A note on hyperharmonic and polyharmonic functions

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Abstract

We show that the spaces of harmonic functions with respect to the Poincaré metric in the unit ball $B^N$ in $\mathbb{R}^N$ have many different properties depending upon whether $N$ is even or odd.

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1. Introduction

For a $C^2$ function $u$ on a domain $G$ in $\mathbb{R}^N$ ($N \geq 2$), let

$$\Delta h u(x) = \left(1 - |x|^2\right)\left[\Delta u(x) + 2(N - 2)Ru(x)\right],$$

where $\Delta u = \sum_{j=1}^N (\partial^2 u / \partial x_j^2)$ and $Ru = \sum_{j=1}^N x_j (\partial u / \partial x_j)$.

We put $h(G) = \{u \in C^2(G) : \Delta h u = 0\}$. We say $u$ is hyperharmonic in $G$ if $u \in h(G)$.

It is well known that the uniform limit of a sequence of hyperharmonic functions in the Poincaré ball $B^N = \{x \in \mathbb{R}^N : |x| < 1\}$ is hyperharmonic. In this paper we show that any function that is hyperharmonic and real analytic on a domain $G \subset \mathbb{R}^2$ is $N$-polyharmonic. Consequently, the uniform limit of a sequence of hyperharmonic and real analytic functions on $G$ is hyperharmonic and real analytic on $G$.

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On the other hand, if $N$ is odd and $0 \in G \subset \mathbb{R}^N$, then there are no functions on $G$, except constants, that are hyperharmonic, real analytic and polyharmonic of any order $k \geq 1$. There is another reason for which the case $N = 2n$ is interesting. Namely, there is an infinite-dimensional class of hyperharmonic polynomials on $B^{2n}$, and moreover, this class is dense in $h(B^{2n})$, the class of hyperharmonic functions on $B^{2n}$, in the topology of uniform convergence on compact sets in $B^{2n}$. On the other hand, $h(B^{2n+1})$ does not contain any non-constant polynomial and, moreover, if $u \in h(B^{2n+1}) \cap C^{2n}(\overline{B^{2n+1}})$ then $u$ is a constant. In fact, we show that a much weaker condition than this implies that a hyperharmonic function on $B^{2n+1}$ must be constant.

We show that functions in the hyperharmonic Hardy space $h^p(B^N)$ have boundary value in the sense of distributions.

2. Polyharmonic and hyperharmonic functions

For a domain $G \subset \mathbb{R}^N$ let $H_m(G)$, $m \geq 1$, be the class of functions polyharmonic of degree $m$ ($m$-polyharmonic) in $G$, i.e., solutions of the equation $\Delta^m u = 0$, where $\Delta^m$ is the Euclidean Laplacian iterated $m$ times (cf. [3]). In particular, $H(G) = H_1(G)$ is the class of functions harmonic in $G$. It is known that $H_m(G) \subset A(G)$, where $A(G)$ is the class of real analytic functions on $G$ (see [3]).

We note that if $G \cap S^{N-1}$ ($S^{N-1} = \partial B^N$) is not empty, then there are solutions of the equation $\Delta_h u = 0$ which are not real analytic in $G$.

For example, if $N = 4$, then the function $u$ defined by

$$u(x) = \begin{cases} |x|^2 - |x|^{-2} - 4 \log |x|, & |x| \geq 1, \\ 0, & |x| \leq 1, \end{cases}$$

is $C^2$ on $\mathbb{R}^4$, and $\Delta_h u = 0$ for all $x$; clearly $u$ is not real analytic.

It is clear that $h(G) = H(G)$ if $N = 2$. If $N \geq 3$ and $G$ contains the origin, then $h(G) \cap H(G) = \{\text{constants}\}$.

Theorem 2.1. Let $G$ be a domain in $\mathbb{R}^N$, $N \geq 3$, and $u \in h(G) \cap A(G)$.

(i) If $N$ is even, then $u \in H_{N/2}(G)$. If in addition $u \neq \text{const}$ and $0 \in G$, then $u \in H_{N/2}(G) \setminus H_{N/2-1}(G)$.

