ADDENDA TO “FUNCTION CLASSES”

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Abstract. This text contains a conjecture on $(C, \alpha)$-summability of Fourier series, and also new proofs which I was not able to find during the writing the book.

1. A CONJECTURE ON $(C, \alpha)$ CONVERGENCE

On page 192, Problem 4.1, I posed the question: Whether there exists a function $f \in H^p(\mathbb{T})$, $1 < p \leq 2$, such that $\sigma_n^{1/p-1} f(\zeta)$ diverges for a.e. $\zeta \in \mathbb{T}$? Here we add:

**Conjecture.** Let $1 < p \leq 2$ and $\alpha = 1/p - 1$.

(i) If $\beta > \alpha$ then $\sigma_n^\beta f(\zeta) \to f(\zeta)$ for a.e. $\zeta \in \mathbb{T}$.

(ii) There is a function $f \in H^p(\mathbb{T})$ such that $\sigma_n^\alpha f(\zeta)$ diverges a.e.

(iii) There exists a function $f \in C(\mathbb{T})$ such that $\sigma_n^{-1/2} f(\zeta)$ diverges a.e.

Of course, (iii) imply (ii) for $p = 1/2$.

We recall that if $f \in H^p$, $p < 1$, then $\sigma_n^\alpha(\zeta) \to f(\zeta)$ [Zygmund]. On the other hand, there exists a function $f \in H^1(\mathbb{T})$ such that $\sigma_n^0 f(\zeta)$ diverges a.e. [Kolmogorov–Hardy–Rogosinski].

The above conjecture is based on the relation [3, Theorem 4.13]

\[
\lim \inf_{n \to \infty} |\sigma_n^{\alpha} f(\zeta) - f(\zeta)| = 0, \quad \text{for a.e. } \zeta \in \mathbb{T},
\]

which is obtained from

\[
\lim \inf_{n \to \infty} \|\sigma_n^{\alpha} f - f\|_p = 0
\]

by means of Fatou’s lemma, whereas (2) follows from

\[
\lim_{n \to \infty} \frac{1}{L_n} \sum_{k=0}^{n} \frac{1}{k+1} \|\sigma_k^{\alpha} f - f\|_p = 0, \quad \text{where } L_n = \sum_{k=0}^{n} \frac{1}{k+1}.
\]

The latter can be “easily” deduced from

\[
\frac{1}{L_n} \sum_{k=0}^{n} \frac{1}{k+1} \|\sigma_k^{\alpha} f\|_p^p \leq C_p \|f\|_p^p,
\]

and the deduction is left to the reader.

The “reader’s” proof of (3). Let $f \in H^p$, $\varepsilon > 0$, and choose a polynomial $P$ of degree $s$ such that $\|f - P\| < \varepsilon$. Then we have

\[
S_n f := \frac{1}{L_n} \sum_{k=0}^{n} \frac{1}{k+1} \|\sigma_k^{\alpha} f - f\|_p = \frac{1}{L_n} \sum_{k=0}^{m} \ldots + \frac{1}{L_n} \sum_{k=m+1}^{n} \ldots =: Q_n + R_n, \quad n > m > s,
\]
where $m$ is fixed integer which will be chosen later on. It is easily checked that $\lim_{n \to \infty} Q_n = 0$. In order to estimate $R_n$ we start from the inequality

$$R_n \leq 2^{p-1} \frac{1}{E_n} \sum_{k=m+1}^{n} \frac{1}{k+1} \left( ||\sigma_k^p(f - P)||^p + ||\sigma_k^p P - P||^p + ||f - P||^p \right) = R'_n + R''_n + R'''_n.$$  

By (4), we have $R'_n \leq C2^{p-1} ||f - P||^p < C2^{p-1} \varepsilon^p$, and, obviously, $S'''_n \leq 2^{p-1} \varepsilon^p$, so it remains to deal with $S''_n$. Since $P$ is a polynomial, we have

$$\lim_{k \to \infty} ||\sigma_k^p P - P|| = 0.$$  

Now choose $m$ so that $||\sigma_k^p P - P|| < \varepsilon$ for $k > m$, which implies $R'''_n \leq 2^{p-1} \varepsilon^p$, concluding the proof.  

2. Proof of Theorem 10.10

In the book, I prove this theorem under the hypothesis that $d \arg f(z) > 0$ and forgot to say that the reader should read the proof in the general case in the Spencer’s paper [4]. However, a proof can be created by analysis of Hörmander’s proof [2, pp. 160–161] of Prawitz’s theorem. It is shown there that if $f(0) = 0$, then, for a fixed $r$,

$$\begin{align*}
D(r) := \frac{2\pi r}{p} \frac{d}{dr} I_p(r, f) &= \frac{2\pi r}{p} \int_{\partial f(r\mathbb{D})} |w|^p \, d(\arg w) 
\end{align*}$$

