LACUNARY SERIES IN MIXED NORM SPACES
ON THE BALL AND THE POLYDISK

Miroljub Jevtić and Miroslav Pavlović

Abstract
We characterize lacunary series in mixed norm spaces on the unit ball $B^n$ in $\mathbb{C}^n$ and on the unit polydisk $D^n$ in $\mathbb{C}^n$.

Introduction and main results
Let $n$ be a positive integer. Two domains will be used in the paper: the open unit ball $B^n$ in $\mathbb{C}^n$,

$$B^n = \{ z \in \mathbb{C}^n : |z| < 1 \},$$

and the open unit polydisk $D^n$ in $\mathbb{C}^n$,

$$D^n = \{ z = (z_1, ..., z_n) \in \mathbb{C}^n : |z_1| < 1, ..., |z_n| < 1 \}.$$

We write $D = B^1 = D^1$.

Denote by $T^n$ the Shilov boundary of $D^n$, by $\partial B^n$ the boundary of $B^n$, by $d\sigma_n$ the normalized surface measure on $\partial B^n$, and define the measure $d\mu_n$ on $T^n$ by

$$d\mu_n(e^{i\theta_1}, \ldots, e^{i\theta_n}) = d\theta_1 \cdots d\theta_n.$$

Lacunary series on the unit ball $B^n$
The mixed norm space $H^{p.q,\alpha}(B^n)$, $0 < p, q \leq \infty$ $0 < \alpha < \infty$, consists of all functions $f$ holomorphic in $B^n$, $f \in H(B^n)$, such that

$$||f||_{p,q,\alpha}^q = \int_0^1 (1-r)^{\alpha-1} M_p(r, f)^q dr < \infty,$$

and

$$||f||_{p,\infty,\alpha} = \sup_{0<r<1} (1-r)^\alpha M_p(r, f) < \infty.$$

2010 Mathematics Subject Classifications. 32A36, 32A37.

Key words and Phrases. Integral means, Bergman spaces, mixed norm spaces.

Received: March 4, 2010
Communicated by Dragan S. Djordjević
The research was supported by a grant from MNS ON144010, Serbia.
Here, as usual,

\[ M_p(r, f) = \left( \int_{\partial B^n} |f(r\xi)|^p d\sigma_n(\xi) \right)^{1/p}, \quad 0 < p < \infty, \]

and

\[ M_\infty(r, f) = \sup_{|\xi|=1} |f(r\xi)|. \]

We write \( ||f||_p = \sup_{0<r<1} M_p(r, f) \).

Note that when \( 0 < p = q < \infty \), then \( H^{p,p,(\alpha+1)/p}(\mathbb{B}^n) \), where \( \alpha > -1 \), coincides, as a topological linear space, with the weighted Bergman space \( A^{p,\alpha}(\mathbb{B}^n) \), consisting of those \( f \in H(\mathbb{B}^n) \) for which

\[ \int_{\mathbb{B}^n} |f(z)|^p (1 - |z|^2)^\alpha dV_n(z) < \infty, \]

where \( dV_n \) is the normalized volume measure on \( \mathbb{B}_n \).

We say that a holomorphic function \( f \) on \( \mathbb{B}^n \) has a lacunary expansion if its homogeneous expansion is of the form

\[ f(z) = \sum_{k=1}^{\infty} f_{m_k}(z), \]

where \( m_k \) satisfies the condition

\[ \inf_{1 \leq k < \infty} \frac{m_{k+1}}{m_k} = \lambda > 1. \]

The series \( \sum_{k=1}^{\infty} f_{m_k}(z) \) as well as the sequence \( \{m_k\} \) are then said to be lacunary.

