NOTE

Series Expansion and Reproducing Kernels for Hyperharmonic Functions

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First we show that any hyperbolically harmonic (hyperharmonic) function in the unit ball $B$ in $\mathbb{R}^n$ has a series expansion in hyperharmonic functions, and then we construct the kernel that reproduces hyperharmonic functions in some $L^1(B)$ space. We show that the same kernel also reproduces harmonic functions in $L^1(B)$.

Key Words: harmonic and hyperharmonic functions; hypergeometric functions.

1. INTRODUCTION

For $m > n - 1$ we define the measures $d\nu_m$ on the unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ by

$$d\nu_m(x) = \frac{2(1 - |x|^2)^{m-n}}{nB(n/2, m + 1 - n)} d\nu(x),$$

where $d\nu$ is the normalized Lebesgue measure on $B$ and $B(\cdot, \cdot)$ denotes the Euler beta function.

For a $C^2$ function $u$ on $B$ let

$$\Delta_h u(x) = (1 - |x|^2)[(1 - |x|^2)\Delta u(x) + 2(n - 2)Ru(x)];$$

here

$$Ru(x) = \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} \quad \text{and} \quad \Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$
A function $u \in C^2(B)$ is harmonic in $B$, $u \in H(B)$, if $\Delta u = 0$, and hyper-harmonic in $B$, $u \in h(B)$, if $\Delta_n u = 0$ (see [2]). (Note that if $n = 2$, then $H(B) = h(B)$, but if $n \geq 3$, then $h(B) \cap H(B) = \{\text{constants}\}$.)

The main purpose of this paper is to construct the kernel $K_m(x, y)$, $m \geq n$, that reproduces harmonic and hyperharmonic functions in the following sense:

(1.1) If $f \in H(B) \cap L^1(B, d\nu_m)$, $m \geq n$, then

$$f(y) = \int_B K_m(x, y) f(x) d\nu_m(x), \quad y \in B.$$ 

(1.2) If $f \in h(B) \cap L^1(B, d\nu_m)$, $m \geq n$, then

$$f(x) = \int_B K_m(x, y) f(y) d\nu_m(y), \quad x \in B.$$ 

We first prove (1.1) for an arbitrary harmonic polynomial $f$ by using the fact that it is a sum of harmonic homogeneous polynomials. If $f$ is arbitrary we can use the easily proved fact that harmonic polynomials are dense in $H(B) \cap L^1(B, d\nu_m)$.

To prove (1.2) we show that if $f \in h(B)$ then there exists a unique sequence of harmonic homogeneous polynomials $f_k$ of degree $k$, $f_k \in \mathcal{H}_k(\mathbb{R}^n)$, such that

(1.3) $f(x) = \sum_{k=0}^{\infty} F_k(x)f_k(x), \quad x \in B,$

where $F_0(x) = 1$ and $F_k(x) = F(k, 1 - n/2, k + n/2; |x|^2)$, $k \geq 1$ (as usual, $F(a, b, c; \cdot)$ denotes the hypergeometric function with parameters $a, b, c$ (see [5, Chap. III])). From this (1.2) follows easily.

A motivation for a series expansion (1.3) is paper [1] of Ahern et al., who considered the case of generalized $\mathcal{M}$-harmonic functions on the complex unit ball.

2. HYPERHARMONIC EXPANSIONS

In [1] it is shown that every generalized $\mathcal{M}$-harmonic function on the complex unit ball in $\mathbb{C}^n$ has a series expansion in homogeneous polynomials. In this section we prove an analogous result for hyperharmonic functions.

THEOREM 2.1. If $u \in h(B)$, then there exists a unique sequence of harmonic homogeneous polynomials $h_k \in \mathcal{H}_k(\mathbb{R}^n)$ such that

(2.1) $u(x) = \sum_{k=0}^{\infty} F_k(x)h_k(x), \quad x \in B,$
the series converging uniformly and absolutely on compact subsets of $B$. Conversely, the sum of any such series that converges uniformly on compact subsets of $B$ is hyperharmonic in $B$.

**Proof.** Assume, first, that the series $\sum_{k=0}^{\infty} F_k(x)h_k(x)$ converges uniformly on compact subsets of $B$. To prove that the sum is hyperharmonic in $B$, it is enough to show that $\Delta_h(F_kh_k) = 0$, $k = 1, 2, \ldots$ (see [2]).

