ON A LITTLEWOOD-PALEY TYPE INEQUALITY

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Abstract. It is proved the following: If \( u \) is a function harmonic in the unit ball \( B \subset \mathbb{R}^N \), and \( 0 < p \leq 1 \), then there holds the inequality

\[
\int_{\partial B} u^*(y)^p \, d\sigma \leq C_{p,N} \left( |u(0)|^p + \int_B (1 - |x|)^{p-1} |\nabla u(x)|^p \, dV(x) \right),
\]

where \( u^* \) is the non-tangential maximal function of \( u \). This improves a recent result of Stoll [19]. This inequality holds for polyharmonic and hyperbolically harmonic functions as well.

Let \( \mathbb{R}^N (N \geq 2) \) denote the \( N \)-dimensional Euclidean space. In [18], Stević proved that if \( u \) is a function harmonic in the unit ball \( B \subset \mathbb{R}^N \), and \( \frac{N-2}{N-1} \leq p < 1 \), then there holds the inequality

\[
(1) \quad \sup_{0 < r < 1} M_p^p(r, u) \leq C_1 |u(0)|^p + C_2 \int_B (1 - |x|)^{p-1} |\nabla u(x)|^p \, dV(x).
\]

Here \( dV \) denotes the Lebesgue measure in \( \mathbb{R}^N \) normalized so that \( V(B) = 1 \), and as usual

\[
M_p^p(r, u) = \int_{\partial B} |u(ry)|^p \, d\sigma(y),
\]

where \( d\sigma \) is the normalized surface measure on the sphere \( \partial B \). The strange condition \( (N-2)/(N-1) \leq p \leq 1 \) appears in [18] because the proof in that paper is based on the fact, due Stein and Weiss [17, 16], that \( |\nabla u|^p \) is subharmonic for \( p \geq (N-2)/(N-1) \).

In the case \( N = 2 \), inequality (1) was proved by Flett [2]. It holds for \( 1 < p < 2 \) as well, while if \( p > 2 \), then there holds the reverse inequality; these inequalities

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are due to Littlewood and Paley [7]. Elementary proofs of the Littlewood-Paley inequalities are given in [13] and [8, 15] \((p > 2)\).

In a recent paper [19], Stoll proved a very general theorem which says, in particular, that (1) holds for every \(p \in (0, 1]\). Here we improve Stoll’s theorem by proving the following result. Here \(u^*\) denotes the nontangential maximal function of \(u\), i.e.

\[
u^*(y) = \sup_{|x-y|<c(1-|x|)} |u(x)|, \quad y \in \partial B,
\]

where \(c > 1\) is a constant.

**Theorem 1.** If \(u\) is a function harmonic in \(B\), and \(0 < p \leq 1\), then there holds the inequality

\[
\int_{\partial B} u^*(y)^p \, d\sigma(y) \leq C_{p,N} \left( |u(0)|^p + \int_B (1 - |x|)^p-1 |\nabla u(x)|^p \, dV(x) \right). 
\]

where \(C\) is a constant depending only on \(p, c, \) and \(N\).

A well known theorem of Fefferman and Stein [1] enables us to replace \(u^*\) in (2) by the radial maximal function \(u^+\),

\[
u^+(y) = \sup_{0<r<1} |u(ry)|, \quad y \in \partial B;
\]

namely:

**Theorem A.** If \(U \geq 0\) is a function subharmonic in \(B\), and \(p > 0\), then there is a constant \(C = C_{p,N,c}\) such that

\[
\int_{\partial B} U^*(y)^p \, d\sigma(y) \leq C \int_{\partial B} U^+(y)^p \, d\sigma(y).
\]

The proof of this theorem (see [3, Theorem 3.6]), as well our proof of Theorem 1, is based on a fundamental result of Hardy and Littlewood [4] and Fefferman and Stein [1] on subharmonic behavior of \(|u|^p\). We state this result in the following way.
Lemma A. If $U \geq 0$ is a function subharmonic in $B(a, 2\varepsilon)$ ($a \in \mathbb{R}^N$, $\varepsilon > 0$), then there holds the inequality

$$\sup_{x \in B(a, \varepsilon)} U(x)^p \leq C \varepsilon^{-N} \int_{B(a, 2\varepsilon)} U^p \, dV,$$

where $C$ depends only on $p, N$.

Here $B(a, r)$ denotes the ball of radius $r$ centered at $a$.

For simple proofs of Lemma A we refer to [10, 14], and for generalizations to various classes of functions, we refer to [5, 6, 9, 11, 12].

Let $P(x, y)$ denote the Poisson kernel,

$$P(x, y) = \frac{1 - |x|^2}{|x - y|^N}.$$

Since $\int_{\partial B} P(x, y) \, d\sigma(y) = 1$, we see that Theorem 1 is a direct consequence of Theorem A and the following:

**Proposition 1.** If $u$ is a function harmonic in $B$, and $0 < p \leq 1$, then there holds the inequality

$$u^+(y)^p \leq C_{p, N} \left( |u(0)|^p + \int_B (1 - |x|)^{p-1} |\nabla u(x)|^p P(x, y) \, dV(x) \right),$$

$|y| = 1$, where $C$ is a constant depending only on $p$ and $N$.

Let

$$u^+(py) = \sup_{0 < r < \rho} |u(ry)| = \sup_{0 < t < 1} |u(tpy)|, \quad 0 < \rho \leq 1, \ y \in \partial B,$$

and $u^+(0) = |u(0)|$.