In this paper we characterize holomorphic functions with lacunary expansions in mixed norm spaces \( H^{p,q,\alpha}(\mathbb{B}^n) \). More precisely, we prove

**Theorem 1.** Let \( 0 < p, q \leq \infty \), \( 0 < \alpha < \infty \) and let \( f(z) = \sum_{k=1}^{\infty} f_{m_k}(z) \) be a holomorphic function on \( \mathbb{B}^n \) with a lacunary expansion. Then \( f \in H^{p,q,\alpha}(\mathbb{B}^n) \) if and only if

\[ \sum_{k=1}^{\infty} \frac{||f_{m_k}||_p^q}{m_k^\alpha} < \infty \quad \text{if} \quad 0 < q < \infty, \]

or

\[ \sup_{1 \leq k < \infty} m_k^{-\alpha} ||f_{m_k}||_p < \infty, \quad \text{if} \quad q = \infty. \]

Lacunary series in \( H^{p,q,\alpha}(\mathbb{D}) \) are characterized in [MP]. (See also [JP]).

Our work was motivated by characterizations of lacunary series in weighted Bergman spaces \( A^{p,\alpha}(\mathbb{B}^n) \), see [Ch], [YO], and [St]. Case \( q = \infty \) of Theorem 1 also follows from [ZZ, Proposition 63]. We note that in [St] lacunary series in mixed norm spaces \( H^{p,q,\alpha}(\mathbb{B}^n) \) are considered and some partial results have been obtained.
Lacunary series on the unit polydisk in $\mathbb{C}^n$

For any Lebesgue measurable function $f$ in $\mathbb{D}^n$, we define
\[
M_p(r, f) = \left( \int_{\mathbb{T}^n} |f(r\xi)|^p d\mu_n(\xi) \right)^{1/p}, \quad 0 < p < \infty,
\]
and
\[
M_\infty(r, f) = \sup_{\xi \in \mathbb{T}^n} |f(r\xi)|,
\]
where $r = (r_1, \ldots, r_n)$.

If $0 < p \leq \infty$, $0 < q < \infty$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j > 0$, $j = 1, \ldots, n$, let
\[
||f||_{p,q,\alpha} = \int_{I^n} \left( \prod_{j=1}^n (1 - r_j)^{\alpha_j - 1} M_p(r, f)^q \right) dr,
\]
where $I^n = [0,1)^n$ and $dr = dr_1 \cdots dr_n$. The mixed norm space $H^{p,q,\alpha}(\mathbb{D}^n)$ is then defined to be the space of functions $f$ holomorphic in $\mathbb{D}^n$, $f \in H(\mathbb{D}^n)$, such that $||f||_{p,q,\alpha} < \infty$.

The mixed norm space $H^{p,\infty,\alpha}(\mathbb{D}^n)$, $0 < p \leq \infty$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_1 > 0$, ..., $\alpha_n > 0$, is the set of those functions $f \in H(\mathbb{D}^n)$ for which
\[
||f||_{p,\infty,\alpha} = \sup_{r \in I^n} \prod_{j=1}^n (1 - r_j)^{\alpha_j} M_p(r, f)
\]
is finite.

Our second result is a characterization of lacunary series in mixed norm spaces $H^{p,q,\alpha}(\mathbb{D}^n)$.

**THEOREM 2.** Let $0 < p \leq \infty, 0 < q \leq \infty, \alpha_j > 0$, $j = 1, \ldots, n$, and
\[
f(z) = \sum_{k_1,\ldots,k_n \geq 1} a_{k_1,\ldots,k_n} z_1^{m_{1,k_1}} \cdots z_n^{m_{n,k_n}}
\]
be a holomorphic function on $\mathbb{D}^n$ such that there is $\lambda > 1$ satisfying the condition
\[
m_{j,k_{j+1}} / m_{j,k_j} \geq \lambda \quad \text{for all} \quad k_j \in \mathbb{N}, \ j = 1, \ldots, n.
\]

If $0 < q < \infty$, then the following statements are equivalent:

(i) $f \in H^{p,q,\alpha}(\mathbb{D}^n)$;

(ii) $\sum_{k_1,\ldots,k_n \geq 1} |a_{k_1,\ldots,k_n}|^q (\prod_{j=1}^n m_{j,k_j})^{q/q} < \infty$. 
If \( q = \infty \), then the following statements are equivalent:

(iii) \( f \in H^{p,\infty,\alpha}(\mathbb{D}^n) \);

(iv) \( \sup_{k_1,\ldots,k_n \geq 1} \prod_{j=1}^n m_{j,k_j}^{\alpha_j} \prod_{j=1}^n |a_{k_1,\ldots,k_n}| < \infty \).

