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ON LIPSCHITZ CONTINUITY OF HARMONIC QUASIREGULAR MAPS ON THE UNIT BALL IN $\mathbf{R}^n$

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Abstract. We show that Lipschitz continuity of $\phi\colon S^{n-1}\to\mathbf{R}^n$ implies Lipschitz continuity of its harmonic extension $u=P[\phi]\colon B^n\to\mathbf{R}^n$, provided u is a quasiregular map.

1. Introduction and notations

It is known, even for $n=2$, that Lipschitz continuity of $\phi\colon T\to\mathbf{C}$, where $T=\{z\in\mathbf{C}\colon |z|=1\}$, does not imply Lipschitz continuity of $u=P[\phi]$. In fact $u=P[\phi]$ is Lipschitz continuous iff the Hilbert transform of $\frac{d}{d\theta}\phi(e^{i\theta})$ (which is defined almost everywhere and bounded since ϕ is Lipschitz) is also in $L^\infty(T)$ (see [7]).

Here, for any $n\ge 2$,

$$P[\phi](x)=\int_{S^{n-1}}P(x,\xi)\phi(\xi)\,d\sigma(\xi),\quad x\in B^n,$$

where $P(x,\xi)=\frac{1-|x|^2}{|x-\xi|^n}$ is the Poisson kernel for the unit ball $B^n=\{x\in\mathbf{R}^n\colon |x|<1\}$, $d\sigma$ is the normalized surface measure on the unit sphere S^{n-1} and $\phi\colon S^{n-1}\to\mathbf{R}^n$ is a continuous mapping.

The situation is different for C^α (or Hölder) continuous $\phi\colon S^{n-1}\to\mathbf{R}^n$, $0<\alpha<1$, i.e., for ϕ satisfying $|\phi(\xi)-\phi(\eta)|\le C|\xi-\eta|^\alpha$. In that case Hölder continuity of ϕ implies Hölder continuity of its harmonic extension $u=P[\phi]$, (see [2, 5]). In the case $n=2$ it is a classical result, following from Privalov's theorem (see [7]).

Our aim is to show that Lipschitz continuity is preserved by harmonic extension, if the extension is quasiregular. The analogous statement is, as noted, true for Hölder continuity without assumption of quasiregularity.

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2. Result

Theorem 1. *Assume* $\phi\colon S^{n-1} \to \mathbf{R}^n$ *satisfies the Lipschitz condition*

$$|\phi(\xi) - \phi(\eta)| \le L|\xi - \eta|, \quad \xi, \eta \in S^{n-1},$$

and assume $u = P[\phi]\colon B^n \to \mathbf{R}^n$ *is* K*-quasiregular. Then*

$$|u(x) - u(y)| \le C'|x - y|, \quad x, y \in B^n,$$

where C' *depends on* L*,* K *and* n *only.*

Kalaj obtained a related result, but under additional assumption of $C^{1,\alpha}$ regularity of ϕ, (see [3]).

The main part of the proof is the estimate of the tangential derivatives of u, and in that part quasiregularity plays no role. We choose $x_0 = r\xi_0 \in B^n$, $r = |x|$, $\xi_0 \in S^{n-1}$. Let $T = T_{x_0} rS^{n-1}$ be the $n-1$-dimensional tangent plane at x_0 to the sphere rS^{n-1}. We want to prove that

$$\|D(u|_T)(x_0)\| \le C(n)L. \tag{1}$$

Without loss of generality we can assume $\xi_0 = e_n$ and $x_0 = re_n$. By a simple calculation

$$\frac{\partial}{\partial x_j} P(x, \xi) = \frac{-2x_j}{|x - \xi|^n} - n(1 - |x|^2)\frac{x_j - \xi_j}{|x - \xi|^{n+2}}.$$

Hence, for $1 \le j < n$ we have

$$\frac{\partial}{\partial x_j} P(x_0, \xi) = n(1 - |x_0|^2)\frac{\xi_j}{|x_0 - \xi|^{n+2}}.$$

It is important to note that this kernel is odd in ξ (with respect to reflection $(\xi_1, \dots \xi_j, \dots, \xi_n) \mapsto (\xi_1, \dots, -\xi_j, \dots, \xi_n)$), a typical fact for kernels obtained by differentiation. This observation and differentiation under integral sign gives, for any $1 \le j < n$,

$$\begin{aligned}\frac{\partial u}{\partial x_j}(x_0) &= n(1 - r^2)\int_{S^{n-1}} \frac{\xi_j}{|x_0 - \xi|^{n+2}}\phi(\xi)\, d\sigma(\xi)\\ &= n(1 - r^2)\int_{S^{n-1}} \frac{\xi_j}{|x_0 - \xi|^{n+2}}(\phi(\xi) - \phi(\xi_0))\, d\sigma(\xi).\end{aligned}$$

Using the elementary inequality $|\xi_j| \le |\xi - \xi_0|$, ($1 \le j < n$, $\xi \in S^{n-1}$) and Lipschitz continuity of ϕ we get

$$\begin{aligned}\left|\frac{\partial u}{\partial x_j}(x_0)\right| &\le Ln(1 - r^2)\int_{S^{n-1}} \frac{|\xi_j||\xi - \xi_0|}{|x_0 - \xi|^{n+2}}\, d\sigma(\xi)\\ &\le Ln(1 - r^2)\int_{S^{n-1}} \frac{|\xi - \xi_0|^2}{|x_0 - \xi|^{n+2}}\, d\sigma(\xi).\end{aligned}$$

