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Theorems about quadrilaterals and conics

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We study quadrilaterals inscribed and circumscribed about conics and prove interesting theorems. Theorems are discovered by experimenting with dynamical geometry software. The Poncelet theorem for quadrilaterals is proved by elementary means together with Poncelet’s grid property.

Keywords: quadrilaterals; conics; projective geometry; dynamical geometry software

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1. Introduction

Our research is motivated by the conjecture of Schwartz and Tabachnikov [22, Theorem 4.c]. While searching for the proof we encountered several interesting results about conics and quadrilaterals.

Conjecture 1 (Conjecture of Tabachnikov and Schwartz) Let \( A_1A_2 \ldots A_{12} \) be a 12-gon inscribed in a conic \( C \). Let \( \pi \) map 12-gon \( X_1X_2 \ldots X_{12} \) onto a new 12-gon according to the rule \( \pi(X_i) = l(X_iX_{i+3}) \cap l(X_{i+1}X_{i+4}) \). Then, 12-gon \( A_1A_2 \ldots A_{12} \) is mapped with \( \pi^{(3)} = \pi \circ \pi \circ \pi \) onto a 12-gon inscribed in a conic.

It seemed that this conjecture is a perfect candidate to use the technique illustrated in the paper Illumination of Pascal’s Hexagrammum and Octagrammum Mysticum by Baralić and Spasojević [2,3]. The problems we study are strongly influenced by the very inspiring paper [13]. Many important questions in dynamical systems and combinatorics have their equivalents in the terms of algebraic curves. Schwartz and Tabachnikov originally formulated their conjecture in [22, Theorem 4.c] in terms of a pentagram map [20].

We will explain Figure 1 carefully. We start with a 12-gon \( A_1A_2 \ldots A_{12} \) (the green points lying on the violet conic) inscribed in a conic and define the (yellow) points obtained by \( \pi \), (blue and violet lines), \( \pi^{(2)} \) the red points (green and orange lines) and \( \pi^{(3)} \) the violet points (black and yellow lines). It turns out that at each step we have a \( 6 \times 6 \) cage of curves, see [13]. But instead of dealing

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with 24 points at the second step we take only 12 of them. It is not possible to catch the curves we want in the cage. By the Mystic Octagon theorem, we could catch three interesting conics and one quartic in the blue-violet cage. What to do with the curves at other steps? Definitely we should try to add some new points and then apply Bézout’s theorem or a similar statement. But what are those points and how to find them? If we look more carefully, three quadrilaterals inscribed in a conic can be noticed ($A_1A_4A_7A_{10}$, $A_2A_5A_8A_{11}$ and $A_3A_6A_9A_{12}$) and usually the steps are always defined as the certain intersection points of the side lines of quadrilaterals. Thus, we thought that if we want to overcome the problems we faced, we should understand the quadrilaterals inscribed in conics better.

Theorems about quadrilaterals and conics are usually known like degenerate cases of Pascal and Brianchon theorems. Baralić and Spasojević [2] proved some new results about two quadrilaterals inscribed in a conic. However, in this paper we study more complicated structures involving both tangents at the vertices and the side lines of quadrilateral. We start from the degenerate form of Pascal and Brianchon theorems for the quadrilateral and then we discover new interesting points, conics and loci. Classical projective geometry from the nineteenth century studied extensively these objects, leading to the founding of new mathematical disciplines such as algebraic geometry. Development of computer graphics, dynamical geometry, dynamical systems, etc. during the second half of the twentieth century renewed the interest of researchers for the classical projective geometry. Recently, two excellent book on this topic were published [15,19]. Theorems 3.1–3.3 we present here extend the known results about geometry of quadrilaterals inscribed in a conic.

The objects are studied by elementary means. Some of the results are in particular the corollary of the Great Poncelet Theorem for the case when $n$-gon is quadrilateral. Here we give a short proof for this case. Some special facts about this special case are explained as well.

