CORRIGENDA AND ADDENDA TO “FUNCTION CLASSES”
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MIROSLAV PAVLOVIĆ

Abstract. We discuss some inaccuracies that arose mainly because of the author’s disregard. A question is added concerning \((C, \alpha)\)-summability of the Taylor series on the unit circle. A full proof of Spencer’s area theorem for univalent functions is given. In this new version we prove a Hardy–Stein identity for univalent harmonic mappings and a Holland–Twomey–Spencer area theorem for QC harmonic mappings.

Although I read the book several times, errors of various kinds (mathematical, linguistic, stylistic, etc.) will doubtless persist. When you find them, I will appreciate it if you call them to my attention.

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1. Corrigendum: Definition of admissible spaces

In the book, admissible spaces are defined by the sentence: “A quasinormed space \(X \subset h(\mathbb{D})\) will be called admissible (or \(h\)-admissible) if it is complete, \(h(\mathbb{D}) \subset X\), and the inclusion \(X \subset h(\mathbb{D})\) is continuous”. This sentence should be continued with: “and additionally, in contrast to the paper [70], \(\sup_{0 < r < 1} \|f_r\|_X \leq C_X \|f\|_X\), for \(f \in X\).”

Here \(h(\mathbb{D})\), resp. \(h(\overline{\mathbb{D}})\), is the class of all complex-valued functions harmonic in \(\mathbb{D}\), resp. in a neighborhood of the closed disc \(\overline{\mathbb{D}}\). Thus the three conditions occur:

(1) \(X \subset h(\mathbb{D})\) continuously;
(2) \(h(\overline{\mathbb{D}}) \subset X\);
(3) \(\sup_{0 < r < 1} \|f_r\|_X \leq C_X \|f\|_X\).

Condition (3) seems to be indispensable in proving the implication

\[
\text{if the harmonic polynomials are dense in } X, \text{ then } \|f - f_r\|_X \to 0 \text{ as } r \uparrow 1, \quad (†)
\]

The addition of (3) to the definition does not affect any result\(^1\) of the book because all the spaces satisfying (1)&(2) satisfy a condition stronger than (3): \(\sup_{w \in \overline{\mathbb{D}}} \|f_w\| \leq C_X \|f\|_X\).

I think that (†) cannot be deduced from (1)&(2) [without appealing to (3)]. On the other hand, the implication (1)&(2) \(\implies\) (3) is not true, which can be seen by considering the space of sequences of bounded \(p\)-variation \((p < 1)\).

\(^1\)Lines 11 and 12 on page 13 are to be deleted.
2. Corrigendum: Proof of Theorem 1.25

The phrase “by the subharmonicity of $f$” should, of course, be replaced with “...of $\log|f|$”. The last four lines on pp. 26, 27 are superfluous. The following two inequalities (stated on p. 26) are sufficient:

\[
\log |f(0)| \leq \frac{1}{T} \int_\mathbb{T} \log |f(re^{i\theta})| \, d\theta, \\
\log |f(re^{i\theta})| \leq \frac{1}{p} \sup_{0 < r < 1} |f(re^{i\theta})|^p.
\]

By the complex maximal theorem, we have that (a) the integral \( \int_\mathbb{T} \log |f_r(e^{i\theta})| \, d\theta \) exists (as finite or \(-\infty\)), and (b) the family \( \log |f_r| \) \((0 < p < 1)\) has an integrable majorant. Because of (b) we can apply the “limesup” variant of Fatou’s lemma along with (\(\dagger\)) to obtain

\[
\infty < \log |f(0)| \leq \limsup_{r \uparrow 1} \frac{1}{T} \int_\mathbb{T} \log |f(re^{i\theta})| \, d\theta \\
\leq \frac{1}{T} \limsup_{r \uparrow 1} \int_\mathbb{T} \log |f(re^{i\theta})| \, d\theta = \frac{1}{T} \int_\mathbb{T} \log |f(e^{i\theta})| \, d\theta.
\]

This and (a) imply that \( \log |f_r| \) is in \(L^1(\mathbb{T})\). In the general case, we consider the functions \( f \circ \sigma_a \), where \( \sigma_a = \frac{a - z}{1 - \overline{a}z} \). This should be all.

