On multipliers from $H^p$ to $l^q$, $0 < q < p < 1$

By

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1. Introduction. Let $H^{p,\alpha}$, $0 < p < \infty$, $0 \leq \alpha < \infty$, denote the space of functions

$$f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$$

holomorphic in the unit disc $U$ (abbreviated $f \in H(U)$) for which

$$\|f\|_{p,\alpha} = \sup_{0 < r < 1} (1 - r)^\alpha M_p(r, f) < \infty,$$

where, as usual,

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}.$$

If $\alpha = 0$, then $H^{p,0}$ is the Hardy class $H^p$, and we write $\| \|_p$ instead of $\| \|_{p,0}$. The space $H^{p,\alpha}$, $\alpha > 0$, has been called the space of functions of slow mean growth [7].

Let $g(z) = \sum_{k=0}^{\infty} \hat{g}(k) z^k$ be holomorphic in $U$. We define the multiplier transformation $D^\beta g$ of $g$, where $\beta$ is any real number, by

$$D^\beta g(z) = \sum_{k=0}^{\infty} (k + 1)^\beta \hat{g}(k) z^k.$$

A function $f \in H(U)$ is said to belong to the space $H^{p,\alpha}_\beta$, $0 < p, \alpha < \infty$, $\beta \in \mathbb{R}$, if $\|D^\beta f\|_{p,\alpha} < \infty$.

A complex sequence $\{a_n\}$ is of class $l(q,s)$, $0 < q, s \leq \infty$, if

$$\|\{a_n\}\|_q^s = \sum_{n=0}^{\infty} \left( \sum_{k \in I_n} |a_k|^q \right)^{s/q} < \infty,$$

where $I_0 = \{0\}$, $I_n = \{k \in \mathbb{N} : 2^{n-1} \leq k < 2^n\}$, $n = 1, 2, \ldots$. In the case where $q$ or $s$ is infinite, replace the corresponding sum by a supremum. Note that $l^q = l(q,q)$.

It is easily checked that if $\{a_n\}$ is in $l(q,s)$ then the function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is holomorphic in $U$. Therefore we may regard $l(q,s)$ as a space of holomorphic functions. We write

$$\|\{a_n\}\|_{q,s} = \|f\|_{q,s}.$$
Let $A$ and $B$ be two vector spaces of functions holomorphic in $U$. A function $g(z) = \sum_{k=0}^{\infty} \hat{g}(k) z^k$ is said to be a multiplier from $A$ to $B$ if whenever $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ belongs to $A$ then $(f \ast g)(z) = \sum_{k=0}^{\infty} \hat{f}(k) \hat{g}(k) z^k$ belongs to $B$. The set of all multipliers from $A$ to $B$ will be denoted by $(A, B)$.

Multipliers of Hardy spaces have been studied by many authors (see [1]). In particular P. Duren and A. Shields described the multipliers from $H^p$ to $l^q$ $(0 < p < 1, p \leq q \leq \infty)$ ([2], p. 100). In this note we calculate multipliers $(H^p, l^q)$ in the case $0 < q < p < 1$ that, up to now, had been left unsettled. More precisely, we prove

**Theorem.** Suppose $0 < q < p < 1$. Then $g \in H(U)$ is a multiplier of $H^p$ into $l^q$ if and only if $D^{(1/p)-1} g \in l(q, p \ast q)$, where $\frac{1}{p} = \frac{1}{q} - \frac{1}{p}$.

The proof of Theorem occupies Section 3. This proof depends on a construction of polynomials $w_n, \ n \geq 0$, presented in Section 2, satisfying

(1.1) $f = \sum_{k=0}^{\infty} w_k \ast f, \ \text{for all } f \in H(U),$

(1.2) $w_0(z) = 1 + z, \ \hat{w}_n(k) = 0, \ \text{if } k \notin (2^{n-1}, 2^{n+1}), \ n = 1, 2, \ldots,$

(1.3) $\|w_n \ast f\|_p \leq C \|f\|_p, \ f \in H^p, \ 0 < p \leq 1, \ n = 0, 1, 2, \ldots.$

We use $C$ to denote a positive constant, depending on the particular parameters $p, q, s, \ldots, \alpha, \beta, \ldots$, concerned in the particular problem in which it appears. It is not necessarily the same on any two occurrences.