Finally, we compare two theorems – the Mystic Octagon theorem for the case of two quadrilaterals and the Poncelet Theorem for the quadrilaterals. They have in common that they state that certain 8 points coming from two quadrilaterals inscribed in a conic lie on the same conic. While the first one is a pure algebro-geometric fact, the latter involves much deeper structure of the space and cannot be seen naturally as a special case of the first one. Thus, we could not find ‘The theorem of all theorems for conics in projective geometry’ and elementary surprises in projective geometry like those in [22,23] could come as the special cases of different general statements.
2. **From Pascal to Brocard theorem**

In this section we show how the Pascal theorem for hexagon [16] (1639) inscribed in a conic degenerates to the Brocard theorem for the quadrilateral inscribed in a circle. All results here are well known and are part of the standard olympiad problem solving curriculum, but our aim is to illustrate the power of degeneracy tool and prepare the background for the next sections.

**Lemma 2.1** Let $ABCD$ be a quadrilateral inscribed in a conic $C$ and let $M$ be the intersection point of the lines $AD$ and $BC$, $N$ be the intersection point of the lines $AB$ and $CD$, $P$ be the intersection point of the tangents to $C$ at $A$ and $C$, and $Q$ be the intersection point of the tangents to $C$ at $B$ and $D$. Then, the points $M$, $N$, $P$ and $Q$ are collinear (Figure 2).

**Proof** Apply the Pascal theorem to degenerate hexagon $AABCCD$ and we get the points $M$, $N$ and $P$ are collinear. Apply the Pascal theorem to degenerate hexagon $ABBCD$ and we get that the points $M$, $N$ and $Q$ are collinear. 

Dual statement to Lemma 2.1 is the following:

**Lemma 2.2** Let conic $C$ touch the sides $AB$, $BC$, $CD$ and $DA$ of a quadrilateral $ABCD$ in the points $M$, $N$, $P$ and $Q$, respectively. Then the lines $AC$, $BD$, $MP$ and $NQ$ pass through the same point $O$ (Figure 3).

Lemmas 2.1 and 2.2 will be used to prove other interesting relations among the lines and points that naturally occur in a quadrilateral inscribed in conics configurations. Many points are going to be introduced so we are going to organize labels of our points.

Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a conic $C$ and let $M_1$ be the intersection point of the lines $A_1A_2$ and $A_3A_4$, $M_2$ of $A_2A_3$ and $A_4A_1$ and $M_3$ of $A_3A_1$ and $A_2A_4$. Let $N_3$ be the intersection point of the tangent lines to the conic at $A_1$ and $A_3$, $P_3$ of the tangents at $A_2$ and $A_4$, $N_2$ of the tangents at $A_1$ and $A_4$, $P_2$ of the tangents at $A_2$ and $A_3$, $N_1$ of the tangents at $A_1$ and $A_2$ and $P_1$ of

![Figure 2. Lemma 2.1.](image-url)
Lemma 2.1 states that the points $M_1, M_2, N_3, P_3$ are collinear, as well as the points $M_2, M_3, N_1, P_1$ and $M_3, M_1, N_2$ and $P_2$. Denote these three lines by $m_3, m_1$ and $m_2$, respectively. We are going to prove that $U_1$ and $U_2$ lie on the line $m_3$, $V_1$ and $V_2$ on the line $m_2$ and $W_1$ and $W_2$ on $m_1$—so that $m_1, m_2$ and $m_3$ are the polar lines of the points $M_1, M_2$ and $M_3$ with respect to $C$.

The following lemma is a well-known result about poles and polars. A classical proof using harmonic division could be found in [10]. However, for the reader’s convenience we give a proof based on a different, well-known idea. Indeed, moving the configuration into a special position will be a central idea in the proof of Theorem 3.3.

**Lemma 2.3**  The points $U_1, M_2, U_2$ and $M_3$ are collinear (Figure 4).

**Proof**  There is a projective transformation $\varphi$ that maps the points $A_1, A_2, A_3$ and $A_4$ onto the vertices of a square. Thus, $\varphi(M_3)$ is the centre of a square with vertices $\varphi(A_1), \varphi(A_2), \varphi(A_3)$ and $\varphi(A_4)$. The points $\varphi(M_1)$ and $\varphi(M_2)$ are at infinity. There is a unique way to inscribe a square into a conic, and the lines $\varphi(A_1)\varphi(A_2)$ and $\varphi(A_1)\varphi(A_4)$ are parallel to the axes of the conic $\varphi(C)$, see [1]. The points $\varphi(U_1)$ and $\varphi(U_2)$ must be mapped onto the axis parallel to the line $\varphi(A_1)\varphi(A_4)$.