We note that the equivalence (iii) and (iv) also follows from [Av, Theorem 3]. The equivalence (i) \( \iff \) (ii) for \( 0 < p = q < \infty \) was proved in [St].

1 Preliminaries

In this section we gather several well-known lemmas that will be used in the proofs of our results.

Lemma 1. [P] Let \( \alpha > -1, 0 < q < \infty \) and \( I_n = \{k \in \mathbb{N} : 2^n \leq k < 2^{n+1}\} \) for \( n \geq 1 \), \( I_0 = \{0, 1\} \). If \( \{a_n\}_n \) is a sequence of non-negative numbers such that the series \( G(r) = \sum_{n=0}^\infty a_n r^n \) converges for every \( r \in (0, 1) \), then the following two conditions are equivalent and the corresponding quantities are “proportional”:

(i) \( \int_0^1 (1 - r)^\alpha G(r)^q dr < \infty; \)

(ii) \( \sum_{n=0}^\infty 2^{-n(\alpha + 1)} \left( \sum_{k \in I_n} a_k \right)^q < \infty. \)

In the case of the function \( G(r) = \sup_{n \geq 0} a_n r^n \) in (i) the expression \( \sum_{k \in I_n} a_k \) in (ii) should be replaced by \( \sup_{k \in I_n} a_k \).

Lemma 2. If \( \{n_k\} \) is a lacunary sequence of positive integers, that is \( \inf_k \frac{n_{k+1}}{n_k} = \lambda > 1 \), and \( \{a_k\} \) is a sequence of nonnegative real numbers, then the following conditions are equivalent and the corresponding quantities are “proportional”:

(i) \( \int_0^1 (1 - r)^\alpha (\sum_{k=1}^\infty a_k r^{n_k})^q dr < \infty; \)

(ii) \( \int_0^1 (1 - r)^\alpha (\sup_{k \geq 1} a_k r^{n_k})^q dr < \infty; \)

(iii) \( \sum_{k=1}^\infty \frac{|a_k|^q}{n_k^\lambda} < \infty. \)

Proof. By Lemma 1,

\[
\int_0^1 (1 - r)^\alpha (\sum_{k=1}^\infty a_k r^{n_k})^q dr \approx \sum_{k=1}^\infty 2^{-k(\alpha + 1)} \left( \sum_{n_j \in I_k} a_j \right)^q.
\]
Lacunary series in mixed norm spaces on the ball and the polydisk

Since $\frac{n_{j+1}}{n_j} \geq \lambda > 1$, for all $j \in N$, the number of $a_j$ when $n_j \in I_k$ is at most $[\log_2 \lambda] + 2$. Using this and the fact that $n_j \simeq 2^k$ when $n_j \in I_k$, we see that

$$\sum_{k=1}^{\infty} 2^{-k(\alpha+1)} \left( \sum_{n_j \in I_k} a_j \right)^q \simeq \sum_{k=1}^{\infty} \frac{a_k^q}{n_k^{\alpha+1}}.$$ 

**Lemma 3.** [Zy, Du, P] Let $0 < p < \infty$. If $\{n_k\}$ is an increasing sequence of positive integers satisfying $n_{k+1}/n_k \geq \lambda > 1$ for all $k$, then there is a positive constant $C$ depending only on $p$ and $\lambda$ such that

$$C^{-1} \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \leq \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} a_k e^{im_k \theta} \right|^p d\theta \right)^{1/p} \leq C \left( \sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2}.$$ 

These Paley’s inequalities were extended to the unit polydisk $\mathbb{D}^n$ in [Av]:

**Lemma 4.** Let $\{m_j,k_j\}_{j=1}^{\infty}, j = 1, \ldots, n,$ be arbitrary lacunary sequences and $f(z)$ be a holomorphic function in $\mathbb{D}^n$ given by

$$f(z) = \sum_{k_1,\ldots,k_n \geq 1} a_{k_1,\ldots,k_n} z_1^{m_{k_1}} \cdots z_n^{m_{k_n}}, \quad z = (z_1, \ldots, z_n) \in \mathbb{D}^n.$$ 