Polynomials $w_{n,N}$ like $w_n$ have already been used in [9] and [10] to provide characterization of multipliers from certain mixed norm spaces of holomorphic functions in $U$ to $l^q$. They also satisfy (1.1) and (1.3) and $\hat{w}_{n,N}(k) = 0, \ \text{if } k \notin (2^{n-1}, 2^{n+N}), \ \text{where } N > 1 - \frac{1}{p}$ $(0 < p < 1)$, instead of (1.2). The proof of (1.3) for $w_{n,N}$ is based on Hardy-Littlewood theorem on Cesaro means of analytic functions. Our construction is simpler in that it avoids Hardy-Littlewood theorem. In the proof of (1.3) we use an inequality on a Hadamard product recently proved in [8].

2. Construction of polynomials $w_n$. To construct a sequence of polynomials $\{w_n\}$ satisfying (1.1), (1.2) and (1.3) some lemmas will be needed.

**Lemma 2.1** ([4], [12]). Let $f(z) = \sum_{k=m}^{n} a_k z^k, \ 0 \leq m \leq n$. Then $r^m \|f\|_p \leq M_p(r, f) \leq r^m \|f\|_p$.

**Lemma 2.2** ([8]). If $f, g \in H^p, \ 0 < p \leq 1$, then

(2.1) $M_p(r, f \ast g) \leq (1 - r)^{1 - (1/p)} \|f\|_p \|g\|_p, \ 0 < r < 1.$
Lemma 2.3. Let \( 0 < p \leq 1 \). Then there exists a constant \( C_p \) depending only on \( p \) such that

\[
\| T_n \ast f \|_p \leq C_p n^{(1/p) - 1} \| T_n \|_p \| f \|_p,
\]

for any polynomial \( T_n \) of degree at most \( n \), \( n \geq 1 \), and any \( f \in H^p \).

Proof. Since \( T_n \ast f \) is a polynomial of degree at most \( n \),

\[
\| T_n \ast f \|_p \leq r^{-n} M_p(r, T_n \ast f),
\]

by Lemma 2.1.

Now (2.2) follows from (2.1) and (2.3) by taking \( r = 1 - (1/n) \).

We are grateful to the referee for pointing out to us that Lemma 2.3 could also be proved by using Formula (3.3.3/4) in [11].

Lemma 2.4. Let \( 0 < r, s, u, v \leq \infty \). Then

\[
(l(r, s), l(u, v)) = l(r \ast u, s \ast v),
\]

where

\[
\frac{1}{r \ast u} = \frac{1}{r} - \frac{1}{u} \quad \text{if} \quad r > u, \quad r \ast u = \infty \quad \text{if} \quad r \leq u,
\]

\[
\frac{1}{s \ast v} = \frac{1}{v} - \frac{1}{s} \quad \text{if} \quad s > v, \quad s \ast v = \infty \quad \text{if} \quad s \leq v.
\]

Lemma was proved in [3] in the case \( 1 \leq r, s, u, v \leq \infty \). The proof shows that it holds for all \( 0 < r, s, u, v \leq \infty \).

We show that polynomials \( w_n \) defined by

\[
w_0(z) = 1 + z, \quad w_n(z) = \sum_{k=0}^{2^n-1} \phi \left( \frac{k}{2^{n-1}} \right) z^k, \quad n = 1, 2, \ldots,
\]

here \( \phi(t) = \omega \left( \frac{t}{2} \right) - \omega(t) \), and \( \omega : R \to R \) is any infinitely differentiable function satisfying \( \omega(t) = 1 \), if \( t \leq 1 \), \( \omega(t) = 0 \), if \( t \geq 2 \), and \( 0 \leq \omega(t) \leq 1 \), if \( 1 \leq t \leq 2 \), satisfies (1.1), (1.2) and (1.3).