Let $\Delta_h$ be the Laplace–Beltrami operator on the unit sphere $S := \partial B$. Then there holds the formula

\begin{equation}
(2.2) \quad \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{n - 1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \Delta_S u,
\end{equation}

where $u_r(y) = u(ry)$ for $y \in S$, $0 \leq r < 1$ (see [2]). A consequence is the formula

\begin{equation}
(2.3) \quad \Delta_S h_k(y) = -k(k + n - 2)h_k(y), \quad y \in S, k \geq 1.
\end{equation}

(See also [6, Section 3.1.4].)

The function $\phi_k(t) = F(k, 1 - n/2, k + n/2; t)$ satisfies the hypergeometric equation

\[ t(1-t)\phi''_k(t) + [k + n/2 - (2 + k - n/2)t] \phi'_k(t) - k(1 - n/2)\phi_k(t) = 0. \]

Then a straightforward, although tedious, computation based on (2.2) and (2.3) shows that $\Delta_h(F_kh_k) = 0$.

Now let $u \in h(B)$. Since $u_r$, $0 < r < 1$, is continuous on $S$ we have

\[ u(ry) = u_r(y) = \sum_{k=0}^{\infty} u_k(ry), \quad y \in S, 0 \leq r < 1, \]

where the series converges in the norm of $L^2(S, d\sigma)$ and

\[ u_k(ry) = \int_S u(ry)Z_k(y, \eta) d\sigma(\eta). \]

Here, $Z_k$ are the zonal harmonics (see [4]), and $d\sigma$ is the normalized surface measure on $S$.

Applying (2.3) $n$ times with $Z_k(y, \eta)$ instead of $h_k(\eta)$, and using the Green formula on the compact manifold $S$, we obtain

\[ u_k(ry) = [-k(k + n - 2)]^{-n} \int_S Z_k(y, \eta)\Delta^2_S u(ry) d\sigma(\eta), \quad \text{for } k \geq 1. \]

Now the uniform convergence follows from the estimate

\[ |Z_k(y, \eta)| \leq C(k + 1)^{n-2}, \quad \text{for } y, \eta \in S \text{ (see [4]).} \]
For fixed \( k \) and \( y \in S \) consider the function \( f(r) = u_k(ry) \). Since \( \Delta_k u = 0 \), we have

\[
\frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r} \frac{\partial u}{\partial r} + \frac{2(n-2)}{1-r^2} r \frac{\partial u}{\partial r} = -\frac{1}{r^2} \Delta_S u_r.
\]

Hence,

\[
f''(r) + \frac{n-1}{r} f'(r) + \frac{2(n-2)}{1-r^2} rf'(r) = -\frac{1}{r^2} \int_S Z_k(y, \eta)(\Delta_S u_r)(\eta) \, d\sigma(\eta).
\]

Applying Green's formula again, we get

\[
\int_S Z_k(y, \eta)(\Delta_S u_r)(\eta) \, d\sigma(\eta) = \int_S \Delta_S Z_k(y, \eta) u_r(\eta) \, d\sigma(\eta),
\]

where \( \Delta_S Z_k \) is taken with respect to \( \eta \).

As in (2.3) we find that

\[
\Delta_S Z_k(y, \eta) = -k(k+n-2) Z_k(y, \eta).
\]

Thus,

(2.4) \quad f''(r) + \frac{n-1}{r} f'(r) + \frac{2(n-2)}{1-r^2} rf'(r) = \frac{k(k+n-2)}{r^2} f(r).

Now we define the function \( g(t), 0 < t < 1 \), by \( f(t) = t^k g(t^2) \). After a direct computation, we find from (2.4) that

\[
t(1-t)g''(t) + [k+n/2 - (2 + k - n/2)r]g'(t) - k(1-n/2)g(t) = 0.
\]

This is the hypergeometric equation with parameters \( a = k, b = 1 - n/2, c = k + n/2 \). The general solution is given by

\[
g(t) = C_1 F(a, b, c; t) + C_2 t^{1-c} F(a', b', c'; t),
\]

where \( a' = a + 1 - c, b' = b + 1 - c, c' = 2 - c \) (see [5]).

