks

It was proved in [14, Theorem 2] that condition (4), where \( u \) is a locally Lipschitz function, implies
\[
\| \nabla u(z) \| \leq \frac{C}{\varepsilon} \text{osc}_u(u, z, \varepsilon).
\]
This fact, of course, implies (5): It is enough to take \( u = |f| \). However, in the case \( p \geq 1 \), the subharmonicity of \( u^+ \) in the above proof gives a slightly stronger result, namely:

Theorem 2. If \( f \in H^p \), \( 1 \leq p < \infty \), then
\[
|f'(0)|^p \leq C_p \int_0^{2\pi} (\max \{ |f(e^{i\theta})| - |f(0)|, 0\})^p d\theta,
\]
and consequently
\[
|f'(0)|^p \leq C_p \int_0^{2\pi} \| f(e^{i\theta}) - f(0) \|^p d\theta.
\]
This improves the well-known and easily proved inequality
\[
|f'(0)|^p \leq C_p \int_0^{2\pi} |f(e^{i\theta}) - f(0)|^p d\theta.
\]

3. Hardy–Lipschitz spaces

Let \( \omega \) be a majorant, and let \( 0 < p < \infty \). We define the \( \Lambda^p_\omega (\mathbb{D}) \) to be the class of all complex-valued Borel functions \( f \) on \( \mathbb{D} \) such that
\[
\| f_w - f_z \|_p \leq C \omega (|w - z|) \quad (z, w \in \mathbb{D})
\]
for some constant \( C \). The class \( \Lambda^p_\omega (\mathbb{D}) \cap H (\mathbb{D}) (\subset H^p) \) is sometimes called a Hardy–Lipschitz space. Clearly, if \( f \) is in \( \Lambda^p_\omega (\mathbb{D}) \), then so is \( |f| \). The converse does not hold in the general case; however:

Theorem 3. Let \( \omega \) be a fast majorant, \( f \in H^p \) \((0 < p < \infty)\), and
\[
\| f_z - f_\zeta \| \leq \omega (|z - \zeta|) \quad \text{for all } z \in \mathbb{D}, \zeta \in \mathbb{T},
\]
then \( f \in \Lambda^p_\omega (\mathbb{D}) \). Consequently, if \( f \in H (\mathbb{D}) \) and \( |f| \in \Lambda^p_\omega (\mathbb{D}) \), then \( f \in \Lambda^p_\omega (\mathbb{D}) \).
We need another lemma:

**Lemma 3.** Let $f \in H(\mathbb{D})$. If $0 < p \leq 1$, then there is a constant $C_p$ such that

$$
\{ \Omega_p(f, \delta) \}^p \leq C_p \int_{1-\delta}^{1} (1-t)^{p-1} M_p^p(t, f') \, dt, \quad 0 < \delta < 1. 
$$

(9)

If $1 \leq p \leq \infty$, then

$$
\Omega_p(f, \delta) \leq C_p \int_{1-\delta}^{1} M_p(t, f') \, dt, \quad 0 < \delta < 1.
$$

(10)

**Proof.** We shall discuss the case $p < 1$; the rest is similar and even somewhat simpler. Let

$$
I_p(\delta) = \int_{1-\delta}^{1} (1-t)^{p-1} M_p^p(t, f') \, dt < \infty.
$$

We can assume that $\delta \leq 1/4$. Otherwise we have

$$
I_p(\delta) \geq \int_{3/4}^{1} (1-t)^{p-1} M_p^p(t, f') \, dt \geq M_p^p(3/4, f') \int_{3/4}^{1} (1-t)^{p-1} \, dt 
\geq c_p \int_{0}^{3/4} (1-t)^{p-1} M_p^p(t, f') \, dt,
$$

and hence

$$
\{ \Omega_p(f, \delta) \}^p \leq 2 \| f \|_p^p \leq C_p I_p(1) \leq I_p(\delta).
$$

Let $0 < \delta < 1/4$. We have to prove that $|z - \rho| < \delta$, $|z| < 1$, $\rho \in (0, 1)$, implies

$$
\| f_z - f_{\rho} \|_p^p \leq C_p I_p(\delta).
$$

If $|z| < 1/4$, then $\rho < 1/2$, whence, by the Lagrange theorem and the Hardy–Littlewood complex maximal theorem,

$$
| f_z(e^{i\theta}) - f_{\rho}(e^{i\theta}) | \leq \sup_{|u| < 1/2} |f'(u)| \leq C_p M_p^p(3/4, f') \leq C_p I_p(\delta).
$$