(ii) If $N$ is odd, $0 \in G$ and $u \in H_m(G)$ for some $m$, then $u$ is constant.

Theorem 2.1 is easily proved by successive applications of the following two lemmas.

Lemma 2.2. Let $u \in A(G)$, where $G$ is a domain in $\mathbb{R}^N$ containing the origin. If $R_s u := su + Ru = 0$, for some $s > 0$, then $u = 0$. If $Ru = 0$, then $u = \text{const}$.

Proof. This follows from the formula $R_s u = |x|^{-s} R(|x|^s u)$ and the uniqueness theorem.

Lemma 2.3. If $u \in h(G) \cap A(G)$, where $G$ is a domain in $\mathbb{R}^N$, then
\((1 - |x|^2) \Delta^m u = 2(2m - N) R_{m-1} \Delta^{m-1} u, \quad m \geq 1. \quad (2.1)\)

(If \(m = 1\), then \(\Delta^{m-1} u = u.\))

**Proof.** If \(m = 1\), then (2.1) follows from the equation \(\Delta h u = 0\). On the other hand, by differentiation and using the formula \(\Delta R_s = R_{s+2} \Delta\), we deduce from (2.1) that

\((1 - |x|^2) \Delta^{m+1} u - 4R \Delta^m u - 2N \Delta^m u = 2(2m - N) R_{m+1} \Delta^m u.\)

This implies, after a little work,

\((1 - |x|^2) \Delta^{m+1} u = 2(2m + 2 - N) R_m \Delta^m u.\)

Now the lemma is proved by induction on \(m\). \(\square\)

**Corollary 2.4.** Let \(N\) be even and \(G \subset \mathbb{R}^N\). Then the uniform limit of a sequence of hyperharmonic and real analytic functions in \(G\) is hyperharmonic and real analytic in \(G\).

This is easily deduced from Theorem 2.1(i) and the analogous fact for polyharmonic functions.

Corollary 2.4 is interesting only when \(G \cap S^{N-1}\) is not empty, since otherwise the operator \(\Delta h\) is elliptic in \(G\).

Note the following consequence of the proof of Lemma 2.3.

**Corollary 2.5.** If \(u \in C^2(G), \ G \subset \mathbb{R}^{2n}, \) and \(\Delta h u = 0\) in \(G\), then \(u \in A(G)\).

### 3. Hyperharmonic functions having a distribution value

In [7] it is shown that if \(f \in h(B^N)\) then there exists a unique sequence of harmonic homogeneous polynomials \(f_k\), of degree \(k\), \(f_k \in \mathcal{H}_k(\mathbb{R}^N)\), such that

\[f(x) = \sum_{k=0}^{\infty} F_k(x) f_k(x), \quad x \in B^N,\]

where \(F_k(x) = F(k, 1 - N/2, k + N/2; |x|^2), \ k \geq 0\) (as usual \(F(a, b, c; \cdot)\) denotes the hypergeometric function with parameters \(a, b, c\) (see [6, Chapter II])). Note that if \(N\) is even, then \(F_k\), \(k \geq 1\), is a polynomial of degree \(N - 2\), while if \(N\) is odd, \(F_k\) is only of class \(C^{N-2}\) on \(B^N\). More precisely the following theorem was proven.

**Theorem 3.1.** If \(u\) is a hyperharmonic function in \(B^N\), then there exists a unique sequence of harmonic homogeneous polynomials \(f_k \in \mathcal{H}_k(\mathbb{R}^N)\) such that

\[u(x) = \sum_{k=0}^{\infty} F_k(x) f_k(x), \quad x \in B^N,\]

the series converging uniformly and absolutely on compact subsets of \(B^N\). Conversely, the sum of any such series that converges uniformly on compact subsets of \(B^N\) is hyperharmonic in \(B^N\).
As a corollary of Theorem 3.1 we have that the class of hyperharmonic polynomials is dense in \( h(B^{2n}) \). The situation is completely different when \( N \) is odd. Since every polynomial is a polyharmonic function, it follows from Theorem 2.1(ii) that there exists no non-constant hyperharmonic polynomial on \( \mathbb{R}^N \) when \( N \) is odd. Our next theorem shows that a more general fact is true.

