GEOMETRIC FUNCTION THEORY 1

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Ahlfors-Schwarz lemma, Hyperbolic geometry, the Carathéodory, Kobayashi Metrics, Denjoy-Wolff theorem and Applications in Complex Analysis

1. Introduction

This is a working version. Throughout this paper, $\mathbb{U}$ will denote the unit disc $\{z : |z| < 1\}$, $\mathbb{T}$ the unit circle, $\{z : |z| = 1\}$ and we will use notation $z = x + iy$ and $z = re^{i\theta}$, where $r = |z|$ and $\theta \in \mathbb{R}$ are polar coordinates. For a function h, we use notation $\partial h = \frac{1}{2}(h'_x - ih'_y)$ and $\overline{\partial} h = \frac{1}{2}(h'_x + ih'_y)$; we also use notations $D^c h$ and $\overline{D}^c h$ instead of ∂h and $\overline{\partial} h$ respectively when it seems convenient. By h'_x and h'_y we denote partial derivatives with respect to x and y respectively. We write $D^2_{z\overline{z}} h = D(\overline{D} h)$, where $Dh = D^c h$ and $\overline{D} h = \overline{D}^c h$.

Probably the best known equivalent of Euclid's parallel postulate, contingent on his other postulates, is Playfair's axiom, named after the Scottish mathematician John Playfair, which states:

In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point. Hyperbolic geometry was created in the first half of the nineteenth century in the midst of attempts to understand Euclid's axiomatic basis for geometry. It is one type of non-Euclidean geometry, that is, a geometry that discards one of Euclid's axioms (Euclid's parallel postulate). The development of non-Euclidean geometry caused a profound revolution, not just in mathematics, but in science and philosophy as well. Einstein and Minkowski found in non-Euclidean geometry a geometric basis for the understanding of physical time and space.

Hyperbolic geometry is tightly related to the function theory of one and several complex variables. Using Schwarz's lemma it is proved that
(A) holomorphic functions do not increase the corresponding hyperbolic distances between the corresponding hyperbolic domains.

The Caratheodory and Kobayashi metrics have proved to be important tools in the function theory of several complex variables. In particular, we have:

(B) If G_1 and G_2 are domains in C^n and $f : G_1 \to G_2$ holomorphic function, then f does not increase the corresponding Caratheodory(Kobayashi) distances.

But they are less familiar in the context of one complex variable. Krantz [36] gathers in one place the basic ideas about these important invariant metrics for domains in the plane and provides some illuminating examples and applications.

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In [64], Wong proved:

(a) If G is a hyperbolic manifold in the sense of Kobayashi and the differential Kobayashi metric K_G is of class C^2, then the holomorphic curvature of K_G is greater than or equal to -4.

(b) If G is Carathéodory-hyperbolic and the differential Caratheodory metric C_G is of class C^2, then the holomorphic curvature of C_G is less than or equal to -4. With this result the author obtain an intrinsic characterization of the unit ball. For (b) see also Burbea [13].

In [28], Earle, Harris, Hubbard and Mitra discuss the Carathéodory and Kobayashi pseudometrics and their infinitesimal forms on complex Banach manifolds. Their discussion includes a very elementary treatment of the Kobayashi pseudometric as an integral of its infinitesimal form. They also prove new distortion theorems for the Carathéodory pseudometric under holomorphic maps from the open unit disk to a complex Banach manifold.

Although this is mainly review paper we treat known results with novelty and outline a few new results. The content of the paper is as follows. In Section 2, we outline how to introduce hyperbolic distances from the point of complex analysis (more precisely, using Schwarz's lemma). We also shortly consider versions of Ahlfors-Schwarz lemma related to ultrahyperbolic metric and the comparison principle related to curvatures and distances. In Section 5 we consider Denjoy-Wolff Theorem. In Section 6 we consider hyperbolic geometry, Möbius transformations and Cayley-Klein model in several variables. It is supposedly classical and can be found in the literature that the restriction of the Beltrami-Klein metric on the ball of $\mathbb{R}^n$ to any minimal surface (minimal with respect to the flat metric) has curvature ≤ -1. Using a heuristic argument we outline an application to minimal surfaces.

XX In Sections 7 and 8 we present some result from [48] related to Schwarz lemmma in the ball, and related to contraction properties of holomorphic functions with respect to Kobayashi distances respectively. Note here that several years ago, the author communicated at Belgrade seminar (probably around 1980 -1990), some results related to the Carathéodory and Kobayashi pseudometrics and their infinitesimal forms on complex Banach spaces (see also [42]) and that our approach in Section 8 is probably known to the experts in the subject (in particular see Theorems 8.3 and 8.4).

The properties (a) and (b) of holomorphic curvature of Kobayashi and Caratheodory metric are considered in Section 9. XX New results related to distortion and boundary Schwarz lemma for harmonic functions are announced in Section 10. Short review of the results related to the Ahlfors-Schwarz lemma for holomorphic maps,Kobayashi distance,holomorphic dynamics and related subject to it, is given in 11.

We plan further to examine carefully this paper and to work on an extension of this paper. We excuse to the reader for numerous mistakes which probably appears in the text. [1]. The text is based on [50] (see also a version published in VII Simpozizijum Matematika i primene, p.1-17,Beograd, November 2017).

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[1]Because of limited time author could not settle some details related to this paper.

2. Schwarz lemma and Hyperbolic geometry

2.1. **The Schwarz lemma, Introduction.** Throughout this paper by $\mathbb{S}(a,b)$ we denote the set $(a,b)\times\mathbb{R}$, $-\infty\le a<b\le\infty$, and in particular $\mathbb{S}_0=\mathbb{S}(-1,1)$. Note that $\mathbb{S}(a,b)$ is a strip if $-\infty<a<b<\infty$ and $\mathbb{S}(a,+\infty)$ is a half-plane if a is a real number, and $\mathbb{S}(-\infty,+\infty)=\mathbb{C}$.

If w is complex number by $\Re w$ (or $u=\mathrm{Re}w$) we denote the corresponding real part, and in a similar way if f is complex-valued function by $\Re f$ (or $u=\mathrm{Re}f$) we denote the corresponding real valued function and by $\nabla f(z)=(f'_x,f'_y)$ the gradient of f.

Occasionally by λ_0 and ρ_0 we denote respectively hyperbolic metric on the unit disk and on the strip $\mathbb{S}_0$. See [71] and also [72] for discussion in this subsection.

The following result is a corollary of the maximum modulus principle:

Proposition 2.1 (classical Schwarz lemma 1-the unit disk)**.** Suppose that $f:\mathbb{D}\to\mathbb{D}$ is an analytic map and $f(0)=0$. The classic Schwarz lemma states : $|f(z)|\le|z|$ and $|f'(0)|\le 1$.

It is interesting that this result(which looks simple and elementary at first glance) has far reaching applications and forms. XX In this paragraph we follow [45, **?**]. We will see below, if we do not specify the value of $f(0)$, we get Picks lemma.

Pick's lemma. Suppose $f:\mathbb{D}\to\mathbb{D}$ is holomorphic. Then

$$|f'(z)|\le\frac{1-|f(z)|^2}{1-|z|^2}$$

for any $z\in\mathbb{U}$. Equality holds for some $z\in\mathbb{D}$ if and only if f is a conformal self-map of $\mathbb{D}$ (and in that case equality holds everywhere). Picks lemma leads naturally to the hyperbolic metric on $\mathbb{D}$ (see also Section 14).

2.1.1. *the subordination principle.* (A) Let $f(z)$ and $g(z)$ be analytic functions in $\mathbb{U}$, $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$ written, or $f\prec g$ $(z\in\mathbb{U})$, if there exists a function $w(z)$, analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$, $(z\in U)$ such that (i) $f(z)=g(w(z))$, $(z\in\mathbb{U})$.

(B) In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalent to (ii) $f(0)=g(0)$ and $f(\mathbb{U})\subset g(\mathbb{U})$.

In the setting (B), then $f(U_r)\subset g(U_r)$ for all $0<r<1$ and $|f'(0)|\le|g'(0)|$.

Since g is one-to-one, g is in fact a conformal map from $\mathbb{U}$ to $g(\mathbb{U})$. Let $h=g^{-1}$ be the inverse of g. Then $h\circ f$ is holomorphic and maps $\mathbb{U}$ into $\mathbb{U}$ with $(h\circ f)(0)=0$ since $f(0)=g(0)$. By Schwarzs lemma, we have $|(h\circ f)'(0)|\le 1$ and $|(h\circ f)(z)|\le|z|$ for all $z\in\mathbb{D}$. Hence $f(U_r)\subset g(U_r)$. By the chain rule, $(h\circ f)'(0)=\frac{f'(0)}{g'(0)}$. So the first inequality gives $|f'(0)|\le|g'(0)|$.

As an exercise, formulate the condition for equality. Here we give a simple example.

Example 1. (i) Let $R=\{z:\mathrm{Re}(z)>0\}$ be the right half plane. Let $f:\mathbb{D}\to R$ be holomorphic. We claim that $|f'(0)|\le 2\mathrm{Re}(f(0))$.

(ii) Let $f:\mathbb{D}\to\mathbb{S}_0$ be holomorphic. Then $|f'(0)|\le\frac{4}{\pi}$.

(iii) Let $f:\mathbb{D}\to\mathbb{D}\setminus\{0\}$ be holomorphic. Then $|f'(0)|\le\frac{2}{e}$.

Proposition 2.2. Suppose (a) ϕ is univalent in $\mathbb{U}$, f holomorphic in $\mathbb{U}$ and $f(\mathbb{U})\subset\phi(\mathbb{U})$.

(b) $\phi(z_0)=f(z_0)$, $z_0\in\mathbb{U}$.
Then $|f'(z_0)|\le|\phi'(z_0)|$.

2.2. **The Schwarz lemma.** If $|a|<1$ define the Möbius transformation

$$\varphi_a(\zeta)=\frac{\zeta-a}{1-\overline{a}\,\zeta}\,. \tag{2.1}$$

Example 2. Fix $a\in\mathbb{D}$. Then $\varphi_a(0)=-a$, $\varphi_a(a)=0$, φ_a is a one-to-one mapping which carries $\mathbb{T}$ onto $\mathbb{T}$, $\mathbb{D}$ onto $\mathbb{D}$. The inverse of φ_a is φ_{-a}.

Check that $\varphi_a'(z)=(1-|a|^2)(1-\bar{a}z)^{-2}$ and in particular $\varphi_a'(0)=(1-|a|^2)$, $\varphi_a'(a)=(1-|a|^2)^{-1}$.

In the literature the notation T_a is also used instead of φ_a. Here we define $T_a=-\varphi_a$ and therefore we have $T_a^{-1}=T_a$.

Proposition 2.3 (Schwarz lemma 1)**.** Suppose that $f:\mathbb{D}\to\mathbb{D}$ is an analytic map and $f(0)=0$. The classic Schwarz lemma states : $|f(z)|\le|z|$ and $|f'(0)|\le1$.

Proof. A standard proof is based on an application of the Maximum Modulus Theorem to the function g defined by $g(z)=\frac{f(z)}{z}$ for $z\neq0$ and $g(0)=f'(0)$.

Now we shall drop the assumption $f(0)=0$. Let $f:\mathbb{D}\to\mathbb{D}$ is an arbitrary analytic map. Fix an arbitrary point $z\in\mathbb{D}$ and consider the mapping $F=\varphi_w\circ f\circ\varphi_{-z}$, where $w=f(z)$. Since $\varphi_{-z}(0)=z$, $F(0)=0$. By an application of Schwarz lemma,

$$|F(\zeta)|=|\varphi_w\circ f\circ\varphi_{-z}(\zeta)|\le|\zeta|,\quad \zeta\in\mathbb{D}, \tag{2.2}$$

and $|F'(0)|\le1$. Hence, since $F'(0)=\varphi_w'(w)f'(z)\varphi_{-z}'(0)$, we find

$$\frac{|f'(z)|}{1-|f(z)|^2}\le\frac{1}{1-|z|^2},\quad z\in\mathbb{D}\,, \tag{2.3}$$

with equality only if $F=e^{i\alpha}Id$, that is $f=\varphi_{-w}\circ(e^{i\alpha}\varphi_z)$.

Hence, equality holds in (2.3) if and only if f is a Möbius transformation of $\mathbb{D}$ onto itself. □

Let ω be an arbitrary point in $\mathbb{D}$ and $\zeta=\varphi_z(\omega)$, then $\varphi_{-z}(\zeta)=\omega$, and by (2.2), we find $|\varphi_w(f(\omega))|\le|\varphi_z(\omega)|$.

It is convenient to introduce a pseudo-distance

$$\delta(z,\omega)=|\varphi_z(\omega)|=\left|\frac{z-\omega}{1-\overline{\omega}\,z}\right|, \tag{2.4}$$

which is a *conformal invariant.* Thus

Proposition 2.4.

$$\delta(f(z),f(\omega))\le\delta(z,\omega) \tag{2.5}$$

with equality only if f is a Möbius transformation of $\mathbb{D}$ onto itself.

This shows that the Riemannian metric whose element of length is

$$\mathrm{d}s=\lambda(z)|\mathrm{d}z|=\frac{2\,|\mathrm{d}z|}{1-|z|^2} \tag{2.6}$$

is invariant under conformal self-mappings of the disk.
In this metric every rectifiable arc γ has length

$$|\gamma|_{\rm hyp} = \int\limits_{\gamma} \frac{2\,|{\rm d}z|}{1-|z|^2}$$

and $|f\circ\gamma|_{\rm hyp} = |\gamma|_{\rm hyp}$ if f is a Möbius transformation of $\mathbb{D}$ onto itself.

We call the distance determined by this metric the non-Euclidean distance (hyperbolic) and denote by λ; we also use notation $\lambda(z) = \frac{2}{1-|z|^2}$ for metric density and $||h||_\lambda = \lambda(z)|h|$ for $h \in T_z$.

The fact that the hyperbolic distance is invariant under self-mapping of the disk we can state in the form: If $h \in T_z$, $A \in {\rm Aut}(\mathbb{D})$ and $h_* = A'(z)h$, then $||h_*||_\lambda = ||h||_\lambda$ for every $z \in \mathbb{D}$ and every $h \in T_z$.

The shortest arc from 0 to any other point is along a radius. Hence the geodesics are circles orthogonal to $\mathbb{T} = \{|z| = 1\}$. The non-Euclidean distance from 0 to r is

$$\lambda(0,r) = \int\limits_0^r \frac{2\,{\rm d}t}{1-t^2} = \ln\frac{1+r}{1-r}\,. \tag{2.7}$$

Since $\delta(0,r) = r$ it follows that non-Euclidean distance λ is connected with δ through $\delta = \tanh\frac{\lambda}{2}$.

Hence, the hyperbolic distance on the unit disk $\mathbb{D}$ is

$$\lambda(z,\omega) = \ln\frac{1+\left|\frac{z-\omega}{1-z\bar\omega}\right|}{1-\left|\frac{z-\omega}{1-z\bar\omega}\right|}. \tag{2.8}$$

If $f : \mathbb{D} \to \mathbb{D}$ is an arbitrary analytic map, then

$$\lambda(fz, f\omega) \le \lambda(z,\omega).$$

Exercise 1. Check the formula (2.7).

Hint. $f(t) = \frac{2}{1-t^2}$, $f(t) = \frac{1}{1-t} + \frac{1}{1+t}$, $F = \int f(t) = -\ln(1-t) + \ln(1+t)$. Hence $\lambda(0,r) = \int_0^r f(t) = -\ln(1-t)|_0^r + \ln(1+t)_0^r = \ln(1+r) - \ln(1-r) = \ln\frac{1+r}{1-r}$.

Exercise 2. If γ is a piecewise continuously differentiable path in $\mathbb{D}$, whether $|\gamma|_{\rm hyp} = |\gamma|_\delta$?

2.3. **The upper half plane.** A region G is conformally equivalent to a region D if there is an analytic bijective function f mapping G to D; we call f conformal isomorphism. Conformal equivalence is an equivalence relation. Conformal isomorphism of a domain onto itself is called conformal automorphism. Conformal automorphisms of a domain D form a group which we denote by ${\rm Aut}D$.

If $f_0 : G \to D$ is a fixed conformal isomorphism, then every conformal isomorphism $f : G \to D$ can be represented in the form

$$f = \phi\circ f_0, \quad \phi \in {\rm Aut\,D}\,. \tag{2.9}$$

The cross-ratio of a 4-tuple of distinct points on the real line with coordinates z_1, z_2, z_3, z_4 is given by

$$(z_1, z_2; z_3, z_4) = \frac{(z_1-z_3)(z_2-z_4)}{(z_2-z_3)(z_1-z_4)}.$$

Example 3. Describe Aut($\mathbb{H}$). If $A \in$ Aut($\mathbb{H}$), there is a point $x_0 \in \mathbb{R}$ such that $A(x_0) = \infty$. We consider two cases.
Case (i) $x_0 = \infty$. Then $A = L$, where $L(z) = \lambda z + s$, $\lambda > 0$ and $s \in \mathbb{R}$.
Case (ii) $x_0 \in \mathbb{R}$. Define $w = T(z) = -\frac{1}{z} + x_0$. Then $T^{-1}(w) = \frac{1}{x_0 - w}$ and $A \circ T$ maps ∞ to ∞. Hence $A \circ T = L$ for some $\lambda > 0$ and $s \in \mathbb{R}$ and therefore $f = L \circ T^{-1}$, that is $A(w) = \lambda T^{-1}(w) + s = \lambda \frac{1}{x_0 - w} + s = \frac{a_1 z + b_1}{x_0 - w}$, where $a_1 = -s$ and $b_1 = \lambda + s x_0$.
Therefore $D(A) = \lambda$ it is readable that every $A \in$ Aut($\mathbb{H}$) can represented in the form

$$f(z) = \frac{az+b}{cz+d}\,, \quad \text{where } a,b,c,d \in \mathbb{R} \text{ and } \mathrm{D} = \mathrm{D}(f) = ad - bc = 1\,. \tag{2.10}$$

If A is represented by (2.10), then

$$Az - \overline{Az} = \frac{z - \bar z}{|cz+d|^2}. \tag{2.11}$$

Hence it is clear that $A \in$ Aut($\mathbb{H}$).

There is another way to describe Aut($\mathbb{H}$) using $(w, 1; 0, \infty) = w$. Namely, if A carries points x_2, x_3, x_4 ($x_2 > x_3 > x_4$) into $1, 0, \infty$, then $w = (z, x_2, x_3, x_4)$ □

If $L \in$ Aut($\mathbb{H}$), then L is Möbius transformation and maps $\mathbb{R}$ onto itself and symmetric points with respect to $\mathbb{R}$ onto symmetric points with respect to $\mathbb{R}$. Hence, if $z_1, z_2 \in \mathbb{H}$ and $w_1 = Lz_1$ and $w_2 = Lz_2$, then $\overline{w_1} = L\overline{z_1}$ and $\overline{w_2} = L\overline{z_2}$. Since the cross-ratio is invariant under Möbius transformation, we get

$$(z_1, \overline{z_1}; z_2, \overline{z_2}) = (Lz_1, L\overline{z_1}; Lz_2, L\overline{z_2}) = (w_1, \overline{w_1}; w_2, \overline{w_2})\,. \tag{2.12}$$

Set $Tz = \dfrac{z - z_2}{z - \overline{z_2}}$. Then $(z_1, \overline{z_1}; z_2, \overline{z_2}) = T(z_1)/T(\overline{z_1})$. T maps $\mathbb{H}$ onto $\mathbb{D}$ and symmetric points z_1 and $\overline{z_1}$ with respect to $\mathbb{R}$ onto points $T(z_1)$ and $T(\overline{z_1})$ symmetric with respect to $\mathbb{T}$ respectively. Hence $T(\overline{z_1})\overline{T(z_1)} = 1$ and therefore $(z_1, \overline{z_1}; z_2, \overline{z_2}) = |T(z_1)|^2$. The pseudo-hyperbolic distance on $\mathbb{H}$ can be defined by

$$\delta_{\mathbb{H}}(z_1, z_2) = \left|\frac{z_1 - z_2}{z_1 - \overline{z_2}}\right|.$$

It is invariant with the group Aut($\mathbb{H}$) because of (2.12) and $\delta_H^2(z_1, z_2) = (z_1, \overline{z_1}, z_2, \overline{z_2})$. We will give another proof of this fact in subsection on Schwarz lemma(below).

For a fixed $z \in \mathbb{H}$, moving on to the limit value of $\delta_H(z, w)/e(z, w)$, where e is Euclidean distance, when $w \to z$ we get an infinitesimal invariant $\mathrm{d}s = |\mathrm{d}z|/y$ (we drop multiple 2), where $y = \mathrm{Im} z$. For a piecewise continuously differentiable path $\gamma(t) = (x(t), y(t))$, $0 \le t \le 1$, in $\mathbb{H}$, we define $|\gamma|_{\mathrm{hyp}} = \int_\gamma |\mathrm{d}\, z|/y = \int_0^1 \frac{|\gamma'(t)|}{y(t)}\, dt$. We use this infinitesimal form to obtain Poincaré distance between two points p and q in $\mathbb{H}$ by putting

$$d_{\mathrm{hyp}}(p, q) = \inf |\gamma|_{\mathrm{hyp}} = \inf \int_\gamma |\mathrm{d}\, z|/y\,,$$

where the infimum is taken over all paths γ joining p to q. The curve for which infimum is attained we call geodesic. We also use shorter notation λ ($\lambda_{\mathbb{H}}(p, q)$) instead of $d_{\mathrm{hyp}} = d_{\mathrm{hyp},\mathbb{H}}$ if it is clear that our considerations is related to $\mathbb{H}$.