Remark. In fact, in proving (\(\dagger\)) we need not use the subharmonicity of \( \log |f| \) because the function \( f_r \) \((0 < r < 1)\) has a finite number of zeroes in \( \mathbb{D} \) so we have

\[
\log |f_r(z)| = \log |A(z)| + \log |g(z)|, \quad \text{where } A(z) = \prod_{k=1}^m \sigma_{a_k}(z), \quad a_k \in \mathbb{D},
\]

and \( g \) is zero-free in \( \mathbb{D} \) and hence \( \log |g| \in h(\mathbb{D}) \). Although \( g \) can have zeroes on \( \mathbb{T} \) (the case which can also be avoided) we still have

\[
\frac{1}{T} \int_\mathbb{T} \log |g(e^{i\theta})| \, d\theta = \log |g(0)|,
\]

which leads to (\(\dagger\)) immediately.

3. What is misprinted or omitted

- p. 24, line \(-1\): Section 1.2, p. 13, line 1
- p. 34, line \(-6\): Remove “there exists \( u \in h_p^\infty \) such that”
- p. 50, line \(-2\): Replace “If \( S \) is inner” with “If \( S \) is singular and nonconstant”
- p. 62, Exercise 2.10: “from Exercise 2.5”
- p. 119, Lemma 4.1: such that \( f = g + h \) and
- p. 132, lines 2.3: Replace \( \sigma_n \) with \( \sigma_k \)
- p. 132, Remark 4.4: Replace “Theorem 4.13” with “(4.20)” and “\( \alpha = 1/2 \)” with “\( \alpha = -1/2 \)”
- p. 177, line \(-1\): Section 11.1, p. 319, line \(-4\)
- p. 190, line \(-4\): Section 11.1, p. 319, line \(-4\)
- p. 194, line 3: (log \( x \))^{1/2}
- p. 249, formula (8.9): \( \frac{1}{T^2} \ldots \)
- p. 267, line 6: “See Proposition 3.11”
- p. 299, Theorem 10.10: Holland–Twomey–Spencer
- p. 309, line \(-1\): “That is all” should, of course, be continued with “if \( \arg w > 0 \)” and “For the general case, see [456].” For the proof of Theorem 10.10, see Section 5 below.
- p. 314, Corollary 10.8: Replace “\( 0 < p < \sigma \)” with “\( 0 < p \leq 2 \)”
- p. 319, line \(-6\): (Section 1.1, p. 12, line \(-9\)) ... (Section 1.2, p. 13, line 1)
- p. 313, line 7: Replace ”Using the fact that the zero of \( f \) isolated” with “Using the quasiconformality of \( f \)”
- p. 313, first line below (10.35): Replace ”quasiconformality” with “injectivity”
- p. 313, (10.35): “\( g/h \)” should be replaced with “\( h/g \)”
- p. 325, line \(-9\): Replace \( (x - 1/2) \) with \( (x - 1/2) \log x \)
4. Addendum: A conjecture on \((C, \alpha)\) convergence

On page 192, Problem 4.1, I posed the question: Whether there exists a function \(f \in H^p(\mathbb{T})\), \(1 < p \leq 2\), such that \(\sigma_n^{1/p-1} f(\zeta)\) diverges for a.e. \(\zeta \in \mathbb{T}\)? Here we add:

**Conjecture.** Let \(1 < p \leq 2\) and \(\alpha = 1/p - 1\).

(i) If \(\beta > \alpha\) then \(\sigma_n^\alpha f(\zeta) \to f(\zeta)\) for a.e. \(\zeta \in \mathbb{T}\).

(ii) There is a function \(f \in H^p(\mathbb{T})\) such that \(\sigma_n^\alpha f(\zeta)\) diverges a.e.

(iii) There exists a function \(f \in C(\mathbb{T})\) such that \(\sigma_n^{-1/2} f(\zeta)\) diverges a.e.

Of course, (iii) imply (ii) for \(p = 1/2\).