**Theorem 3.2.** Let \( u \in h(B^{2n+1}) \). If
\[
\int_{S^{2n}} R^{2n} u(r\xi)\phi(\xi) \, d\sigma(\xi) = o\left(\log \frac{1}{1-r}\right), \quad r \to 1,
\]
for every \( \phi \in C^\infty(S^{2n}) \), then \( u \) is constant.

**Proof.** From (3.1) and the orthogonality in \( L^2(S^{2n}, d\sigma) \) of the sequence \( f_k(y) \) we get
\[
F_k(r) r^k \int_{S^{2n}} f_k(y)^2 \, d\sigma(y) = \int_{S^{2n}} u(r y) f_k(y) \, d\sigma(y),
\]
where we write \( F_k(r) = F_k(ry) \), \( y \in S^{2n} \).

It is well known that
\[
F^{(j)}(k, 1 - \frac{2n+1}{2}, k + \frac{2n+1}{2}; r^2), \quad k \geq 1,
\]
is bounded for \( j = 0, 1, 2, \ldots, 2n-1 \), and
\[
F^{(2n)}(k, 1 - \frac{2n+1}{2}, k + \frac{2n+1}{2}; r^2) \sim C \log \frac{1}{1-r}, \quad r \to 1.
\]
If \( k \geq 1 \), we find from (3.2) and (3.3) that
\[
\left( r \frac{d}{dr} \right)^{(2n)} \left( F_k(r)^r \int_{S^{2n}} f_k(y)^2 \, d\sigma(y) \right)
= \int_{S^{2n}} R^{2n} u(r y) f_k(y) \, d\sigma(y) = o\left(\log \frac{1}{1-r}\right).
\]
Hence, \( f_k = 0 \), for \( k = 1, 2, \ldots \), i.e., \( u = \text{const.} \)

In particular, the following is true.

**Corollary 3.3.** If \( u \in h(B^{2n+1}) \cap C^2(B^{2n+1}) \), then \( u \) is constant.

We say that a function \( u \) defined on \( B^N \) has a distribution value on \( S^{N-1} \) if
\[
\lim_{r \to 1} \int_{S^{N-1}} u(ry) \phi(y) \, d\sigma(y)
\]
exists for each test function $\phi \in C^\infty(S^{N-1})$.

Recall that the tangential derivatives of $u \in C^1(B^N)$ are defined by

$$T_{i,j} u(x) = x_i \frac{\partial u}{\partial x_j}(x) - x_j \frac{\partial u}{\partial x_i}(x), \quad 1 \leq i, j \leq N.$$ 

The following result is closely related to Theorem 3.2.

**Theorem 3.4.** If $u$ is a hyperharmonic function on $B^N$ having a distribution boundary value and $X = R^k Y$, with $Y$ tangential, then $Xu$ has a distribution value when $k \leq N - 2$.

If $k = N - 1$, then

$$\int_{S^{N-1}} R^{N-1} Y u(ry) \phi(y) \, d\sigma(y) = O \left( \log \frac{1}{1-r} \right)$$

for each $\phi \in C^\infty(S^{N-1})$.

**Proof.** By direct calculation, since $(1 - |x|^2) \Delta u + (2N - 4) Ru = 0$, we have

$$(N - 2)(1 + |x|^2)Ru + (1 - |x|^2) R^2 u = (|x|^2 - 1) \sum_{i<j} T^2_{i,j} u. \quad (3.4)$$

Apply $R^{k-1}$ to both terms, noticing that $R(|x|^2) = 2|x|^2$. One gets

$$(N - 2)(1 + |x|^2)R^k u + (N - 2) \sum_{j=1}^{k-1} \binom{k-1}{j} 2^j |x|^2 R^{k-j} u + (1 - |x|^2)R^{k+1} u$$

$$- 2(k - 1)|x|^2 R^k u - \sum_{j=2}^{k-1} \binom{k-1}{j} 2^j |x|^2 R^{k-j} u$$

$$= \sum_{j=0}^{k-1} \binom{k-1}{j} R^j (|x|^2 - 1) R^{k-1-j} \sum_{i<j} T^2_{i,j} u. \quad (3.5)$$