Now the points $\varphi(U_1), \varphi(U_2), \varphi(M_2)$ and $\varphi(M_3)$ lie on the axis of the conic $\varphi(C)$. Consequently, the points $U_1, M_2, U_2$ and $M_3$ then lie on the same line.
Lemma 2.3 clearly implies the analogous statement for the lines $m_2$ and $m_3$. This is the classical theorem of the projective geometry and a very useful tool (Figure 5).

We treat one very special case – when the conic $C$ is a circle. Projective geometry gives us a plenty of techniques. For example, in the proof of Lemma 2.3 we used the projective transformation. We have already described degeneracy tool when we take some limit cases of polygons inscribed (or circumscribed) in a conic. It is good to keep in mind that a conic could degenerate itself for example to the two lines. This is a way to get interesting configurations of points and lines.

The configuration 5 in the case of a circle has a nice property which is known as the Brocard theorem. Let $O$ be the centre of a circle $C$. Then the quadrilateral $M_1U_1OU_2$ is deltoid and we get $M_1O \perp m_1$. Similarly, $M_2O \perp m_2$. Thus:
Theorem 2.1 (Brocard theorem) Let $O$ be the centre of circumscribed circle of a cyclic quadrilateral $A_1A_2A_3A_4$. Then $O$ is the orthocentre of triangle $\triangle M_1M_2M_3$ (Figure 6).

Theorem 2.2 (Mystic Octagon Theorem) Let $ABCDEFGH$ be an octagon inscribed in a conic $C$ and let the lines $AB$, $CD$, $EF$ and $GH$ intersect the lines $BC$, $DE$, $FG$ and $HA$ in the points $K$, $L$, $M$, $N$, $O$, $P$, $Q$ and $R$. Then the eight points $K$, $L$, $M$, $N$, $O$, $P$, $Q$ and $R$ lie on the same conic (Figure 7).

Proof This theorem was formulated by Wilkinson [25]. The proof we present uses only the Pascal theorem and is given by Evans and Rigby [11]. It could also be found in the monograph [4] by Bix.

Let $U$ be the intersection point of the lines $BC$ and $EH$. By Pascal’s theorem the points $K$, $R$ and $U$ are collinear, see Figure 7. Then by the converse of Pascal’s theorem, the points $K$, $L$, $M$, $N$, $O$, $Q$ and $R$ lie on the same conic. Analogously, we prove that the points $L$, $M$, $N$, $O$, $Q$ and $R$ lie on the same conic. There is a unique conic through some five points, so the points $K$, $L$, $M$, $N$, $O$, $Q$ and $R$ lie on the same conic. In the same manner we can prove that the point $P$ also belongs to this conic.

3. More lines, pencils of lines and surprising conics

We continue in the same manner. The lines and the pencils of lines we study came from various degenerations of the vertices of hexagon inscribed in a conic. Let us note that configuration associated with 60 Pascal lines has been described in [2,14,24]. All results from this section could be obtained as certain degenerate cases. But we are going to treat them by elementary means.

Let $T_1$ be the point of intersection of the line $A_3A_4$ and the tangent at $A_1$ to $C$, $T_2$ of $A_4A_1$ and the tangent at $A_2$, $T_3$ of $A_1A_2$ and the tangent at $A_3$ and $T_4$ of $A_2A_3$ and the tangent at $A_4$. Let $X_1$ be the point of intersection of the line $A_2A_3$ and the tangent at $A_1$ to $C$, $X_2$ of $A_3A_4$ and the tangent at...
Figure 8. Lemma 3.1.