In our case we have \( F(a', b', c'; 0) = 1 \) and \( g(t) = O(t^{-k/2}), t \to 0 \). It follows that \( u_k(ry) = A_k(y)F(a, b, c; r^2) r^k \). Clearly, \( A_k(y) \) is a spherical harmonics (i.e., the restriction to \( S \) of a harmonic homogeneous polynomial of degree \( k \)) because so is \( y \to Z_k(y, \eta) \). We define \( h_k \) by \( h_0(x) = u(0) \) and \( h_k(ry) = A_k(y) r^k \), whence \( u_k(x) = F_k(x) h_k(x) \). This finishes the proof of Theorem 2.1.
3. REPRODUCING KERNELS FOR HYPERHARMONIC FUNCTIONS

Following [4] let \( P(x, y) = (1 - |x|^2|y|^2)(1 - 2x \cdot y + |x|^2|y|^2)^{-n/2}, \) \( x \in B, y \in B, \) where \( x \cdot y \) stands for the inner product in the \( n \)-space. The kernel \( P(x, y) \) has the series expansion \( P(x, y) = \sum_{k=0}^{\infty} Z_k(x, y), \) \( x \in B, y \in B, \) where, now, \( Z_k(x, y) \) are the extended zonal harmonics (see [4]).

For an integer \( m \geq n \) let

\[
K_m(x, y) = 1 + \frac{1}{\Gamma(m)} \int_0^1 [(1 - t)(1 - t|x|^2)]^{m-1} \frac{dm}{dt} [t^{m-1}P(tx, y)] \, dt.
\]

**Lemma 3.1.** For \( x \in B \) and \( y \in \overline{B} \) we have

\[
K_m(x, y) = \frac{\Gamma(n/2)}{\Gamma(m)} \sum_{k=0}^{\infty} \frac{\Gamma(m + k)}{\Gamma(n/2 + k)} F(k, 1 - n/2, n/2 + k; |x|^2) Z_k(x, y).
\]

Here, as usual, \( \Gamma(\cdot) \) denotes the Euler gamma function.

**Proof.** Using the integral representation of hypergeometric functions,

\[
F(a, b, c; x) = \frac{1}{B(a, c - a)} \int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{(1-t)^b} \, dt, \quad c > a > 0 \text{ (see [5])},
\]

we find that

\[
F(k, 1 - n/2, n/2 + k; |x|^2) = \frac{1}{B(k, n/2)} \int_0^1 t^{k-1}[(1 - t)(1 - t|x|^2)]^{n/2-1} \, dt.
\]

Hence

\[
K_m(x, y) = 1 + \frac{1}{\Gamma(m)} \int_0^1 [(1 - t)(1 - t|x|^2)]^{m-1} \frac{dm}{dt} \left( \sum_{k=1}^{\infty} t^{m+k-1} Z_k(x, y) \right) \, dt
\]

\[
= 1 + \frac{1}{\Gamma(m)} \sum_{k=1}^{\infty} \frac{\Gamma(k + m)}{\Gamma(k)} \frac{dm}{dt} \left( \sum_{k=1}^{\infty} \frac{\Gamma(k + m)}{\Gamma(k)} t^{k-1} Z_k(x, y) \right) \, dt
\]

\[
= 1 + \frac{1}{\Gamma(m)} \sum_{k=1}^{\infty} \frac{\Gamma(k + m)}{\Gamma(k)} \int_0^1 t^{k-1} F(k, 1 - n/2, k + n/2; |x|^2) \, dt
\]

\[
\times \int_0^1 t^{m+k-1} Z_k(x, y) \, dt
\]

\[
= \frac{\Gamma(n/2)}{\Gamma(m)} \sum_{k=0}^{\infty} \frac{\Gamma(k + m)}{\Gamma(k + n/2)} F(k, 1 - n/2, k + n/2; |x|^2) Z_k(x, y).
\]

**Theorem 3.2.** Let \( m \geq n \) be an integer.

(i) If \( f \) is a harmonic function in \( L^1(B, dv_m) \) then

\[
f(y) = \int_B K_m(x, y) f(x) \, dv_m(x), \quad y \in B.
\]
(ii) If \( f \) is a hyperharmonic function of class \( L^1(B, dv_m) \) then

\[
f(x) = \int_B K_m(x, y) f(y) \, dv_m(y), \quad x \in B.
\]

**Proof.**

(i) Assume first that \( f \) is a harmonic polynomial of degree \( k \). Then \( f(x) = \sum_{j=0}^k f_j(x) \), where \( f_j \in \mathcal{H}(\mathbb{R}^n) \).