**Lemma 1.** Let $r_j = 1 - 2^{-j}$ for $j \geq 0$. If $0 < p \leq 1$ and $u$ is of class $C^1(B)$, then there holds the inequality

$$u^+(y)^p \leq C |u(0)|^p + C \sum_{j=0}^{\infty} 2^{-jp} \sup_{r_j < r < r_{j+1}} |\nabla u(ry)|^p, \quad y \in \partial B,$$

where $C$ depends only on $p$ and $N$. 
Proof. We start from the inequality
\[
 u^+(r_{j+1}y)^p - u^+(r_jy)^p \leq \sup_{0<t<1} |u(tr_{j+1}y) - u(tr_jy)|^p.
\]
By Lagrange's theorem,
\[
|u(tr_{j+1}y) - u(tr_jy)| \leq (r_{j+1} - r_j) \sup_{0<r<r_{j+1}} |\nabla u(ry)|
\]
and hence
\[
 u^+(r_{j+1}y)^p - u^+(r_jy)^p \leq 2^{-j} \sup_{0<r<r_{j+1}} |\nabla u(ry)|^p,
\]
which implies
\[
 u^+(y)^p - |u(0)|^p = \sum_{j=0}^{\infty} (u^+(r_{j+1}y)^p - u^+(r_jy)^p)
\]
\[
\leq \sum_{j=0}^{\infty} 2^{-jp} \sup_{0<r<r_{j+1}} |\nabla u(ry)|^p.
\]
On the other hand, by summation by parts we see that if \(\{A_j\}_{0}^{\infty}\) is a nondecreasing sequence of real numbers, then
\[
\sum_{j=0}^{\infty} 2^{-jp} A_{j+1} \leq CA_0 + C \sum_{j=0}^{\infty} 2^{-jp} (A_{j+1} - A_j),
\]
where \(C\) depends only on \(p\). By taking \(A_j = \sup_{0\leq r\leq r_j} |\nabla u(ry)|^p\), and using the inequalities
\[
A_{j+1} - A_j \leq \sup_{r_j<r<r_{j+1}} |\nabla u(ry)|^p,
\]
we get the desired result. \(\square\)

Proof of Proposition 1. By Lemma A with \(U = |\nabla u|\), \(a = a_j := (r_j + r_{j+1})y/2\) and \(\varepsilon = (r_{j+1} - r_j)/2 = 2^{-j-2},\)
\[
2^{-jp} \sup_{r_j<r<r_{j+1}} |\nabla u(ry)|^p \leq C2^{-jp}2^{jn} \int_{B(a_j, 2^{-j-1})} |\nabla u(x)|^p dV(x).
\]
On the other hand, simple calculation shows that \(|x - a_j| \leq 2^{-j-1}\) implies
\[
2^{-j-2} \leq 1 - |x|, \quad |x - y| \leq 2^{-j+1}.
\]
Hence
\[ 2^{-j}2^j N \leq 2^{N+2} P(x, y), \quad \text{for } x \in B(a_j, 2^{-j-1}). \]

From this and (6) we get
\[
2^{-jp} \sup_{r_j < r < r_{j+1}} \|\nabla u(r_jy)\|^p \leq C2^{-j(p-1)} \int_{r_{j-1} \leq |x| \leq r_{j+2}} P(x, y) |\nabla u(x)|^p dV(x)
\]
\[
\leq 2^{1-p} C \int_{r_{j-1} \leq |x| \leq r_{j+2}} (1 - |x|)^{p-1} P(x, y) |\nabla u(x)|^p dV(x)
\]
(r_{-1} = 0) where we have used the inclusion
\[ \{x: |x - a_j| \leq 2^{-j-1}\} \subset \{x: r_{j-1} \leq |x| \leq r_{j+2}\}. \]

Now the desired conclusion is easily obtained by using Lemma 1. \[\square\]

Remarks.

**Remark 1.** As the above proofs show, the validity of Proposition 1 depends only on the inequality
\[
(7) \quad \sup_{x \in B(a, \varepsilon)} |\nabla u(x)|^p \leq C\varepsilon^{-N} \int_{B(a, 2\varepsilon)} |\nabla u|^p dV.
\]
It was proved in [10] that (7) is implied by
\[
(8) \quad |\nabla u(x)| \leq K\varepsilon^{-1} \sup_{z \in B(a, \varepsilon)} |u(z) - u(x)|.
\]
More precisely:

Let \( u \in C^1(B) \) and let \( 0 < p < \infty \). If there is a constant \( K \) such that there holds (8) whenever \( B(x, \varepsilon) \subset B \), then there is a constant \( C = C_{p,N,K} \) such that there holds (7) whenever \( B(x, 2\varepsilon) \subset B \).

**Remark 2.** Condition (8), and hence (7), is satisfied in a wide class of functions containing, in particular, polyharmonic, hyperbolically harmonic, and convex functions [11]. Recall that \( u \) is called polyharmonic if \( \Delta^k u \equiv 0 \) for some integer \( k \geq 1 \),
and $u$ is hyperbolically harmonic if

$$\Delta_h u(x) := (1 - |x|^2)^2\Delta u(x) + 2(N - 2)(1 - |x|^2)^{-1}x \cdot \nabla u(x) \equiv 0.$$ 

Note also that the class of hyperbolically harmonic functions is invariant under Möbius transformations of the ball.

**Remark 3.** The proof of Theorem A given in [3] (see also [14, theorem 7.1.8]) shows that (3) is implied by (4). On the other hand, as was proved in [10], condition (8) implies (4).

It follows from the above remarks that there holds the following generalization of Theorem 1.

**Theorem 2.** If $u$ is a function polyharmonic, hyperbolically harmonic, or convex in $B$, and $0 < p \leq 1$, then there holds inequality (1).

**References**


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