Since \( \sum_{n=0}^{\infty} \hat{w}_n(k) = 1, \ k = 0, 1, 2, \ldots \), (1.1) holds for polynomials. From this it follows easily (1.1) for all \( f \in H(U) \). Trivially polynomials \( w_n \) satisfy (1.2). It remains to show that (1.3) holds.

Since \( 0 \leq \phi(t) \leq 1 \), we have

\[
|w_n(z)| \leq 2^{n+1}, \quad z \in U, \quad n = 0, 1, 2, \ldots.
\]
Choose an integer $N$ so that $Np > 1$. Note that $\varphi\left(\frac{k}{2^{n-1}}\right) = 0$ if $k$ is an integer such that $k \leq 2^{n-1}$ or $2^{n+1} \leq k$. Hence,

\begin{equation}
(1 - e^{i\pi})^N w_n(e^{i\pi}) = \sum_{k = -\infty}^{\infty} \varphi\left(\frac{k}{2^{n-1}}\right) (1 - e^{i\pi})^N e^{i\pi t}
= \sum_{k = -\infty}^{\infty} \varphi\left(\frac{k}{2^{n-1}}\right) \sum_{m = 0}^{N} \binom{N}{m} (-1)^m e^{i(m+k)\pi}
= \sum_{m = 0}^{N} (-1)^m \binom{N}{m} \sum_{k = -\infty}^{\infty} \varphi\left(\frac{k - m}{2^{n-1}}\right) e^{i\pi t}
= \sum_{m = 0}^{N} \binom{N}{m} (-1)^m \varphi\left(\frac{k - m}{2^{n-1}}\right) e^{i\pi t}.
\end{equation}

By the Lagrange theorem for symmetric differences, for each $k$ there exists $\xi_{k,N}$ such that

\begin{equation}
\sum_{m = 0}^{N} (-1)^m \binom{N}{m} \varphi\left(\frac{k - m}{2^{n-1}}\right) = 2^{n(1-\alpha)} \varphi^{(N)}(\xi_{k,N}).
\end{equation}

It follows from (2.5) and (2.6) that

\begin{equation}
|w_n(e^{i\pi})| \leq C t^{-n} 2^{n(1-N)}.
\end{equation}

Using Lemma 2.3, (2.4) and (2.7) we get

\[\|w_n * f\|_p \leq C 2^{n((1/p)-1)} \|f\|_p \|w_n\|_p \leq C 2^{n((1/p)-1)} \|f\|_p \left(\int_0^{2^{-n}} 2^{n(t)} dt + \int_{2^{-n}}^{\pi} t^{-n} 2^{n(1-N)} dt\right) \leq C \|f\|_p.\]

(Note that here we needed $N p > 1$.)

3. Proof of Theorem. The following theorem due to Matheson [7] (see also [5]) will also be needed.

**Theorem M.** Let $0 < p < 1$, $0 < \alpha$, $q < \infty$. Then

\[\left(H^{p, \alpha}, l^q\right) = \{g \in H(U); D^{\alpha + (1/p)-1} g \in l^q\}.\]

Note that this implies

\[\left(H^{p, \alpha}_1, l^q\right) = \{g \in H(U); D^{\alpha + (1/p)-2} g \in l^q\}.\]

**Proof of Theorem.** If $g \in (H^p, l^q)$, then by the closed graph theorem

\[\|f * g\|_q \leq C \|f\|_p, \quad f \in H^p.\]
Since \( |\hat{w}_n(k)| \leq 1, k, n \in N \cup \{0\} \), we have

\[
\sum_{n=0}^{\infty} \| w_n \ast f \ast g \|_q^q \leq 2 \| f \ast g \|_q^q.
\] (3.2)

If \( f \in H(U) \) and \( \sum_{n=0}^{\infty} \| w_n \ast f \|_p^p < \infty \), then integrating the identity (1.1) and using inequality \( \sum_{n=0}^{\infty} \| w_n \ast f \|_p^p \leq \sum_{n=0}^{\infty} |w_n \ast f|^p \), (note that \( 0 < p \leq 1 \)), we get

\[
\| f \|_p^p \leq \sum_{n=0}^{\infty} \| w_n \ast f \|_p^p.
\] (3.3)

After elementary calculations we find that

\[
\| w_n \ast f \|_p \leq C \| w_n \ast D^1 f \|_{p,1} \leq C \| D^1 f \|_{p,1}.
\] (3.4)

(for the first inequality see [4], the second one follows from (1.3)).