It follows that we can assume that

$$
1/4 \leq |z| \leq \rho, \quad |z - \rho| < \delta < 1/4.
$$

Since

$$
|\rho - re^{i\theta}|^2 = (\rho - r)^2 + \rho r |e^{i\theta} - 1|^2,
$$

we have

$$
\rho - r \leq \delta, \quad |e^{i\theta} - 1| \leq 4\delta \quad (z = re^{i\theta}).
Hence
\[ \|f_z - f_\rho\|_p \leq \|f_z - f_r\|_p + \|f_\rho - f_r\|_p \]
\[ \leq \{\omega_p(f, 4\delta)\}^p + C_p (\rho - r)^p M_\rho^p (\rho, f') \]
\[ \leq C_p \{\omega_p(f, \delta)\}^p + C_p \delta^p M_\rho^p (1 - \delta, f') \]
\[ \leq C_p I_p (\delta). \]

In the last step we used the inequality (see [12]):
\[ \omega_p^p (f, \delta) \leq C_p \int_{1-\delta}^1 M_\rho^p (r, f') (1 - r)^{p-1} dr \quad (0 < \delta < 1), \]
where \( \omega_p (f, \delta) \) stands for the \( L^p \)-modulus of continuity of the boundary function,
\[ \omega_p (f, \delta) = \sup \{ \|f_w - f_z\|_p : |z - w| < \delta, |z| = |w| = 1 \} \]
\[ = \sup \{ \|f_w - f_z\|_p : |z - w| < \delta, |z| = |w| \leq 1 \}. \]

The proof is completed. \( \square \)

**Proof of Theorem 3.** Again, we consider only the case \( p < 1 \). Let \( |w - z| \leq 1 - r, |z| = r < 1 \). Let \( \zeta = z/r \). Then \( |z - \zeta| = 1 - r \). From this and (8) it follows that
\[ \| |f_z| - |f_\zeta| \| \leq \omega (1 - r). \]

Therefore
\[ \| |f_w| - |f_z| \|_p \leq C_p \| |f_w| - |f_\zeta| \|_p + C_p \| |f_z| - |f_\zeta| \|_p \]
\[ \leq C_p \omega (|w - \zeta|) + C_p \omega (1 - r). \]

Since
\[ |w - \zeta| \leq |w - z| + |z - \zeta| \leq 2 (1 - r), \]
and
\[ \omega (2 (1 - r)) \leq 2 \omega (1 - r), \]
we see that
\[ \| |f_w| - |f_z| \|_p \leq C \omega (1 - r). \]

Hence, by Corollary 2,
\[ M_\rho^p (r, f') \leq C \frac{\omega (1 - r)}{1 - r}. \]

Thus we can apply Lemma 3 to conclude that
\[ \{\Omega_p (f, \delta)\}^p \leq C_p \int_0^\delta \frac{\omega(t)^p}{t} dt, \quad 0 < \delta < 1. \]

Finally, the desired conclusion follows from Corollary 4 below. \( \square \)

As a consequence of the above proof we have:
Theorem 4. For a majorant \( \omega \) and \( 0 < p \leq \infty \), let
\[
\psi(x) = \begin{cases} 
\int_0^x \frac{\omega(t)}{t} \, dt, & \text{if } 1 \leq p \leq \infty, \\
\left( \int_0^x \frac{\omega(t)^p}{t} \, dt \right)^{1/p}, & \text{if } 0 < p < 1.
\end{cases}
\]
If \( \psi(1) < \infty \), then \( \psi \) is a majorant, and there holds the implication
\[ |f| \in \Lambda^p_\omega(\mathbb{D}) \Rightarrow f \in \Lambda^p_\psi(\mathbb{D}). \]

4. Properties of majorants

Bernstein [2] introduced the notion of almost increasing and almost decreasing functions. A nonnegative real function \( \varphi(x) \) is almost increasing if there is a constant \( C > 0 \) such that \( x < y \) implies \( \varphi(x) \leq C \varphi(y) \). An almost decreasing function is defined similarly.

Proposition 2. A majorant \( \omega \) is fast if and only if there exists a constant \( \alpha > 0 \) such that the function \( \omega(x)/x^\alpha \) (\( 0 < x < 2 \)) is almost increasing.

A majorant \( \omega \) is slow if and only if there exists a constant \( \beta < 1 \) such that the function \( \omega(x)/x^\beta \) (\( 0 < x < 2 \)) is almost decreasing.

Corollary 4. If \( \omega(x) \) is a fast majorant, then so are \( \omega(x)^p \) and \( \omega(x^p) \) for \( 0 < p < 1 \). The same holds if “fast” is replaced by “slow”.