Proposition 2.5. In half-plane model, geodesics are the arcs of circles orthogonal to the real axis. The pseudo-hyperbolic distance and the hyperbolic distance are related by

$$\delta = \tanh(\lambda/2).$$

Proof. To find geodesic which joins p and q we use $A \in \mathrm{Aut}(\mathbb{H})$ which maps z_1 and z_2 to iy_1 and iy_2. It is easy to conclude that a minimum is attained along the vertical segment I_0 that connects iy_1 and iy_2. If γ is a path which joins iy_1 and iy_2, using obvious geometric interpretation, we find $|\gamma|_{\mathrm{hyp}} \geq |I_0|_{\mathrm{hyp}}$ and hence

$$\lambda_{\mathbb{H}}(iy_1, iy_2) = |I_0|_{\mathrm{hyp}} = |\ln(y_2/y_1)|.$$

It is interesting to prove this inequality directly (without geometric interpretation). We outline a proof. Suppose that $y_1 \leq y_2$. Since

$|\gamma|_{\mathrm{hyp}} = \int\limits_{\gamma} |\mathrm{d}\, z|/y = \int\limits_{0}^{1} \frac{|\gamma'(t)|}{y(t)}\, dt$, we have

$|\gamma|_{\mathrm{hyp}} \geq \int\limits_{0}^{1} \frac{|y'(t)|}{y(t)}\, dt \geq \int\limits_{0}^{1} \frac{y'(t)}{y(t)} \;= \ln y(t)|_0^1 = \ln(y_2/y_1)$.

Hence it follows that geodesics are the arcs of circles orthogonal to the real axis.

There is circular arc K perpendicular to the real axis that contains z_1 and z_2 and connects real points a_1 and a_2. We can compute $\omega = (p, q, a_1, a_2)$. Suppose that $a_1 > a_2$ and define $A(z) = \dfrac{z - a_1}{z - a_2}$, then $\det(1, -a_1; 1, -a_2) = a_1 - a_2 > 0$ and therefore $A \in \mathrm{Aut}(\mathbb{H})$. Hence it maps K on one half of the imaginary axis. If $A(p) = iy_1$ and $A(q) = iy_2$, the cross ratio ω equals

$$(iy_2, iy_1, 0, \infty) = y_2/y_1.$$

Hence $\lambda_H(p, q) = |\ln(y_2/y_1)| = \ln(p, q, a_1, a_2)$. Since, for $y_2 \geq y_1$, we get

$$\delta(iy_1, iy_2) = \frac{y_2 - y_1}{y_2 + y_1} = \frac{e^{\lambda} - 1}{e^{\lambda} + 1}$$

and since $\delta = \delta_{\mathbb{H}}$ is invariant with respect to $\mathrm{Aut}(\mathbb{H})$, we find $\delta(iy_1, iy_2) = \delta((p, q)$.

Hence $e^{\lambda} = \frac{1+\delta}{1-\delta}$, i.e.

$$\delta = \tanh(\lambda/2).$$

In a similar way one can prove that this formula is valid if $y_2 < y_1$. $\square$

We consider the canonical Möbius transformation T of $\mathbb{H}$ onto $\mathbb{D}$ that maps the points $0, i, \infty$ onto the points $-1, 0, 1$, respectively, and let S denote the inverse of T. Then we find

$$w = Tz = \frac{z - i}{z + i},\;\; z = Sw = i\frac{1 + w}{1 - w}.$$

Note that if $z, a \in \mathbb{H}$, $b = Ta$, then $(z, \bar{z}, a, \bar{a}) = (w, w^*, b, b^*) = |\varphi_b(w)|^2$.

It is convenient to introduce the mapping $\phi_a = T^{-1} \circ \varphi_b \circ T$ and the pseudo-distance

$$\delta(z, \omega) = |\varphi_z(\omega)| = \left| \frac{z - \omega}{1 - \overline{\omega}\, z} \right|, \tag{2.13}$$

which is a *conformal invariant*. It is easy to check that $\delta_{\mathbb{H}}(a, z) := |\phi_a(z)| = \delta_{\mathbb{U}}(T(a), T(z))$.

Moving on to the limit value when $\omega \to z$ we get infinitesimal invariant $\mathrm{d}s = \lambda(z)|\mathrm{d}z|$, where $\lambda(z) = 2(1 - |z|^2)^{-1}$ is the hyperbolic density (we add multiple 2 so that the Gaussian curvature of the hyperbolic density is -1 see below).

The shortest arc from 0 to any other point is along a radius. Hence the geodesics are circles orthogonal to $\mathbb{T}$.

Since $\delta(0,r)=r$ it follows that non-Euclidean distance λ is connected with δ through $\delta=\tanh\dfrac{\lambda}{2}$.

There is another way of calculating that exhibits additivity.

Let γ be a circular arc (geodesic), orthogonal to T at the points w_1 and w_2, that contains the points z_1 and z_2 of the unit disk (suppose that the points w_1,z_1,z_2,w_2 occur in this order). Since $(r,0,-1,1)=(1+r)/(1-r)$, we find

$$\lambda(z_1,z_2)=\ln(z_2,z_1,w_1,w_2).$$

We leave to the interested reader to check that $\{z_1,z_2\}=(z_2,z_1,w_1,w_2)>0$ if the points are in the order indicated above.

In this form we can consider λ as the oriented distance which changes the sign of the permutation z_1 and z_2. Additivity of the distance on geodesics follow from $(z_2,z_1,w_1,w_2)=(z_2,z_3,w_1,w_2)(z_3,z_1,w_1,w_2)$.

We summarize

Theorem 1.

$$\lambda_{\mathbb{U}}=\ln\frac{1+\delta_{\mathbb{U}}}{1-\delta_{\mathbb{U}}},\quad \lambda_{\mathbb{H}}=\ln\frac{1+\delta_{\mathbb{H}}}{1-\delta_{\mathbb{H}}}\,. \tag{2.14}$$

2.4. **Ahlfors-Schwarz lemma.** It was noted by Pick that result can be expressed in invariant form. We refer the following result as Schwarz-Pick lemma.

Theorem 2.1 (Schwarz-Pick lemma)**.** *Let F be an analytic function from a disk B to another disk U. Then F does not increase the corresponding hyperbolic (pseudo-hyperbolic) distances.*

2.5. **Convex Functions.** A set is convex if it contains the line segment between any two of its points. We wish to characterize the analytic functions f that define a one-to-one conformal map of the unit disk on a convex region. For simplicity such functions will be called convex univalent (Hayman [27]). An analytic function f in $B(a;R)$ is convex univalent if and only if $C[f](z)=\mathrm{Re}\frac{(z-a)f''(z)}{f'(z)}+1>0$, $z\in B(a;R)$.

Theorem 2.2. *An analytic function f in $\mathbb{U}$ is convex univalent if and only if $C[f](z)=\mathrm{Re}\frac{zf''(z)}{f'(z)}+1>0$, $z\in\mathbb{U}$. When this is true the stronger inequality also holds*
$|\frac{zf''(z)}{f'(z)}-\frac{2|z|^2}{1-|z|^2}|\le\frac{2|z|}{1-|z|^2}$.

Although this could be made into a rigorous proof, we much prefer an idea due to Hayman. We may assume that $f(0)=0$. If f is convex univalent, the function $g(z)=f^{-1}[(f(\sqrt{z})+f(-\sqrt{z}))/2]$ is well defined, analytic, and of absolute value <1 in $\mathbb{U}$. Hence $|g'(0)|\le1$. But if $f(z)=a_1z+a_2z^2+...$, then $g(z)=(a_2/a_1)z+...$, and we obtain $|a_2/a_1|\le1$, $|f"(0)/f'(0)|\le2$. This is (1) for $z=0$. We apply this result to $F(z)=f[(z+c)/(1+\overline{c}z)]$, $|c|<1$, which maps $\mathbb{U}$ on the same region. Simple calculations give
$\frac{F''(0)}{F'(0)}=\frac{f''(c)}{f'(c)}(1-|c|^2)-2\overline{c}$,
and we obtain (1-9) and its consequence (1-8).

The condition $|f''(0)/f'(0)|\le2$ has an interesting geometric interpretation. Consider an arc γ in $\mathbb{U}$ that passes through the origin and whose image is a straight line. The curvature of γ is measured by $d(\arg dz)/|dz|$. By assumption $d(argdf) = 0$ along γ so that $d(\arg dz) = -d\arg f'$. The curvature is thus a directional derivative of $argf'$, and as such it is at most $|f''/f'|$ in absolute value. We conclude that the curvature at the origin is at most 2.

Proposition 2.6. Let G be a convex domain in $\mathbb{C}$ different from $\mathbb{C}$ and $H_1 = H(b;r_1)$ an hyperbolic disk in G. Then H_1 is convex wrt euclidean metric

Proof. Then there is a conformal map ϕ of $\mathbb{U}$ onto G such that $\phi^{-1}(H)$ is an hyperbolic disk $H_2 = H(a;r_2)$ in $\mathbb{U}$. Then $\varphi_a(H_2) = B := B(0;r)$ and $\phi\circ\varphi_a(B) = H_1$. By th conv, H_1 is convex. □

A1) Let γ be a curve and $\tilde\gamma = f\circ\gamma$, T and $\tilde T$ tangent vectors at point $\gamma(t)$ and $\tilde\gamma(t)$ respectively, $\theta = \arg T$ and $\tilde\theta = \arg\tilde T$. Then $\tilde T = f'(\gamma(t))T$ and therefore $\tilde\theta = \arg\tilde T = \arg f'(\gamma(t)) + \arg T$. Hence $\tilde\theta'_t = \mathrm{Im}(\frac{f''}{f'}T) + \theta_t$.

A2) In particular if $w = \gamma(t) = a + re^{it}$, then $T = \gamma'(t) = ire^{it}$, $\theta = t + \pi/2$ and therefore $\tilde\theta'_t = \mathrm{Re}\frac{f''(w)}{f'(w)}z + 1$.

A3) If $f = \ln$, then $\tilde\theta'_t = -\mathrm{Re}\frac{z}{z+a} + 1 = 1 - \mathrm{Re}\frac{w-a}{w} = \mathrm{Re}\frac{a}{w}$.

A4) If a=0 in A3), $\tilde\theta'_t = R[f](re^{it})$.

There is a branch $\tilde\theta$ of $Arg\tilde T$ along the path K_r defined by $K_r(t) = re^{it}$

The assumption (1-8) implies that $\tilde\theta = \arg df$ increases with t on $|z| = r$.

Since f' is never zero, the change of $\arg df$ is 2π. Therefore, we can find t_1 and t_2 such that arg df increases from 0 to π on $[t_1,t_2]$ and from π to 2π on $[t_2,t_1+2\pi]$. If $f(re^{it}) = u(t)+iv(t)$, it follows that v increases on the first interval and decreases on the second. Let v_0 be a real number between the minimum $v(t_1)$ and the maximum $v(t_2)$. Then $v(t)$ passes through v_0 exactly once on each of the intervals, and routine use of winding numbers shows that the image of U_r intersects the line $v = v_0$ along a single segment. The same reasoning applies to parallels in any direction, and we conclude that the image is convex.

Let A denote the class of functions $f = z+a_2z^2+a_3z^3+\cdots$ which are holomorphic in the unit disc $\mathbb{U}$. Conformal mappings of the unit disk onto convex domains have been studied for a long time and are known to have many special properties. They are described by the analytic condition $R[f](z) = \mathrm{Re}\frac{zf''(z)}{f'(z)} + 1 > 0$, $z\in\mathbb{U}$, which essentially expresses the monotonic turning of the tangent vector at the boundary (see, for instance, Duren [18]).

Implicit in this description is the hereditary property: if an analytic function maps the unit disk univalently onto a convex domain, then it also maps each concentric subdisk onto a convex domain. It is natural to ask to what extent the special properties of conformal mappings will generalize to harmonic mappings of the disk onto convex domains.

In Euclidean geometry, a convex quadrilateral with at least one pair of parallel sides is referred to as a trapezoid.

Example 4. Let f be holomorphic on U_R, $w = H_R(z) = Rz$, and $F = f\circ H_R$. Then $F'(z) = Rf'(w)$, $F''(z) = R^2f''(w)$ and $R[F](z) = R[f](w)$.

Let $A = 0$, $B = 1$, $C = 1+4\pi i$, $D = 4\pi i$, $E = 2\pi i$ and let R be rectangle $ABCD$, T trapezoid $ABCE$ and $f(z) = e^z$. Check that $R' = f(R) = f(T) = \{1 < |w| < e\}$ and that f is bi-valent on R.

Whether the following is true? Question. Let γ be a simple closed smooth positively oriented curve in $\mathbb{C}$, $G = Int(\gamma)$ and f analytic function on $\overline{G}$.
Suppose, in addition, that f' has no zeros on ∂G.
Define $\Gamma = f \circ \gamma$ and suppose that $\mathbf{i}_\Gamma = 1$.

Whether then (i): Γ is a closed Jordan curve and f is injective mapping of G onto $Int(\Gamma)$.

Covering Theorems and Coefficient Bounds The classical Koebe one-quarter theorem says that each function $f \in A$ analytic and univalent in the unit disk $\mathbb{U}$ contains the entire disk $|w| < 1/4$ in its range $f(\mathbb{U})$. The Koebe function $k(z) = \frac{z}{(1-z)^2}$ maps $\mathbb{U}$ conformally onto the full plane minus the portion of the negative real axis from $-1/4$ to infinity, showing that the radius $1/4$ is the best possible. The celebrated Bieberbach conjecture, now a theorem, asserts that the coefficients of each such function f satisfy the sharp inequalities $|a_n| \le n$, $n = 2, 3, \ldots$. Both of these results can be improved under the additional assumption that the range of f is convex. Let C denote the class of functions in A that map the unit disk conformally onto a convex region. It is known that the range of each function $f \in C$ contains the larger disk $|w| < 1/2$, and its coefficients satisfy the better bound $|a_n| \le 1$. The function $\ell(z) = \frac{z}{(1-z)}$ which maps $\mathbb{U}$ conformally onto the half-plane $\mathrm{Re} w > -1/2$, shows that both results are again best possible (see Duren [18], p. 45).

Lemma 2.1. *If* $g(z) = c_0 + c_1 z + \cdots$ *is analytic with* $\mathrm{Reg(z)} > 0$ *in* $\mathbb{U}$*, then* $|c_n| \le 2\mathrm{Re}\{c_0\}$*,* $n = 1, 2, \ldots$.

Proof. The Herglotz representation shows that $g(z) = 2\int_0^{2\pi} S(z,t)d\mu(t)$ $c_n = 2\int_0^{2\pi} e^{-int}d\mu(t)$, $n = 1, 2, \ldots$, so that $|c_n| \le ||\mu|| = 2Re\{c_0\}$. □

We denote by $S^*(\alpha)$ the subclass of A consisting of functions $f(z)$ which are starlike of order α in $\mathbb{U}$. Thus, if $w \notin f(\mathbb{U})$, a suitable rotation will give $\mathrm{Re(e^{i\alpha}[f(z)-w])} > 0$

Proposition 2.7. If g is analytic in $\mathbb{U}$ and $g \prec f$ for some $f \in C$, then $|g_n| \le 1$ for $n = 1, 2, \ldots$.

Proof. $\varphi(z^n) = \frac{1}{n}\sum_{k=1}^n \frac{1}{n} g(\varepsilon_k z) = g_n z^n + \cdots$
$|g(z)| \le \sum_{k=1}^\infty |g_k||z^k| \sum_{k=1}^\infty |r^k| = \frac{r}{1-r} = k(r)$, where $r = |z|$.

This last expression is an analytic function of z^n, since $\sum_{k=1}^n (\varepsilon_k)^m = 0$ unless m is a multiple of n. Thus, $\varphi \prec f$, and $|g_n| = |\varphi'(0)| \le |f'(0)| = 1$, by the Schwarz lemma. □

A sense-preserving harmonic mapping $f \in S_H$ is said to be starlike if its range is starlike with respect to the origin. This means that the whole range can be "seen" h from the origin. In other words, if some point $w_0 = f(z_0)$ is in the range of f , then so is the entire radial segment from 0 to w_0. If f has a smooth extension to the closed disk, an equivalent requirement is that $\arg(f(e^{it}))$ be a nondecreasing function of t, or that $\frac{d}{dt}\arg(f(e^{it})) \ge 0$.

If $f = h + \overline{g} \in S_H$ is a starlike function, and if H and G are the analytic functions defined by $zH'(z) = h(z)$, $zG'(z) = -g(z)$, $H(0) = G(0) = 0$, then $F = H + \overline{G}$ is a convex function of class C_H.

3. Harmonic functions

In this section we present some material from Duren's [18].

3.1. **Euclidean harmonic function.** Let D be a domain in C and f complex valued function defined na D such that $\Delta f = 0$ and $\Delta f^2 = 0$. Then f is analytic or anti-analytic.

Since $\Delta f^2 = 2(f_x^2 + f_y^2)$, we find $f_z f_{\overline{z}} = 0$. Set $A = \{z \in D : f_z = 0\}$ and $B = \{z \in D : f_{\overline{z}} = 0\}$. A and B are closed in D and $D = A \cup B$; hence $A = D$ or $B = D$.

Connections with complex function theory

The real and imaginary part of any holomorphic function yield harmonic functions on R^2 (these are said to be a pair of harmonic conjugate functions). Conversely, any harmonic real function u on an open set $D \subset R^2$ is locally the real part of a holomorphic function. This is immediately seen observing that, writing $z = x + iy$, the complex function $g(z) := u_x - iu_y$ is holomorphic in D, because it satisfies the Cauchy-Riemann equations. Therefore, g has locally a primitive f, and u is the real part of f up to a constant, as u_x is the real part of $f' = g$. In a simply connected domain $D \subset \mathbb{C}$, a complex-valued harmonic function $f = u + iv$ has the representation $f = h + \overline{g}$, where h and g are analytic in D; this representation is unique up to an additive constant and we call it local representation with analytic functions h and g. For a proof, consider first the case when D is the unit disk. Then $u = \Re h_1$ and $v = \Re g_1$, where h_1 and g_1 are analytic on $\mathbb{D}$ and therefore we have the representation with $h = (h_1 + ig_1)/2$ and $g = (h_1 - ig_1)/2$. In general case we can use conformal mapping ψ of $\mathbb{D}$ onto D and the fact that $f \circ \psi$ is a complex-valued harmonic function which has the representation $f \circ \psi = h_2 + \overline{g_2}$. Hence we have the representation on D with $h_2 \circ \phi$ and $g_2 \circ \phi$, where ϕ is the inverse of ψ.

For a harmonic mapping f of the unit disk $\mathbb{D}$, it is convenient to choose the additive constant so that $g(0) = 0$. The representation $f = h + \overline{g}$ is then unique and is called the canonical representation of f. Sometimes we denote by h harmonic function and write the representation in the form $h = f + \overline{g}$.

Although the above correspondence with holomorphic functions only holds for functions of two real variables, still harmonic functions in n variables enjoy a number of properties typical of holomorphic functions. They are (real) analytic; they have a maximum principle and a mean-value principle; a theorem of removal of singularities as well as a Liouville theorem one holds for them in analogy to the corresponding theorems in complex functions theory.

complex-valued harmonic univalent functions whose real and imaginary parts are not necessarily conjugate. They have a maximum and a maximum principle and a mean-value principle: A nonconstant harmonic function u has nether a maximum nor a minimum in its region of definition D. If $F \subset D$ is a closed bounded set u attains the maximum and the minimum on the boundary of F. The proof is the same as for the maximum principle for analytic functions. It applies to the minimum because $-u$ is harmonic together with u. Note that the minimum principle for analytic functions requires an extra hypothesis (non-vanishing of the function).

If f is a complex-valued harmonic function in D and $|f|$ has a local maximum at $a \in D$, then f is constant. Let $M = |f(a)|$ and choose $\lambda \in \mathbb{T}$ such that $\lambda f(a) = M$. Then the real-valued harmonic function $\Re(\lambda f)$ attains the maximum value M at a.

In general, the minimum modulus principle for complex-valued harmonic functions is not valid; see the following examples:

2. If $f(z) = x + i$, then $|f(z)|^2 = x^2 + 1$ and $|f|$ attains minimum which is 1 for every points on y axis

2. If $f_c(z) = x + i(x^2 - y^2 + c)$, then $J_f = -2y$. Whether $|f(0)| = |c|$ is the minimum value for $|f|$ if $c < 0$? The answer is yes. Let $d(z) = |z|$, $g = f_i$, $C(x) = x + i(x^2 + i)$ and $D = \{(x, y) : y < x^2 - 1\}$. Then d attains minimum on C at some point w_0 and there is a real point x_0 such that $g(x_0) = w_0$, g maps $\mathbb{C}$ onto D and $|g|$ attains minimum at x_0.

As soon as analyticity is abandoned, serious obstacles arise. Analytic functions are preserved under composition, but harmonic functions are not. A harmonic function of an analytic function is harmonic, but an analytic function of a harmonic function need not be harmonic. The analytic functions form an algebra, but the harmonic functions do not. Even the square or the reciprocal of a harmonic function need not be harmonic. The inverse of a harmonic mapping need not be harmonic. The boundary behavior of harmonic mappings may be much more complicated than that of conformal mappings. It will be seen, nevertheless, that much of the classical theory of conformal mappings can be carried over in some way to harmonic mappings. The Jacobian of a function

$$J_f(z) = \begin{vmatrix} u'_x(z_0) & v'_x(z_0) \\ u'_y(z_0) & v'_y(z_0) \end{vmatrix} = u_x v_y - u_y v_x = |h'|^2 - |g'|^2.$$

where the subscripts indicate partial derivatives.