Using Lemma 3.1 and the orthogonality of spherical harmonics of different degree, we find that

\[
\int_B K_m(x, y) f(x) \, dv_m(x)
\]

\[
= \frac{\Gamma(n/2)}{\Gamma(m)} \sum_{j=0}^k \frac{\Gamma(m + j)}{\Gamma(n/2 + j)} \int_0^1 r^{n-1}(1 - r^2)^{m-n} F(j, 1 - n/2, j + n/2; r^2) \, dr
\]

\[
\times \frac{2}{nB(n/2, m + 1 - n)} \int_S Z_j(r, \eta) f_j(r\eta) \, d\sigma(\eta)
\]

\[
= \frac{2\Gamma(n/2)}{\Gamma(m)} \sum_{j=0}^k \frac{\Gamma(m + j)}{\Gamma(n/2 + j)B(n/2, m + 1 - n)}
\]

\[
\times \int_0^1 r^{n-1+2j}(1 - r^2)^{m-n} F(j, 1 - n/2, j + n/2; r^2) \, dr.
\]

Using the equality

\[
\int_0^1 F(a, b, c; \xi t) t^{c-1}(1 - t)^{d-c-1} \, dt = \frac{\Gamma(c)\Gamma(d - c)}{\Gamma(d)} F(a, b; d; \xi),
\]

which holds provided that \( d > c > 0 \) (see [5]), we find that the last integral is equal to

\[
\frac{1}{2} \int_0^1 r^{n/2 - 1 + j}/(1 - r)^{m-n} F(j, 1 - n/2, n/2 + j; r) \, dr
\]

\[
= \frac{1}{2} \frac{\Gamma(n/2 + j)\Gamma(m - n + 1)}{\Gamma(j + m - n/2 + 1)} F(j, 1 - n/2, m - n/2 + j + 1; 1)
\]

\[
= \frac{\Gamma(m - n + 1)}{2} \frac{\Gamma(n/2 + j)\Gamma(m - n + 1)}{\Gamma(m + j)}.
\]

Hence,

\[
\int_B K_m(x, y) f(x) \, dv_m(x) = \sum_{j=0}^k f_j(y) = f(y).
\]

If \( f \) is arbitrary, we can use the easily proved fact that harmonic polynomials are dense in the harmonic \( H(B) \cap L^1(B, dv_m) \) space.
(ii) By Theorem 2.1,

\[ f(y) = \sum_{k=0}^{\infty} F(k, 1 - n/2, k + n/2; |y|^2) f_k(y), \]

where \( f_k \in \mathcal{H}^k(\mathbb{R}^n) \). Assume that the series converges uniformly on \( B \). Then

\[ \int_{B} K_m(x, y) f(y) \, dv_m(y) \]

\[ = \sum_{k=0}^{\infty} \frac{\Gamma(n/2)}{\Gamma(m)} \frac{\Gamma(k + m)}{\Gamma(k + n/2)} F(k, 1 - n/2, k + n/2; |x|^2) \]

\[ \times \int_{B} F(k, 1 - n/2, k + n/2; |y|^2) f_k(y) Z_k(x, y) \, dv_m(y). \]

Here, we have used again the orthogonality of spherical harmonics of different degree.

The last integral equals

\[ \int_{B} F(k, 1 - n/2, k + n/2; |y|^2) f_k(y) Z_k(x, y) \, dv_m(y) \]

\[ = \frac{2n}{nB(n/2, m + 1 - n)} \int_{0}^{1} r^{n-1} F(k, 1 - n/2, k + n/2; r^2) r^k \]

\[ \times \int_{S} f_k(\eta) Z_k(x, \eta) \, d\sigma(\eta) \]

\[ = \frac{f_k(x)}{B(n/2, m + 1 - n)} \int_{0}^{1} F(k, 1 - n/2, k + n/2; r) r^{k+n/2-1} (1 - r)^{m-n} \, dr. \]

Since

\[ \int_{0}^{1} F(k, 1 - n/2, k + n/2; r) r^{k+n/2-1} (1 - r)^{m-n} \, dr \]

\[ = \frac{\Gamma(n/2 + k) \Gamma(m - n + 1) \Gamma(m)}{\Gamma(m - n/2 + 1) \Gamma(m + k)}, \]

we obtain

\[ \int_{B} K_m(x, y) f(y) \, dv_m(y) = \sum_{k=0}^{\infty} \frac{\Gamma(n/2) \Gamma(m + k)}{\Gamma(m) \Gamma(n/2 + k)} F(k, 1 - n/2, n/2 + k; |x|^2) \]

\[ \times \frac{\Gamma(n/2 + k) \Gamma(m - n + 1) \Gamma(m) f_k(x)}{B(n/2, m - n + 1) \Gamma(m - n/2 + 1) \Gamma(m + k)} \]

\[ = \sum_{k=0}^{\infty} F(k, 1 - n/2, k + n/2; |x|^2) f_k(x) = f(x). \]

If \( f(x) = \sum_{k=0}^{\infty} F_k(x) f_k(x) \) is hyperharmonic on \( B \) and \( 0 < \rho < 1 \), we define a function \( F_{\rho} \) by \( F_{\rho}(x) = \sum_{k=0}^{\infty} F_k(x) \rho^k f_k(x) \), \( x \in B \).