Then for any $p$, $0 < p < \infty$, $f$ is in the Hardy space $H^p(\mathbb{D}^n)$, i.e. $||f||_p = \sup_{r \in \mathbb{R}} M_p(r,f) < \infty$, if and only if $\sum_{k_1,\ldots,k_n \geq 1} |a_{k_1,\ldots,k_n}|^2 < \infty$. Moreover,

$$C^{-1} ||f||_p \leq \left( \sum_{k_1,\ldots,k_n \geq 1} |a_{k_1,\ldots,k_n}|^2 \right)^{1/2} \leq C ||f||_p,$$

where $C$ is a constant independent of $f$.

**2 Proof of Theorem 1**

Let

$$\sum_{k=1}^{\infty} \frac{||f_{n_k}||_p^q}{n_k^{\alpha q}} < \infty, \quad 0 < p \leq \infty, \quad 0 < q < \infty.$$ 

If $1 \leq p < \infty$, then by using Minkowski’s inequality we obtain

$$M_p(r,f) \leq \sum_{k=1}^{\infty} ||f_{n_k}||_p r^{n_k}.$$  \hspace{1cm} (1)

If $p = \infty$, then

$$M_\infty(r,f) \leq \sum_{k=1}^{\infty} ||f_{n_k}||_\infty r^{n_k}. \hspace{1cm} (2)$$
An application of Lemma 2 gives

\[ ||f||^q_{p,q,\alpha} \leq \int_0^1 (1 - r)^{q\alpha - 1} \left( \sum_{k=1}^{\infty} ||f_k||_p r^{n_k} \right)^q dr \]

\[ \leq C \sum_{k=1}^{\infty} \frac{||f_k||_p}{n_k^{q\alpha}}. \]

If 0 < p < 1, then

\[ M^p_p(r, f) \leq \sum_{k=1}^{\infty} ||f_k||_p r^{p n_k}. \]

Hence,

\[ ||f||^q_{p,q,\alpha} \leq \int_0^1 (1 - r)^{q\alpha - 1} \left( \sum_{k=1}^{\infty} ||f_k||_p r^{p n_k} \right)^{q/p} dr \]

\[ \leq C \int_0^1 (1 - r)^{q\alpha - 1} \left( \sum_{k=1}^{\infty} ||f_k||_p r^{p n_k} \right)^{q/p} dr \]

\[ \leq C \sum_{k=1}^{\infty} \frac{||f_k||_p^{q/p}}{n_k^{q\alpha}}, \]

by Lemma 2.

If \( \alpha > 0 \) and \( \{n_k\} \) is a lacunary sequence of positive integers, then

\[ \sum_{k=1}^{\infty} n_k^{q\alpha} = O\left( \frac{1}{(1 - r)^{\alpha}} \right), \quad \text{see [Du].} \]

Using this, (1), (2), and (3) we find that

\[ ||f||_{p,\infty,\alpha} = \sup_{0 < r < 1} (1 - r)^{\alpha} M_p(r, f) \leq C \sup_{k \geq 1} \frac{||f_k||_p}{n_k^{q\alpha}}. \]

Conversely, let \( ||f||_{p,q,\alpha} < \infty. \)

If 0 < p < \( \infty \), then by using the slice integration formula [Ru2, Proposition 1.4.7] and Lemma 3 we find that

\[ M_p(r, f) = \left( \int_{\mathbb{D}^n} \left| \sum_{k=1}^{\infty} f_n(\xi) r^{n_k} \right|^p d\sigma(\xi) \right)^{1/p} \]

\[ = \left( \int_{\mathbb{D}^n} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} f_n(\xi) r^{n_k} e^{in_k \theta} \right|^p d\theta \right) d\sigma(\xi) \right)^{1/p} \]

\[ \cong \left( \int_{\mathbb{D}^n} \left( \sum_{k=1}^{\infty} |f_n(\xi)|^{2r^{2n_k}} ||f_k||_p^2 d\sigma(\xi) \right)^{1/p}, \right] \]
and consequently

\[ M_p(r, f) \geq C \| f_{n_k} \|_p r^{n_k}, \quad \text{for all} \quad k \geq 1. \]

If \( p = \infty \), also we have \( M_{\infty}(r, f) \geq \| f_{n_k} \|_{\infty} r^{n_k}, \) for all \( k \geq 1 \).