$A_2, X_3$ of $A_4A_1$ and the tangent at $A_3$ and $X_4$ of $A_1A_2$ and the tangent at $A_4$. Let $Y_1$ be the point of intersection of the line $A_2A_3$ and the tangent at $A_1$, $Y_2$ of $A_1A_4$ and the tangent at $A_2$, $X_3$ of $A_3A_4$ and the tangent at $A_3$ and $Y_4$ of $A_2A_3$ and the tangent at $A_4$.

**Lemma 3.1** The points $X_1, X_2, X_3, X_4, T_1, T_2, T_3$ and $T_4$ lie on the same conic $C_1$; $Y_1, Y_2, Y_3, Y_4, X_1, X_3, T_2$, and $T_4$ lie on the same conic $C_2$; $T_1, T_3, X_2, X_4, Y_1, Y_2, Y_3$ and $Y_4$ lie on the same conic $C_3$ (Figure 8).

**Proof** This statement is a special case of the Mystic Octagon theorem. The first conic appears when we consider degenerate octagon $A_1A_2A_3A_1A_4A_1$, the second for $A_1A_3A_2A_2A_4A_1$, and the third for $A_1A_3A_3A_4A_2A_2A_1$.

**Proposition 3.1** The following 16 triples of points are collinear: $(M_1, Y_1, Y_2)$, $(M_1, Y_3, Y_4)$, $(M_1, X_3, T_4)$, $(M_1, X_1, T_2)$, $(M_2, Y_1, Y_4)$, $(M_2, Y_2, Y_3)$, $(M_2, X_4, T_1)$, $(M_2, T_2, T_3)$, $(M_3, T_1, T_3)$, $(M_3, X_2, X_4)$, $(M_3, X_1, X_3)$, $(M_3, T_2, T_4)$, $(M_3, Y_2, T_3)$, $(X_1, Y_2, T_3)$, $(X_3, Y_4, T_1)$, $(X_4, Y_1, T_2)$ (Figure 9).

**Proof** The collinearity of the points $M_1, X_3$ and $T_4$ follows from Pascal’s theorem for degenerate hexagon $A_1A_2A_4A_3A_3A_2$, the collinearity of the points $M_1, Y_3$ and $Y_4$ from degenerate hexagon $A_1A_3A_4A_4A_2$ and the collinearity of the points $X_2, Y_3$ and $T_4$ from degenerate hexagon $A_2A_3A_4A_4A_2$. The proof for the rest is analogous.

**Proposition 3.2** The following six triples of lines are concurrent: $(M_2M_3, X_2Y_3, X_3Y_4)$, $(M_1M_4, X_1Y_2, X_2Y_3)$, $(M_1M_2, X_1Y_2, X_3Y_4)$, $(M_2M_3, X_1Y_2, X_4Y_1)$, $(M_1M_3, X_4Y_1, X_3Y_4)$, $(M_1M_2, X_4Y_1, X_2Y_3)$ (Figure 9).

**Proof** By Lemma 3.1 the points $X_1, X_2, X_3, X_4, T_1, T_2, T_3$ and $T_4$ lie on the same conic. From the Pascal theorem for the hexagon $T_1X_3X_1T_2X_4X_2$ we get that lines $M_1M_3$, $X_4Y_1$ and $X_3Y_4$ are concurrent. Analogously for other triples.

Define the points as the intersections of the lines: $B_1 = l(A_2V_1) \cap l(A_1V_2)$, $C_1 = l(A_1V_1) \cap l(A_2V_2)$, $D_1 = l(A_3V_1) \cap l(A_4V_2)$, $E_1 = l(A_4V_1) \cap l(A_3V_2)$, $B_3 = l(A_4V_1) \cap l(A_2V_2)$,
Figure 9. Propositions 3.1 and 3.2.