At this point we need a lemma.
LEMMA 3.3. Let $f \in L^1(B, d\nu_m) \cap h(B)$. Then
\[
\lim_{\rho \to 1} \| F_\rho - f \|_{L^1(B, d\nu_m)} = 0.
\]

Proof. Since $\phi_f(x) = \phi_f(\rho y) = \sum_{k=0}^{\infty} F_k(r^k f_k(\rho y)$, $0 \leq \rho \leq 1$, $y \in S$, is a harmonic function on $B$ for fixed $0 < \rho < 1$, we have
\[
\int_{S} | F_\rho(r^k(y)) | d\sigma(y) = \int_{S} | \phi_f(\rho y) | d\sigma(y) \leq \int_{S} | \phi_f(y) | d\sigma(y) = \int_{S} | f(r^k(y)) | d\sigma(y).
\]
Hence, $\int_{S} | f(\rho y) - F_\rho(\rho y) | d\sigma(y) \leq 2 \int_{S} | f(r^k(y)) | d\sigma(y)$, and the dominated convergence theorem shows that $\lim_{\rho \to 1} \| f - F_\rho \|_{L^1(B, d\nu_m)} = 0$.

Now let $f(x) = \sum_{k=0}^{\infty} F_k(x)f_k(x)$ be an arbitrary function in $L^1(B, d\nu_m) \cap h(B)$. Assuming that the series $\sum_{k=0}^{\infty} F_k(x)\rho^k f_k(x)$ converges uniformly on $B$ for any fixed $0 < \rho < 1$, we see that
\[
\int_{B} K_m(x, y)F_\rho(y) d\nu_m(y) = F_\rho(x).
\]
Using this and Lemma 3.3, we find that
\[
\int_{B} K_m(x, y)f(y) d\nu_m(y) = f(x).
\]

So, to finish the proof of Theorem 3.2(ii) it remains to show that the series $\sum_{k=0}^{\infty} F_k(x)\rho^k f_k(x)$ converges uniformly on $B$ for fixed $0 < \rho < 1$.

Let $x = r y$, $0 < r < 1$, $y \in S$. Using the same argument as in the proof of Theorem 2.1, we find that
\[
F_k(x)\rho^k f_k(x) = \frac{F_k(x)}{F_k(px)} F_k(px) f_k(px)
= \frac{F_k(x)}{F_k(px)} \left[-k(k+n-2)\right]^{-n} \int_{S} Z_k(y, \eta) \Delta^2_{x} f(rp \eta) d\sigma(\eta).
\]

The integral representation of hypergeometric functions, already used in the proof of Lemma 3.1, shows that
\[
F_k(x) = \frac{1}{B(k, n/2)} \int_{0}^{1} t^{k-1}(1 - t)(1 - t|x|^2)^{n/2-1} dt.
\]
Thus, $F_k(x) \leq F_k(px)$, $x \in B$, $k \geq 1$, $0 < \rho < 1$. Now the uniform convergence of the series $\sum_{k=0}^{\infty} F_k(x)\rho^k f_k(x)$ follows from this estimate and the estimates
\[
|Z_k(y, \eta)| \leq C(k+1)^{n-2}, \quad y, \eta \in S,
\]
and
\[
|\Delta^2_{x} f(rp \eta)| \leq C_f(\rho, n), \quad 0 < r < 1, \eta \in S.
\]

Remark. A similar argument shows that if $f \in L^1(B, d\nu_m) \cap h(B)$ and $P_h f_\rho$, $0 < \rho < 1$, is a hyperharmonic Poisson integral of a function $f_\rho$ defined by $f_\rho(y) = f(\rho y)$, $y \in S$, then $\lim_{\rho \to 1} \| f - P_h f_\rho \|_{L^1(B, d\nu_m)} = 0$. So, to finish the proof of Theorem 3.2 (ii), we can use the functions $P_h f_\rho$ instead of the functions $F_\rho$. 
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