Thus, if \( 0 < q < \infty \), then

\[ \| f \|_{p,q,\alpha}^q \geq \int_0^1 (1 - r)^{q \alpha - 1} \left( \sum_{k \geq 1} \| f_{n_k} \|_p r^{n_k} \right)^q dr \]

by Lemma 2.

If \( q = \infty \), then

\[ \| f \|_{p,\infty,\alpha} \geq \sup_{0 < r < 1} (1 - r)^{\alpha} \sup_{k \geq 1} \| f_{n_k} \|_p r^{n_k} \]

\[ \geq \sup_{k \geq 1} \| f_{n_k} \|_p \frac{1}{n_k^\alpha} \left( 1 - \frac{1}{n_k} \right)^{n_k} \]

\[ \geq e \sup_{k \geq 1} \| f_{n_k} \|_p \frac{1}{n_k^\alpha}. \]

This finishes the proof of Theorem 1.

3 Proof of Theorem 2

In order to avoid too much calculations we will assume that \( n = 2 \).

Proof of implications (ii) \( \implies \) (i) and (iv) \( \implies \) (iii)

Let \( 0 < p \leq \infty \), \( r = (r_1, r_2) \) and \( \alpha = (\alpha_1, \alpha_2), \alpha_1 > 0, \alpha_2 > 0 \). Then

\[ M_p(r, f) \leq \sum_{k_1, k_2 \geq 1} |a_{k_1, k_2}| r_1^{m_1 k_1} r_2^{m_2 k_2}. \]

If \( 0 < q < \infty \) then by applying Lemma 2 twice we obtain

\[ \| f \|_{p,q,\alpha}^q = \int_0^1 (1 - r_2)^{q_2 \alpha_2 - 1} dr_2 \int_0^1 (1 - r_1)^{q_1 \alpha_1 - 1} M_p(r, f)^q dr_1 \]

\[ \leq \int_0^1 (1 - r_2)^{q_2 \alpha_2 - 1} dr_2 \int_0^1 (1 - r_1)^{q_1 \alpha_1 - 1} \]

\[ \times \left( \sum_{k_1 \geq 1} \left( \sum_{k_2 \geq 1} |a_{k_1, k_2}| r_2^{m_2 k_2} r_1^{m_1 k_1} \right)^q \right) dr_2 \]

\[ \leq C \int_0^1 (1 - r_2)^{q_2 \alpha_2 - 1} \left( \sum_{k_1 \geq 1} \frac{1}{n_1^{q_1 k_1}} \left( \sum_{k_2 \geq 1} |a_{k_1, k_2}| r_2^{m_2 k_2} \right)^q \right) dr_2 \]
\( \sum_{k_1 \geq 1, k_2 \geq 1} m_{1,k_1}^{-q_{\alpha_1}} \int_0^1 (1 - r_2)^{q_{\alpha_2} - 1} (\sum_{k_2 \geq 1} |a_{k_1,k_2}| r_2^{m_{2,k_2}})^q dr_2 \)
\( \leq C \sum_{k_1 \geq 1, k_2 \geq 1} m_{1,k_1}^{-q_{\alpha_1}} m_{2,k_2}^{-q_{\alpha_2}} |a_{k_1,k_2}|^q. \)