can be represented as

$$\begin{align*}
\sum_{j=1}^{n} \int_{A_j} \left( R_{1,j}(\alpha)^p + \sum_{k=1}^{m_j} \left[ R_{2k+1,j}(\alpha) - R_{2k,j}(\alpha)^p \right] \right) d\alpha \left( \sum_{k=1}^{m_j} \int_{A_j} d\alpha \right) &= 0, 
\end{align*}$$

where $A_j \subset [0, 2\pi]$ are disjoint intervals such that $|A_1| + \ldots + |A_n| = 2\pi$, and

$$R_{1,j} < R_{2,j} < \ldots < R_{2m_j+1,j}.$$  

Assuming that $p > 2$, we apply the reverse Hölder inequality with exponent $2/p, 2/(2 - p)$ to obtain

$$D(r) \geq \left( \sum_{j=1}^{n} \int_{A_j} \left( R_{1,j}(\alpha)^2 + \sum_{k=1}^{m_j} \left[ R_{2k+1,j}(\alpha) - R_{2k,j}(\alpha)^2 \right] \right) d\alpha \right)^{p/2} \left( \sum_{j=1}^{n} \int_{A_j} d\alpha \right)^{1-p/2}.$$  

Since $(a - b)^\gamma \geq a^\gamma - b^\gamma$ when $a > b > 0$ and $0 < \gamma < 1$, we conclude that

$$D(r) \geq \left( \sum_{j=1}^{n} \int_{A_j} \left( R_{1,j}(\alpha)^2 + \sum_{k=1}^{m_j} \left[ R_{2k+1,j}(\alpha) - R_{2k,j}(\alpha)^2 \right] \right) d\alpha \right)^{p/2} (2\pi)^{1-p/2} = \left( 2\pi r \frac{d}{dr} I_2(r, f) / 2 \right)^{p/2} (2\pi)^{1-p/2} = 2^{p/2} A(r, f)^{p/2} (2\pi)^{1-p/2}.$$  

Hence

$$\frac{d}{dr} I_p(r, f) \geq \frac{p}{2\pi r} 2^{p/2} A(r, f)^{p/2} (2\pi)^{1-p/2}$$

just in the case when $d \arg f(z) > 0$.

Remark 1. Since, by Green’s formula,

$$\int_{\partial G} |w|^p \, d(\arg w) = \text{Im} \int_{\partial G} |w|^{p-2} \overline{w} \, dw = p \int_{G} |w|^{p-2} \, dA(w),$$

we see that the Hörmander’s proof yields the inequality

$$\begin{align*}
p \int_{G} |w|^{p-2} \, dA(w) &\geq 2^{p/2} (2\pi)^{1-p/2} |G|^{p/2}, 
\end{align*}$$

where $G = f(r\mathbb{D}), 0 < r < 1$, and $f$ is a real-analytic homeomorphism from $\mathbb{D}$ onto $f(\mathbb{D})$ such that $f(0) = 0$.  

3. A **Hardy–Stein identity for locally univalent harmonic mappings**

A function \( f \in h(\mathbb{D}) \) is said to be locally univalent if for every \( a \in \mathbb{D} \) it is univalent in some neighborhood of \( a \).

**Theorem 2.** If \( p > 0 \) and \( f \) is locally univalent, then the function \( M_p(r, f) \), \( 0 \leq r < 1 \), is of class \( C^1 \) and the formula

\[
d\ell_{M_p}(r, f) = \frac{1}{2\pi r} \int_{\partial \mathbb{D}} \left( p^2 |f|^{p-2}(|h'|^2 + |g'|^2) + 2p(p-2)|f|^{p-4} \text{Re}(f^2h'g') \right) \, dA(z)
\]

holds, where the integral is absolutely convergent.

**Proof.** The absolute convergence of the integral is a consequence of its absolute convergence on the closed discs \( D \subset \mathbb{D} \) in which \( f \) is univalent. In order to prove the latter, it suffices to note that \( D \) can be chosen so that \( f \) is quasiconformal on \( D \). This implies

\[
\int_D |\ldots| \, dA \leq C \int_D |f|^{p-2} \, dA, \quad \text{where } J_f \text{ is the Jacobian of } f,
\]

i.e.,

\[
\int_D |\ldots| \, dA \leq C \int_{f(D)} |w|^{p-2} \, dA < \infty.
\]

Because of that we may assume that \( r \neq |a_j| \) for all \( j \). By Lewi’s theorem, we have that \( J_f \neq 0 \) in \( \mathbb{D} \), and the zeroes of \( f \) are isolated; denote the zeroes by \( a_j, j \geq 1 \), and assume that the sequence \( |a_j| \) is increasing. Let \( |a_j| < r \) for \( j \leq k \). Let \( \Omega = \Omega_{r, \rho} \) denote the domain bounded by the circles \( |z| = r \) and \( |z - a_j| = \rho \), where \( \rho \) is chosen to be “infinitesimally” small. Then apply the (Green) formula

\[
\int_{\Omega} \Delta u \, dA = \oint_{\partial \Omega} \frac{\partial u}{\partial n} \, d\ell,
\]

where \( d\ell \) is the arc-length element, and \( \frac{\partial u}{\partial n} \) the derivative of \( u \) in the direction of the unit vector \( \vec{n} \) oriented accordingly with the orientation of \( \partial \Omega \). We get

\[
\int_{\Omega} \Delta u \, dA = r \frac{d}{dr} \int_0^{2\pi} u(r e^{i\theta}) \, dt - \sum_{j=1}^{k} \rho \frac{d}{d\rho} \int_0^{2\pi} u(a_j + \rho e^{i\theta}) \, d\theta, \quad \text{where } u = |f|^p, \ p > 0.
\]