We proceed by induction on $k$. Fix a test function $\phi \in C^\infty(S^{N-1})$ and let

$$\varphi(r) = \int_{S^{N-1}} R^k Y u(r \xi) \phi(\xi) \, d\sigma(\xi), \quad 0 < r < 1.$$ 

Applying $Y$ to formula (3.5) and using the fact that $R$ and $Y$ commute, we find that the induction hypothesis implies that the function

$$g(r) = 2(N - k - 1) \varphi(r) + (1 - r^2)(R \varphi(r) + (2k - N) \varphi(r))$$

has a limit $L$ as $r \to 1$. Solving the differential equation yields

$$\varphi(r) = \frac{(1 - r^2)^{N-k-1}}{r^{N-2}} \int_0^r g(t)t^{N-3} (1 - t^2)^{k-N} \, dt.$$
If \( k \leq N - 2 \), it follows from above that \( \varphi(r) \) has limit \( L/2(N - k - 1) \). If \( k = N - 1 \), then \( \varphi(r) \) has a logarithmic growth. \( \Box \)

For an analogous result for \( \mathcal{M} \)-harmonic functions on the unit ball \( B^N \) in \( \mathbb{C}^N \) see [5].

4. The Dirichlet problem for hyperharmonic functions

Let \( \phi \in C(S^{N-1}) \). In this section we look in detail at the solvability of the Dirichlet problem: \( \Delta_h u = 0 \) in \( B^N \) and \( u = \phi \) on \( S^{N-1} \). More precisely we prove the following theorem.

**Theorem 4.1.** Let \( \phi = \sum_{k=0}^{\infty} \phi_k \) be the spherical harmonic expansion of \( \phi \in C(S^{N-1}) \). Then the Dirichlet problem has a unique solution. It is given by

\[
\begin{align*}
  u(x) &= \int_{S^{N-1}} P_h(x, \eta)\phi(\eta)\,d\sigma(\eta), \\
  & \quad (4.1)
\end{align*}
\]

where

\[
P_h(x, \eta) = \left( \frac{1 - |x|^2}{|x - \eta|^2} \right)^{N-1}, \quad x \in B^N, \; \eta \in S^{N-1},
\]

or, alternatively, by

\[
\begin{align*}
  u(x) &= u(ry) = \sum_{k=0}^{\infty} \frac{F_k(r)}{F_k(1)} r^k \phi_k(y), \quad 0 \leq r < 0, \; y \in S^{N-1}.
  & \quad (4.2)
\end{align*}
\]

**Proof.** For the statement (4.1) see [2]. Let \( u \) be a solution of the Dirichlet problem. By Theorem 3.1,

\[
\begin{align*}
  u(x) &= u(ry) = \sum_{k=0}^{\infty} F(k, 1 - N/2, k + N/2; r^2) r^k \phi_k(y).
\end{align*}
\]

The proof of Theorem 3.1 given in [7] shows that

\[
F_k(r)r^k f_k(y) = \int_{S^{N-1}} u(r\eta)Z_k(y, \eta)\,d\sigma(\eta), \quad k \geq 0.
\]

Here, \( Z_k \) are the zonal harmonics (see [4]). Letting \( r \to 1 \) we see that \( F_k(1)f_k(y) = \phi_k(y) \).