If f is analytic, its Jacobian takes the form $J_f(z) = (u_x)^2 + (v_x)^2 = |f'(z)|^2$. For analytic functions f , it is a classical result that $J_f(z) \neq 0$ if and only if f is locally univalent at z. Hans Lewy showed in 1936 that this remains true for harmonic mappings. A relatively simple proof will be given in XXX

In view of Lewy's theorem, harmonic mappings are either sense-preserving (or orientation-preserving) with $J_f(z) > 0$, or sense-reversing with $Jf(z) < 0$ throughout the domain D where f is univalent. If f is sense-preserving, then $\bar{f}$ is sense-reversing. Conformal mappings are sense-preserving. The simplest examples of harmonic mappings that need not be conformal are the affine mappings $f(z) = az + b\bar{z} + c$ with $|a| \neq |b|$. Affine mappings with $b = 0$ are linear mappings. It is important to observe that every composition of a harmonic mapping with an affine mapping is again a harmonic mapping: if f is harmonic, then so is $af + b\bar{f} + c$. Another important example is the function $f(z) = z + \frac{1}{2}\bar{z}^2$, which maps the open unit disk D onto the region inside a hypocycloid of three cusps inscribed in the circle $|w| = 3/2$. To verify its univalence, suppose $f(z_1) = f(z_2)$.

Now let f be a complex-valued function defined in a domain D having continuous second partial derivatives. Suppose that f is locally univalent D, with Jacobian $J_f(z) > 0$. Let $\omega = \overline{f_{\bar{z}}}/f_z$ be its second complex dilatation; then $|\omega(z)| < 1$ in D.

A1) f is harmonic

A2) ω is analytic

Differentiating the equation $\overline{f_{\bar{z}}} = \omega f_z$ with respect to z, one finds $\overline{f_{\bar{z}z}} = \omega_{\bar{z}} f_z + \omega f_{z\bar{z}}$. Now if f is harmonic in D, then $f_{z\bar{z}} = 1/4\triangle f = 0$ there. Thus it follows that $\omega_{\bar{z}} = 0$ in D, so that ω is analytic.

There is also another way to prove A1) implies A2)

Now if f is harmonic in D, using local representation of f with analytic functions h and g, we find $|h'| > |g'|$, $f_z = h'$, $f_{\bar{z}} = \overline{g'}$ and $\overline{f_{\bar{z}}} = g'$. Hence ω is analytic.

Conversely, if ω is analytic, then $\overline{f_{\bar{z}z}} = \omega f_{z\bar{z}}$. But since $|\omega(z)| < 1$, this implies that $f_{z\bar{z}} = 0$, and f is harmonic. Thus, f is harmonic if and only if ω is analytic. In particular, the second complex dilatation ω of a sense-preserving harmonic mapping f is always an analytic function of modulus less than one. This function ω will be called the analytic dilatation of f , or simply the dilatation when the context allows no confusion. Note that $\omega \equiv 0$ if and only if f is analytic. The analytic dilatation has some nice properties. For instance, if f is a sense-preserving harmonic mapping with analytic dilatation ω and it is followed by an affine mapping $A(w) = aw+c+b\bar{w}$ with $|b| < |a|$, then the composition $F = A \circ f$ is a sense-preserving harmonic mapping with analytic dilatation

$$\omega_F = \frac{\overline{F_{\bar{z}}}}{F_z} = \frac{\bar{a}\omega + \bar{b}}{b\omega + a},$$

where $\omega = \omega_f$.

If $J_f \equiv 0$ on some open set $U \subset \mathbb{D}$, then $f = a + bu$, where u real harmonic on $\mathbb{D}$.

$h' \equiv 0$

$h' \neq 0$ by maximum principle $\omega = e^{i\alpha} = const$

$e^{i\alpha}h' = g'$ $c + e^{i\alpha}h = g$ $f = h + \bar{c} + e^{-i\alpha}\bar{h}$

$p = e^{i\alpha/2}h$

If $z_0 \in J$ is an isolated zero, then there is an open neighborhood U such that J_f does not change sing in $U\{z_0\}$ and $h'(z_0) = g'(z_0) = 0$

If $z_0 \in J$ is an isolated zero n_h the order of zero of h

$h' = a(z - z_0)^{n_h} h_0(z)$, $g' = b(z - z_0)^{n_g} g_0(z)$

If $J_f > 0$ in some neighborhood of z_0, then $n_h < n_g$ or

$n_h = n_g$ and $|a| > |b|$. In this setting z_0 is called a critical point of f of order $m = n_h$.

$z_0 \in J$ is not an isolated zero iff $|\omega(z_0)| = 1$.

Denote by J_0 the set of isolated zeros and by J_1 the set of all zeros which are not isolated zeros.

Let $h(z) = z + z^n$ and $g(z) = z - z^n$. Then $h'(z) = 1 + nz^{n-1}$, $g'(z) = 1 - nz^{n-1}$, $X_\alpha(z) = e^{i\alpha}(1 + nz^{n-1}) - 1 + nz^{n-1} = (1 + e^{i\alpha})nz^{n-1} + e^{i\alpha} - 1$ and in particular $X_0(z) = 2nz^{n-1}$

Let $\omega(z_0) = e^{i\alpha}$ $\omega(z) = e^{i\alpha} + (z - z_0)^n\omega_0(z)$, where $\omega_0(z_0) \neq 0$. Then the set J_1 in a neighborhood V of z_0 consists of $2n$ analytic curves emanating from z_0 at equal angles and which divide V on $2n$ sectors.

Define $X_\alpha(z) = e^{i\alpha}h'(z) - g'(z)$, $L_c = \{\Im w = c\}$, $L^c_\alpha = e^{-i\alpha/2}L_c$, $l(w) = \Im(e^{i\alpha/2}w)$, $f_0 = f^\alpha = l \circ f$ and $F_c = \{z : f_0(z) = c\}$. Note that F_c is the inverse image under f of line L^c_α.

We now consider a point $z_0 \in J_1$ for which $\omega(z_0) = e^{i\alpha}$ and $f^\alpha(z_0) = c$.

If $z_0 \in F_c$ is zero of X_α of order n, then

P1) $f_0 = 2\Re B$, $B(z) = \frac{c}{2} + (z - z_0)^{n+1}C$, $C(z_0) \neq 0$,

and then

P2) there is an open neighborhood U of z_0 such that $F_c \cap U$ consists of $2(n+1)$ analytic curves emanating from z_0 at equal angles and which divide U on $2(n+1)$ sectors U_k, $k = 1, \cdots, n+1$.

Let $U^+_k = U_k \cap J^+$ and $U^-_k = U_k \cap J^+$

P3) f maps homeomorphically each U^+_k, U^-_k

If U_k is subset of J^+ or J^-, then f maps U_k homeomorphically onto the half disk XX.

see example **??**

XX A. Lyzzaik, Local properties of light harmonic mappings, Cand. J. Math. 35 (1992), 119.

Let f be a sense-preserving harmonic function in a Jordan domain D with boundary C. Suppose f is continuous in $\overline{D}$ and $f(z) \neq= 0$ on C. Then $\Delta_C argf(z) = 2\pi N$, where N is the total number of zeros of f in D, counted according to multiplicity.

Every harmonic function h in $\mathbb{D}$ can be written in the form $h = f + \bar{g}$, where f and g are holomorphic functions in $\mathbb{D}$. ? Let $h = f + \bar{g}$. An easy calculation shows ??

$\partial_\theta h(z) = i(zf'(z) - \overline{zg'(z)})$, $h_r = e^{i\theta}f' + \overline{e^{i\theta}g'}$, $\quad h_\theta + irh_r = 2izf'$ and therefore rh_r is harmonic conjugate of h_θ. Let

$$P(r,t) = \frac{1 - r^2}{(1 - 2r\cos(t) + r^2)}$$

denote the Poisson kernel.

If $\gamma \in L^1[0, 2\pi]$ and

$$h(z) = P[\gamma] = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\,\gamma(t)\,dt,$$

then the function $h = P[\gamma]$ so defined is called Poisson integral of γ.

If γ is of bounded variation, define $T_\gamma(x)$ as variation of γ on $[0, x]$; and let $V(\gamma)$ denote variation of γ. If ?? γ (see, for example, [**?**] p.171).

Define

$$h_*(\theta) = h^*(e^{i\theta}) = \lim_{r\to 1} h(re^{i\theta})$$

when this limit exists.

If γ is continuos, then h is harmonic on $\mathbb{D}$, continuos $\overline{\mathbb{D}}$

and $h_* = \gamma$. Thus, h solves the Dirichlet problem for the unit disk.

The Dirichlet problem is to find a function harmonic in a domain D and continuous in $\overline{D}$ that agrees with a prescribed continuos function on the boundary ∂D. A solution exists (for prescribed continuos function on the boundary) iff the boundary of D has no degenerate components; in particular, if D is a Jordan domain.

Suppose that u is harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.

For $z \in \mathbb{D}$ define $T_z(w) = \frac{w-z}{1-\bar{z}w}$ and $U = u \circ T_z$. By the mean value theorem , $u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(s)ds$ and $u(z) = U(0) = \frac{1}{2\pi}\int_0^{2\pi} U(s)ds$. using the change of variables $t = T_z(s)$, $s = T_z(t)$, and $s'(t) = \frac{1-|z|^2}{|1-ze^{-it}|^2} = P(z,t)$, we find

$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,t)u(t)dt = P[u](z)$.

Since $C(z,t)dt = C(w,z)dw$, $C(\bar{z}, -t) = -1 + \overline{C(z,t)}$, we find $P[1] = 1$.

A1) If f is continuous at t_0, then $P[f]$ tends to $f(t_0)$ z approaches e^{it_0}.

For a given t_0 define $I_1 = I_1(t_0, \delta) = \{|t - t_0| < \delta\}$, $I_2 = I_2(t_0, \delta) = \{|t - t_0| > \delta\}$, $f_{t_0}(t) = f(e^{it}) - f(e^{it_0})$ and $M = \max|f|$; set

$\epsilon_1(\delta) = \max_{t\in I_1}\{|f_{t_0}(t)|\}$ and $\epsilon_2(\delta) = \int_{I_2} P(z,t)dt$. Using

$|P[f](z) - f(z_0)| = |P[f_{t_0}](z)| \le \epsilon_1(\delta) + M\epsilon_2(\delta)$, one can verify A1) easily.

Define $A(z) = i\frac{1-z}{1+z}$, $U_1(z) = \frac{a}{\pi}argz$, $U_2(z) = b(1 - \frac{argz}{\pi})$ and $U = U_1 + U_2$.

More generally, as z approaches 0 along a linear segment (curve) at an angle α, $0 < \alpha < \pi$, with the tangent line, one can show that $U_1(z)$, $U_2(z)$ and U tend to a/π, $b(1-\frac{\alpha}{\pi})$ and $a/\pi + b(1-\frac{\alpha}{\pi})$ respectively.

$u_1 = U_\circ A$ as z approaches 0 along a linear segment (curve) at an angle α, $0 < \alpha < \pi$, with the tangent line, one can show that $u_1(z)$ tends to a/π

If $u(z) = a\ z \in T^+$

$$u(z) = \begin{cases} b & z \in T^+ \\ a & z \in T^- \end{cases}$$

A1) as z approaches 0 along a linear segment (curve) at an angle α, $0 < \alpha < \pi$, with the tangent line, one can show that $u(z)$ tends to $a/\pi + b(1-\frac{\alpha}{\pi})$.

A2) If γ has a finite number of jump discontinuites, $\gamma(\theta-) \neq \gamma(\theta+)$

$$h_*(\theta) = \frac{1}{2}(\gamma(\theta-) + \gamma(\theta+))\,.$$

More generally, as z approaches $e^{i\theta}$ along a linear segment at an angle α, $0 < \alpha < \pi$, with the tangent line, one can show that $h(z)$ tends to

$$\frac{\alpha}{\pi}\gamma(\theta-) + (1-\frac{\alpha}{\pi})\gamma(\theta+)\,.$$

$a = \gamma(\theta-)$ $b = \gamma(\theta+)$ $K(t) = a$ for $t \in (\theta - \pi, \theta)$ and $K(t) = b$ for $t \in (\theta, \theta + \pi)$ $u = \gamma - K$ $u(\theta) = 0$ is continuous at θ

z approaches $e^{i\theta}$ along a linear segment at an angle α, $0 < \alpha < \pi$ then $P[u]$ tend 0 and A2) follows from A1).

Harnack's inequality states that

$$\frac{R-r}{R+r}u(0) \le u(z) \le \frac{R+r}{R-r}u(0), \quad |z| = r$$

for a positive harmonic function u in the disk $|z| < R$.

A critical point of u is a point where u'_x and u'_y both vanish.

All critical points of a nonconstant harmonic function are isolated.

The level set of a nonconstant harmonic function u through a critical point z_0 consists locally of two or more analytic arcs intersecting with equal angles at z_0. Let $f = u + iv$ be an analytic completion of u near z_0. It follows from the C R equations that $f'(z_0) = 0$.

Suppose for convenience that $z_0 = 0$ and that $f(z_0) = 0$. Then f has a local structure $f(z) = \varphi^m$, where $m \geq 2$ and φ is univalent near the origin.

the elementary function $f(x, y) = (x, y^3)$ maps $\mathbb{R}^2$ univalently onto $\mathbb{R}^2$, yet its Jacobian $J_f(z) = 3y^2$ vanishes on $\mathbb{R}$. The Jacobian of a locally univalent analytic function cannot vanish at any point; the same principle holds more generally for harmonic functions in the plane.

Lewy's theorem

Theorem 3.1. *If a complex- valued harmonic function is locally univalent in a domain D, then its Jacobian is different from 0 for all $z \in D$.*

Let $h = (u, v)$ and suppose that $detJ(h)$ is zero at z_0, that is

$$\begin{vmatrix} u'_x(z_0) & u'_y(z_0) \\ v'_x(z_0) & v'_y(z_0) \end{vmatrix} = 0.$$

Thus vectors $(u'_x(z_0), u'_y(z_0))$ and $(v'_x(z_0), v'_y(z_0))$ are ?? linearly dependent and therefore there exists $(\alpha, \beta) \neq (0,0)$ such that $U'_x = 0$, $U'_y = 0$ at z_0, where $U = \alpha u + \beta v$. Let $L = \{z : U(z) = U(z_0)\}$. The level-set L consists locally of two or more analytic arcs intersecting with equal angles at z_0. h maps this level-set into the line. But h is locally univalent and the assumption has led to a contradiction.

For a generalization of Lewy's theorem we refer to [**?**], p.78:

Theorem 3.2 (Heinz)**.** *Suppose $u : \mathbb{D} \to (S, \rho)$ is univalent (i.e. injective) ρ-harmonic map.Then $J_u(z) \neq 0$ for all $z \in \mathbb{D}$.*

Radó there is no harmonic mapping of $\mathbb{D}$ onto $\mathbb{C}$.

Recall that by S we denote the class of functions $f(z) = z + a_2 z^2 + \cdots$ analytic and univalent in the unit disk $\mathbb{D}$.

The classical Koebe one-quarter theorem says that each function $f \in S$ contains the entire disk $|w| < 1/4$ in its range $f(\mathbb{D})$. The celebrated Bieberbach conjecture, now a theorem, asserts that the coefficients of each function $f \in S$ satisfy the sharp inequalities $|a_n| \leq n$, $n = 2, 3, \cdots$.

Let C denote the class of functions that map the unit disk conformally onto a convex region.

It is known that each function $f \in C$ contains the entire disk $|w| < 1/2$ in its range $f(\mathbb{D})$ and its coefficients satisfy the better bound $|a_n| \leq 1$, $n = 2, 3, \cdots$.

The function $\ell(z) = \frac{z}{1-z}$ which maps $\mathbb{D}$ conformally onto the half-plane $\Re w > -1/2$, shows that both results are again best possible.

The above results on convex conformal mappings extend nicely to convex harmonic mappings. Before stating the theorems, we need to introduce some terminology. The class C_H consists of all sense-preserving harmonic mappings $f = h + \bar{g}$ of the unit disk onto convex domains, with the normalization $h(0) = g(0) = 0$ and $h'(0) = 1$. Note that $|g'(0)| < |h'(0)| = 1$, since f preserves orientation and thus has positive Jacobian. Let $b_1 = g'(0)$, $a = (1 - |b_1|^2)^{-1}$ and $b = \overline{b_1} a$. Postcomposing a function $f = h + \bar{g} \in C_H$ by the sense-preserving affine mapping $A(w) = aw + b\bar{w}$, which preserves convexity, the further normalization $g'(0) = 0$ can be achieved. The resulting class of functions will be denoted by C_H^0. Thus, a sense-preserving harmonic function $f = h + \bar{g}$ belongs to C_H^0 if it maps the unit disk univalently onto a convex region with the normalization $h(0) = g(0) = g'(0) = 0$ and $h'(0) = 1$.

A primary example of a function of class C_H^0 is

$L(z) = \Re \ell(z) + \Im k(z)$

Although L maps $\mathbb{T} \setminus \{1\}$ onto a point $-1/2$, it maps the unit disk univalently onto the convex region $\{w : \Re w > 1/2\}$.

Theorem 2. Each function $h \in C_H^0$ contains $\mathbb{D}_{1/2}$ in its range.

The proof depends on Herglotz representation

$$p(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t) + i\gamma$$

of a function p analytic in $\mathbb{D}$ with positive real part; here $d\mu$ is a positive measure and γ is a real constant.

Since $C(z,t) = (e^{it} + z)(e^{it} - z)^{-1} = 1 + 2\sum_1^\infty e^{-ikt} z^k$, we find $p_n = 2\int_0^{2\pi} e^{-ikt} d\mu(t)$, $n \geq 1$. Hence $|p_n| \leq 2\Re p_0$.

By hypothesis, the range $f(\mathbb{D})$ is convex. Thus, if $w \notin h(\mathbb{D})$, a suitable rotation will give $\Re F > 0$ on $\mathbb{D}$, where $F(z) = e^{i\alpha}(f - w) + e^{-i\alpha} g$. Hence $F_0 = -e^{i\alpha} w$, $F_1 = e^{i\alpha}$ and therefore $1 = |e^{i\alpha}| = |F_1| \leq 2|F_0| = 2|w|$, that is $|w| \geq 1/2$. This proves the theorem.

Now we will give a generalization of Theorem 2 using a more geometric approach. We need first the following result:

Lemma 1. Suppose that F is analytic on $\mathbb{D}$, $G = F(\mathbb{D})$ is convex and $m = dist(F(0), \partial G)$. Then $|F'(0)|/2 \leq m$.

Proof. We can suppose that $F(0) = 0$ and that F is not a constant and after a suitable rotation that
(I1) $\Re F + m > 0$ on $\mathbb{D}$.
By the subordination principle $F = 2m\ell \circ \omega$. Hence $F'(0) = 2m\omega'(0)$ and the result follows. □

Theorem 3. Suppose that f is harmonic on $\mathbb{D}$, $G = F(\mathbb{D})$ is convex, $g'(0) = 0$ and $m = dist(F(0), \partial G)$. Then $|h'(0)|/2 \leq m$.

Proof. After a suitable rotation, the function $e^{i\alpha} f$ satisfies $(I1)$. Note that $\Re e^{i\alpha} f = \Re F_\alpha$, where $F_\alpha = e^{i\alpha} h + e^{-i\alpha} g$. Since $h'(0) = F'_\alpha(0)$ an application of Lemma 1 to the the function F_α yields the result.

□

Lemma 3.1. *If $h = f + \bar{g} \in C_H$, then exists α and β such that*

$$\operatorname{Re}\left((e^{i\alpha} f'(z) + e^{-i\alpha} g'(z)(e^{i\beta} - e^{-i\beta} z^2)\right) > 0$$

for all $z \in \mathbb{U}$.

Proof. It suffices to assume that h has a smooth extension to the boundary.

Define $\underline{h}(t) = h(e^{it})$, $\lambda(t) = \underline{h}'(t)$ and $T(t) = \frac{\lambda(t)}{|\lambda(t)|}$.

There is a continuous real valued function φ such that $T(t) = e^{i\varphi(t)}$, with $\varphi(t + 2\pi) = \varphi(t)$. For each t there is a unique $t^* = \alpha(t)$ with $t < t^* < t + 2\pi$ for which $\varphi(t^*) = \varphi(t) + \pi$.

geometric interpretation: if $L(t)$ denotes tangent line of C at ponit $A(t) = \underline{h}(t)$, where $C = h(T)$, then lines $L(t)$ anf $L(t^*)$ are paralallel and if $t^* - t < \pi$ then $t^{**} - t^* > \pi$. Since the function $t^* - t$ is continuous, one can show that $t_0^* = t_0 + \pi$ for some t_0.

Since t^* is a continuous and $t^{**} = \alpha(\alpha(t)) = t + 2\pi$, one can show $t_0^* = t_0 + \pi$ for some t_0. Define $V(t) = T(t)/T(t_0) = e^{i\left(\varphi(t) - \varphi(t_0)\right)}$.