If \( q = \infty \), then we have
\[
\|f\|_{p,\infty,\alpha} = \sup_{0<r_1<1, 0<r_2<1} (1 - r_1)^{\alpha_1} (1 - r_2)^{\alpha_2} M_p(r, f)
\leq \sup_{0<r_1<1, 0<r_2<1} (1 - r_1)^{\alpha_1} (1 - r_2)^{\alpha_2} \sum_{k_1,k_2 \geq 1} |a_{k_1,k_2}| r_1^{m_{1,k_1}} r_2^{m_{2,k_2}}
\leq \sup_{0<r_1<1} (1 - r_1)^{\alpha_1} \sum_{k_1 \geq 1} (\sup_{0<r_2<1} (1 - r_2)^{\alpha_2} \sum_{k_2 \geq 1} |a_{k_1,k_2}| r_2^{m_{2,k_2}})^{m_{1,k_1}} r_1
\leq C \sup_{0<r_1<1} (1 - r_1)^{\alpha_1} \sum_{k_1 \geq 1} \sum_{k_2 \geq 1} |a_{k_1,k_2}| r_1^{m_{1,k_1}} m_{2,k_2}^{m_{1,k_1}}
\leq C \sup_{k_1 \geq 1, k_2 \geq 1} \frac{|a_{k_1,k_2}|}{m_{1,k_1}^{\alpha_1} m_{2,k_2}^{\alpha_2}}.
\]

**Proof of implications** (i) \( \implies \) (ii) and (iii) \( \implies \) (iv)

By Lemma 4 we have
\[
M_p(r, f) \cong \left( \sum_{k_1,k_2 \geq 1} |a_{k_1,k_2}|^2 r_1^{2m_{1,k_1}} r_2^{2m_{2,k_2}} \right)^{1/2}.
\]

Thus
\[
M_p(r, f) \geq \sup_{k_1,k_2 \geq 1} |a_{k_1,k_2}| r_1^{m_{1,k_1}} r_2^{m_{2,k_2}}, \quad 0 < p < \infty.
\]

This holds also for \( p = \infty \). Hence, if \( 0 < q < \infty \), by applying Lemma 2 twice we get
\[
\|f\|_{p,q,\alpha}^q \geq \int_0^1 (1 - r_1)^{q_{\alpha_1} - 1} dr_1 \int_0^1 (1 - r_2)^{q_{\alpha_2} - 1} dr_2 
\times \left( \sup_{k_1 \geq 1} (\sup_{k_2 \geq 1} |a_{k_1,k_2}| r_1^{m_{1,k_1}} r_2^{m_{2,k_2}}) \right)^q dr_2
\geq C \int_0^1 (1 - r_1)^{q_{\alpha_1} - 1} \sum_{k_2 \geq 1} m_{2,k_2}^{-q_{\alpha_2}} (\sup_{k_1 \geq 1} |a_{k_1,k_2}| r_1^{m_{1,k_1}})^q dr_1
\geq C \sum_{k_1 \geq 1} m_{2,k_2}^{-q_{\alpha_2}} \int_0^1 (1 - r_1)^{q_{\alpha_1} - 1} (\sup_{k_1 \geq 1} |a_{k_1,k_2}| r_1^{m_{1,k_1}})^q dr_1
\geq C \sum_{k_1 \geq 1} \sum_{k_2 \geq 1} m_{2,k_2}^{-q_{\alpha_2}} m_{1,k_1}^{-q_{\alpha_1}} |a_{k_1,k_2}|^q.
\]
If \( q = \infty \), then

\[
||f||_{p,\infty,\alpha} = \sup_{0<r_1<1} \sup_{0<r_2<1} (1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2} M_p(r, f) \\
\geq \sup_{0<r_1<1} \sup_{0<r_2<1} (1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2} \sup_{k_1,k_2 \geq 1} |a_{k_1,k_2}| r_1^{m_1,k_1} r_2^{m_2,k_2} \\
\geq C \sup_{k_1,k_2 \geq 1} \left| a_{k_1,k_2} \right| r_1^{m_1,k_1} r_2^{m_2,k_2}.
\]

This finishes the proof of Theorem 2.

References


[YO] W. Yang, C. Oyang, Exact location of \( \alpha \)-Bloch spaces in \( L^p_0 \) and \( H^p \) of a complex unit ball, *Rocky Mountain J. Math.* 30(2000), 1151-1169.


Miroljub Jevtić:
Matematički Fakultet, p.p. 550, 11000 Belgrade, Serbia
E-mail: jevtic@matf.bg.ac.rs

Miroslav Pavlović:
Matematički Fakultet, p.p. 550, 11000 Belgrade, Serbia
E-mail: pavlovic@matf.bg.ac.rs