Abstract

Let \( u(x) = \sum_{j=0}^{k-1} (1 - |x|^2)^j u_j(x) \) be a polyharmonic function in the unit ball \( B \subset \mathbb{R}^N \). Then, for \( p > 0 \),
\[
\int_{\partial B} \sup_{0 < r < 1} |u(ry)|^p \, d\sigma(y) < \infty
\]
if and only if
\[
\int_{\partial B} \sup_{0 < r < 1} [(1 - r^2)^j |u_j(ry)|]^p \, d\sigma(y) < \infty
\]
for all \( j \in \{0, 1, \ldots, k - 1\} \).

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Almansi representation theorem [1], if \( u \in \mathcal{H}_k(B) \), then there exist unique harmonic functions \( u_j \) such that
\[
u(x) = \sum_{j=0}^{k-1} (1 - |x|^2)^j u_j(x), \quad x \in B.
\]
Conversely, if \( u_j \in \mathcal{H}(B) := \mathcal{H}_1(B) \), then the function \( u \) defined by (1) is polyharmonic of order \( k \).

Our starting point here is the definition of the Hardy space \( H^p(B), 0 < p < \infty \), via the radial maximal function \( u^+ \) on \( \partial B \),
\[
u^+(y) = \sup_{0 \leq r < 1} |u(ry)|, \quad |y| = 1.
\]
Namely \( H^p(B) \) consists of those function \( u \) harmonic in \( B \) for which
\[
\|u\|_{H^p} := \|u^+\|_{L^p} = \left( \int_{\partial B} |u^+(y)|^p d\sigma(y) \right)^{1/p} < \infty,
\]
where \( d\sigma \) denotes the normalized surface measure on \( \partial B \). We extend this definition to the polyharmonic case: The space \( H^p_k(B) \) consists of those \( f \in \mathcal{H}_k(B) \) for which \( \|f\|_{H^p_k} := \|u^+_f\|_{L^p} < \infty \). Our main result is that \( u \) belongs to \( H^p_k(B) \) if and only if for every \( j \) the function \( (1 - |x|^2)^j u_j(x) \) belongs to \( H^p_{j+1}(B) \). In order to state the result in a more precise form we introduce the space \( H^{p,\alpha}(B), \alpha \geq 0 \): it consists of all harmonic functions \( u \) on \( B \) for which
\[
\|u\|_{H^{p,\alpha}} := \left( \int_{\partial B} \sup_{0 \leq r < 1} \left[ (1 - r^2)^\alpha |u(ry)| \right]^p d\sigma(y) \right)^{1/p} < \infty.
\]

**Theorem 1.** Let \( u \) be defined by (1), \( k \geq 2 \), and \( p > 0 \). Then \( u \in H^p_k(B) \) if and only if \( u_j \in H^{p-j}(B) \) for all \( j \in \{0, 1, \ldots, k-1\} \). Moreover, we have
\[
\|u\|_{H^p} \asymp \|u_0\|_{H^p} + \sum_{j=1}^{k-1} \|u_j\|_{H^{p-j}}, \quad u \in H^p_k(B).
\]
(We write \( A(s) \asymp B(s), s \in S \), to denote that \( A(s)/B(s) \) lies between two positive constants independent of \( s \).)

1. Subharmonic behavior

For the proof of Theorem 1 we need a number of facts concerning subharmonic behavior of \( |u|^p \) and \( |\nabla u|^p \). It is convenient to state these facts for wider classes of functions.

1.1. The class \( QNS(B) \)

Let \( QNS(B) \) denote the class of nonnegative measurable functions \( u \) on \( B \) for which there exists a constant \( Q = Q(u) \) such that
\[
u(a) \leq Qe^{-N} \int_{B(a, \varepsilon)} u \, dV
\]
whenever
\[ B(a, \varepsilon) := \{ x : |x - a| < \varepsilon \} \subset B. \]

Here \( dV \) denotes the normalized Lebesgue measure on \( B \). Members of \( QNS(B) \) are called quasi-nearly subharmonic functions \([9,10]\). The class \( QNS \) contains nonnegative subharmonic functions.

**Theorem A.** (See \([5,10]\).) Let \( p > 0 \). If \( u \in QNS(B) \), then \( u^p \in QNS(B) \), and \( Q(u^p) \leq C_{p,N} Q(u) \), where \( C_{p,N} \) is a constant.

1.2. The class \( HC^1(G) \)

This class consists of all locally Lipschitz functions \( u \) on \( B \) for which there exists a constant \( Q' = Q'(u) \) such that
\[
|\nabla u(a)| \leq \frac{Q'}{\varepsilon} \sup_{B(a, \varepsilon)} |u|, \quad B(a, \varepsilon) \subset B. \tag{3}
\]

1.3. The class \( OC^1(B) \)

This is the subclass of \( HC^1(B) \) consisting of those \( u \) for which
\[
|\nabla u(a)| \leq \frac{Q''}{\varepsilon} \sup \{|u(x) - u(a)| : x \in B(a, \varepsilon)\},
\]
where \( Q'' = Q''(u) \) is a constant independent of \( \varepsilon \) and \( a \).