Figure 10. Propositions 3.3.

\[ C_3 = l(A_4V_2) \cap l(A_2V_1), \quad D_3 = l(A_1V_1) \cap l(A_3V_2), \quad E_3 = l(A_1V_2) \cap l(A_3V_1), \quad D_2 = l(A_4U_1) \cap l(A_1U_2), \quad E_2 = l(A_1U_1) \cap l(A_4U_2), \quad B_2 = l(A_3U_1) \cap l(A_2U_2), \quad C_2 = l(A_2U_1) \cap l(A_3U_2), \quad F_3 = l(A_4U_1) \cap l(A_2U_2), \quad H_3 = l(A_4U_2) \cap l(A_2U_1), \quad G_3 = l(A_1U_1) \cap l(A_3U_2), \quad I_3 = l(A_1U_2) \cap l(A_3U_1), \quad E_1 = l(A_2W_1) \cap l(A_1W_2), \quad F_1 = l(A_1W_1) \cap l(A_2W_2), \quad G_1 = l(A_3W_1) \cap l(A_4W_2), \quad H_1 = l(A_4W_1) \cap l(A_3W_2), \quad H_2 = l(A_4W_1) \cap l(A_1W_2), \quad I_2 = l(A_4W_2) \cap l(A_1W_1), \quad F_2 = l(A_2W_1) \cap l(A_3W_2) \quad \text{and} \quad G_2 = l(A_2W_2) \cap l(A_3W_1). \]

**Proposition 3.3** The points \( B_1, C_1, D_1, E_1, F_1, G_1, H_1, I_1 \) lie on the line \( M_2M_3 \). Similarly, the points \( B_2, C_2, D_2, E_2, F_2, G_2, H_2, I_2 \) lie on the line \( M_3M_1 \) and the points \( B_3, C_3, D_3, E_3, F_3, G_3, H_3, I_3 \) lie on the line \( M_1M_2 \).

**Proof** Consider the quadrilateral formed by the tangent lines to the conic \( C \) at the points \( A_4, A_2, V_1 \) and \( V_2 \). Applying Lemma 2.2, we get that the point \( B_3 \) lies on the line \( M_1M_2 \). Analogously for other points. \[ \blacksquare \]

We introduced many points and showed that some of them are collinear while some are the intersections of certain lines. But some of them lie on the conics that we are going to introduce (Figure 10).
Let $J_{2i-1}$ be the intersection point of the tangents at $X_{i-2}$ and $T_i$ on the conic $C_1$, and $J_{2i}$ be the intersection point of the tangents at $X_{i-1}$ and $T_i$ (modulo 4), for $i = 1, 2, 3, 4$. Then the following claim is true:

**Theorem 3.1**

- The lines $J_iJ_{i+4}$, for $i = 1, 2, 3, 4$ intersect at the point $M_3$.
- The lines $J_1J_7$, $J_2J_6$ and $J_3J_8$ intersect at $M_1$ and the lines $J_1J_3$, $J_4J_5$ and $J_5J_7$ intersect at $M_2$.
- The lines $J_1J_4$ and $J_2J_5$ intersect at $A_1$, the lines $J_4J_7$ and $J_3J_6$ at $A_2$, the lines $J_6J_1$ and $J_5J_8$ at $A_3$ and the lines $J_3J_8$ and $J_2J_7$ at $A_4$.
- The intersection points $l(J_4J_8) \cap l(J_5J_7)$, $l(J_5J_7) \cap l(J_2J_5)$, $l(J_2J_5) \cap l(J_3J_8)$, $l(J_3J_8) \cap l(J_1J_4)$, $l(J_1J_4) \cap l(J_4J_6)$, $l(J_4J_6) \cap l(J_2J_7)$, $l(J_2J_7) \cap l(J_1J_3)$, $l(J_1J_3) \cap l(J_4J_8)$ and $l(J_iJ_{i+1}) \cap l(J_{i+4}J_{i+5})$ for $i = 1, 2, 3, 4$ lie on the same line $M_1M_2$.
- The intersection points $l(J_3J_6) \cap l(J_2J_9)$ and $l(J_3J_6) \cap l(J_4J_8)$ lie on the same line $M_1M_3$, and the intersection points $l(J_2J_7) \cap l(J_3J_5)$, $l(J_1J_2) \cap l(J_6J_7)$ lie on the same line $M_2M_3$.
- The point $P_3$ lies on the line $J_3J_7$ and the point $N_3$ on the line $J_1J_5$.
- Three lines $J_{2i}J_{2i+4}$, $J_{2i+1}J_{2i-2}$ and $J_{2i-1}J_{2i+2}$ (modulo 8) are concurrent for $i = 1, 2, 3, 4$ (Figure 11).