Let \( P_n = w_{n-1} + w_n + w_{n+1}, n \geq 0, (w_{-1} = 0) \).

Since \( w_n \ast w_k = 0 \), for \( |n - k| \geq 2 \), we have

\[
P_n \ast w_n = w_n, \quad n \geq 0,
\] (3.5)

and

\[
w_n \ast P_k = 0, \quad \text{if} \quad |k - n| \geq 3.
\] (3.6)

Combining all the above results with (1.3) yields

\[
\left( \sum_{k=0}^{\infty} \| w_{3k+m} \ast \left( \sum_{n=0}^{\infty} P_{3n+m} \ast f_n \right) \ast g \|_q^q \right)^{1/q} \leq C \left( \sum_{k=0}^{\infty} \| w_k \ast \left( \sum_{n=0}^{\infty} P_{3n+m} \ast f_n \right) \ast g \|_q^q \right)^{1/q} \leq C \left( \sum_{n=0}^{\infty} \| D^1 f_n \|_{p,1}^p \right)^{1/p},
\] (3.7)

for any finite sequence \( f_n \) of functions holomorphic in \( U \) and any \( m, 0 \leq m \leq 2 \).

By (3.6) \( w_{3k+m} \ast P_{3n+m} = 0 \), if \( k \neq n \). Hence

\[
w_{3k+m} \ast \sum_{n=0}^{\infty} (P_{3n+m} \ast f_n) = w_{3k+m} \ast P_{3k+m} \ast f_k = w_{3k+m} \ast f_k
\] (3.8)

by (3.5).

From (3.7) and (3.8) we conclude that

\[
\left( \sum_{k=0}^{\infty} \| w_{3k+m} \ast g \ast f_k \|_q^q \right)^{1/q} \leq C \left( \sum_{k=0}^{\infty} \| D^1 f_k \|_{p,1}^p \right)^{1/p}.
\] (3.9)

Fix \( m, 0 \leq m \leq 2 \). Obviously, \( w_{3k+m} \ast g \in (H^1_p, l^q) \) for every \( k \geq 0 \). Hence,

\[
\| w_{3k+m} \ast g \ast f \|_q \leq \| w_{3k+m} \ast g \|_{(H^1_p, l^q)} \| D^1 f \|_{p,1}, \quad \text{for all} \ f \in H^1_p,
\]

where \( \| w_{3k+m} \ast g \|_{(H^1_p, l^q)} \) denote the operator norm of \( w_{3k+m} \ast g \).
By the remark following Theorem M we see that there is an absolute constant $C$ such that

$$C^{-1} \| w_{3k+m} \ast D^{(1/p)-1} g \|_q \leq \| w_{3k+m} \ast g \|_{(H^p_t, 1)^q} \leq C \| w_{3k+m} \ast D^{(1/p)-1} g \|_q.$$  

Now fix $\varepsilon < 1$. For every $k \geq 0$ choose $h_k$ so that $\| D^1 h_k \|_{p,1} = 1$ and

$$\varepsilon \| w_{3k+m} \ast g \|_{(H^p_t, 1)^q} \leq \| w_{3k+m} \ast g \ast h_k \|_q.$$  

Let $\{a_k\}$ be an arbitrary sequence in $l^p$ and let $f_k = a_k h_k$, $k = 1, 2, \ldots, n$. Using (3.9), (3.10) and (3.11) we find that

$$\left(\sum_{k=0}^n |a_k|^q \| w_{3k+m} \ast D^{(1/p)-1} g \|_q^q\right)^{1/q} \leq C \left(\sum_{k=0}^n |a_k|^p\right)^{1/p} \leq C \| \{a_k\} \|_p,$$

where the constant $C$ doesn’t depend on $n$. This gives

$$\left\{ \| w_{3k+m} \ast D^{(1/p)-1} g \|_q \right\} \in (l^p, l^q) = l^{p \ast q},$$

for $m = 0, 1, 2$, by Lemma 2.4.