$V(t)$ lies in the upper half-plane and $\sin\left(\varphi(t) - \varphi(t_0)\right) \geq 0$ for $t_0 < t < t_0 + \pi$, whereas it lies in the lower half-plane and $\sin\left(\varphi(t) - \varphi(t_0)\right) \leq 0$ for $t_0 + \pi < t < t_0 + 2\pi$. Hence

$$\begin{aligned} A(t) &= \operatorname{Re}\left((e^{i(t_0 - t)} - e^{i(t - t_0)}) e^{i\left(\varphi(t) - \varphi(t_0)\right)}\right) \\ &= 2\sin(t - t_0)\sin\left(\varphi(t) - \varphi(t_0)\right) \geq 0 \end{aligned}$$

A simple calculation shows $\lambda(t) = ie^{it} f'(e^{it}) - ie^{-it}\, \overline{g'(e^{it})}$, $\lambda(t)/\lambda(t_0) = i(B - \overline{C})$, and $A(t) = \operatorname{Re}\left((e^{i(t_0 - t)} - e^{i(t - t_0)}(iB - iC))\right)$, where $B = e^{it} f'(e^{it})/\lambda(t_0)$ and $C =$

$e^{it}\,g'(e^{it})/\overline{\lambda(t_0)}$. Hence

$$A(t) = \mathrm{Re}\left((e^{it_0} - e^{-it_0}e^{2i(t)})(i\frac{f'(e^{it})}{\lambda(t_0)} - i\frac{\overline{g'(e^{it})}}{\overline{\lambda(t_0)}})\right) \geq 0$$

or

$$\mathrm{Re}\left((e^{i\alpha}f'(z) + e^{-i\alpha}g'(z)(e^{it_0} - e^{-it_0}z^2)\right) > 0$$

on the unit circle, where $e^{i\alpha} = i|\lambda(t_0)|/\lambda(t_0)$. The result follows from the maximum principle for harmonic functions. □

Lemma 3.2 (Hopf's lemma)**.** *Let D be a Jordan domain with smooth boundary. Let u be noncostant harmonic function in D that has a smooth extension to $\overline{D}$. If u has a local minimum at some point $\zeta \in \partial D$, then its inner normal derivative is strictly positive at that point: ?.*

Suppose $u(\zeta) = 0$. Then $u > 0$ in some disk $B \subset D$ whose boundary is tangent to ∂D at ζ. Suppose, after rotation and translation, that $B = B(0;r)$ and $\zeta = r$.

By Harnack's inequality,

$$\frac{r-x}{x+r}u(0) \leq u(x), \quad 0 < x < r\,.$$

Hence, since $u(r) = 0$,

$$\frac{u(0)}{x+r}u(0) \leq \frac{u(x) - u(r)}{r - x}, \quad 0 < x < r\,.$$

Now let x tend to r to conclude that

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x} \geq \frac{u(0)}{2r} > 0\,.$$

The following theorem helps to explain the concavity of image persistently apparent in harmonic mappings with dilatations of unit modulus on the boundary.

Theorem. Let f be a sense-preserving harmonic mapping of $\mathbb{D}$ onto a domain G. Suppose that f has a C^1 extension to some open arc $I \subset \mathbb{T}$ that it maps univalently onto a convex arc $L \subset \partial G$. Let s denote arclength along L as a function of t for $e^{it} \in I$. Suppose $s'(\alpha) \neq 0$ at some point $a = e^{i\alpha} \in I$. Then the dilatation of f has a continuous extension to a subarc of I containing a , and $|\omega(a)| < 1$.

Corollary. Let f be a sense-preserving harmonic mapping of $\mathbb{D}$ onto a convex domain G and suppose f has a C^1 extension to a homeomorphism of $\overline{\mathbb{D}}$ onto G with $s'(t) > 0$ at every boundary point. Then f is quasiconformal.

Deduction of Corollary. The dilatation $\omega = \overline{f_{\bar z}}/f_z$ has the property $|\omega(z)| < 1$ in $\mathbb{D}$, and it has a continuous extension to $\overline{\mathbb{D}}$ which, by the theorem, still satisfies $|\omega(z)| < 1$ everywhere on $\mathbb{T}$. Thus, $|\omega(z)| < k$ for some constant $k < 1$, and f is quasiconformal in $\mathbb{D}$.

If $\psi \in L^1[0, 2\pi]$ we define the Cauchy transform

$$C(\psi)(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\psi(t)e^{it}}{e^{it} - z}\,dt\,.$$

It is convenient to use notation

$$C(z,t) = \frac{e^{it}}{e^{it} - z}\,.$$

An easy calculation shows

$$C(z,t)+\overline{C(z,t)}-1=P_r(\theta-t)\,.$$

3.2. **Schwarz's Lemma for harmonic mappings.** Throughout this paper, $\mathbb{U}$ will denote the unit disc $\{z:|z|<1\}$, $\mathbb{T}$ the unit circle, $\{z:|z|=1\}$ and we will use notation $z=x+iy$ and $z=re^{i\theta}$, where $r=|z|$ and $\theta\in\mathbb{R}$ are polar coordinates. For a function h, we use notation $\partial h=\frac{1}{2}(h'_x-ih'_y)$ and $\overline{\partial}h=\frac{1}{2}(h'_x+ih'_y)$; we also use notations D^ch and $\overline{D}^ch$ instead of ∂h and $\overline{\partial}h$ respectively when it seems convenient. By h'_x and h'_y we denote partial derivatives with respect to x and y respectively. We write $D^2_{z\overline{z}}h=D(\overline{D}h)$, where $Dh=D^ch$ and $\overline{D}h=\overline{D}^ch$.

Example 5. Let $\mathbb{S}=\{w:|\mathrm{Re}w|<1\}$ and $\mathbb{S}_1=\{w:|\mathrm{Re}w|<\pi/4\}$. tan maps $\mathbb{S}_1$ onto $\mathbb{D}$. Let $B(w)=\frac{\pi}{4}w$ and $f_0=\tan\circ B$, ie. $f_0(w)=\tan(\frac{\pi}{4}w)$. Then f_0 maps $\mathbb{S}$ onto $\mathbb{D}$.

$u=\frac{\sqrt{2}}{\pi}\arg A_0(iz), v=\frac{\sqrt{2}}{\pi}\arg A_0(z), t=\frac{\sqrt{2}}{\pi}\ln|A_0(z^2)|$

Let $r<1$, $A_0(z)=\frac{1+z}{1-z}$, and let $\phi=i\dfrac{2}{\pi}\ln A_0$; that is $\phi=\phi_0\circ A_0$, where $\phi_0=i\frac{2}{\pi}\ln$. Let $\hat{\phi}$ be defined by $\hat{\phi}(z)=-\phi(iz)$. Note that $\hat{\phi}=\frac{4}{\pi}\arctan$ is the inverse of f_0.

Then $\mathrm{Re}\hat{\phi}(z)=\frac{2}{\pi}\arg\frac{1+iz}{1-iz}$ and $|\mathrm{Re}\phi(z)|\leq\frac{4}{\pi}\tan^{-1}|z|$.

Let F be analytic such that $\mathrm{Re}h=\mathrm{Re}F$ on $\mathbb{D}$ with $F(0)=0$.

By subordination, we show that $|\mathrm{Re}\,F(z)|\leq\frac{4}{\pi}\arctan(|z|)$.

Example $h(z)=\frac{4}{\pi}\mathrm{Re}(\arctan z)+iay$, $a\in\mathbb{R}$ shows that we can not control growth of h in general at 0. h maps $\mathbb{D}$ into $\mathbb{S}$, but $|(Imh)_y(0)|=|a|$ can be arbitrarily large.

But if $f:\mathbb{D}\to\mathbb{D}$ is harmonic and $f(0)=0$, then the maximal distortion (i) $L_f(0)\leq\frac{4}{\pi}$. (ii) In particular, if f is conformal at 0, then $|f'(0)|\leq\frac{4}{\pi}$. The estimate (i) is sharp.

It seems that if f is conformal at 0, then $|f'(0)|<\frac{4}{\pi}$.

By a normal family argument there is extremal function for the problem: (iii) $D(0)=\sup\{|L'_f(0)|\}$, where supremum is taken over all harmonic maps $f:\mathbb{D}\to\mathbb{D}$ with $f(0)=0$. But extremal functions $f_0(z)=\frac{2}{\pi}\arg\frac{1+z}{1-z}$ maps $\mathbb{D}$ onto $(-1,1)$.

Lemma 3.3. *Let $\mathbb{S}=\{w:|\mathrm{Re}w|<1\}$ and let $h:\mathbb{U}\to\mathbb{U}$ be a harmonic mapping with $h(0)=0$. Then $|\mathrm{Re}h(z)|\leq\frac{4}{\pi}\tan^{-1}|z|$.*

Proof. Let $M_h(r)=\max\{|h(z)|:z\in T_r\}$. Then there is $z_r=re^{i\alpha}$ such that $R=M_h(r)=|h(z_r)|$. If $h(z_r)=Re^{i\beta}$ and $H(z)=e^{-i\beta}h(e^{i\alpha}z)$, then $M_h(r)=M_H(r)=R$. By Lemma 3.4, $M_H(r)\leq\frac{4}{\pi}\arctan r$ and the proof follows. □

Lemma 3.4 (Schwarz lemmma for harmonic functions [18])**.** *Let $\mathbb{S}=\{w:|\mathrm{Re}w|<1\}$ and let $h:\mathbb{D}\to\mathbb{S}$ be a harmonic mapping with $h(0)=0$. Then $|\mathrm{Re}\,h(z)|\leq\frac{4}{\pi}\tan^{-1}|z|$ and this inequality is sharp for each point $z\in\mathbb{D}$.*

Proof. Let A_1 be defined by $z\mapsto A_0(iz)$. Then A_1 carries the segment $[-i,i]$ onto half circle $T^+=\{w:|w|=1,\mathrm{Re}w\geq0\}$ and $\frac{4}{\pi}\arctan r=\phi_1(r)=\frac{2}{\pi}\arg A_1(r)=-\frac{2}{\pi}\alpha$.

Observe now that the linear fractional mapping $w=\frac{1+z}{1-z}$ carries the circle K_r : $|z|=r<1$ onto the circle $K_R:|w-w_0|=R$ with center $w_0=(1+r^2)/(1-r^2)$ and

radius $R = 2r/(1-r^2)$. Let $r<1$, $A_0(z) = \frac{1+z}{1-z}$, $s = A_0(r)$, $R = \dfrac{s-s^{-1}}{2} = \dfrac{2r}{1-r^2}$ and α be the maximum of $|\arg w|$ on K_R; therefore since $w_0^2 - R^2 = 1$, $\tan\alpha = R$ and $\alpha(r) = \arctan R = 2\arctan r$; recall $\mathbb{S}_0 = \{w : |\mathrm{Re} w| < 1\}$ and let $\phi = i\dfrac{2}{\pi}\ln A_0$; that is $\phi = \phi_0 \circ A_0$, where $\phi_0 = i\frac{2}{\pi}\ln$.
We prove: if $h:\mathbb{D}\to\mathbb{S}$ is harmonic, $h(0)=0$, then $|h(z)| \le \frac{2}{\pi}\alpha(|z|)$.

The linear fractional transformation A maps the circle $|z| = r$ onto the circle $K(a,R)$, where $a = (1+r^2)/(1-r^2)$ and $R = \frac{2r}{1-r^2}$; and therefore the disk $\mathbb{D}_r$ onto the disk $B(a;R)$ of radius R with the center at a.
Thus
(1) $|\arg\, A_1|$ is bounded by $\alpha(r) = 2\arctan r$ on $\mathbb{D}_r$ and therefore
since $\mathrm{Re}\,\phi = -\frac{2}{\pi}\arg\, A_0$,
(2) $|\mathrm{Re}\phi|$ is bounded by $\alpha(r) = \frac{4}{\pi}\arctan r$ on $\mathbb{D}_r$.

Thus, (1) says that A_0 maps $\mathbb{D}_r$ in the angle of opening $2\alpha(r) = 4\arctan r$.

Let F be analytic such that $Reh = ReF$ on $\mathbb{D}$ with $F(0)=0$. By subordination $F(\mathbb{D}_r) \subset \phi_0(B(a;R))$. Hence, by (2) (recall $|\arg z| \le 2\arctan r$ on $B(a;R)$),
(3) $|Re\,\phi|$ is bounded by $\alpha(r) = \frac{4}{\pi}\arctan r$ on $\mathbb{D}_r$.
From (3), it follows

$$|Re\,h(z)| = |Re\,F(z)| \le \frac{4}{\pi}\arctan|z|.$$

$\square$

Example 6. Let $C(t) = t + iA(t)$ and $R(t) = |C(t)|^2 = t^2 + A^2(t)$. Check that $R''(t) = 2[1 + A'^2 + AA'']$.
If $A(t) = \ln B(t)$, then $A'(t) = \frac{B'(t)}{B(t)}$, $A''(t) = \frac{B''(t)B - B'^2(t)}{B^2(t)}$ and $R''(t) = 2[B^2 + A(B''B - B'^2) + B'^2]/B^2$.

Example 7. Let ln a branch of Ln on Π determined by $\ln(1) = 0$ and $K(a,R)$, where $a>0$ and $0<R<a$. Check that $M(R) = M(a,R) = \{|\ln z| : z \in K(a,R)\} = \ln(a+R)$.

Outline. If $z = \rho e^{i\varphi} \in K(a,R)$, then $(\rho - a\cos\varphi)^2 = R^2 - a^2\sin^2\varphi$. Set $A = \ln\rho$ and $\lambda = \rho - a\cos\varphi$. Check $\lambda\rho' = -a\rho\sin\varphi$, $A' = -a\frac{\sin\varphi}{\lambda}$, $\lambda' = \rho' + a\sin\varphi$ and

$$A'' = -a\frac{\lambda\cos\varphi - \lambda'\sin\varphi}{\lambda^2} = a\frac{\rho'\sin\varphi - \rho\cos\varphi + a}{\lambda^2} = -aR^2\frac{\cos\varphi}{\lambda^3}.$$

Since $A'' \le 0$, $\varphi \in I_0 = [0,\varphi_0]$, and $A'(0) = 0$, A' is non-positive on I_0 and A is decreasing. Compute $R(\varphi) = 2[1 + A'^2 + AA'']$.

N. Mutavdzic suggested the following approach. Set $f(r,\theta) = \arctan(re^{i\theta})$ and

$$M(r,\theta) = \arctan^2\frac{2r\sin\theta}{1-r^2} + \frac{1}{4}\log^2\frac{1+2r\cos\theta+r^2}{1-2r\cos\theta+r^2}, 0<r<1, 0\le\theta\le\frac{\pi}{2}.$$

$$\frac{\partial M}{\partial\theta}(r,\theta) = \frac{2}{r}\frac{2(1-r^2)\cos\theta\arctan\frac{2r\sin\theta}{1-r^2} - (1+r^2)\sin\theta\log\frac{1+2r\cos\theta+r^2}{1-2r\cos\theta+r^2}}{4\sin^2\theta + (r-\frac{1}{r})^2}.$$

Dokazac1emo da je $\frac{\partial M}{\partial\theta}(r,\theta) \le 0, 0<r<1, 0\le\theta\le\frac{\pi}{2}$, odakle c1e slediti da je $\max\limits_{0\le\theta\le\frac{\pi}{2}} M(r,\theta) = M(r,0)$.

Ako je $P(r,\theta) = 2(1-r^2)\cos\theta \arctan\frac{2r\sin\theta}{1-r^2} - (1+r^2)\sin\theta\log\frac{1+2r\cos\theta+r^2}{1-2r\cos\theta+r^2}$ imac1emo da je sgn$\frac{\partial M}{\partial\theta}(r,\theta) =$ sgn$P(r,\theta)$. Posto je $P(0,\theta) = 0, 0 \le \theta \le \frac{\pi}{2}$ dovoljno c1e biti da pokazhemo da je $\frac{\partial P}{\partial r}(r,\theta) \le 0, 0 < r < 1, 0 \le \theta \le \frac{\pi}{2}$. I konachno, izrachunavshi

$$\frac{\partial P}{\partial r}(r,\theta) = -2r(2\cos\theta\arctan\frac{2r\sin\theta}{1-r^2} + \sin\theta\log\frac{1+2r\cos\theta+r^2}{1-2r\cos\theta+r^2}), 0<r<1, 0\le\theta\le\frac{\pi}{2}$$

dobili smo trazhenu nejednakost.

Proposition 3.1 ([74]). Let F be holomorphic map from $\mathbb{U}$ into $\mathbb{S}_0$ with $F(0) = 0$. Then $M_F(r) \le \frac{2}{\pi}\ln\frac{1+r}{1-r}$, $0 < r < 1$.

3.2.1. *Rado-Kneser-Choquet.*

Example 8. Let $h(z) = x + i(x^2 - y^2)$. Then $J_h = -2y$, $h(i) = -1$, $h(H) = D = \{(x,y) : y < x^2\}$ and $h(z) = h(\overline{z})$; Since $h(z_1) = h(z_2)$ implies $x_1 = x_2$ and $y_1 = \pm y_2$, h is univalent on the upper half plane. Consider triangle Δ_1 with vertices $A = 0$, $B = 2$ and $C = 1 + i$ and
Δ_2 with vertices $A = 0$, $B = 2$ and $D = 1 - i/2$;
quadrilateral $\Delta_3 = \Delta_1 \cup \Delta_2$, $\overline{D} = 1 + i/2$ and let quadrilateral Δ_4 consist of points $A = 0$,$\overline{D}$, $B = 2$ and C.
Verify that $\mathbf{L_0} = h(\partial\Delta_3) = h(\partial\Delta_4)$, $h(\Delta_3) \ne int(h(\partial\Delta_3))$ and that $h(\partial\Delta_3)$ is not a convex set.
Consider a conformal mapping φ of the unit disk onto Δ_3 and $\check{h} = h\circ\varphi$. Check that a $\check{h}$ is harmonic on $\mathbb{D}$, a homeomorphism of $\mathbb{T}$ onto $\mathbf{L_0}$, but $\check{h}$ is not $1-1$ onto $\mathbb{U}$.

Choquet showed For every Jordan domain D which is not convex, there exists a homeomorphism $\phi : \mathbb{T} \to \partial D$ such that $h = P[\phi]$ is not a homeomorphism in $\mathbb{D}$.

Theorem 3.3. *Assume that $\Omega \subset \mathbb{R}^2$ is a convex domain with smooth boundary $\partial\Omega$. Given any homeomorphism $\varphi : S^1 \to \partial\Omega$, there exists a unique harmonic map $h : \mathbb{U} \to \Omega$ such that $h = \varphi$ on S^1 and h is a diffeomorphism.*

Proof. Let $h = (u, v)$. It suffices to show that $\det J(h) \ne 0$. Suppose that $\det J(h)$ is zero at z_0, that is

$$\begin{vmatrix} u'_x(z_0) & u'_y(z_0) \\ v'_x(z_0) & v'_y(z_0) \end{vmatrix} = 0\,.$$

Thus vectors $X = (u'_x(z_0), u'_y(z_0))$ and $Y = (v'_x(z_0), v'_y(z_0))$ are linearly dependent and therefore there exists $(\alpha,\beta) \ne (0,0)$ such that $\alpha X + \beta Y = 0$, that is $U'_x = 0$, $U'_y = 0$ at z_0, where $U = \alpha u + \beta v$. Let $L = \{z : U(z) = U(z_0)\}$. Since U is a real harmonic function, there is an analytic function F such that $U = \mathrm{Re}F$ in $\mathbb{U}$ and that $F'(z_0) = 0$. By the maximum principle for harmonic functions, no pair of the arcs of L emanating from z_0 can rejoin elsewhere in $\overline{\mathbb{U}}$. Since a neighborhood of z_0 consists of at least four arcs emanating from z_0, and each of these arcs must extend out to the boundary, which means that L must meet T in at least four distinct points (that is $L \cap \Delta$ contains at least 4 points). On the other hand, h maps L into the line, which meets $\partial\Omega$ in exactly two points because of the assumption that Ω is convex. It follows that h maps at least four points on T onto two points in $\partial\Omega$, contradicting the hypothesis (φ being $1-1$). This contradiction proves that the Jacobian cannot vanish in $\mathbb{U}$, so h is locally univalent. An application of the argument principle(see Proposition 3.3 below) completes the proof. $\square$

Choquet's Proof. We now turn to Choquet's more analytic proof of the Rado-Kneser-Choquet theorem. He begins by observing, as had Kneser, that it is enough to establish the local univalence of f in $\mathbb{U}$. Still with Kneser, he argues that the vanishing of the Jacobian of $f = u + iv$ at some point z_0 in $\mathbb{U}$ would imply that some nondegenerate linear combination $\psi = au + bv$ has a critical point at z_0. Diverging from Kneser's proof (of which he was unaware), Choquet then appeals to the following lemma to reach a contradiction.

Since ψ is a real valued harmonic on $\mathbb{U}$, there is an analytic function F on $\mathbb{U}$ such that $\psi = F + \overline{F}$. Hence $\psi = \sum a_k z^k + \sum \overline{a_k z}^k$. Since $\psi_z = F_z$, $a_1 = F'(0) = \psi_z(0)$, we have the formula $2\pi\psi_z(0) = \int_{-\pi}^{\pi} e^{-it}\psi(e^{it})dt$ and therefore $-2\pi \mathrm{Im}\psi_z(0) = \int_0^{\pi} \chi(t)\sin t dt$, where $\chi(t) = \psi(e^{it}) - \psi(e^{-it})$.

Since $2\pi\overline{a_1} = \int_{-\pi}^{\pi} e^{-it}\psi(e^{-it})dt$, we have $4\pi i \mathrm{Im} a_1 = \int_{-\pi}^{\pi} \chi(t)e^{-it}dt$ and hence $-4\pi \mathrm{Im} a_1 = \int_{-\pi}^{\pi} \chi(t)\sin t dt$.

Let ψ be a real-valued function harmonic in $\mathbb{U}$ and continuous in $\mathbb{U}$. If ψ is at most bivalent on $\mathbb{T}$, then ψ has no critical points in $\mathbb{U}$.

On the other hand, the bivalence hypothesis says that ψ can have only one local maximum and one local minimum on the circle, and that ψ is monotonic on each of the arcs joining those points. After a rotation of coordinates, which does not change $|\psi_z(0)|$, we may conclude from the bivalence hypothesis that ψ increases from a minimum at some point $e^{-i\alpha}$ to a maximum at $e^{i\alpha}$, where $\alpha \in [0, \pi)$.