**Proof** Consider the quadrilateral formed by tangents to $C_1$ at $J_2$ and $J_6$. By Lemma 2.2 and Proposition 3.1 the points $M_3$ and $M_2$ lie on the line $J_2J_6$ (we could take different orders of points). Analogously, the lines $J_1J_5$, $J_3J_7$ and $J_4J_8$ pass through the point $M_3$. In a similar manner we prove other statements for the points $M_1$ and $M_2$, as well as the points $N_3$ and $P_3$.

Lemma 2.2 applied to the quadrilateral formed by the tangents to $C_1$ at $J_2$ and $J_5$ implies that the line $J_2J_5$ passes through $A_1$. Similarly, $A_1$ belongs to the line $J_1J_4$. Analogously, we prove the corresponding statements for the points $A_2$, $A_3$ and $A_4$.

From Lemma 2.1 applied to the quadrilateral $T_2X_1T_4X_3$ and Proposition 3.1, it follows that the intersection point of the lines $J_3J_4$ and $J_7J_8$ and the intersection point of the lines $J_4J_5$ and $J_8J_1$
lie on the line $M_1M_2$. Then by Brianchon’s theorem for the hexagon formed by the tangents to $C_1$ at $T_2, X_1, T_3, T_1, X_3$ and $T_4$ the intersection point of the lines $J_1J_4$ and $J_5J_8$ lies on the line $M_1M_2$. Analogously, we prove the same statement for other points.

The Brianchon theorem for the hexagon formed by the tangents to $C_1$ at $T_2, X_1, X_4, T_1, X_3$ and $T_4$ gives the concurrency of the lines $J_2J_6, J_1J_4$ and $J_5J_8$. We use the similar argument for the rest of the proof.

Let $K_i$ be the intersection point of the lines $J_iJ_{i+1}$ and $J_{i+2}J_{i+3}$ (modulo 8) for $i = 1, \ldots, 8$.

**Theorem 3.2** The points $K_i$ lie on the same conic $D_1$ (Figure 12).

**Proof** It is not hard to prove that the lines $K_1K_5, K_2K_6, K_3K_7$ and $K_4K_8$ pass through the point $M_3$, the lines $K_2K_3, K_1K_4, K_4K_8$ and $K_6K_7$ pass through the point $M_1$ and the lines $K_3K_7, K_1K_8, K_3K_6$ and $K_4K_5$ pass through the point $M_2$. From the collinearity of the points $M_1, J_2$ and $l(J_4J_5) \cap l(J_7J_8)$ the points $K_1, K_2, K_4, K_5, K_7$ and $K_8$ lie on the same conic. Using the similar argument we show that $K_2, K_4, K_5, K_6, K_7$ and $K_8$ lie on the same conic as well. Because there is a unique conic that passes through some 5 points, the points $K_1, K_2, K_4, K_5, K_6, K_7$ and $K_8$ are on the same conic. Then it is easy to prove that $K_3$ also lies on the conic.

Let $Z_1 = l(M_1U_1) \cap l(M_2V_1)$, $Z_2 = l(M_1U_1) \cap l(M_2V_2)$, $Z_3 = l(M_1U_2) \cap l(M_2V_2)$ and $Z_4 = l(M_1U_2) \cap l(M_2V_1)$.

**Theorem 3.3** The points $N_1, N_2, P_1, P_2, Z_1, Z_2, Z_3$ and $Z_4$ lie on the same conic.

**Proof** There exists a projective transformation $\varphi$ that maps the vertices $A_1, A_2, A_3$ and $A_4$ onto the vertices of a square. Then the point $\varphi(M_3)$ is mapped onto the centre of a conic $\varphi(C)$ and the lines $\varphi(N_1)\varphi(P_1)$ and $\varphi(N_2)\varphi(P_2)$ are the axes of this conic. The points $\varphi(U_1), \varphi(U_2), \varphi(V_1)$ and $\varphi(V_2)$ also lie on the axes. As we could see from Figure 13, everything is symmetric with respect to the axes and it is easy to conclude that there is a conic through $\varphi(Z_1), \varphi(Z_2), \varphi(Z_3), \varphi(Z_4), \varphi(N_1), \varphi(N_2), \varphi(P_1)$ and $\varphi(P_2)$.