Proof of Proposition 2. The implications (11) \( \Rightarrow \) (1) and (12) \( \Rightarrow \) (2) are obvious. To prove the implication (1) \( \Rightarrow \) (11) let
\[
\psi(x) = \int_0^x \frac{\omega(t)}{t} \, dt, \quad 0 < x < 2.
\]
Since \( \omega(t)/t \) decreases, we have \( \psi(x) \geq \omega(x) \) and so (1) can be written as
\[
\omega(x) \leq \psi(x) \leq \omega(x)/\alpha,
\]
where \( \alpha \) is a positive constant. This implies \( x \psi'(x) \geq \alpha \psi(x) \), whence \( (\psi(x)/x^\alpha)' \geq 0 \). Thus \( \psi(x)/x^\alpha \) is increasing, which together with (13) implies that \( \omega(x)/x^\alpha \) is almost increasing.

The proof of the implication (2) \( \Rightarrow \) (12) is similar (see, e.g., [11, Lemma 3]).

5. Multiplication by inner functions

Let \( \Lambda^p_\omega(\mathbb{T}) \) denote the class of those functions \( h \in L^p(\mathbb{T}) \) for which
\[
\|h - h_\xi\|_p \leq C \omega(|1 - \xi|), \quad \xi \in \mathbb{T}.
\]
If \( I \) is an inner function, then \( |I(\xi)| = 1 \) a.e. on \( \mathbb{T} \). As is easily seen, this together with Theorem 3 implies that \( I \) belongs to \( \Lambda^p_\omega(\mathbb{D}) \) if and only if
\[
\|1 - |I_r|\|_p \leq C \omega(1 - r),
\]
where \( \omega \) is fast. More generally:
Theorem 5. Let \( f \in H^p \cap \Lambda^p_\omega(\mathbb{D}) \), where \( \omega \) is fast, and let \( I \) be an inner function. Then \( fI \) belongs to \( \Lambda^p_\omega(\mathbb{D}) \) if and only if
\[
\| f_r | (1 - |I_r|) \|_p \leq C \omega(1 - r), \quad 0 < r < 1.
\]
(14)

In the case where \( p \geq 1 \) and \( \omega(x) = x^\alpha, \quad 0 < \alpha < 1 \), this follows from a result of Böe [3, Corollary 3.2]. Böe’s approach is based on using the Poisson integral of \( |f_1| \) and therefore cannot be applied in the case \( p < 1 \).

Proof. Let \( g = fI \). From the hypothesis \( f \in \Lambda^p_\omega(\mathbb{D}) \) it follows that
\[
|g_1| \in \Lambda^p_\omega(\mathbb{T})
\]
(because \( |g_1| = |f_1| \) on \( \mathbb{T} \) and
\[
\| f_1 - |f_r| \|_p \leq C \omega(1 - r).
\]
(16)

Assuming (14) we have
\[
\| g_1 - |g_r| \|_p = \| f_1 - |f_r| I_r | \|_p \leq K \| f_1 - |f_r| \|_p + K \| |f_r|(1 - |I_r|) \|_p,
\]
where \( K = \max\{2^{1/p} - 1, 1\} \). Hence, (14) and (16) imply
\[
\| g_1 - |g_r| \|_p \leq C \omega(1 - r).
\]
This and (15) imply
\[
\| g_1 - |g_z| \|_p \leq C \omega(|1 - z|), \quad z \in \mathbb{D}.
\]

Now Theorem 3 shows that \( g \in \Lambda^p_\omega(\mathbb{D}) \).
Conversely, assume that \( g \in \Lambda^p_\omega(\mathbb{D}) \). Then
\[
\| f_r | (1 - |I_r|) \|_p = \| f_r | - |g_r| \|_p \leq K \| f_1 - |f_r| \|_p + K \| g_1 - |g_r| \|_p
\]
\[
\leq C \omega(1 - r) + C \omega(1 - r).
\]

This completes the proof. \( \square \)

As an application of Theorem 5 we prove the following.

Theorem 6. Let \( f \in H^p \cap \Lambda^p_\omega(\mathbb{D}) \), where \( \omega \) is fast, be such that \( 1/f \in H^\infty \), and let \( I \) be a singular inner function. If \( fI \) belongs to \( \Lambda^p_\omega(\mathbb{D}) \), then
\[
\omega(x) \geq \begin{cases} 
  cx^{1/(2p)}, & \text{for } p > 1/2, \\
  cx(\log \frac{2}{x})^2, & \text{for } p = 1/2,
\end{cases}
\]
(17)
where \( c \) is a positive constant.