Set $m_0 = \underline{\psi}(-\alpha)$ and $M_0 = \underline{\psi}(\alpha)$. One can check that $\underline{\psi}$ is decreasing on $[-\pi, -\alpha] \cup [\alpha, \pi]$ and increasing on $[-\alpha, \alpha]$. If $t \in [\alpha, \pi]$, then $-t \in [-\pi, -\alpha]$, and $\underline{\psi}(t) > \underline{\psi}(\pi) = \underline{\psi}(-\pi) > \underline{\psi}(-t)$. Thus, ψ is strictly increasing as e^{it} moves in either direction from $e^{-i\alpha}$ to $e^{i\alpha}$, and so $\chi(t) = \psi(e^{it}) - \psi(e^{-it})$ is odd and $\chi(t) > 0$, $0 < t < \pi$. Consequently, $-2\pi \mathrm{Im}\psi_z(0) = \int_0^{\pi} \chi(t)\sin t dt > 0$ proving that $\psi_z(0) \neq 0$.

To say that ψ is at most bivalent on $\mathbb{T}$ means that ψ takes any given value at most twice on $\mathbb{T}$. Let us first observe that the function $\psi = au + bv$ just constructed does have this property if G is strictly convex; i.e., if the boundary Γ contains no line segments. Indeed, no line $au + bv = c$ can then intersect Γ in more than two points, and each of these points has a unique preimage under f, since f is assumed to map T in one-to-one fashion onto Γ. Thus, $\psi(z) = c$ for at most two points z on T. The lemma then says that ψ can have no critical points in D, which is a contradiction. In other words, the Rado- Kneser-Choquet theorem follows from the lemma under the extra assumption that G is strictly convex. As we shall see, the strict convexity is inessential and, in fact, the argument can be generalized to establish a much stronger form of the Rado-Kneser-Choquet theorem.

Theorem 3.4 (Summa of degrees of mappings)**.** *Let γ be a planar curve Jordan curve which enclosed domain G Suppose that*
a. $\varphi : \overline{G} \to \mathbb{C}$ *is continuous and C^1 on G.*
b. $\varphi^{-1}(0) = \{a_1, a_2, ..., a_m\}$ *is a finite subset of$G$*
c. $\mathcal{K}_\nu = \mathcal{K}_\nu(r)$ *positively oriented circles with center a_ν, radius r, such that closed circles $\overline{B_\nu}$, which are enclosed by $\mathcal{K}_\nu$, $\nu = 1, 2, ..., m$, are disjoint; and let $\gamma_\nu = \varphi \circ \mathcal{K}_\nu$, $\nu = 1, 2, ..., m$ and in particular $\gamma_0 = \varphi \circ \gamma$.*
Then

$$Ind_{\gamma_0}(0) = \sum_{k=1}^{m} Ind_{\gamma_k}(0)\,. \tag{1}$$

Proposition 3.2. If f is C^1 on $\mathbb{C}$ and $J_f(z) > 0$ for every $z \in \mathbb{C}$ and $f(z) = z + 0(1)$, when $z \to \infty$, then f homeomorphizm S^2 onto itself.

Neka je $w \in \mathbb{C}$ fiksirana tačka i $g = f - w$. Proveriti da je $VarArg_{K_r}(f - w) = 1$ za dovoljno veliko r. Otuda je $Ind_{\Gamma_r}(w) = 1$, gde je $\Gamma_r = f \circ K_r$.

Neka je k_r pozitivna kružnica sa središtem u a i $f(a) = w$, tada je $Ind_{\gamma_r}(w) = 1$, gde je $\gamma_r = f \circ k_r$ za dovoljno malo r. Otuda, na osnovu Teoreme 3.4 postoji tačno jedno a tako da je $f(a) = w$.

Proposition 3.3. Let γ and γ_0 be two planar Jordan curves which enclosed domain G and G_0 respectively. Suppose that $f : \overline{G} \to C$ is continuous and C^1 on G. If the restriction f_* of f is homeomorphism of $tr(\gamma)$ onto $tr(\gamma_0)$. and the Jacobian of J_f of f has no zeroes on G, then f is a homeomorphism of $\overline{G}$ onto $\overline{G_0}$.

Proof. Take $w_0 \in G_0$. Using homotopy, one can show that $w_0 \in f(\overline{G})$. Since f is locally injective and f_* is homeomorphism of $tr(\gamma)$ onto $tr(\gamma_1)$, $f^{-1}(w_0)$ has no point of accumulation in G or $tr(\gamma_0)$. Thus $f^{-1}(w_0)$ is a finite set, say $z_1, z_2, ..., z_m$. By Theorem 3.4, $I := Ind_{\gamma_0}(w_0) = \sum_{k=1}^{m} Ind_{\gamma_k}(w_0)$. If $J_f > 0$, then $I = m$ and if $J_f < 0$, then $I = -m$. Since $I = \pm 1$, we conclude that $m = 1$. Hence if f_* preserves orientation then $J_f > 0$ on G. □

(A-1) If ϕ conformal mapping of a planar domain D onto $\mathbb{U}$, we define the ϕ-hyperbolic density on D by $\mathrm{Hyp}_{\mathbb{U}}(\phi z)|\phi'(z)| = \lambda_{\phi,D}(z)$. If another ϕ_1 conformal mapping of the domain D onto $\mathbb{U}$, set $w = \phi(z)$, $w_1 = \phi_1(z)$, $\omega = \phi_1 \circ \phi^{-1}$ and $w_1 = \omega(w)$. Then $\phi_1 = \omega \circ \phi$, and by the composition rule $\phi_1'(z) = \omega'(w)\phi'(z)$. Since $\omega \in \mathrm{Aut}(\mathbb{U})$, $1 - |w_1|^2 = |\omega'(w)|(1 - |w|^2)$ and hence $1 - |w_1|^2 = |\phi_1'(z)/\phi'(z)|(1 - |w|^2)$.

Therefore $\lambda_{\phi,D} = \lambda_{\phi_1,D}$ and the definition of the hyperbolic density is independent of conformal maps from D onto $\mathbb{U}$; and we write $\mathrm{Hyp}_D(z)$ for the hyperbolic density on D at z.

Exercise 3. (I-1) If G and D are simply connected domains different then $\mathbb{C}$ and f conformal mapping of D onto G, then $\mathrm{Hyp}_G(fz)|f'(z)| = \mathrm{Hyp}_D(z)$.
Outline. Let ψ be conformal mappings of the domain D onto $\mathbb{U}$, $g = f^{-1}$ and $\psi_1 = \psi \circ g$; set also $z_1 = f(z)$. Then $\lambda_D(z) = \lambda_0(\psi z)|\psi' z|$ and $\lambda_G(z) = \lambda_0(\psi_1 z_1)|\psi' z_1|$.

Exercise 4. Suppose that D is a simply connected domain different then $\mathbb{C}$ and ω holomorphic map from D into self with $\omega(z_0) = z_0$ for some $z_0 \in D$. Then $|\omega'(z_0)| \leq 1$.

If $G_1 = kG$, then $H(z) = kz$ maps G onto G_1 and $H^{-1}(w) = w/k$. Hence $\mathrm{Hyp}_{G_1}(w) = \frac{1}{k}\mathrm{Hyp}_G(w/k)$.

Example 9. 1. If $\Pi = \{w : \mathrm{Re} w > 0\}$, then $B_0(w) = \frac{1-w}{1+w}$ maps Π on $\mathbb{U}$. Since $1 - |B_0(w)|^2 = \frac{4u}{|1+w|^2}$, where $u = \mathrm{Re} w$, $B_0'(w) = 2(1 + w)^{-2}$, and $\mathrm{Hyp}_\Pi(w) = \lambda_0(B_0(w))|B_0'(w)|$, we find

$$\mathrm{Hyp}_\Pi(w) = \frac{1}{\mathrm{Re} w}.$$

2. Since $e = \exp$ maps $S = \{y : |y| < \pi/2\}$ onto Π, we have $\lambda_S(z) = \mathrm{Hyp}_\Pi(e^z)|e^z| = \frac{1}{\cos y}$.

3. Let λ_0 be a hyperbolic density on $\mathbb{S}_0$. Then

$$\lambda_0(w) = \mathrm{Hyp}_{\mathbb{S}_0}(w) = \frac{\pi}{2} \frac{1}{\cos(\frac{\pi}{2} u)}. \tag{3.1}$$

4. $\lambda_0(iy_1, iy_2) = \frac{\pi}{2}|y_2 - y_1|$, $y_1, y_2 \in \mathbb{R}$.

5. If $a, b \in \mathbb{R}$, $a < b$, the linear map L defined by $L(w) = \frac{2w-(a+b)}{b-a}$, maps $\mathbb{S}(a,b)$ onto $\mathbb{S}_0$ and $\rho(w) = \rho_0(Lw)\frac{2}{b-a}$. Hence

$$\rho(w) = \mathrm{Hyp}_{\mathbb{S}(a,b)}(w) = \frac{\pi}{(b-a)} \frac{1}{\cos\left(\frac{\pi}{2}[(2u-(a+b))/(b-a)]\right)}. \tag{3.2}$$

Example 10. Check $\arctan(\frac{2r}{1-r^2} = 2\arctan(r)$, $0 < r < 1$.
Outline. If $tg\beta = r$, then $tg(2\beta) = \frac{2tg\beta}{1-(tg\beta)^2}$.

Example 11. For $w_1, w_2 \in \mathbb{S}_0$, $\rho_0(u_1, u_2) \le \rho_0(w_1, w_2)$.

A plane region D whose complement has at least two points we call a hyperbolic plane domain. On a hyperbolic plane domain there exists a unique maximal ultrahyperbolic metric, and this metric has constant curvature -1.

Using holomorphic covering $\pi : \mathbb{U} \to D$, one can define the pseudo-hyperbolic and the hyperbolic metric on D.

3.3. **hyperbolic domains.** Now we outline how we can use powerful tools which yield the uniformization theorem to get hyperbolic version of Ahlfors-Schwarz lemma.

XX Let W and W^* be surfaces and $f : W^* \to W$ a continuous surjective map such that for every $p \in W$, there exists an open neighborhood V of p, such that is a union of disjoint open sets in W^* and every component of $p^{-1}(V)$ is in one-to-one correspondence with V. When this is so The map f is called the covering map and the pair (W^*, f) is called a covering surface of W. A deck transformation or automorphism of a cover $f : W^* \to W$ is a homeomorphism $A : W^* \to W^*$ such that $f \circ A = f$. The set of all deck transformations of A forms a group under composition, the deck transformation group $Aut(f)$. Deck transformations are also called covering transformations.

In particular, if W and W^* are Riemann surfaces and $f : W^* \to W$ holomorphic, we call f the holomorphic covering map. If W^* is simply connected, we call the pair (W^*, f) a universal covering.

The uniformization theorem says that every simply connected Riemann surface is conformally equivalent to a disk, the complex plane, or the Riemann sphere. In particular it implies that every Riemann surface admits a Riemannian metric of constant curvature. Every Riemann surface is the quotient of the deck transformation group (a free, proper and holomorphic action of a discrete group on its universal covering) and this universal covering is holomorphically isomorphic (one also says: "conformally equivalent" or "biholomorphic") to the Riemann sphere,the complex plane and the unit disk in the complex plane. If the universal covering of a Riemann surface S is the unit disk we say that S is hyperbolic. XX Using holomorphic covering $\pi : \mathbb{U} \to S$, one can define the pseudo-hyperbolic and the hyperbolic metric on S. In particular, if $S = G$ is hyperbolic planar domain we can use

(I-2) $\mathrm{Hyp}_G(\pi z)|\pi'(z)| = \mathrm{Hyp}_{\mathbb{D}}(z)$ and

(I-3) If G and D are hyperbolic domains and f conformal mapping of D onto G, then $\mathrm{Hyp}_G(fz)|f'(z)| = \mathrm{Hyp}_D(z)$.

XX

Proposition 3.4 (Schwarz lemma 1- planar hyperbolic domains, Ahlfors-Schwarz (hyperbolic version))**.** (a) If G and D are conformally isomorphic to $\mathbb{U}$ and $f \in$

$\mathrm{Hol}(G, D)$, then

$$\delta_D(fz, fz') \le \delta_G(z, z'), \quad z, z' \in G.$$

(b) The result holds more generally: if G and D are hyperbolic domains and $f \in \mathrm{Hol}(G, D)$, then

$$\mathrm{Hyp}_D(fz, fz') \le \mathrm{Hyp}_G(z, z'), \quad z, z' \in G.$$

(c) If $z \in G$, $\mathbf{v} \in T_z\mathbb{C}$ and $\mathbf{v}^* = df_z(\mathbf{v})$, then

$$|\mathbf{v}^*|_{\mathrm{Hyp}} \le |\mathbf{v}|_{\mathrm{Hyp}}.$$

For a hyperbolic planar domain G the Carathéodory distance $C_G \le \lambda_G$ with equality if and only if G is a simply connected domain.

3.4. **Khavinson extremal problem.** We follow [52], arXiv:1805.02979v1. For a hyperbolic plane domain D, we denote by ρ_D(or λ_D) the hyperbolic density and by abusing notation the hyperbolic metric occasionally.

Lemma 2. If G and D are simply connected domains different from $\mathbb{C}$ and $\omega \in \mathrm{Hol}(G, D)$, then $\rho_D(\omega z)|\omega'(z)| \le \rho_G(z)$, $z \in G$ and

$$\rho_D(\omega z, \omega z') \le \rho_G(z, z'), \quad z, z' \in G.$$

We denote the right half plane by Π.

Proposition 3.5. If ω is holomorphic from Π into itself, then

$$|\omega'(z)| \le \frac{\mathrm{Re}\omega(z)}{\mathrm{Re} z}.$$

If in addition ω maps $\mathbb{R}^+$ into itself, then $|\omega'(1)| \le \mathrm{Re}\omega(1) = \omega(1)$ and therefore $\omega'(1) \le \omega(1)$.

Definition 3.5. By $\mathbb{C}$ we denote the complex plane by $\mathbb{U}$ the unit disk and by $\mathbb{T}$ the unit circle. For $z_1 \in \mathbb{U}$, define

$$T_{z_1}(z) = \frac{z - z_1}{1 - \overline{z_1} z},$$

$\varphi_{z_1} = -T_{z_1}$.

d1) Throughout this paper by $\mathbb{S}(a, b)$ we denote the set $(a, b) \times \mathbb{R}$, $-\infty \le a < b \le \infty$, and in particular we write $\mathbb{S}_0$ for $\mathbb{S}(-1, 1)$. Note that $\mathbb{S}(a, b)$ is a strip if $-\infty < a < b < \infty$ and $\mathbb{S}(a, +\infty)$ is a half-plane if a is a real number, and $\mathbb{S}(-\infty, +\infty) = \mathbb{C}$. By λ_0 and ρ_0 we denote hyperbolic metrics on $\mathbb{U}$ and $\mathbb{S}_0$ respectively.

d2) Set $I_0 = (-1, 1)$, and for $a \in I_0$ define

$$s = s(a) = \tan(\frac{\pi}{4}(a + 1)), \quad e = e(a) = \cot\big(\frac{\pi}{4}(a + 1)\big), \quad \text{and}$$

$$X(r) = X^+(r, a) = \frac{4}{\pi}\arctan(s\frac{1 + r}{1 - r}) - 1, \quad X^-(r, a) = 1 - \frac{4}{\pi}\arctan\left(e\frac{1 + r}{1 - r}\right), z \in \mathbb{U}.$$

Further it is convenient to introduce the functions A, B, A_s and B_s by $A(r) = (1 + r)(1 - r)^{-1}, B(r) = (1 - r)(1 + r)^{-1}$, $A_s(r) = sA(r), B_s(r) = sB(r)$, and $Y(r) = X^+(|z|, |a|)$.

d3) Set $c = (a + 1)/2$, $\overline{c} = 2\pi c$, $\alpha = \alpha(c) = \alpha(a) = \overline{c}/2 = (a + 1)\pi/2$.

It is convenient to write $f_y(x) = f(x, y)$.

(A0) It is straightforward to check

$$X_a^-(r) = \frac{4}{\pi}\arctan(B_s(r)) - 1, \quad X_a^-(r) \le a \le X_a^+(r),$$

$X^+(r,a)$ (respectively $X^-(r,a)$) is increasing (respectively decreasing) in both variables r and a, $X_1^+ = 1$ and $X_{-1}^- = -1$.

Suppose that f is harmonic map from $\mathbb{U}$ into $I_0 = (-1,1)$ with $f(0) = a$. Using a version of Schwarz lemma [48], we will show

$$\rho_0(fz,a) = |\ln \frac{s(fz)}{s(a)}| \le \ln \frac{1+r}{1-r}, z \in \mathbb{U}. \tag{3.3}$$

This inequality is equivalent to $X^-(|z|,a) \le f(z) \le X(|z|) = X^+(|z|,a), z \in \mathbb{U}$.

Theorem 4. If $u_1, u_2 \in (-1,1)$, then

$$\rho_0(u_1,u_2) = |\ln \frac{s(u_2)}{s(u_1)}|. \tag{3.4}$$

Let h be a real valued harmonic map from $\mathbb{U}$ into $I_0 = (-1,1)$ with $h(0) = a$, $a \in I_0$. Then

$$X^-(|z|,a) \le h(z) \le X(|z|) = X^+(|z|,a), z \in \mathbb{U}, \tag{3.5}$$

$$\text{and} \quad |(dh)_0| \le X'(0) = \frac{4}{\pi}\sin\alpha. \tag{3.6}$$

If $a = 0$, then $a_1 = \tan\frac{\pi}{4} = 1$ and $X(|z|,0) = \frac{4}{\pi}\arctan|z|$. Hence we get classical Schwarz lemma for harmonic maps which states $|h(z)| \le X(|z|) = X(|z|,0) = \frac{4}{\pi}\arctan|z|$.

Definition 3.6. d1) For $a \in (-1,1)$, let Har^a denote the family of all real valued harmonics maps f from $\mathbb{U}$ into $(-1,1)$ with $f(0) = a$.

d2) For $a \in \mathbb{U}$ and $b \in (-1,1)$, set $L(a,b) = L(a,b) = \sup|(du)_a|$, where the supremum is taken over all real valued harmonics maps u from $\mathbb{U}$ into $(-1,1)$ with $u(a) = b$.

d3) For $a \in \mathbb{U}$ and $\ell \in T_a\mathbb{C}$ a unit vector, set $L(a) = \sup|(du)_a|$ and $L(a,\ell) = \sup|(du)_a(\ell)|$, where the supremum is taken over all real valued harmonics maps from $\mathbb{U}$ into $(-1,1)$.

Now, we can restate and strength the part of Theorem 4:

Theorem 5. If $a \in (-1,1)$ and $h \in \mathrm{Har}^a$, then
(3.7)

$$(i) \quad h(z) \le X(|z|), \quad (ii) \quad |(dh)_0| \le X'(0) = \frac{4}{\pi}\sin\alpha \quad \text{and} \quad (iii) \quad L(0,a) = \frac{4}{\pi}\sin\alpha(a).$$

Proof. We need only to prove (iii). There is a conformal mapping f of $\mathbb{U}$ onto $\mathbb{S}_0$ with $f(0) = a$ and $f'(0) > 0$; then for harmonic function $u_0 = \mathrm{Re} f$ the equality holds in (iii). □

Theorem 6. Let h be a real valued harmonics map from $\mathbb{U}$ into $(-1,1)$ with $f(a) = b$, $a \in \mathbb{U}$. Then

$$h(z) \le \frac{4}{\pi}\arctan\left(\frac{1+|\varphi_a(z)|}{1-|\varphi_a(z)|}\tan\frac{\alpha(|b|)}{2}\right) - 1, \tag{3.8}$$

$$|(dh)_a| \le \frac{4}{\pi}\frac{\sin\alpha(|b|)}{1-|a|^2}. \tag{3.9}$$

Proof. Set $w = \varphi_a(z)$. Apply Theorem 5 on $h^a = h \circ \varphi_a$, we find $h^a(z) \le X(|z|)$. Hence $h(w) = h^a(z) \le X(|\varphi_a(w)|)$. Since we can identify $(d\varphi_a)_0$ with $1 - |a|^2$, using $(dh^a)_0 = (dh)_a \circ (d\varphi_a)_0$ and Theorem 5 we prove (3.9). □

Further set

$$A_0(z) = \frac{1+z}{1-z}, \quad \text{and let} \quad \phi = i\frac{2}{\pi}\ln A_0;$$

that is $\phi = \phi_0 \circ A_0$, where $\phi_0 = i\dfrac{2}{\pi}\ln$. Let $\hat{\phi}$ be defined by $\hat{\phi}(z) = -\phi(iz)$. Note that ϕ maps $I_0 = (-1,1)$ onto y-axis and $\hat{\phi}$ maps I_0 onto itself.

If $\hat{u} = \mathrm{Re}\hat{\phi}$, then

$$\hat{u} = \frac{2}{\pi}\arg\left(\frac{1+iz}{1-iz}\right) \tag{3.10}$$

and $\hat{u}$ maps $I_0 = (-1,1)$ onto itself.

Let $a \in (0,1)$ and $\ell \in T_a\mathbb{C}$. There is a conformal mapping $f = f_\ell$ of $\mathbb{U}$ onto $\mathbb{S}_0$ with $f(a) = 0$ and $f'(a)\ell > 0$. We will show that $u = u_\ell = \mathrm{Re} f_\ell$ is extremal. In particular, there is a conformal mapping f of $\mathbb{U}$ onto $\mathbb{S}_0$ with $f(a) = 0$ and $f'(a) > 0$; set $u_0 = \mathrm{Re} f$.