**Theorem B.** (See \([5]\).)

(a) If \( u \in HC^1(B) \), then \( |u| \in QNS(B) \), and \( Q(|u|) \leq C_N Q'(u) \).

(b) If \( u \in OC^1(B) \), then both \( |u| \) and \( |\nabla u| \) belong to \( QNS(B) \), and \( Q(|\nabla u|) \leq C_N Q''(u) \).

**Theorem C.** (See \([6,7]\).) If \( u \) is a function polyharmonic in \( B \), then \( u \in OC^1(B) \) and consequently \( |u|^p \) and \( |\nabla u|^p \) are in \( QNS(B) \). Moreover if \( u \in H_k(B) \), then \( Q''(u) \leq C_{k,N} \) (where \( C_{k,N} \) depends only on \( k \) and \( N \)).

In the case where \( u \) is harmonic, this theorem is due to Hardy and Littlewood \([4]\) \((N = 2)\) and Fefferman and Stein \([2]\) \((N \geq 3)\).

2. Inequalities for maximal functions

For \( y \in \partial B \) and \( c > 1 \), let
\[
S_{c,y} = \{ x \in B : |x - y| < c(1 - |x|) \}
\]
(Stolz domain). For a function \( u \) on \( B \) let \( u^* = u^*_c \) denote the nontangential maximal function associated with \( S_{c,y} \), i.e.
\[
u^*(x) = \sup\{|u(x)| : x \in S_{c,y}\}.\]
It is clear that \( u^+(x) \leq u^*(x) \). Although the estimate \( u^*(x) \leq \text{const} u^+(x) \) need not hold we have the following:
Theorem D. If $u \in \text{QNS}(B)$ and $p > 0$, then
\[
\int_{\partial B} u^*(y)^p \, d\sigma(y) \leq C \int_{\partial B} u^+(y)^p \, d\sigma(y),
\]
where $C$ depends only on $p$, $N$ and $Q(u)$.

This fact is essentially due to Fefferman and Stein [2], who proved it in the case where $u$ is the modulus of a harmonic function on a half-space; see [3, Theorem III.3.6] and [8, Theorem 7.1.8] (the case of the unit disk).

Theorem 2. If $u \in \text{HC}^1(B)$ and $p > 0$, then
\[
\int_{\partial B} \sup_{0 \leq r < 1} \left[ (1 - r) |\nabla u(ry)| \right]^p \, d\sigma(y) \leq C \int_{\partial B} u^+(y)^p \, d\sigma(y),
\]
where $C$ depends only on $p$, $N$ and $Q'(u)$.

Proof. From the definition of $\text{HC}^1$ it follows that
\[
(1 - r) |\nabla u(ry)| \leq 2Q' \sup_{|x - ry| < \varepsilon} |u(x)|, \quad \varepsilon = (1 - r)/2.
\]
But if $|x - ry| < (1 - r)/2$, then $|x - y| < (3/2)(1 - r)$ and $1 - r < 2(1 - |x|)$ and therefore $x \in S_{3,y}$. Hence
\[
\sup_{0 \leq r < 1} (1 - r) |\nabla u(ry)| \leq 2Q'u_3^*(x).
\]
Now the desired result is obtained by integration and using Theorem D. \(\square\)

The following theorem is a generalization of Theorem 2.

Theorem 3. If $u \in \text{HC}^1(B)$, $p > 0$, and $-\infty < \alpha < \infty$, then
\[
\int_{\partial B} \sup_{0 \leq r < 1} \left[ (1 - r)^{\alpha + 1} |\nabla u(ry)| \right]^p \, d\sigma(y) \leq C \int_{\partial B} \sup_{0 \leq r < 1} \left[ (1 - r)^\alpha |u(ry)| \right]^p \, d\sigma(y),
\]
where $C$ depends only on $p$, $N$, $\alpha$ and $Q'(u)$.

Lemma 1. If $u \in \text{HC}^1(B)$ and $v(x) = (1 - |x|^2)^\alpha u(x)$, then $v \in \text{HC}^1(B)$.

Proof. Let $a \in B$ and $0 < \varepsilon < (1 - |a|)/2$. Since
\[
\nabla v(a) = -2a(1 - |a|^2)^{\alpha - 1} u(a) + (1 - |a|^2)^\alpha \nabla u(a),
\]
we have
\[
|\nabla v(a)| \leq C(1 - |a|)^{\alpha - 1} |u(a)| + C \frac{(1 - |a|)^\alpha}{\varepsilon} \sup_{x \in B(a, \varepsilon)} |u(x)|.
\]
Now use the inequalities $(1 - |a|)^{-1} < 2/\varepsilon$ and
\[
1 - |a| < 2, \quad |x| \leq \frac{3(1 - |a|)}{2}, \quad x \in B(a, \varepsilon),
\]
to obtain
\[
|\nabla v(a)| \leq \frac{C}{\varepsilon} \sup_{x \in B(a, \varepsilon)} |v(x)|
\]
for \(\varepsilon < (1 - |a|)/2\). This implies \(v \in HC^1(B)\).