As a special case we have the following result which, in the case \( p > 1/2 \) and \( \omega(x) = x^\alpha \), is due to Ahern [1].

Corollary 5. If a singular inner function belongs to \( \Lambda^p_\omega(\mathbb{D}) \), then (17) holds.
Conclusion (17) is the best possible because the so-called atomic function

\[ S(z) = \exp \left( -\frac{1+z}{1-z} \right) \]

belongs to the space \( \Lambda^p_{\omega_p}(\mathbb{D}) \), where

\[ \omega_p(x) = \begin{cases} 
  x^{1/(2p)}, & \text{for } p > 1/2, \\
  x(\log \frac{2}{x})^2, & \text{for } p = 1/2. 
\end{cases} \]

Namely, it was proved in [10] that

\[ M_p(r, S') \asymp \omega_p(1-r)^{1-r}. \]

In the case \( p > 1/2 \), Theorem 5 is an immediate consequence of the following result of Ahern [1].

**Theorem B.** If \( I \) is a singular inner function, then for every \( p > 0 \) we have

\[ \frac{1}{2\pi} \int_0^{2\pi} (1 - |I(re^{i\theta})|)^p d\theta \geq c_p(1-r)^{1/2}, \]

where \( c_p \) is a positive constant.

Ahern’s proof was based on a highly nontrivial analysis of singular measures, which enabled him to treat the case \( p > 1 \). In [15, Theorem 4.4.5] the subordination principle is used to improve Theorem B in the case \( p = 1/2 \).

**Theorem C.** With the hypotheses of Theorem B, we have

\[ \frac{1}{2\pi} \int_0^{2\pi} (1 - |I(re^{i\theta})|)^{1/2} d\theta \geq c(1-r)^{1/2} \log \frac{2}{1-r}, \]

where \( c \) is a positive constant.

In the case \( p = 1/2 \), Theorem 5 is a direct consequence of Theorem C.

### 6. The Poisson integral of \(|f|\)

For \( f \in H^p \), \( p \geq 1 \), let \( h \) denote the Poisson integral of \(|f_1|\), where \( f_1 \) is the boundary function of \( f \). It is well known that \( h \) is the smallest harmonic majorant of \(|f|\). It is another result of Dyakonov [5] that if \( f \in A(\mathbb{D}) \) and \( \omega \) is regular, then \( f \) is in \( \Lambda_\omega(\mathbb{D}) \) if and only if

\[ h(z) - |f(z)| \leq C \omega(1-|z|), \quad z \in \mathbb{D}. \]

This can be extended in the following way:

**Theorem 7.** Let \( \omega \) be a regular majorant. Let \( f \in H^p \), \( p \geq 1 \), and \(|f_1| \in \Lambda^p_{\omega} \). Then \( f \in \Lambda^p_{\omega}(\mathbb{D}) \) if and only if

\[ \|h_r - |f_r|\|_p \leq C \omega(1-r), \quad 0 < r < 1. \]
We omit the proof because the case $p < \infty$ is discussed in the same way as the case $p = \infty$; see the proof of Theorem B in [13]. In the case where $\omega(x) = x^\alpha, 0 < \alpha < 1$, Theorem 7 is due to Böe [3].

**Remark 2.** After completing the paper the author has learned of two papers by Dyakonov [7,8]. In [7], the formula $|\nabla f| = |f'|$ is systematically used to derive new information on the moduli of holomorphic functions. Paper [8] contains interesting observations concerning harmonic functions.

**References**

Corrigendum to “Remarks on $L^p$-oscillation of the modulus of a holomorphic function”  

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Four misprints should be corrected:

- Inequality (3) should be stated as
  
  \[ |f'(z)| \leq \frac{2}{\varepsilon} \sup \{|f(w)| - |f(z)| : w \in D_\varepsilon(z) \}. \]

- The oscillation $\text{osc}(u; z, \varepsilon)$ (after Eq. (4)) should be written as
  
  $\text{osc}(u; z, \varepsilon) = \sup \{|u(w) - u(z)| : w \in D_\varepsilon(z) \}.$

- The inequality at the bottom of page 3 should be written as
  
  \[ |f'(z)|^p \leq \frac{4p+1}{\varepsilon^{p+2}} \int_{D_\varepsilon(z)} (u^+)^p \, dm. \]

- The integrand in Proposition 1 should be equal
  
  \[ \| |f_w| - |f_z| \|^p. \]