Theorem 7. If $a \in (-1,1)$ and $\ell \in T_a\mathbb{C}$, then

(1) $L(a) = (u_0)'_r(a) = \frac{4}{\pi}(1-|a|^2)^{-1}$ and

(2) $L(a,\ell) = L(a) = (du_\ell)_a(\ell) = \frac{4}{\pi}(1-|a|^2)^{-1}$.

This yields solution of D. Khavinson extremal problem for harmonic functions in planar case, cf. [34, 37, 8].

Proof. (1) By hypothesis $\rho_0(f(a))|f'(a)| = 2(1-|a|^2)^{-1}$, $\rho_0(f(a)) = \rho_0(0) = \frac{\pi}{2}$, $(u_0)'_r(a) = f'(a)$ and $|(du_0)_a| = |\nabla u_0(0)| = \frac{4}{\pi}(1-|a|^2)^{-1}$.

(2) Recall there is a conformal mapping $f = f_\ell$ of $\mathbb{U}$ onto $\mathbb{S}_0$ with $f(a) = 0$ and $f'(a)\ell > 0$. If $u = u_\ell = \mathrm{Re} f_\ell$, then $(du)_a(\ell) = \mathrm{Re}\big(f'(a)\ell\big)$. We leave the interested reader to fill details. □

Theorem 8. Let h be a real valued harmonics map from $\mathbb{U}$ into $(-1,1)$ with $f(a) = b$, $a \in \mathbb{U}$. Then

$$(i) \quad |(dh)_a| \le \frac{4}{\pi}\frac{\sin\alpha(|b|)}{1-|a|^2}, \quad (ii) \quad L(a,b) = \frac{4}{\pi}\frac{\sin\alpha(|b|)}{1-|a|^2}. \tag{3.11}$$

Proof. There is a conformal mapping of $\mathbb{U}$ onto $\mathbb{S}_0$ with $f(a) = b$. We leave the interested reader to show that $u_0 = \mathrm{Re} f$ is extremal for (i) and therefore (ii) holds. □

3.5. **Schwarz lemma at the boundary for holomorphic functions.** It this subsection we discuss some known results related to the subject.

3.5.1. *Jack's lemma.* In connection with Jack's lemma we state:
(T1) Let f be analytic function on the unit disk. Then for given $r \in (0,1)$, $|f|$ attains maximum at a point $z_0 \in T_r$. Then $z_0 f'(z_0) = k w_0$, where $w_0 = f(z_0)$. Using homothety, rotation and translation we can reduce it to the following.

(T2) Let $B = B(a_0; a_0)$, $a_0 > 0$, f be analytic function on $\overline{B}$, $f(B) \subset B$ and $f(0) = 0$. Then $f'(0) = k$, where $k > 0$. Contrary, if $f'(0) = ke^{i\alpha}$, where $0 < \alpha < 2\pi$, then by the little o technique we can show that there is a small arc L on the boundary of B centered at the origin such that $f(L)$ is out of $\overline{B}$.

Q1. If D is domain and f is analytic function on $\overline{D}$, $f(D) \subset D$ and there is $z_0 \in \partial D$ such that $f(z_0) = z_0$ whether $f'(z_0) > 0$. It seems that using similar approach as the above we can prove that answer to Q1 is positive if ∂D is smooth at z_0.

For $r > 0$, set $M_f(r) = \max\{|f(z)| : |z| = r\}$.

Proposition 3.6. a) Let $f : \mathbb{U} \to \mathbb{U}$. Assume that (H0): there is a point $b \in \mathbb{T}$ so that f extends continuously to b, $|f(b)| = 1$ (say that $f(b) = c$), and and f is $\mathbb{R}$-differentiable at b.

b) Further assume that there is a function A such that $A : [0, 1] \to [0, 1]$, $A'(1)$ exists and $M_f(r) \leq A(r)$.

Then $|\Lambda_f(b)| \geq A'(1)$.

Proposition 3.7. Under the above hypothesis, if there exists $f'(b)$, then
(i) $|f'(b)| \geq A'(1)$.

In [57], R. Osserman offered the following boundary refinement of the classical Schwarz lemma. It is very much in the spirit of the sort of result that we wish to consider in the future.

Theorem 3.7. *Let $f : \mathbb{U} \to \mathbb{U}$ be holomorphic. Assume that $f(0) = 0$.*
a1) *Further assume that there is a point $b \in \mathbb{T}$ so that f extends continuously to b, $|f(b)| = 1$(say that $f(b) = c$), and $f'(b)$ exists. Then* (i) $|f'(b)| \geq \frac{2}{1+|f'(0)|}$.
a2) *If f has a zero of order p at 0, then* (ii) $|f'(b)| \geq p$.

Proof. Let $f : \mathbb{U} \to \mathbb{U}$ be holomorphic and satisfy $f(0) = 0$. Then $|f(\zeta)| \leq |\zeta| \frac{|\zeta|+|f(0)|}{1+|f(0)||\zeta|}$ for $|\zeta| < 1$.
Set $r = |z|$ and $k = |f'(0)|$. Then $1 - |f(z)| \geq 1 - r\frac{r+k}{1+rk} = \frac{1-r^2}{1+rk}$ and therefore $\frac{1-|f(z)|}{1-r} \geq \frac{1+r}{1+rk}$. Hence $|f'(b)| \geq \frac{2}{1+|f'(0)|}$. Without loss of generality we reduce the proof to the case $b = 1$ and $f(1) = 1$. By Schwarz lemma $|f(z)| \leq |z|$. Hence $|1 - f(x)| \geq |1 - x|$.
a2) $M_f(r) \leq A(r) := r^p$. Hence, since $A'(1) = p$, we have (ii). ☐

We also outline the following proof of (i). Set $k = |f'(0)|$, $g(z) = f(z)/z$ and $F = T_k \circ g$. Then $g = T_{-k} \circ F$, $M_{T_{-k}}(r) \leq T_{-k}(r) = \frac{r+k}{1+rk}$ and therefore $M_f(r) \leq A(r) := r\frac{r+k}{1+rk}$. Hence, since $A'(1) = \frac{2}{1+k}$, we have (i).

Theorem 3.8. *Let $f : \mathbb{U} \to \mathbb{U}$ be holomorphic function. Suppose that there is an extension of f at $b \in T$ such that $|f(b)| = 1$ and there exists $f'(b)$. Then*

$$|f'(b)| \geq \frac{2(1 - |f(0)|)^2}{1 - |f(0)|^2 + |f'(0)|} \tag{3.12}$$

Now we state a version of Lowner and Velling result.

Proposition 3.8. Let $f : \mathbb{U} \to \mathbb{U}$ be holomorphic and $f(0) = 0$. Let $S \subset \mathbb{T}$ be a nontrivial arc, and suppose that f extends continuously to S. Further assume that $f(S)$ lies in $\mathbb{T}$. Let s denote the length of S and σ the length of $f(S)$ (which is also necessarily an arc, since it is a connected subset of the circle). Then

$$\sigma \geq s\frac{2}{1 + |f'(0)|}.$$

Proof: By Schwarz reflection, we may take it that f is analytic on the interior of the arc S. Hence it certainly satisfies the hypotheses of the first lemma at each point of the interior of S. The conclusion of that lemma then holds, and integration yields the desired result.

Theorem 3.9. *Let $f : \mathbb{U} \to \mathbb{U}$ be holomorphic. Further assume that there is a point $b \in \mathbb{T}$ so that f extends continuously to b, $|f(b)| = 1$ (say that $f(b) = c$), and $f'(b)$ exists. Then*

$$|f'(b)| \geq \frac{2(1-|f(0)|)^2}{1-|f(0)|^2+|f'(0)|}\,.$$

Suppose f is an analytic map of $\mathbb{U}$ into itself. If $|b| < 1$, we say b is a fixed point of f if $f(b) = b$. If $|b| = 1$, we say b is a fixed point of f if $lim_{r\to 1-} f(rb) = b$. Julia-Caratheordory Theorem implies If b is a fixed point of f with $|b| = 1$, then $lim_{r\to 1-} f'(rb)$ exists (call it $f'(b)$) and $0 < |f'(b)| \leq \infty$.

Perhaps, we can restate the hypothesis. The following boundary version of the Schwarz lemma was proved in 1938 by Unkelbach in [61] and then rediscovered and partially improved by Osserman [59] in 2000.

Theorem 3.10. *In addition to hypothesis of Theorem 3.7, suppose that $f = c_p z^p + o(z^p)$ if z tends 0. Then* (ii) $|f'(c)| \geq p + \frac{1-|c_p|}{1+|c_p|}$. *The equality in* (ii) *holds if and only if f is of the form $f = -z^p\varphi_a$ on $\mathbb{U}$ for some constant $a \in (-1;0]$.*

Outline: Set $c = c_p$, $k = |c_p|$, $g(z) = f(z)/z^p$ and $F = T_c \circ g$. Then $g = T_{-c} \circ F$, $M_{T_{-c}}(r) \leq T_{-k}(r) = \frac{r+k}{1+rk}$ and therefore $M_f(r) \leq A(r) := r^p \frac{r+k}{1+rk}$.

Hence, since $A'(r) = pr^{p-1}T_{-k}(r) + r^pT'_{-k}(r)$ and $T'_{-k}(r) = (1-k^2)(1+rk)^{-2}$, we have $A'(1) = p + \frac{1-k}{1+k}$, and therefore (iii.1).

The inequality (ii) is a particular case of a result due to Dubinin in (see [17]), who strengthened the inequality $|f'(c)| \geq 1$ by involving zeros of the function f.

Suppose (iii): Let $f(z) = b + c_p(z-a)^p + c_{p+1}(z-a)^{p+1} + ...$, $c_p > 0$, $p \geq 1$ be a holomorphic function in the disc $\mathbb{U}$ satisfying $f(a) = b$, $|a| \leq 1$ and
(c) $|f(z) - \alpha| < \alpha$ for $|z| < 1$, where α is a positive real number and $1/2 < \alpha \leq 1$, and $f(z) - b$ has no zeros in $\mathbb{U}$ except $z = a$. Assume that, for some $c \in \mathbb{T}$, f has an angular limit $f(c)$ at c, $f(c) = 2\alpha$.

There are several papers by B. Örnek and Örnek-Akyel(see for example [56] and [55]) related to the subject. In [55], under the hypothesis (iii.2) the optimal lower estimate for $|f'(c)|$ are obtained, and the following functions are used. Let $z_1, z_2, .., z_n$ be zeros of the function $f(z) - b$ in $\mathbb{U}$ that are different from $z = a$. Set

$$B(z) = z^p \prod_{k=1}^{n} \frac{z-z_k}{1-\overline{z_k}z},$$

$\phi = \varphi_d$, where $d = f(0)$, $\Upsilon = \phi/B$ and $\kappa = \varphi_e$, where $e = \Upsilon(0)$. Set $B_0(z) = z^p$, $p = \phi/B_0$ and

$$\Theta(z) = \frac{\ln p(z) - \ln p(0)}{\ln p(z) + \ln p(0)}.$$

Set $F = F_\alpha = f - \alpha$. Then the hypothesis (c) can be rewritten as (c1): F maps the unit disc into the disc of radius α and $F(c) = \alpha$. The auxiliary function Θ is a holomorphic in the unit disc $\mathbb{T}$, $|\Theta(z)| < 1, \Theta(0) = 0$ and $|\Theta(b)| = 1$ for $b \in \mathbb{T}$.

Apply

$$\frac{2}{1+|\Theta'(0)|} \leq |\Theta'(b)|.$$

Theorem 3.11 (Burns/Krantz [12]). *Let g be an analytic function of the unit disk $\mathbb{U}$ into self which satisfy* (i) $g(z) = z + O(1-z)^4$ *when z approaches* 1 *throughout $\mathbb{U}$. Then $g = Id$.*

Then $g = Id$. The result is the sharpest possible. Indeed, since $g(z) = z - (z-1)^3/10$ maps the unit disc to itself, this example shows that the exponent 4 in the theorem cannot be replaced by 3. The proof in fact shows that 4 can be replaced by $o((z-1)^3)$.

In [26], a new theory of regular functions over the skew field of Hamilton numbers (quaternions) and in the division algebra of Cayley numbers (octonions) has been recently introduced by Gentili and Struppa (Adv. Math. 216(2007) 279-301). For these functions, among several basic results, the analogue of the classical Schwarz'Lemma has been already obtained. In this paper, following an interesting approach adopted by Burns and Krantz in the holomorphic setting, they prove some boundary versions of the Schwarz Lemma and Cartans Uniqueness Theorem for regular functions. We are also able to extend to the case of regular functions most of the related "rigidity" results known for holomorphic functions.

4. Cartan theorem and FPT

The following is a result of Cartan, Kaup, Caratheodory, and Wu.

Theorem 4.1 (Theorem 3.3 [35]). *Let M be a hyperbolic manifold and o a point of M. Let $f : M \to M$ be a holomorphic such that $f(o) = o$. Then The eigenvalues of df_o have absolute value ≤ 1; If df_o is the identity linear transformation, then f is the identity transformation of M; If $|det f_o| = 1$, then f is a biholomorphic mapping.*

Recall: Let G be bounded connected open subset of complex Banach space, $p \in G$ and $v \in T_pG$. We define $k_G(p,v) = \inf\{|h|$, where infimum is taking over all $h \in T_0\mathbb{C}$ for which there exists a holomorphic function such that $f : \mathbb{U} \to G$ such that $f(0) = p$ and $df(h) = v$.

Recall one can prove

Theorem 4.2. *Suppose that G and G_1 are bounded connected open subset of complex Banach space and $f : G \to G_1$ is holomorphic. Then $k_G(fz, fz_1) \leq k_G(z, z_1)$ for all $z, z_1 \in G$.*

Theorem 4.3. *Suppose that G is bounded connected open subset of complex Banach space and $f : G \to G$ is holomorphic, $s_0 = dist(f(G), G^c)$, $d_0 = diam(G)$ and $q_0 = \frac{d_0}{d_0+s_0}$. Then $k_G(fz, fz_1) \leq q_0 k_G(z, z_1)$.*

Perhaps there are applications of this result in the Teichmüller theory.

One can prove for example

Theorem 4.4. *Suppose that G is bounded connected open subset of complex Banach space and $f : G \to G_*$ is holomorphic, $G_* \subset G$, $s_0 = dist(G_*, G^c)$, $d_0 = diam(G)$ and $q_0 = \frac{d_0}{d_0+s_0}$. Then $k_{G_*}(fz, fz_1) \leq q_0 k_{G_*}(z, z_1)$ for $z, z_1 \in G_*$.*

Hence we have

Theorem 4.5. *Let $D \subset C^n$ domain for which Carthéodory pseudo-distance is distance and $f : D \to D$ holomorphic mapping such that $\overline{f(D)}$ is a compact subset of D. Then f is contraction with respect to Carthéodory metric on D. In particular f has fixed points in D.*

There is the theory of holomorphic functions with domain and range contained in a complex Banach space. We then review some well-known fixed point theorems for holomorphic functions. Perhaps the most basic is the Earle-Hamilton fixed point theorem, which may be viewed as a holomorphic formulation of Banach's contraction mapping theorem.

A set S is said to lie strictly inside a subset G of a Banach space if there is some $\epsilon > 0$ such that $B = (x, \epsilon) \subset G$ whenever $x \in S$. The following theorem may be viewed as a holomorphic version of the Banach's contraction mapping theorem.

Theorem 4.6 (Earle-Hamilton theorem)**.** *Let G be a nonempty domain in a complex Banach space X and let $h : G \to G$ be a bounded holomorphic function. If $h(G)$ lies strictly inside G, then h has a unique fixed point in G.*

The Earle-Hamilton theorem still applies in cases where the holomorphic function does not necessarily map its domain strictly inside itself. In fact, the following interesting fixed point theorem is a consequence of two applications of the Earle-Hamilton theorem.

Theorem 4.7 (Khatskevich-Reich-Shoikhet theorem)**.** *Let G be a nonempty bounded convex domain in a complex Banach space and let $h : G \to G$ be a holomorphic function having a uniformly continuous extension to $\overline{G}$. If there exists an $\epsilon > 0$ such that $|h(x) - x| \geq \epsilon$ whenever $x \in \partial G$, then h has a unique fixed point in G.*

The Hahn-Banach Theorem is a central tool in functional analysis. It allows the extension of bounded linear functionals defined on a subspace of some vector space to the whole space, and it also shows that there are "enough" continuous linear functionals defined on every normed vector space to make the study of the dual space "interesting".

Let S be a vector space over the real numbers, or, more generally, some ordered field. This includes Euclidean spaces. A set C in S is said to be convex if, for all x and y in C and all t in the interval $[0, 1]$, the point $(1 - t)x + ty$ also belongs to C. In other words, every point on the line segment connecting x and y is in C. This implies that a convex set in a real or complex topological vector space is path-connected, thus connected. Furthermore, C is strictly convex if every point on the line segment connecting x and y other than the endpoints is inside the interior of C.

A balanced set, circled set or disk in a vector space (over a field K with an absolute value function $||$) is a set S such that for all scalars a with $|\alpha| \leq 1$ $\alpha S \subseteq S$.

The open and closed balls centered at 0 in a normed vector space are balanced sets. Any subspace of a real or complex vector space is a balanced set. The cartesian product of a family of balanced sets is balanced in the product space of the corresponding vector spaces (over the same field K).

A set C is called absolutely convex if it is convex and balanced.

Theorem 4.8 (Rudin [69])**.** *Let B be unit ball (or completely circular domain) strongly convex $F : B \to B$ holomorphic, $F(0) = 0$. Then F and $A = F'(0)$ have the same fixed point.*

Consider examples $\mathbb{B}_2$, and $\mathbb{U}^2$.

Proof. Let $z_0 \in B$. Then $z_0 = ru_0$, where $|u_0| = 1$ and $0 < r < 1$. By Hahn-Banach there is L such that $Lu_0 = 1$ and $|L| = 1$. Define $g(\lambda) = L(F(\lambda u_0)$. g maps U intoself and $g(0) = 0$. We use (1) $g'(0) = L(Au_0)$. Suppose that $F(z_0) = z_0$. Then $g(r) = r$ and therefore $g = Id$ and in particular $g'(0) = 1$, and $g'(0) = L(Au_0) = 1$. Since B is strongly convex, we conclude $Au_0 = u_0$ and therefore $Az_0 = z_0$ Contrary suppose that $A(z_0) = z_0$. Since A is linear then $Au_0 = u_0$. Hence by (1), $g'(0) = L(Au_0) = L(u_0) = 1$ and therefore $g = Id$. Hence we conclude that $r = g(r) = L(F(z_0))$ and using that rB is strongly convex and $L(z_0) = r$, we get $Fz_0 = z_0$. □

See also Kang-Hyurk Lee, Almost Complex Manifolds and Cartan's Uniqueness Theorem, Transactions of the American Mathematical Society, Vol. 358, No. 5 (May, 2006), pp. 2057-2069.

The group of biholomorphic maps from a domain onto itself is called $Aut(D)$. (Convince yourself that $Aut(D)$ is indeed a group!) Given $a \in D$, one can form the subgroups $\mathrm{Aut}_a(D)$ of biholomorphic maps on D which leave a invariant.

The following example shows that a real version is not true.

Example 12. If $f(x, y) = (x - x^3/3 + 1/3, y - y^3/3 + 1/3)$, then f maps $[0, 1]^2$ onto itself and $f'(0) = Id$.

We now prove Cartan uniqueness theorem (strongly convex pluriharmonic version).

Theorem 4.9. *Let D be a bounded domain in $\mathbb{C}^n$ and given $a \in D$. If $f \in \mathrm{Aut}_a(D)$ satisfies $f'(a) = 1$, then $f(z) = z$ for all $z \in D$.*

Proof. By a translation of coordinates (replace D with $D - a$), we can assume $a = 0$. Because D is bounded, one has $D \subset D^n(0, R)$ for some $R > 0$. Every $f \in Aut_0(D)$ has a Taylor expansion at the origin $f(z) = \sum_n a_n z^n$. Cauchy's estimates give $|a_n| \le Mr^n$, where r is such that $D^n(0, r) \subset D$ and $M = sup_{z \in D}|f(z)|$. By assumption, f has a Taylor expansion $f = z + f_{n_0}(z) + \cdots$, where f_k are n-tuples of homogeneous polynomials of degree k and n_0 is chosen to be the smallest possible. The k'th iterate f^k of f has then the Taylor expansion $f^k = z + kf_{n_0}(z) + \cdots$. $kf_m(z) = \int_{-\pi}^{\pi} f^k(e^{it})e^{-imt}dt$, $kf_m(z) \le R$ which violates the above Cauchy estimate for large k unless $f_{n_0} = 0$. But if $f(z) = z$ in $D(0, r)$, then also f(z) = z in D by the principle of analytic continuation. □

Corollary 1 (Corollary 4.6 (Cartan))**.** . *Let D be a bounded circular domain in $\mathbb{C}^n$ and assume $0 \in D$ and $f \in Aut_0(D)$. Then f is linear.*

If G is a simply connected domain in $\mathbb{C}^n$, then it is clear that a mapping $f : G \to C$ is pluriharmonic if and only if f has a representation $f = h + \overline{g}$, where h, g are holomorphic in G. A vector-valued mapping $f = (f_1, , f_N)^T$ defined in G is said to be pluriharmonic, if each component $f_j(1 \le j \le N$ is a pluriharmonic mapping from G into C, where N is a positive integer and T is the transpose of a matrix. We refer to [4, 6, 7, 11, 12, 20] for further details and recent investigations on pluriharmonic mappings.