It easily seen that

$$\sum_{k=0}^\infty \| w_k \ast D^{(1/p)-1} g \|_q^q = \sum_{m=0}^\infty \sum_{k=0}^\infty \| w_{3k+m} \ast D^{(1/p)-1} g \|_q^q.$$

Thus,

$$\sum_{k=0}^\infty \| w_k \ast D^{(1/p)-1} g \|_q^{p \ast q} < \infty.$$

Using simple inequalities

$$C_\beta^{-1} (a + b)^\beta \leq a^\beta + b^\beta \leq C_\beta (a + b)^\beta, \quad \beta > 0, \ a, b > 0,$$

we show that this implies $D^{(1/p)-1} g \in l(q, p \ast q)$.

Set $h = D^{(1/p)-1} g$ and $p \ast q = s$. Then we have

$$\| h \|_{q, s}^s = |\hat{h}(0)|^s + |\hat{h}(1)|^s + \sum_{n=1}^{\infty} \left( \sum_{k=2^n}^{2^n + 1 - 1} |\hat{h}(k)|^q \right)^{s/q}$$

$$\leq C \left( \| w_0 \ast h \|_q^s + \sum_{n=1}^{\infty} \left( \sum_{k=2^n}^{2^n + 1 - 1} \left| \omega \left( \frac{k}{2^n} \right) \hat{h}(k) \right|^q \right)^{s/q} \right)$$

$$+ \left( \sum_{k=2^n}^{2^n + 1 - 1} \left( 1 - \omega \left( \frac{k}{2^n} \right) \right) \hat{h}(k) \right)^{s/q} \right)$$

$$\leq C \left( \| w_0 \ast h \|_q^s + \| w_1 \ast h \|_q^s \right)$$

$$+ \sum_{n=2}^{\infty} \left( \sum_{k=2^n}^{2^{n-1} - 1} \left( 1 - \omega \left( \frac{k}{2^{n-1}} \right) \right) \hat{h}(k) \right)^{s/q} \right)$$

$$+ \left( \sum_{k=2^n}^{2^n + 1 - 1} \left| \omega \left( \frac{k}{2^n} \right) \hat{h}(k) \right|^q \right)^{s/q} \right).$$
\[ \leq C \left( \| w_0 * h \|^s_q + \| w_1 * h \|^s_q \right) \\
+ \sum_{n=2}^{\infty} \left( \sum_{k=2^n-1}^{2^n-1} \left( 1 - \omega \left( \frac{k}{2^n-1} \right) \right) \| \hat{h}(k) \|^q \right)^{\frac{q}{s}} \\
+ \sum_{k=2^n}^{2^{n+1}} \left( \omega \left( \frac{k}{2^n} \right) \| \hat{h}(k) \|^q \right)^{\frac{q}{s}} \right)^{\frac{1}{q}} = C \sum_{n=0}^{\infty} \| w_n * h \|^s_q. \]

Conversely, let \( D^{1/(p-1)} g \in l(q, p * q) = l(\infty, p), l(q, q) \), by Lemma 2.4. From the theorem of P. L. Duren and A. Shields cited in the Introduction it follows easily that if \( f \in H^p \) then \( D^{1/(p-1)} f \in l(\infty, p) \), (see also [5] and [6]). Hence, if \( f \in H^p \), then \( f * g = D^{1/(p-1)} f * D^{1/(p-1)} g \in l^q \), i.e., \( g \in (H^p, l^q) \). This completes the proof of Theorem.

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