This gives (4.2). \( \Box \)

We note that the uniqueness of the solution for the Dirichlet problem shows that for \( P_h(x, \eta) \) we have

\[
P_h(x, \eta) = \sum_{k=0}^{\infty} \frac{F_k(r)}{F_k(1)} r^k Z_k(y, \eta), \quad x = ry, \; y \in S^{N-1}, \; \eta \in S^{N-1}.
\]
5. Hardy spaces of hyperharmonic functions

A function \( u \in h(B^N) \) is said to belong to the Hardy space \( h^p(B^N) \), \( 0 < p < \infty \), if \( M_\alpha u \in L^p(S^{N-1}) \), for some (any) \( \alpha > 1 \). Here, as usual \( M_\alpha u \) denotes the non-tangential maximal function defined on \( S^{N-1} \) by

\[
M_\alpha u(\xi) = \sup \{|u(x)| : x \in \Gamma_\alpha(\xi)\},
\]

where \( \Gamma_\alpha(\xi) \) denotes the non-tangential approach region

\[
\Gamma_\alpha(\xi) = \{ x \in B^N : |x - \xi| < \alpha(1 - |x|) \}, \quad \alpha > 1.
\]

A function \( u \) on \( B^N \) is said to have an admissible limit \( L \) at \( \xi \in S^{N-1} \) if \( \lim_{\Gamma_\alpha \ni x \to \xi} = L \).

In this section we show that any function in \( h^p(B^N) \) has a distribution value on \( S^{N-1} \).

**Proposition 5.1.** Let \( u \in h(B^N) \). Then

(i) \( u = P_h[f] \) for some \( f \in L^p(S^{N-1}) \), \( 1 < p < \infty \), if and only if

\[
\sup_{0 < r < 1} \int_{S^{N-1}} |u(r\xi)|^p \, d\sigma(\xi) < \infty. \tag{5.1}
\]

In this case, \( u \) has admissible limit \( f \) a.e. and \( u \in h^p(B^N) \).

(ii) \( u = P_h[\mu] \) for some measure \( \mu \) if and only if

\[
\sup_{0 < r < 1} \int_{S^{N-1}} |u(r\xi)| \, d\sigma(\xi) < \infty. \tag{5.2}
\]

In this case, \( u \) has admissible limit \( d\mu/d\sigma \) a.e. Moreover, if \( u \in h^1(B^N) \), then \( d\mu \) is absolutely continuous.

**Proof.** If \( u = P_h[f] \), obviously, by Hölder's inequality

\[
|u(x)|^p \leq \int_{S^{N-1}} P_h(x, \xi)|f(\xi)|^p \, d\sigma(\xi).
\]

Then (5.1) follows from Theorem 4.1.

Conversely, the fact the \( L^p \)-norms are uniformly bounded give the existence of \( \varphi \in L^p(S^{N-1}) \) and a sequence \( r_n \to 1 \) such that \( u(r_n\xi) \to \varphi(\xi) \) as \( n \to \infty \) weakly in \( L^p(S^{N-1}) \). In particular, for each \( x \in B^N \) fixed, by Theorems 3.1 and 4.1,

\[
P_h[\varphi](x) = \lim_{n \to \infty} \int_{S^{N-1}} P_h(x, \xi)u(r_n\xi) \, d\sigma(\xi)
\]

\[
= \lim_{n \to \infty} \sum_{k=0}^{\infty} F_k(r_n)^k \frac{F_k(|x|)}{F_k(1)} f_k(x) = u(x).
\]
From the explicit formula for $P_h$ one easily obtains as in the classical case that $M_\alpha u$ is dominated by the Hardy–Littlewood maximal function of $f$. This implies that $M_\alpha u \in L^p(S^{N-1})$, and the existence of admissible limits is proved in the standard way.

The first part of (ii) is proved similarly. If $u \in h^1(B^N)$, then the convergence of $u_r$, defined by $u_r(\xi) = u(r\xi)$, $\xi \in S^{N-1}$, is dominated, and hence its weak limit $d\mu$ is absolutely continuous.

Now we will show that when $p < 1$, and $u \in h^p(B^N)$, then $u$ has a distribution value on $S^{N-1}$. We will need the technical Lemma 10 from [1].

**Lemma 5.2** [1]. Let $F \in C^2([1/2, 1])$ and $h \in C^1([1/2, 1])$ satisfying $h(1) > -1$. Suppose that

$$(1 - x)F''(x) + h(x)F'(x) = O((1 - x)^{-A}) \quad \text{as } x \to 1.$$ 

Then:

(i) If $A > 1$, $F(x) = O((1 - x)^{-A+1})$.