Corollary 2. *Suppose*
(a) G_1 and G_2 are circular and $0 \in G_1$, $0 \in G_2$

(b) F biholomorphic $F(0) = 0$
(c) G_1 bounded
Then F is a linear transformation.

If $f \in Aut(B)$ fixes a point of B, then the fp set of f is affine. Conversely, XXX

Theorem 4.10 (Hayed-Suffridge). *If $f \in \mathrm{Aut}(\mathbb{B})$ fixes three point of $\mathbb{S}$, then f fixes a point of $\mathbb{B}$.*

An open subset G of $\mathbf{C}^n$ is called Reinhardt domain if $(z_1, \ldots, z_n) \in G$ implies $(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in G$ for all real numbers $\theta_1, \ldots, \theta_n$.

The reason for studying these kinds of domains is that logarithmically convex Reinhardt domain are the domains of convergence of power series in several complex variables. Note that in one complex variable, a logarithmically convex Reinhardt domain is simply a disc. The intersection of logarithmically convex Reinhardt domains is still a logarithmically convex Reinhardt domain, so for every Reinhardt domain, there is a smallest logarithmically convex Reinhardt domain which contains it.

A Reinhardt domain D is called logarithmically convex if the image of the set $D^* = \{z = (z_1, \cdots, z_n) \in D / z_1 \cdots z_n \neq 0\}$ under the mapping
$\lambda : z \to \lambda(z) = (\ln(|z_1|), \cdots, \ln(|z_n|))$ is a convex set in the real space $\mathbb{R}^n$.

A simple example of logarithmically convex Reinhardt domains is a polydisc, that is, a product of disks.

See Wang and Ren[85]. Abstract. In this paper, we generalize a recent work of Liu et al. from the open unit ball $B^n \subset \mathbb{C}^n$ to more general bounded strongly pseudoconvex domains with C^2 boundary. It turns out that part of the main result in this paper is in some certain sense just a part of results in a work of Bracci and Zaitsev. However, the proofs are significantly different: the argument in this paper involves a simple growth estimate for the Caratheodory metric near the boundary of C^2 domains and the well-known Grahams estimate on the boundary behavior of the Caratheodory metric on strongly pseudoconvex domains, while Bracci and Zaitsev use other arguments.

5. Complex Dynamics and Denjoy-Wolff Theorem

5.1. **Application to complex Dynamics.** Let f be a complex function. We define the iterates of f as $f^2 = f \circ f$, $f^n = f^{n-1} \circ f$. Given z_0, the sequence of points $\{z_n\}$ defined by $z_n = f(z_{n-1})$ is called the orbit of z_0.

The Fatou set $\mathcal{F}$ of f is defined to be the set of points $z_0 \in \overline{\mathbb{C}}$ such that f^n is a normal family in some neighborhood of z_0.

The Julia set $\mathcal{J}$ is the complement of the Fatou set. Let R be a rational function. Consider a fixed component U of Fatou set of R.

1. If $R(U) = U$, we call U a *fixed component* of $\mathcal{F}$
2. If $R^n(U) = U$ for some $n > 1$, we call U a *periodic component* of $\mathcal{F}$
3. If $R^m(U)$ is periodic for some $m \geq 1$, we call U a *preperiodic component* of $\mathcal{F}$
4. Otherwise, all $R^n(U)$ are distinct, and we call U a *wandering domain.*

Theorem 5.1. *A rational map has no wanderings domains.*

Assume U_0 is wandering and let $U_n = R^n(U_0), n \geq 1$. We assume ∞ is in some component V of $\mathcal{F}$ other than the $U_n's$. this implies that $area(U_n) < \infty$, which implies that only possible limit functions of R^n on U_0 are constants. In fact,

$(R^n)' \to 0$. Replacing U_n by U_{n+m}, we may assume no U_n contains a critical point of R.

We claim that R maps each U_n one-to-one onto U_{n+1}. For this it suffices to prove that each U_n is simply connected.

We will now construct a family of qc mappings $\{f_t\}$ with $t \in \mathbb{C}^m$, so that $f_t^{-1} \circ R \circ f_t$ is an m-dimensional analytic family of distinct rational maps.

So fix $m > 2d-1$, and suppose $B_{2\varepsilon} \subset U_0$. Let $D_0 = B_\varepsilon$ and $D_n = R^n(D_0)$. For $|t| < \delta$, we define a Beltrami coefficient (ellipse field) μ on D_0 by

$$\mu(z) = \sum_1^m t_\kappa e^{-i\,k\theta}, z = re^{i\theta} \in D_0.$$

we can extend the ellipse field to $\cup D_n$ to be invariant under R. For other z set $\mu = 0$, and then the ellipse field is everywhere invariant.

Let $f_t(z) = f(z,t)$ be the solution of the Beltrami equation, normalized so $f(z,t) = z + o(1)$ at ∞.

From the invariance of the ellipse field, we see that $R_t = f_t^{-1} \circ R \circ f_t$ is analytic. The normalized coefficients of R_t are holomorphic functions of t and agree with those of R at $t = 0$. Let V be the connected component containing 0 of $|t| < \delta$ determined by the equations $a_\nu(t) = a_\nu(0), b_\nu(t) = b_\nu(0)$.

Suppose $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a meromorphic function. The Fatou set of f is defined to be the set of points a such that f^n is a normal family in some neigh of a.
We say that a set E is *completely invariant* if both it and its complement are invariant. Since R is onto, this occurs if and only if $R^{-1}(E) = E$.
$\mathcal{J}(f)$ is the closure of the set of repelling periodic points of f.

Prop. If there is a neighborhood U of a such that f^{n_k} uniformly on U, then a is a Fatou point.

Conversely, let $a \in \mathcal{J}$ there is a repelling periodic point z_0 with period p. Without loss of generality, we can assume that the orbit of z_0 consists of finite point.

Let $g = f^p$ and $\lambda = g'(z_0)$; so that $|\lambda| > 1$. Using chain rule, we can verify that $(f^{n_k})'(z_0)$ tends to ∞. Since $|f^{n_k}(z_0)|$ is bounded, f^{n_k} uniformly converges to an analytic function in some neigh of z_0. This gives a contradiction.

See Bergweler W., *Iteration of meromorphic functions*, Bull. Am. Math. Soc. 29(2) (1993), 151-188.

If $\infty \notin \mathcal{J}$, then R is hyperbolic on $\mathcal{J}$ if there exists $m \geq 1$ such that $|(R^m)'| > 1$ on $\mathcal{J}$.

R is hyperbolic on $\mathcal{J}$ if and only if every critical point belongs to $\mathcal{F}$ and is attracted to an attracting circle.

Suppose that P a polynomial with finite postcritical set, such that no finite critical point is periodic. Then the Julia set $\mathcal{J}$ is a dendrite.

Example 13. (a) $P(z) = z^2 - 2$, $0 \to -2 \to 2 \to 2 \to \dots$, $\mathcal{J} = [-2, 2]$.
(b) $P(z) = z^2 + i$, $0 \to i \to i-1 \to -i \to i-1 \to \dots$.

If there are two completely invariant components of the Fatou set, at least one of which has an attracting fixed point, then the Julia set $\mathcal{J}$ is a simple closed Jordan curve.

Let $CP = CP(f)$ denote the set of critical points of f. The *postcritical set* of f is defined to be the forward orbit $P(f) = \cup_{n\geq 0} f^n(CP(f))$ of the critical points. ? On its complement all branches of f^{-n}, $n \geq 1$, are locally defined and analytic.

Let U_1 and U_2 be bounded, open, simply connected domains with smooth boundaries, such that $\overline{U_1} \subset U_2$ Let f be holomorphic on $\overline{U_1}$ and maps U_1 onto U_2 with d-fold covering, so that f maps ∂U_1 onto ∂U_2; we call the triple $(f; U_1, U_2)$ *polynomial-like.*

If $(f; U_1, U_2)$ is *polynomial-like* of degree d, then there are a polynomial P of degree d and a qc map φ with $\varphi(z) = z + o(1)$ near ∞, such that $f = \varphi \circ P \circ \varphi^{-1}$ on U_1. For qc surgery see T 5.1 [19].

Sullivan: Suppose that the Fatou set of a rational function R has exactly two components and that R is hyperbolic on the Julia set $\mathcal{J}$. Then $\mathcal{J}$ is a quasicircle.

For stability of the Julia set see Theorem 6.2 [86], and for Bers-Royden Theorem 6.8

Example 14. $f(z) = z^2 + i$; $P(f) = \{\infty, i, -1+i, -i\}$.
The Lattès example

$$l(z) = \left(\frac{z-i}{z+i}\right)^2$$

has $P(l) = \{0, \infty, -1, 1\}$; $0 \to 1, \to -1, \to -1$.

Theorem 5.2. *Let f and g be topologically conjugate critically finite rational maps. Then either*
f and g are conformally conjugate; or
f and g are double-covered by integral torus endomorphisms.

Let $Q(f) = f^{-1}(P(f))$ and ϕ topological conjugacy; then ϕ maps $P(f)$ and $Q(f)$ to $P(f)$ and $Q(f)$.

f and g are covering maps between $\overline{\mathbb{C}} \setminus Q(f)$ and $\overline{\mathbb{C}} \setminus P(f)$
deforme ϕ to the Teichmüller mapping

$$\psi_0 : \overline{\mathbb{C}} \setminus P(f) \to \overline{\mathbb{C}} \setminus P(f)$$

in the homotopy class of ϕ. Lifting ψ_0 to ψ_1; $K(\psi_1) = K(\psi_0)$; ψ_0 and ψ_1 are homotopic $rel\, P(f)$. By uniqueness of the Teichmüller mapping, we have $\psi_0 = \psi_1 = \psi$. If ψ is conformal, then it provides a conformal conjugacy.
Now suppose ψ is strictly qc. and let α be its associate quadratic differential. Then $f^*(\alpha) = deg(f)\,\alpha$ because f and ψ comute; so f is affine.

Let $F : S^2 \to S^2$ be a smooth map of positive degree. We say that F is a *branched cover* if near any point p, we can find smooth charts ϕ, ψ ? such that

$$\phi \circ F \circ \psi^{-1} = z^d$$

for some $d \geq 1$.

Theorem 5.3 (Thom)**.** *Any branched covering F between a pair of spheres is equivalent to a rational map f .*

identify S^2 with $\overline{\mathbb{C}}$; solve the Beltramy equation

$$\frac{\overline{D}h}{Dh} = \frac{\overline{D}F}{DF} = \mu.$$

Since h and F have the same complex dilatation $f = F \circ h^{-1}$ is a rational map.

Theorem 5.4 (Sullivan)**.** *A branched cover F is qc conjugate to a rational map iff the iterates of F are uniformly quasiregular; that is, $K(F^n) \leq K_0 < \infty$*

If $F = \phi \circ f \circ \phi^{-1}$ with f rational and ϕ qc, then $F^n = \phi \circ f^n \circ \phi^{-1}$ and then $K(F^n) \leq K(\phi)^2$.
A rational map is *critically finite* if $|P(f)| < \infty$.

Let f and g be critically finite branched covers of the sphere. We say f and g are *combinatorially equivalent* if there are homeomorphisms ϕ_0 and ϕ_1 such that $\phi_0, \phi_1 : (S^2, P(f)) \to (S^2, P(g))$, $\phi_0 \circ f = g \circ \phi_1$ and ϕ_0 is isotopic to ϕ_1 $rel\, P(f)$.

For any finite set $A \subset S^2$, we denote by $Teich(S^2, A)$ the Teichmüller space of the sphere with the points in A marked. Any point in $Teich(S^2, A)$ is represented by a finite set $B \subset \overline{\mathbb{C}}$ together with marking homeomorphism $g : (S^2, A) \to (\overline{\mathbb{C}}, B)$.

Now let $F : S^2 \to S^2$ be a critically finite branched cover. $(\overline{\mathbb{C}}, P_0) \in Teich(S^2, P(F)$ with a marking homeomorphism $g : (S^2, P(F)) \to (\overline{\mathbb{C}}, P_0)$; use the covering

$$F : (S^2, Q(F)) \to (S^2, P(F))$$

we can form a new Riemann surface

$$F^*(\overline{\mathbb{C}}, P_0) = (\overline{\mathbb{C}}, Q_0) \in Teich(S^2, Q(F)).$$

$g \circ F$ is local chart on $(S^2, Q(F))$ and define complex structure $\mathcal{A}$.

Let $h : (S^2, Q(F)) \to (\overline{\mathbb{C}}, Q_0)$ be a qc such that complex structure $h^*(\overline{\mathbb{C}}, Q_0)$ is ? def equivalent with $\mathcal{A}$ and $g \circ F \circ h^{-1}$ is a covering $(\overline{\mathbb{C}}, Q_0)$ onto $(\overline{\mathbb{C}}, P_0)$. Mark a subset $P_1 = h(P(F))$ of Q_0.

The covering $(\overline{\mathbb{C}}, Q_0) \to (\overline{\mathbb{C}}, P_0)$ then prolongs to a rational map

$$f : (\overline{\mathbb{C}}, P_1) \to (\overline{\mathbb{C}}, P_0).$$

Define $T_F(\overline{\mathbb{C}}, P_0) = (\overline{\mathbb{C}}, P_1)$.

Now suppose that $(\overline{\mathbb{C}}, P_0) = (\overline{\mathbb{C}}, P_1)$. Then after adjusting with a Möbius we can assume that $P_0 = P_1$, and thus f is a rational map with $P(f) = P_0$.
f is combinatorial equivalent to F.

The converse is also easy to check, so we have

Theorem 5.5. *F is combinatorially equivalent to a rational map iff T_F has a fixed point on Teichmüller space.*

5.2. **dynamics.** Suppose $f : \mathbb{C} \to \overline{\mathbb{C}}$ is a meromorphic function. The Fatou set of f is defined to be the set of points a such that f^n is a normal family in some neigh of a.

$\mathcal{J}(f)$ is the closure of the set of repelling periodic points of f.

Proposition 5.1 ([74])**.** If there is a neighborhood U of a such that f^{n_k} uniformly converges on U, then a is a Fatou point.

Proof. Conversely, let $a \in \mathcal{J}$ there is a repelling periodic point z_0 with period p. Without loss of generality, we can assume that the orbit of z_0 consists of finite point.

Let $g = f^p$ and $\lambda = g'(z_0)$; so that $|\lambda| > 1$. Using chain rule, we can verify that $(f^{n_k})'(z_0)$ tends to ∞. Since $|f^{n_k}(z_0)|$ is bounded, f^{n_k} uniformly converges to an analytic function in some neigh of z_0. This gives a contradiction. $\square$

Se also Bergweler W., *Iteration of meromorphic functions*, Bull. Am. Math. Soc. 29(2) (1993), 151-188.

The Schwarz Lemma is related to the following result.
See https://en.wikipedia.org/wiki/Denjoy-Wolff_theorem.

Denjoy-Wolff Theorem (1926). Theorem. Let $\mathbb{U}$ be the open unit disk in $\mathbb{C}$ and let f be a holomorphic function mapping $\mathbb{U}$ into $\mathbb{U}$ which is not an automorphism of $\mathbb{U}$ (i.e. a Möbius transformation). Then there is a unique point z in the closure of $\mathbb{U}$ such that the iterates of f tend to z uniformly on compact subsets of $\mathbb{U}$. If z lies in $\mathbb{U}$, it is the unique fixed point of f. The mapping f leaves invariant hyperbolic disks centered on z, if z lies in $\mathbb{U}$, and disks tangent to the unit circle at z, if z lies on the boundary of $\mathbb{U}$.

When the fixed point is at $z = 0$, the hyperbolic disks centred at z are just the Euclidean disks with centre 0. Otherwise f can be conjugated by a Möbius transformation so that the fixed point is zero. An elementary proof of the theorem is given below, taken from Shapiro (1993) and Burckel (1981). Two other short proofs can be found in Carleson & Gamelin (1993).

For the subject se for example:

Beardon, A. F. (1990), "Iteration of contractions and analytic maps", J. London Math. Soc., 41: 141-150,
Burckel, R. B. (1981), "Iterating analytic self-maps of discs", Amer. Math. Monthly, 88: 396-407, doi:10.2307/2321822,
Carleson, L.; Gamelin, T. D. W. (1993), Complex dynamics, Universitext: Tracts in Mathematics, Springer-Verlag, ISBN 0-387-97942-5,
Shapiro, J. H. (1993), Composition operators and classical function theory, Universitext: Tracts in Mathematics, Springer-Verlag, ISBN 0-387-94067-7
See also https://www.math.iupui.edu/ ccowen/Talks/FixPts1005.pdf, Fixed Points of Functions Analytic in the Unit Disk by Carl C. Cowen, IUPUI (Indiana University Purdue University Indianapolis). Conference on Complex Analysis, University of Illinois, May 22, 2010.

Definition. Suppose f is an analytic map of $\mathbb{U}$ into itself. If $|b| < 1$, we say b is a fixed point of f if $f(b) = b$. If $|b| = 1$, we say b is a fixed point of f if $\lim_{r\to 1_-} f(rb) = b$.

Julia-Caratheordory Theorem implies that if b is a fixed point of f with $|b| = 1$, then $\lim_{r\to 1_-} f'(rb)$ exists (call it $f'(b)$) and $0 < f'(b) \leq \infty$.

Denjoy-Wolff Theorem (1926). (a) If f is an analytic map of $\mathbb{U}$ into itself, not the identity map, there is a unique fixed point, a, of f in $\overline{\mathbb{U}}$ such that $|f'((a)| \leq 1$.

(b) If f is not an automorphism of $\mathbb{U}$ (i.e. a Mobius transformation) with fixed point in $\mathbb{U}$, iterates of f tend to a uniformly on compact subsets of $\mathbb{U}$

This distinguished fixed point will be called the Denjoy-Wolff point of f.

The Schwarz-Pick Lemma implies f has at most one fixed point in $\mathbb{U}$ and if f has a fixed point in $\mathbb{U}$, it must be the Denjoy-Wolff point. See
C. C. Cowen, Iteration and the Solution of Functional Equations for Functions Analytic in the Unit Disk. Trans. Amer. Math. Soc. 265 (1981) 69-95.
C. C. Cowen and Ch. Pommerenke, Inequalities for the Angular Derivative of an Analytic Function in the Unit Disk. J. London Math. Soc. 26 (1982) 271-289.

Question 3. Is there a version of this result for quasi-regular mappings? For Danjoy-Wolff theorem see [19], [20]. It seems that using the analityc covering theorem (the uniformization theorem for hyperbolic domains) one can get a version of Theorem Danjoy-Wolff for hyperbolic domains.

Exercise 5. Set $T(z_0,R) = \{|1-\overline{z_0}z|^2 = R(1-|z|^2)\}$ For and $z_0 \in \mathbb{T}$, the set $D_{z_0} = D(z_0,R) = \{z \in \mathbb{U} : |1-\overline{z_0}z|^2 < R(1-|z|^2)\}$ is called a horocycle at z_0 with radius R. This set is a disc in which is internally tangent to $\mathbb{T}$ at z_0. Since $|1-\overline{z_0}z|^2 = |z|^2 - 2\mathrm{Re}\overline{z_0}z + 1$, $z \in T(z_0,R)$ iff $(R+1)|z|^2 - 2\mathrm{Re}\overline{z_0}z = R-1$. Hence $z \in T(z_0,R)$ iff $|z-w_0| = R_0$, where $w_0 = \frac{z_0}{R+1}$ and $R_0 = \frac{R}{R+1}$.

Exercise 6. Check that
a) For $a \in \mathbb{C}$, the function T_a has singularity at $a^* := 1/a$, and for $s \le 1/|a|$, the set $\sigma(w,a) < s$ is Euclidean disk disk, the circle $E(a;s) = \sigma(w,a) = s$ has the center at $a_0 = \tau a$, where $\tau = \frac{1-s^2}{1-s^2|a|^2}$ and radius $R = \frac{s(1-|a|^2)}{1-s^2|a|^2}$;
Thus $E(a;s) = T(a_0;R)$; and if $s = |a| \le 1$, then $|a_0| \le 1/2$.
b) In particular $0 \in E^a := E(a;|a|) = T(\frac{a}{1+|a|^2}; \frac{|a|}{1+|a|^2})$, and $E(1;1) = T(1/2;1/2)$ and $T(1/2;1/2) = \{|w|^2 = 2\mathrm{Re}w\}$.
c) Suppose that $f : \mathbb{U} \to \mathbb{U}$ is continuous and non-expansive wrt pseudo-hyperbolic metric on $\mathbb{U}$. Prove that there is a boundary point z_0 such that f leaves any given disk tangent to the boundary at z_0 invariant.
Outline. c)Set $f_m(z) = \rho_m z$, where $\rho_m \to 1_0$, when tends to ∞, and let z_m be fixed point of f_m. Since $f_m(z_m) = z_m$, $\sigma(f_m(0), z_m) = \sigma(f_m(0), f(z_m)) < \sigma(z_m, 0) = |z_m|$ and therefore $f_m(0) \in E(z_m;|z_m|)$. Next $E^m := E(z_m;|z_m|) = T(O_m;R_m)$, where $O_m = \frac{z_m}{1+|z_m|^2}$ and $R_m = \frac{|z_m|}{1+|z_m|^2}$. Hence $|f_m(0)|^2 \le 2\mathrm{Re}(\overline{w}O_m)$ and since O_m tends to $1/2$ when $m \to \infty$, we conclude $f(0) \in B(1/2;1/2)$.

c1) Visualize the proof in the point c); draw the picture of the circles E^m and give geometric interpretation of the relations f_m leaves E^m invariant and E^m "tends" to $T(1/2;1/2)$.