Theorems 3.1–3.3 associate new conics to the quadrilateral inscribed in a conic. They have interesting properties which will be explained in the following section.
4. Poncelet’s quadrilateral porism

Jean-Victor Poncelet’s famous Closure theorem states that if there exists one $n$-gon inscribed in a conic $C$ and circumscribed about a conic $D$ then any point on $C$ is the vertex of some $n$-gon inscribed in a conic $C$ and circumscribed about a conic $D$. Poncelet published his theorem in [18]. However, this result influenced mathematics until nowadays. In a recent book [9] by Dragovic and Radnovic there are several proofs of the Closure theorem, its generalizations as well as its relations with elliptic functions theory. The proof is not elementary for an arbitrary $n$, although in the case $n = 3$ an elegant proof can be found in almost every monograph in projective geometry, see [5,17].

Theorems 3.2 and 3.3 are the special cases of the Poncelet theorem for $n = 4$. Actually, quadrilaterals and conics in them have a poristic property. An elementary proof using harmonic locus of two conics can be found in [12]. We kept the spirit of elementarity through our paper and our agenda was: At first, we experiment in Cinderella, after that the proof is recovered by elementary tools (again directly guided by Cinderella’s tools). In the same style we continue and offer a direct analytic proof of the Poncelet theorem for quadrilaterals without using differentials and elliptic functions. A synthetic version of this proof is presented in [15].

Lemma 4.1 Let $\lambda$, $\mu$ be such that the conics $C : \lambda x^2 + (1 - \lambda)y^2 - 1 = 0$ and $D : x^2 + \mu xy + y^2 + (\mu^2 - 1)/4 = 0$ are non-degenerate. Let $A$ be a point on $C$ and $B$ and $B'$ be the intersections of the tangent lines from $A$ to $D$ with the conic $C$. Then the points $B$ and $B'$ are symmetric with respect to the origin.

Proof Let $t : y = kx + n$ be a tangent line to the conic $D$ (Figure 14). The condition of tangency between $t$ and $D$ is

$$n^2 = k^2 + mk + 1. \quad (1)$$

The coordinates of the intersection points of $t$ and $C$ are

$$(x_1, y_1) = \left(\frac{-2(1 - \lambda)kn - \sqrt{D}}{2(\lambda + (1 - \lambda)k^2)} , \ k \cdot \left(\frac{-2(1 - \lambda)kn - \sqrt{D}}{2(\lambda + (1 - \lambda)k^2)} \right) + n\right)$$

and

$$(x_2, y_2) = \left(\frac{-2(1 - \lambda)kn + \sqrt{D}}{2(\lambda + (1 - \lambda)k^2)} , \ k \cdot \left(\frac{-2(1 - \lambda)kn + \sqrt{D}}{2(\lambda + (1 - \lambda)k^2)} \right) + n\right),$$
where $D = 4(\lambda - \lambda(1 - \lambda))n^2 + (1 - \lambda)k^2$. It is necessary and enough to prove that a line through the points $(-x_1, -y_1)$ and $(x_2, y_2)$ is tangent to $D$. This line has the equation $y = \tilde{k}x + \tilde{n}$ where $\tilde{k}$ and $\tilde{n}$ can be calculated as

$$\tilde{k} = \frac{-\lambda}{(1 - \lambda)k} \quad \text{and} \quad \tilde{n} = \frac{\sqrt{D}}{2k(1 - \lambda)}.$$  

(2)

We need to check if

$$\tilde{n}^2 = \tilde{k}^2 + m\tilde{k} + 1.$$

It is directly verified that condition (1) multiplied by $\lambda(1 - \lambda)/k^2(1 - \lambda)^2$ finishes our proof.  