In order to estimate the last integral, we split S^{n-1} into two subsets $E = \{\xi \in S^{n-1}\colon |\xi - \xi_0| \le 1 - r\}$ and $F = \{\xi \in S^{n-1}\colon |\xi - \xi_0| > 1 - r\}$. Since $|\xi - x_0| \ge 1 - |x_0|$

for all $\xi \in S^{n-1}$ we have

$$\begin{aligned}\int_E \frac{|\xi-\xi_0|^2}{|x_0-\xi|^{n+2}}\,d\sigma(\xi) &\le (1-r^2)^{-n-2}\int_E |\xi-\xi_0|^2\,d\sigma(\xi)\\ &\le (1-r^2)^{-n-2}\int_0^{1-r}\rho^2\rho^{n-2}\,d\rho\\ &\le \frac{2}{n+1}(1-r)^{-1}.\end{aligned}$$

On the other hand, $|\xi-\xi_0|\le C_n|\xi-x_0|$ for every $\xi\in F$, so

$$\begin{aligned}\int_F \frac{|\xi-\xi_0|^2}{|x_0-\xi|^{n+2}}\,d\sigma(\xi) &\le C_n^{n+2}\int_F |\xi-\xi_0|^{-n}\,d\sigma(\xi)\\ &\le C_n'\int_{1-r}^{2}\rho^{-n}\rho^{n-2}\,d\rho\\ &\le C_n'(1-r)^{-1}.\end{aligned}$$

Combining these two estimates we get

$$\left|\frac{\partial u}{\partial x_j}(x_0)\right| \le LC(n)$$

for $1\le j<n$. Due to rotational symmetry, the same estimate holds for every derivative in any tangential direction. This establishes estimate (1). Finally, K-quasiregularity gives

$$\|Du(x)\| \le LKC(n).$$

Now the mean value theorem gives Lipschitz continuity of u.

We conclude by noting that, for each $n\ge 2$, there is a Lipschitz continuous map $\phi\colon S^{n-1}\to\mathbf{R}^n$ such that $u=P[\phi]$ is not Lipschitz continuous. We first briefly recall a well known example in the plane. Let $f(z)=\sum_{n=1}^{\infty} z^n/n^2$ and let $u(z)=\operatorname{Re} f(z)$. Then

$$u(z)=\sum_{n=1}^{\infty}\frac{r^n\cos n\theta}{n^2} \quad\text{and}\quad zf'(z)=\sum_{n=1}^{\infty}\frac{z^n}{n}=-\log(1-z).$$

So, $zf'(z)=-\log|1-z|-iv(z)$, where $-\pi/2<v(z)<\pi/2$. Hence $d_\theta u(z)=v(z)$ is a bounded harmonic function while its harmonic conjugate $rd_r u(z)=\operatorname{Re} zf'(z)$ is not. This implies that $u(e^{i\theta})$ is Lipschitz continuous on the unit circle, while $u(z)$ is not Lipschitz continuous in the disc. However, u has the following weaker property:

$$(2)\qquad |u(z')-u(z'')|\le C\left(|z'-z''|+\big||z'|-|z''|\big|\log\frac{1}{\big||z'|-|z''|\big|}\right).$$

This gives a counterexample in any dimension $n\ge 2$.

Example 1. Set $\phi(x_1,x_2,\dots,x_n)=(u(x_1+ix_2),x_2,\dots,x_n)$, $x\in S^{n-1}$. Then ϕ is a Lipschitz continuous map on S^{n-1}, while its harmonic extension $U=P[\phi]$ is not Lipschitz continuous on the unit ball.

It is clear that $U(x) = P[\phi](x) = (u(x_1 + ix_2), x_2, \dots, x_n)$, $x \in B^n$, is not Lipschitz continuous since $u(x_1 + ix_2)$ is not Lipschitz continuous on the disc. Proving Lipschitz continuity of ϕ on S^{n-1} reduces to checking Lipschitz continuity of $u(x_1 + ix_2) = u(x_1, x_2)$ on S^{n-1}. Choose $x' = (z', w')$ and $x'' = (z'', w'')$ on S^{n-1} where $z' = (x'_1, x'_2)$, $w' = (x'_3, \dots, x'_n)$, $z'' = (x''_1, x''_2)$ and $w'' = (x''_3, \dots, x''_n)$. Then, using (2),

$$|U(x') - U(x'')| = |u(z') - u(z'')| \le C(d + \delta \log \frac{1}{\delta})$$

where $d = |z' - z''|$ and $\delta = ||z'| - |z''||$. On the other hand, we have

$$|x' - x''|^2 = d^2 + |w' - w''|^2 \ge d^2 + (|w'| - |w''|)^2.$$

But

$$(|w'| - |w''|)^2 = (\sqrt{1 - |z'|^2} - \sqrt{1 - |z''|^2})^2 \ge 2\delta - \delta^2.$$

Therefore, for small $\delta > 0$, $|x' - x''| \ge \sqrt{\delta}$. Since $\delta \log \frac{1}{\delta} = o(\sqrt{\delta})$ and, obviously, $|x' - x''| \ge d$, we have proven Lipschitz continuity of $u(x_1, x_2)$ on the unit sphere S^{n-1}.

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