Now it suffices to prove that

\[
\lim_{\rho \to 0^+} \rho \int_0^{2\pi} u(\rho e^{i\theta}) \, d\theta = 0,
\]

and then translate this identity to the points \( a_j \) to obtain the desired result. We may assume that \( f(0) = 0 \) because otherwise the proof is trivial. Then write \( f \) as \( f(z) = h(z) + g(z) \). Since \( J_f(0) \neq 0 \), we see that \( |h'(0)| \neq |g'(0)| \). Assuming this, we have

\[
\left| \frac{d}{d\rho} \int_0^{2\pi} |f(\rho e^{i\theta})|^p \, d\theta \right| \leq p \int_0^{2\pi} |f(\rho e^{i\theta})|^{p-1} \left| \frac{d}{d\rho} f(\rho e^{i\theta}) \right| \, d\theta
\]

\[
\leq C \int_0^{2\pi} |f(\rho e^{i\theta})|^{p-1} \, d\theta \leq C \int_0^{2\pi} \left| h'(0) - g'(0) \right| \rho - \rho^2 \psi(\rho e^{i\theta}) \, d\theta,
\]

where \( \rho \) is sufficiently small and \( \psi \) is a function continuous in the disc \( \rho \mathbb{D} \). This implies that

\[
\rho \frac{d}{d\rho} \int_0^{2\pi} |f(\rho e^{i\theta})|^p \, d\theta \leq C \rho^p,
\]

which, together with the formula

\[
\Delta(|f|^p) = p^2 |f|^{p-2}(|h'|^2 + |g'|^2) + 2p(p-2)|f|^{p-4} \text{Re}(f^2h'g'),
\]

concludes the proof. \( \square \)
4. **A Holland–Twomey–Spencer theorem for QC harmonic mappings**

The case $p < 2$ of the following theorem is discussed in [3, Corollary 10.8].

**Theorem 3.** Let $f$ be a QC harmonic mapping and $p \geq 2$. Then each of the following quantities is equivalent to $\|f\|_p^p$:

\begin{equation}
\int_0^1 |f(r \mathbb{D})|^{p/2} dr, \quad \int_0^1 \left( \int_{r \mathbb{D}} (|h'|^2 + |g'|^2) \, dA \right)^{p/2} dr, \quad \int_0^1 P(r, f)^p dr,
\end{equation}

where $P(r, f) = \sum_{n \in \mathbb{Z}} |\hat{f}(n)| r^n$.

**Proof.** Denote these quantities by $Q_1$, $Q_2$, and $Q_3$, respectively. Since

\begin{equation}
|f(r \mathbb{D})|^2 = \int_{r \mathbb{D}} (|h'|^2 - |g'|^2) \, dA,
\end{equation}

we have, by the quasiconformality of $f$ that $Q_1 \asymp Q_2$. In proving that $\|f\|_p^p \leq CQ_2$, we start from the inequality

\begin{equation}
M^p_p(r, f) \leq \frac{C}{r} \int_{r \mathbb{D}} |f|^{p-2}(|h'|^2 + |g'|^2) \, dA,
\end{equation}

which follows from (8), and

\begin{equation}
\int_0^1 M^p_\infty(r, f)^p dr \leq C\|f\|_p^p,
\end{equation}

and then proceed exactly as in [1]. It remains to prove that $\|f\|_p^p \geq CQ_1$, since “$Q_2 \asymp Q_3$” can be proved by use of “$L^p$-integrability”.

It follows from Theorem 2 that

\begin{align*}
2\pi r \frac{d}{dr} M^p_p(r, f) &\geq \int_{r \mathbb{D}} (p^2|f|^{p-2}(|h'|^2 + |g'|^2) - 2p(p-2)|f|^{p-2}|h'| |g'|) \, dA \\
&\geq \int_{r \mathbb{D}} 2p|f|^{p-2}(|h'|^2 + |g'|^2) \, dA \geq \int_{r \mathbb{D}} 2p|f|^{p-2}(|h'|^2 - |g'|^2) \, dA \\
&= 2p \int_{f(r \mathbb{D})} |w|^{p-2} dA(w).
\end{align*}

Now, by Remark 1, we see that if $f(0) = 0$, then $\|f\|_p^p \geq CQ_1$. That should be all. □

**References**