**Theorem 4.1** Let $C$ and $D$ be conics such that there exists one quadrilateral inscribed in a conic $C$ and circumscribed about a conic $D$. Then any point on $C$ is the vertex of some quadrilateral inscribed in a conic $C$ and circumscribed about a conic $D$.

**Proof** There exists a projective transformation that maps the vertices of the quadrilateral inscribed in a conic $C$ and circumscribed about a conic $D$ onto the points $(1, 1), (1, -1), (-1, -1)$ and $(-1, 1)$ (in the standard chart). Thus, the conics $C$ and $D$ are transformed in those with the equations as in Lemma 4.1. Now the claim follows.  

In fact, we proved more. All quadrilaterals with poristic property with respect to $C$ and $D$ have a common point of the intersection of diagonals (lines joining opposite vertices) and a common line passing through the intersections of opposite side lines. Our work in previous section now could be reviewed in a new light.

Theorems 3.1–3.3 are obtained after we defined certain points. If we apply the same procedure for defining new points on the points and conics in theorems, again we come to similar conclusions. Thus, by repeating this procedure, we obtain an infinite sequence of conics (Figure 15). Every two consecutive conics in this sequence are Poncelet 4-connected.
Our theorems resemble Darboux’s theorem, see [6–8]. They could be seen as a very special case of Dragović–Radnović theorem 8.38 [9]. Such constructions are also studied in the paper of Schwartz, see [21]. The following result further explains their connection, but first we define 16 points of the intersections $R_1 = l(Z_1Z_2) \cap l(N_1N_2)$, $R_2 = l(Z_1Z_2) \cap l(N_1P_2)$,
$R_3 = l(Z_2Z_3) \cap l(N_1P_2), R_4 = l(Z_2Z_3) \cap l(P_1P_2), R_5 = l(Z_3Z_4) \cap l(P_1P_2), R_6 = l(Z_3Z_4) \cap l(P_1N_2), R_7 = l(Z_1Z_4) \cap l(P_1N_2), R_8 = l(Z_1Z_4) \cap l(N_1N_2), R_9 = l(Z_1Z_2) \cap l(P_1P_2), R_{10} = l(Z_3Z_4) \cap l(N_1P_2), R_{11} = l(Z_2Z_3) \cap l(P_1N_2), R_{12} = l(Z_4Z_4) \cap l(P_1P_2), R_{13} = l(Z_3Z_4) \cap l(N_1N_2), R_{14} = l(Z_2Z_2) \cap l(P_1N_2), R_{15} = l(Z_3Z_4) \cap l(N_1P_2)$ and $R_{16} = l(Z_2Z_3) \cap l(N_1N_2)$ (Figure 16).

**Theorem 4.2** The next groups of 8 points lie on the same conic: \{R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8\}, \{R_9, R_{10}, R_{11}, R_{12}, R_{13}, R_{14}, R_{15}, R_{16}\}, \{R_1, R_2, R_3, R_4, R_5, R_6, R_{11}, R_{12}, R_{15}, R_{16}\}, \{R_3, R_4, R_7, R_8, R_9, R_{10}, R_{13}, R_{14}\}, \{R_1, R_5, R_7, R_9, R_{11}, R_{13}, R_{15}\}$ and \{R_2, R_3, R_4, R_6, R_{10}, R_{12}, R_{14}, R_{16}\}.

The proof of Theorem 4.2 uses the same arguments we used in the previous proofs so we omit it. If we look at the conic $C$ and a conic $F$ through the points \{R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8\} we see they are Poncelet 8-connected and appropriate conics from Theorem 4.2, the conic from Theorem 3.3 with the line $M_1M_2$ form Poncelet-Darboux grid. Two conics \{R_2, R_3, R_4, R_6, R_{10}, R_{12}, R_{14}, R_{16}\} and \{R_1, R_5, R_7, R_9, R_{11}, R_{13}, R_{15}\} are not coming from Poncelet-Darboux grid, but they could be directly obtained from Dragović–Radnović theorem 8.38, [9]. This result improves the result of Schwartz [21] in a particular case.

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