We recall that if \(f \in H^p\), \(p < 1\), then \(\sigma_n^\alpha(\zeta) \to f(\zeta)\) [Zygmund]. On the other hand, there exists a function \(f \in H^1(\mathbb{T})\) such that \(\sigma_n^\alpha f(\zeta)\) diverges a.e. [Kolmogorov–Hardy–Rogosinski].

The above conjecture is based on the relation [4, Theorem 4.13]

\[
\liminf_{n \to \infty} |\sigma_n^\alpha f(\zeta) - f(\zeta)| = 0, \quad \text{for a.e. } \zeta \in \mathbb{T},
\]

which is obtained from

\[
\liminf_{n \to \infty} \|\sigma_n^\alpha f - f\|_p = 0
\]

by means of Fatou’s lemma, whereas (2) follows from

\[
\lim_{n \to \infty} \frac{1}{L_n} \sum_{k=0}^{n} \frac{1}{k+1} \|\sigma_n^\alpha f - f\|_p^p = 0,
\]

where

\[
L_n = \sum_{k=0}^{n} \frac{1}{k+1}.
\]

The latter can be “easily” deduced from

\[
\frac{1}{L_n} \sum_{k=0}^{n} \frac{1}{k+1} \|\sigma_n^\alpha f\|_p^p \leq C_p \|f\|_p^p,
\]

and the deduction is left to the reader.

The “reader’s” proof of (3). Let \(f \in H^p\), \(\varepsilon > 0\), and choose a polynomial \(P\) of degree \(s\) such that \(\|f - P\| < \varepsilon\). Then we have

\[
S_n f := \frac{1}{L_n} \sum_{k=0}^{n} \frac{1}{k+1} \|\sigma_k^\alpha f - f\|_p^p
\]

\[
= \frac{1}{L_n} \sum_{k=0}^{m} \ldots + \frac{1}{L_n} \sum_{k=m+1}^{n} \ldots =: Q_n + R_n, \quad n > m > s,
\]

where \(m\) is fixed integer which will be chosen later on. It is easily checked that \(\lim_{n \to \infty} Q_n = 0\). In order to estimate \(R_n\) we start from the inequality

\[
R_n \leq 2^{p-1} \frac{1}{L_n} \sum_{k=m+1}^{n} \frac{1}{k+1} (\|\sigma_k^\alpha (f - P)\|_p^p + \|\sigma_k^\alpha P - P\|_p^p + \|f - P\|_p^p)
\]

\[
= R'_n + R''_n + R'''_n.
\]

By (4), we have \(R'_n \leq C 2^{p-1} \|f - P\|_p < C 2^{p-1} \varepsilon^p\), and, obviously, \(S'''_n \leq C 2^{p-1} \varepsilon^p\), so it remains to deal with \(S''_n\). Since \(P\) is a polynomial, we have

\[
\lim_{k \to \infty} \|\sigma_k^\alpha P - P\| = 0.
\]

Now choose \(m\) so that \(\|\sigma_k^\alpha P - P\| < \varepsilon\) for \(k > m\), which implies \(R''_n \leq 2^{p-1} \varepsilon^p\), concluding the proof.
5. Addendum: Proof of Theorem 10.10

In the book, I prove this theorem under the hypothesis that $d \arg f(z) > 0$ and forgot to say that the reader should read the proof in the general case in the Spencer’s paper [5]. However, a proof can be created by analysis of Hörmander’s proof [2, pp. 160–161] of Prawitz’s theorem. It is shown there that if $f(0) = 0$, then, for a fixed $r$, $D(r) := 2\pi r (d/dr) I_p (r, f)/p$ can be represented as

\begin{equation}
\sum_{j=1}^{n} \int_{A_j} \left( R_{1,j}(\alpha)^2 + \sum_{k=1}^{m_j} \left[ R_{2k+1,j}(\alpha) - R_{2k,j}(\alpha) \right]^2 \right) d\alpha \left( \sum_{k=1}^{0} = 0 \right),
\end{equation}

where $A_j \subset [0, 2\pi]$ are disjoint intervals such that $|A_1| + \ldots + |A_n| = 2\pi$, and $R_{1,j} < R_{2,j} < \ldots < R_{2m_j+1,j}$.