(ii) If $0 < A < 1$, then there exists $\lim_{x \to 1} F(x)$.

**Theorem 5.3.** Let $u \in h(B^N)$. Assume that for some $p < 1$,

$$\sup_{0 < r < 1} \int_{S^{N-1}} |u(r\xi)|^p d\sigma(\xi) < \infty.$$ 

Then there exists a distribution $\phi$ satisfying:

(i) $\lim_{r \to 1} u(r\xi) = \phi(\xi)$ in the sense of distributions.

(ii) $u = P_\phi[\phi]$.

**Proof.** Suppose $\varphi \in C^\infty(S^{N-1})$. Define

$$F(r) = \int_{S^{N-1}} u(r\xi)\varphi(\xi) d\sigma(\xi).$$

Formula (3.4) gives

$$(1 - r^2) \left( r \frac{d}{dr} \right)^2 F(r) + (N - 2)(1 + r^2) \left( r \frac{d}{dr} \right) F(r) = \int_{S^{N-1}} Xu(r\xi)\varphi(\xi) d\sigma(\xi),$$

where $X = (r^2 - 1) \sum_{i < j} T_{i,j}$ is a tangential derivative. Thus, writing $\psi = X^* \varphi \in C^\infty(S^{N-1})$ with $X^*$ the adjoint operator we have

$$(1 - r^2) F''(r) + \frac{1 - r^2 + (N - 2)(1 + r^2)}{r} F'(r) = \int_{S^{N-1}} u(r\xi)r^{-2}\psi(\xi) d\sigma(\xi).$$
Iterating the process above and writing
\[ L = (1 - r^2) \frac{d^2}{dr^2} + \frac{1 - r^2 + (N - 2)(1 + r^2)}{r} \frac{d}{dr} \]
we deduce that for each \( k = 1, 2, \ldots \) there exists \( \phi_k \in C^\infty(S^{N-1}) \) such that
\[ (L^k F)(r) = \int_{S^{N-1}} u(r\xi) \phi_k(\xi) d\sigma(\xi). \]
Since \( \sup_{0 < r < 1} \int_{S^{N-1}} |u(r\xi)|^p d\sigma(\xi) < \infty \), we have
\[ |u(x)| \leq C \frac{1}{(1 - |x|)^{(N-1)/p}}, \]
and consequently \( L^k F(r) = O((1 - r)^{-(N-1)/p}) \). Applying Lemma 5.2 we find that
\[ L^{k-1} F(r) = O((1 - r)^{-(N-1)/p+1}). \]
Iterating the process we deduce that \( \lim_{r \to 1} F(r) \) exists.
Part (ii) follows similarly.

The proof of Theorem 5.3 shows that a hyperharmonic function \( u \) in \( B^N \) has a distribution value if and only if \( u(x) = O((1 - |x|)^A) \), \( A \in \mathbb{R} \). Consequently, as a corollary of Theorems 3.2, 3.4 and 5.3, we have that if \( u \in h^p(B^{2n+1}) \) (more generally, if \( u \) is hyperharmonic in \( B^{2n+1} \) and \( u(x) = O((1 - |x|)^A) \) and \( u \neq \text{const} \), then \( \int_{S^{2n}} R^{2n} u(r\xi) \varphi(\xi) d\sigma(\xi) \) is \( O(\log(1/(1 - r))) \) but is not \( o(\log(1/(1 - r))) \) for every \( \varphi \in C^\infty(S^{2n}) \).

**Remark.** We note that the results in Sections 4 and 5 are analogous to results on for classical harmonic functions. The Poisson kernel
\[ P(x, \eta) = \frac{1 - |x|^2}{|x - \eta|^{2n}}, \quad x \in B^N, \ \eta \in S^{n-1}, \]
solves the Dirichlet problem for the ordinary Laplacian \( \Delta \); it extends distributions on \( S^{N-1} \) to classical harmonic functions on \( B^N \) in the same way as the “hyperharmonic” Poisson kernel \( P_h \) extends distributions on \( S^{N-1} \) to hyperharmonic functions on \( B^N \).

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