**Proof of Theorem 3.** By (4) we have
\[
(1 - r)^{\alpha + 1} |\nabla u(ry)| \leq C (1 - r) |\nabla v(ry)| + C (1 - r)^{\alpha} |u(ry)|.
\]
Combining this with Theorem 2, applied to the function \(v\), we conclude the proof.

Applying Theorem 3 to the partial derivatives of a polyharmonic function \(u\) and using Theorem C (as well as the fact that if \(u\) is in \(H_k(B)\), then so are the partial derivatives of all orders) we get:

**Theorem 4.** If \(u \in H_k(B)\), \(p > 0\), and \(-\infty < \alpha < \infty\), then
\[
\int_{\partial B} \sup_{0 \leq r < 1} \left[ (1 - r)^{\alpha + 2} |\Delta u(ry)| \right]^p d\sigma(y) \leq C \int_{\partial B} \sup_{0 \leq r < 1} \left[ (1 - r)^{\alpha} |u(ry)| \right]^p d\sigma(y),
\]
where \(C\) depends only on \(p, N, \alpha\) and \(k\).

### 3. Radial derivatives and integrals

For a continuous function \(u\) on \(B\) and \(s > 0\) we define
\[
I_s u(x) = \int_0^1 u(tx) t^{s-1} dt, \quad x \in B,
\]
i.e.,
\[
I_s u(ry) = r^{-s} \int_0^r u(\rho y) \rho^{s-1} d\rho, \quad y \in \partial B, \quad 0 < r < 1.
\]
On the other hand, if \(u \in C^1(B)\), then we define
\[
R_s u = su + Ru,
\]
where
\[
Ru(x) = \sum_{i=1}^N x_i \frac{\partial u}{\partial x_i}, \quad x = (x_1, \ldots, x_N) \in B.
\]
It is easily checked that
\[
R_s I_s u = I_s R_s u = u,
\]
provided \(u \in C^1(B)\). It is clear that \(I_s\) maps \(H_k(B)\) to itself and since the analogous fact is true for \(R\) (see [1, Proposition 1.1]), we have from (5):
Lemma 2. \( I_s \) and \( R_s \) are isomorphisms from \( \mathcal{H}_k(B) \) onto \( \mathcal{H}_k(B) \).

We also need:

Lemma 3. (See [7, Lemma 3].) If \( u \) is given by (1), then

\[
\Delta^{k-1}u = (-1)^{k-1} c_k R_s R_{s+1} \cdots R_{s+k-2} u_{k-1},
\]

where \( s = N/2 \) and \( c_k = (k-1)!4^{k-1} \), and consequently

\[
u_{k-1} = (-1)^{k-1} c_k I_{s+1} \cdots I_{s+k-2} \Delta^{k-1} u.
\]

Lemma 4. If \( \alpha > 0, s > 0 \) and \( u \) is continuous on \( B \), then

\[
\int_{\partial B} \sup_{0 \leq r < 1} \left[ (1-r)^{\alpha} |I_s u(ry)| \right]^p d\sigma(y) \leq C \int_{\partial B} \sup_{0 \leq r < 1} \left[ (1-r)^{\alpha+1} |u(ry)| \right]^p d\sigma(y).
\]

This is true because

\[
\sup_{0 \leq r < 1} (1-r)^\alpha |I_s u(ry)| \leq C \sup_{0 \leq r < 1} (1-r)^{\alpha+1} |u(ry)|.
\]

4. Proof of Theorem 1

Since the inequality

\[
\|u\|_{H^p} \leq C \|u_0\|_{H^p} + C \sum_{j=1}^{k-1} \|u_j\|_{H^{p,j}}
\]

is obvious, we have to prove the reverse inequality. To do this let \( u \in H^p_k(B) \). We use formula (6) and apply Lemma 4 \( k-1 \) times to get

\[
\int_{\partial B} \sup_{0 \leq r < 1} \left[ (1-r)^{k-1} |u_{k-1}(ry)| \right]^p d\sigma(y) \leq C \int_{\partial B} \sup_{0 \leq r < 1} \left[ (1-r)^{2k-2} |\Delta^{k-1} u(ry)| \right]^p d\sigma(y).
\]

Now we apply Theorem 4 \( k-1 \) times and get

\[
\int_{\partial B} \sup_{0 \leq r < 1} \left[ (1-r)^{2k-2} |\Delta^{k-1} u(ry)| \right]^p d\sigma(y) \leq C \int_{\partial B} \sup_{0 \leq r < 1} |u(ry)|^p d\sigma(y).
\]

From these two inequalities we conclude that \( u_{k-1} \in H^{p,k-1}(B) \). But this together with the hypothesis \( u \in H^p_k(B) \) imply that the function

\[
\sum_{m=0}^{k-2} (1-|x|^2)^m u_m(x)
\]

belongs to \( H^p_{k-1}(B) \). Hence, by what we have proved, \( u_{k-2} \in H^{p,k-2}(B) \). Repeating the same argument we conclude that \( u_j \in H^{p,j}(B) \) for every \( j \) and that there holds the required reverse inequality. Thus the theorem is proved.
References