Assuming that $p > 2$, we apply the reverse Hölder inequality with exponent $2/p, 2/(2 - p)$ to obtain

\begin{equation}
D(r) \geq \left( \sum_{j=1}^{n} \int_{A_j} \left( R_{1,j}(\alpha)^2 + \sum_{k=1}^{m_j} \left[ R_{2k+1,j}(\alpha) - R_{2k,j}(\alpha) \right]^2 \right) d\alpha \right)^{2/p} (2\pi)^{1-p/2},\end{equation}

where $\Gamma_r$ is the image of the circle $|z| = r$ under $f$. Since $(a - b)^\gamma \geq a^\gamma - b^\gamma$ when $a > b > 0$ and $0 < \gamma < 1$, we conclude that

\begin{equation}
D(r) \geq \left( \sum_{j=1}^{n} \int_{A_j} \left( R_{1,j}(\alpha)^2 + \sum_{k=1}^{m_j} \left[ R_{2k+1,j}(\alpha) - R_{2k,j}(\alpha) \right]^2 \right) d\alpha \right)^{2/p} (2\pi)^{1-p/2},\end{equation}

just in the case when $d \arg f(z) > 0$.

6. Addendum: A Hardy–Stein identity for locally univalent harmonic mappings

A function $f \in \mathcal{H}(\mathbb{D})$ is said to be locally univalent if for every $a \in \mathbb{D}$ it is univalent in some neighborhood of $a$.\n
**Theorem 1.** If $p > 0$ and $f$ is locally univalent, then the function $M_p(r, f)$, $0 \leq r < 1$, is of class $C^1$ and the formula

\begin{equation}
\frac{d}{dr} M_p(r, f) = \frac{1}{2\pi r} \int_{\partial D} \left( p^2 |f|^{p-2} (|h'|^2 + |g'|^2) + 2p(p - 2)|f|^{p-4} \Re(f^2 h'/g') \right) dA(z)
\end{equation}

holds, where the integral is absolutely convergent.

**Proof.** The absolute convergence of the integral, denote it by, is a consequence of its absolute convergence on the closed discs $D \subset \mathbb{D}$ in which $f$ is univalent. In order to prove the latter, we note that $f$ is quasiconformal on $D$. This implies

\begin{equation}
\int_D |\ldots| dA \leq C \int_D |f|^{p-2} J_f dA, \quad \text{where } J_f \text{ is the Jacobian of } f,
\end{equation}

i.e.,

\begin{equation}
\int_D |\ldots| dA \leq C \int_{f(D)} |w|^{p-2} dA < \infty.
\end{equation}

Because of that we may assume that $r \neq |a_j|$ for all $j$. By Lewy’s theorem, we have that $J_f \neq 0$ in $\mathbb{D}$, and the zeroes of $f$ are isolated; denote the zeroes by $a_j, \ j \geq 1$, and assume that the sequence $|a_j|$ is increasing. Let $|a_j| < r$ for
\( j \leq k \). Let \( \Omega = \Omega_{r, \rho} \) denote the domain bounded by the circles \(|z| = r\) and \(|z - a_j| = \rho\), where \( \rho \) is chosen to be “infinitesimally” small. Then apply the (Green) formula

\[
\int_{\Omega} \Delta u \, dA = \int_{\partial \Omega} \frac{\partial u}{\partial \vec{n}} \, d\ell,
\]

where \( d\ell \) is the arc-length element, and \( \frac{\partial u}{\partial \vec{n}} \) the derivative of \( u \) in the direction of the unit vector \( \vec{n} \) oriented accordingly with the orientation of \( \partial \Omega \). We get

\[
\int_{\Omega} \Delta u \, dA = \int_{0}^{2\pi} u(re^{i\theta}) \, dt - \sum_{j=1}^{k} \frac{d}{d\rho} \int_{0}^{2\pi} u(a_j + \rho e^{i\theta}) \, d\theta, \quad \text{where } u = |f|^p, \ p > 0.
\]

Now it suffices to prove that

\[
\lim_{\rho \to 0^+} \rho \int_{0}^{2\pi} u(\rho e^{i\theta}) \, d\theta = 0,
\]

and then translate this identity to the points \( a_j \) to obtain the desired result. We may assume that \( f(0) = 0 \) because otherwise the proof is trivial. Then write \( f \) as \( f(z) = h(z) + g(z) \). Since \( J_f(0) \neq 0 \), we see that \(|h'(0)| \neq |g'(0)|\).

Now Assuming this, we have

\[
\left| \frac{d}{d\rho} \int_{0}^{2\pi} |f(\rho e^{i\theta})|^p \, d\theta \right| \leq p \int_{0}^{2\pi} |f(\rho e^{i\theta})|^{p-1} \left| \frac{d}{d\rho} f(\rho e^{i\theta}) \right| \, d\theta
\]

\[
\leq C \int_{0}^{2\pi} |f(\rho e^{i\theta})|^{p-1} \, d\theta \leq C \int_{0}^{2\pi} |h'(0) - g'(0)| |\rho - \rho^2 \psi(\rho e^{i\theta})|^{p-1} \, d\rho,
\]

where \( \rho \) is sufficiently small and \( \psi \) is a function continuous in the disc \( \rho \mathbb{D} \). This implies that

\[
\left| \frac{d}{d\rho} \int_{0}^{2\pi} |f(\rho e^{i\theta})|^p \, d\theta \right| \leq C \rho^p,
\]

which, together with the formula

\[
\Delta(|f|^p) = p^2 |f|^{p-2} (|h'|^2 + |g'|^2) + 2p(p-2)|f|^{p-4} \text{Re}(f^2 \bar{g} \bar{f}) \quad (\text{see [3]}),
\]

concludes the proof. \( \square \)


The case \( p < 2 \) of the following theorem is discussed in [4, Corollary 10.8].

**Theorem 2.** Let \( f \) be a QC harmonic mapping and \( p \geq 2 \). Then each of the following quantities is equivalent to \( \|f\|_p^p \):

\[
\int_{0}^{1} |f(r \mathbb{D})|^{p/2} \, dr, \quad \int_{0}^{1} \left( \int_{r \mathbb{D}} (|h'|^2 + |g'|^2) \, dA \right)^{p/2} \, dr, \quad \int_{0}^{1} P(r, f)^p \, dr,
\]

where \( P(r, f) = \sum_{n \in \mathbb{Z}} |\hat{f}(n)| r^n \).

**Proof.** Denote these quantities by \( Q_1, Q_2, \) and \( Q_3 \), respectively. Since

\[
|f(r \mathbb{D})|^2 = \int_{r \mathbb{D}} (|h'|^2 - |g'|^2) \, dA,
\]

we have, by the quasiconformality of \( f \) that \( Q_1 \asymp Q_2 \). In proving that \( \|f\|_p^p \leq C Q_2 \), we start from the inequality

\[
M_p^p(r, f) \leq \frac{C}{r} \int_{r \mathbb{D}} |f|^{p-2} (|h'|^2 + |g'|^2) \, dA,
\]

which follows from (6), and

\[
\int_{0}^{1} M_{\infty}(r, f)^p \, dr \leq C \|f\|_p^p,
\]

and then proceed exactly as in [1]. The most delicate is to prove that \( \|f\|_p^p \geq C Q_1 \).

It follows from Theorem 1 that

\[
2\pi r \frac{d}{dr} M_p^p(r, f) \geq \int_{r \mathbb{D}} (p^2 |f|^{p-2} (|h'|^2 + |g'|^2) - 2p(p-2)|f|^{p-2}|h'| \, dA
\]

\[
\begin{align*}
&\geq \int_{r \in \Omega} 2p|f|^{p-2}(|h'|^2 + |g'|^2) \, dA \\
&\geq \int_{r \in \Omega} 2p|f|^{p-2}(|h'|^2 - |g'|^2) \, dA \\
&= 2p \int_{f(r \in \Omega)} |w|^{p-2} \, dA(w) = \int_{\partial f(r \in \Omega)} |w|^{p-2} \text{Im}(\bar{w} \, dw) \\
&= \int_{\partial f(r \in \Omega)} |w|^p \, d(\text{arg} \, w).
\end{align*}
\]

Now we use Hörmander’s idea as in Section 5 to prove that if \( f(0) = 0 \), then \( \|f\|_p^p \geq CQ_1 \). \( \square \)

References