Theorem Danjoy-Wolff. Let $\mathbb{U}$ be the open unit disk in $\mathbb{C}$ and let f be a holomorphic function mapping $\mathbb{U}$ into $\mathbb{U}$ which is not an automorphism of $\mathbb{U}$ (i.e. a Mobius transformation). Then there is a unique point z_0 in the closure of $\mathbb{U}$ such that the iterates of f tend to z_0 uniformly on compact subsets of $\mathbb{U}$. If z_0 lies in $\mathbb{U}$, it is the unique fixed point of f. The mapping f leaves invariant hyperbolic disks centered on z_0, if z_0 lies in $\mathbb{U}$, and disks tangent to the unit circle at z_0, if z_0 lies on the boundary of $\mathbb{U}$.

Proof. When the fixed point is at $z_0 = 0$, the hyperbolic disks centred at z_0 are just the Euclidean disks with centre 0. Otherwise f can be conjugated by a Mobius transformation so that the fixed point is zero. An elementary proof of the theorem is given below, taken from Shapiro (1993) and Burckel (1981). Two other short proofs can be found in Carleson & Gamelin (1993)[19].

Case 1(Fixed point in the disk). If f has a fixed point z in $\mathbb{U}$ then, after conjugating by a Möbius transformation, it can be assumed that $z = 0$. Let $M(r)$ be the maximum modulus of f on $|z| = r < 1$. By the Schwarz lemma $|f(z)| \le \delta(r)|z|$, for $|z| \le r$, where $\delta(r) = \frac{M(r)}{r}$. Since f is not automorphism of $\mathbb{U}$, $\delta(r) < 1$. It follows by iteration that $|f^n(z)| \le \delta(r)^n$ for $|z| \le r$. These two inequalities imply the result in this case.

Case 2 (No fixed points in the unit disk). When f acts in $\mathbb{U}$ without fixed points, Wolff showed that there is a point z_0 on the boundary such that the iterates of f leave invariant each disk tangent to the boundary at that point. Take a sequence

r_k increasing to 1 and set $f_k(z) = r_k f(z)$. By applying Rouché's theorem to $f_k(z) - z$ and $g(z) = z$, f_k has exactly one zero z_k in D. Passing to a subsequence if necessary, it can be assumed that $z_k \to z_0$. The point z_0 cannot lie in $\mathbb{U}$, because, by passing to the limit, z_0 would have to be a fixed point. The result for the case of fixed points implies that the maps f_k leave invariant all Euclidean disks whose hyperbolic center is located at z_k. We leave the interested reader to fill details for proof of the following(use Exercise 6c)):
(I1)Explicit computations show that, as k increases, one can choose such disks so that they tend to any given disk tangent to the boundary at z_0. By continuity, f leaves each such disk B invariant.

To see that f^n converges uniformly on compacta to the constant z_0, it is enough to show that the same is true for any subsequence f^{n_k}, convergent in the same sense to a function g, say. Such limits exist by Montel's theorem, and if g is non-constant, it can also be assumed that $f^{n_{k+1}-n_k}$ has a limit, h say. Set $m_k = n_{k+1} - n_k$. But then $f^{n_{k+1}} = (f^{n_k})^{m_k}$ and $f^{n_k}(w) \to g(w)$ and $f^{n_{k+1}}(w) \to g(w)$. Hence since f^{m_k} holomorphic function does not increase hyperbolic distance on $\mathbb{U}$, we find
$d\big(f^{m_k}(f^{n_k}(w)), f^{m_k}(g(w))\big) \leq d\big(f^{n_k}(w), g(w)\big)$ and therefore $h(g(w)) = g(w)$ for w in $\mathbb{U}$.

Since h is holomorphic and $g(\mathbb{U})$ open, $h(w) = w$ for all w.

It can also be assumed that f^{m_k-1} is convergent to F say.

But then $f^{m_k}(w) = f^{m_k-1}(fw) = f(f^{m_{k-1}}(w))$ and therefore

$f(F(w)) = w = f(F(w))$, contradicting the fact that f is not an automorphism.

Hence every subsequence tends to some constant uniformly on compacta in $\mathbb{U}$.

The invariance of B implies each such constant lies in the closure of each disk B, and hence their intersection, the single point z_0. By Montel's theorem, it follows that f^n converges uniformly on compacta to the constant z_0.

For the subject see: What are the most recent versions of The Schwarz Lemma at the Boundary? - ResearchGate. Available from: https://www.researchgate.net/post/ $\square$

5.3. **Further results related to Denjoy-Wolff Theorem.** For and $z_0 \in \mathbb{T}$, the set $D_{z_0} = D(z_0, R) = \{z \in \mathbb{U} : |1 - \overline{z_0}z|^2 < R(1 - |z|^2)\}$ is called a horocycle at z_0 with radius R. This set is a disc in which is internally tangent to $\mathbb{T}$ at z_0(see Exercise5).

The classical Denjoy-Wolff theorem is the following one-dimensional result: Let $\mathbb{U}$ be the open unit disc in the complex plane $\mathbb{C}$. If $F \in \mathrm{Hol}(\mathbb{U})$ is not the identity and is not an automorphism of with exactly one fixed point in $\mathbb{U}$, then there is a unique point a in the closed unit disc such that the iterates $\{F^n\}$ of F converge to a, uniformly on compact subsets of $\mathbb{U}$. This result is, in fact, a summary of the following three assertions A)-C) due to A. Denjoy and J. Wolff.

A) The Wolff-Schwarz lemma: If has no fixed point in $\mathbb{U}$, then there is a unique unimodular point a such that every horocycle D_a in $\mathbb{U}$, internally tangent to $\mathbb{T}$ at a, is F-invariant, i.e., $F(D_a) \subset D_a$.

This assertion is a natural complement of the Julia-Wolff-Carathodory theorem.

B) If $F \in \mathrm{Hol}(\mathbb{U}, \mathbb{U})$ has no fixed point in $\mathbb{U}$, then there is a unique unimodular point b such that the sequence $\{F^n\}$ converges to b , uniformly on compact subsets of $\mathbb{U}$.

C) If $F \in \mathrm{Hol}(\mathbb{U}, \mathbb{U})$ is not an automorphism of but has a fixed point c in $\mathbb{U}$, then this point is unique in $\mathbb{U}$, and the sequence $\{F^n\}$ converges to c uniformly on compact subsets of $\mathbb{U}$.

The limit point in B) is sometimes called the Denjoy-Wolff point of F.

When is a complex Hilbert space with the inner product $(\,,\,)$, and B is its open unit ball, the following generalization of the Wolff-Schwarz lemma is due to K. Goebel [78]: If $F \in \mathrm{Hol}(\mathbb{B})$ has no fixed point, then there exists a unique point $a \in \mathbb{S}$ such that for each the set $E(a, R) = \{z \in \mathbb{B} : |1 - (z, a)|^2 < R(1 - |z|^2)\}$ is F-invariant.

Geometrically, the set $E(a, R)$ is an ellipsoid the closure of which intersects the unit sphere at the point a. It is a natural analogue of the horocycle $D(a, R)$.

Another look at the DenjoyWolff theorem is provided by a useful result of P. Yang [83] concerning a characterization of the horocycle in terms of the Poincar hyperbolic metric in (cf. also Poincar model). More precisely, he established the following formula:

$$d^0(z, a) := \lim_{w \to a}[d(z, w) - d(0, w)] = \frac{1}{2} \ln \frac{|1 - \overline{a}z|^2}{1 - |z|^2}. \tag{A2}$$

Outline. Set

$$A(z, w) = \frac{1 + \sigma(z, w)}{1 - \sigma(z, w)} \frac{1 - |w|}{1 + |w|} = \frac{B}{C} \frac{1 - |w|}{1 + |w|},$$

where $B = (|1 - \overline{z}w| + |z - w|)^2$ and $C = |1 - \overline{z}w|^2 - |z - w|^2 = (1 - |z|^2)(1 - |w|^2)$. Hence $A(z, w)$ tends $\frac{B(z,a)}{4(1-|z|^2)}$ when w tends a and since $|1 - \overline{a}z| = |z - a|$, $B(z, a) = 4|1 - \overline{a}z|^2$. Therefore horocycle $E(a, R)$ is given by $\{z \in \mathbb{U} : d^0(z, a) < \frac{1}{2} \ln R\}$.

Since Kobayashi metric (whether a hyperbolic metric exists ?) can be defined in each bounded domain in $\mathbb{C}^n$, one can try to extend this formula and use it as a definition of the horosphere in a domain in $\mathbb{C}^n$. Unfortunately, in general the limit in (A2) does not exist.

To overcome this difficulty, M. Abate [79] introduced two kinds of horospheres. More precisely, he defined the small horosphere $E_{z_0}(a, R)$ of centre a, pole z_0 and radius R by the formula

$$E_{z_0}(a, R) = \{z \in D : \limsup_{w \to a}[K_D(z, w) - K_D(z_0, w)] < \frac{1}{2} \ln R,$$

and the big horosphere of centre a, pole z_0 and radius R by the formula

$$F_{z_0}(a, R) = \{z \in D : \liminf_{w \to a}[K_D(z, w) - K_D(z_0, w)] < \frac{1}{2} \ln R,$$

where is D a bounded domain in $\mathbb{C}^n$ and K_D is its Kobayashi metric (cf. Hyperbolic metric). For the Euclidean ball in $\mathbb{C}^n$, $E_{z_0}(a, R) = F_{z_0}(a, R)$.

Thus, each assertion which states for a domain D in C^n the existence of a point $a \in \partial D$ such that $F(E_z(a, R)) \subset F_z(a, R)$ is a generalization of the Wolff-Schwarz lemma. This is true, for example, for a bounded convex domain in $\mathbb{C}^n$, see [79]. However, in this case B) does not hold in general.Nevertheless, the convergence result does hold for bounded strongly convex domains, and for strongly pseudo-convex hyperbolic domains with a C^2 boundary [79, 80].

In the hyperboloid model, a horosphere is represented by a plane whose normal lies in the asymptotic cone. More recently, Shoiket et al. have also obtained [84] a boundary version of the Earle-Hamilton theorem for the Hilbert ball: If $F : \mathbb{B} \to \mathbb{B}$

is a fixed point free mapping of the open unit ball $\mathbb{B}$ in (complex) Hilbert space such that $F(\mathbb{B})$ is contained in a horosphere in $\mathbb{B}$, then the iterates F^n converge to a boundary point of $\mathbb{B}$.

We are now ready to formulate and establish the main theorem of ?? this section.

In [81], the authors proved the following results:

Theorem 5.6. *If D is a bounded and strictly convex domain in an arbitrary complex Banach space $(X;||)$, and $f : D \to D$ is compact, k_D-nonexpansive and fixed-point-free, then there exists a point $z_0 \in \partial D$ such that the sequence $\{f^n\}$ of the iterates of f converges in the bounded-open topology to the constant map taking the value z_0, that is, the sequence $\{f^n\}$ tends to z_0, uniformly on each k_D -bounded subset C of D.*

Theorem 5.7. *Let X be a complex strictly convex Banach space with an open unit ball B. For each compact, holomorphic and fixed-point-free mapping $f : B \to B$ there exists $z_0 \in \partial B$ such that the sequence $\{f^n\}$ of iterates of f converges locally uniformly on B to the constant map taking the value z_0.*

Abstract. In [82], the authors study the dynamics of fixed point free mappings on the interior of a normal, closed cone in a Banach space that are nonexpansive with respect to Hilbert's metric or Thompson's metric. They establish several Denjoy-Wolff type theorems that confirm conjectures by Karlsson and Nussbaum for an important class of nonexpansive mappings. They also extend and put into a broader perspective results by Gaubert and Vigeral concerning the linear escape rate of such nonexpansive mappings.

5.4. **Curvature.** Let D be a domain in $z = x + iy$-plane and a Riemannian metric be given by the fundamental form

$$ds^2 = \sigma|dz|^2 = \sigma(dx^2 + dy^2)$$

which is conformal with euclidian metric. If $M = (D, \sigma|dz|^2$, then the Gaussian curvature of M is

$$K_M = -\frac{1}{2\,\sigma}\triangle \ln \sigma\,.$$

Instead of K_M it is also convenient to use notation K_σ and call σ shortly metric coefficient.

Often in the literature a Riemannian metric is given by $ds = \rho|dz|$, $\rho > 0$, that is by the fundamental form

$$ds^2 = \rho^2(dx^2 + dy^2)\,.$$

In some situations it is convenient to to call ρ shortly metric density.

If $\rho > 0$ is a C^2 function on D and $M = (D, \rho|dz|)$, the Gaussian curvature of M is expressed by the formula

$$K_M = \bar{K}_\rho := -\rho^{-2}\triangle \ln \rho.$$

We also call the above term the Gaussian curvature of a Riemannian metric density ρ on D. Also we write $K(\rho)$ and $\bar{K}_\rho$ instead of K_ρ and $\bar{K}(\rho)$ respectively. It is clear that $\bar{K}_\rho = K(\rho^2)$.

For $a > 0$, $\bar{K}(a\rho) = a^{-2}\bar{K}(\rho)$.

Recall that a pseudohermitian metric on D is a non-negative upper semicontinuous function ρ such the set $\rho^{-1}(0)$ is discrete in D.

If u is an upper semicontinuous function on D and $\omega \in D$, the *lower generalized Laplacian* of u is defined by ([2], see also [25])

$$\triangle_L\, u(\omega) = 4 \liminf_{r\to 0} \frac{1}{r^2} \Big[\frac{1}{2\pi}\int_0^{2\pi} \big(u(\omega + re^{it}) - u(\omega)\big) dt\Big].$$

When u is a C^2 function, then the lower generalized Laplasian of u reduces to the usual Laplacian

$$\triangle u = u_{xx} + u_{yy}\,.$$

The Gaussian curvature of a pseudohermitian metric density on D is defined by the formula

$$K = \bar{K}_\rho = -\rho^{-2} \triangle_L \ln \rho\,.$$

For all $a > 0$ define the family of functions λ_a

$$\lambda_a(z) = \frac{2}{a(1-|z|^2)}\,.$$

Also, it is convenient to write λ instead of λ_1.

XXX Suppose ρ is a semimetric density on a region G and $f : D \to G$ is a holomorphic function. The pull-back of ρ by f is $f^*(\rho) = \rho(f(z))|f'(z)|$.

Suppose $a \in D$, $f'(a) \neq 0$, $\rho(f(a)) > 0$ and ρ is of class C^2 at $f(a)$. Then $K_{f^*(\rho)}(a) = K_\rho\big(f(a)\big)$.The Gaussian curvature of the density λ_a is $\bar{K}(\lambda_a) = -a^2$. This family of Hermitian metrics on $\mathbb{D}$ is of interest because it allows an ordering of all pseudohermitian metrics on $\mathbb{D}$ in the sense of the following ([2]).

Theorem 5.8. *Let ρ be a pseudohermitian metric density on $\mathbb{D}$ such that*

$$\bar{K}_\rho(z) \le -a^2$$

for some $a > 0$. Then $\rho \le \lambda_a$.

This kind of estimate is similar to Ahlfors-Schwarz lemma.
Ahlfors lemma can be found in Ahlfors [5].

A metric ρ is said to be ultrahyperbolic in a region Ω if it has the following properties :
(a) ρ is upper semicontinuous; and
(b) at every z_0 with $\rho(z_0) > 0$ there exits a supporting metric ρ_0 , defined and class C^2 in a neighborhood V of z_0 , such that $\rho_0 \le \rho$ and $\bar{K}_{\rho_0} \le -1$ in V, while $\rho_0(z_0) = \rho(z_0)$.

Theorem 5.9. ($Ahlfors$ *Lemma* 1). *Suppose ρ is an ultrahyperbolic metric on $\mathbb{D}$. Then $\rho \le \lambda$.*

The version presented in Gardiner [24] has a slightly modified definition of supporting metric. This modification and formulation is due to Earle. This version has been used (see [24]) to prove that *Teichmüller* distance is less than equal to *Kobayashi's* on *Teichmüller* space.
Ahlfors [5] proved a stronger version of Schwarz's lemma and Ahlfors lemma 1.

Theorem 5.10. ($Ahlfors$ *lemma* 2). *Let f be an analytic mapping of $\mathbb{D}$ into a region on which there is given ultrahyperbolic metric ρ. Then $\rho[f(z)]\,|f'(z)| \le \lambda$.*

The proof consists of observation that $\rho[f(z)]\,|f'(z)|$ is ultrahyperbolic metric on $\mathbb{D}$. Observe that the zeros of $f'(z)$ are singularities of this metric.

Note that if f is the identity map on $\mathbb{D}$ we get Theorem 3 (Ahlfors lemma 1)from Theorem 4.

The notation of an ultrahyperbolic metric makes sense , and the theorem remains valid if Ω is replaced by a Riemann surface.

In a plane region Ω whose complement has at least two points, there exists a unique maximal ultrahyperbolic metric, and this metric has constant curvature -1.

The maximal metric is called the *Poincaré metric* of Ω , and we denote it by λ_Ω. It is maximal in the sense that every ultrhyperbolic metric ρ satisfies $\rho \leq \lambda_\Omega$ throughout Ω.

The hyperbolic density (metric) of a disk $|z| < R$ is given by

$$\lambda_R(z) = \frac{2R}{R^2 - |z|^2}\,.$$

If ρ is ultrahyperbolic in $|z| < R$, then $\rho \leq \lambda_R$. In particular , if ρ is ultrhyperbolic in the whole plane, then $\rho = 0$. Hence there is no ultrahyperbolic metric in the whole plane.

The same is true of the punctured plane $C^* = \{z : z \neq 0\}$. Indeed, if ρ were ultrahyperbolic metric in the whole plane, then $\rho(e^z)\ \mid e^z \mid$ would be ultrahyperbolic in the whole plane. These are only cases in which ultrahyperobolic metric fails to exist.

Ahlfors [5] used Theorem 4 to prove Bloch and the Picard theorems. Ultrahypebolic metrics (without the name) were introduced by Ahlfors. They found many applications in the theory of several complex variables.

The comparison principle.

Theorem 9 ([43])**.** If ρ and σ are two metrics (densities) on the disk $\mathbb{D}$, σ is complete and $0 > \bar{K}_\sigma \geq \bar{K}_\rho$ on $\mathbb{D}$, then $\sigma \geq \rho$.

Here, $\bar{K}$ is Gaussian curvature. For the hyperbolic density on the disk we have $\bar{K} = -4$ (or -1, depends of normalization).

Example 15. If σ is the Poincaré metric with $K_\sigma = -1$ and ρ is any other metric with $K_\rho \leq -1$, then $\rho \leq \sigma$. In particular this holds if $\rho = F^*(\sigma)$ for a holomorphic map $F\colon \mathbb{D} \to \mathbb{D}$. (That is, the map F must be conformal with respect to the complex structures induced by the respective metrics.)

5.5. **An inequality opposite to Ahlfors-Schwarz lemma.** Mateljević [43] proved an estimate opposite to Ahlfors-Schwarz lemma.

A metric $H|dz|$ is said to be superhyperbolic in a region Ω if it has the following properties:
(a) H is continuous (more general, lower semicontinuous) on Ω.
(b) at every z_0 there exists a supporting metric (from above) H_0, defined and class C^2 in a neighborhood V of z_0, such that $H_0 \geq H$ and $K_{H_0} \geq -1$ in V, while $H_0(z_0) = H(z_0)$.

Theorem 5.11 ([43])**.** *Suppose H is a superhyperbolic metric on $\mathbb{D}$ for which*
(c) $H(z)$ tends to $+\infty$ when $|z|$ tends to 1_-
Then $\lambda \leq H$.

Proof. XXX Let $\rho_r(z) = 2r(1-|rz|^2)^{-1}$, where $r \in (0,1)$, and let

$$\Psi_r(z) = \log |H(z)| - \log \rho_r(z) .$$

By the hypothesis of theorem Ψ_r has a minimum on $\mathbb{D}$ at a point z_0. Let H_0 be supporting metric density from above to H at z_0 in a neighborhood V and

$$\tau_r(z) = \log |H_0(z)| - \log \rho_r(z) .$$

τ_r has a minimum on V at z_0 and so

$$0 \leq \Delta \tau_r(z_0) = \Delta \log |H_0(z_0)| - \Delta \log \rho_r(z_0) . \tag{1}$$

By the hypothesis, we have

$$\bar{K}_{H_0}(z_0) = -H_0(z_0)^{-2}(\triangle \ln H_0)(z_0) \geq -1 ,$$

that is

$$(\triangle \ln H_0)(z_0) \leq H_0(z_0)^2 ,$$

and

$$(\triangle \ln \rho_r)(z_0) \leq (\rho_r(z_0))^2 .$$

Hence by (1),

$$0 \leq \Delta \tau_r(z_0) = \Delta \log |H_0(z_0)| - \Delta \log \rho_r(z_0) \leq H_0^2(z_0) - (\rho_r(z_0))^2 \tag{2}$$

and therefore $\rho_r(z_0) \leq H_0(z_0)$. Since $H_0(z_0) = H(z_0)$ it follows that Ψ_r has non-negative minimum at z_0 and hence we conclude that $\rho_r \leq H$ for every $z \in \mathbb{D}$. If r tends to 1_-, we find $\rho \leq H$ on $\mathbb{D}$. XXX

□

By applying a method developed by Yau in [68] (or by generalized maximum principle of Cheng and Yau [16]), it follows that this result holds if we suppose instead of (c) that
(d) H is a complete metric on $\mathbb{D}$.

Theorem 5.12. *If ρ and σ are two metrics (density) on $\mathbb{D}$, σ complete and $0 > \bar{K}_\sigma \geq \bar{K}_\rho$ on $\mathbb{D}$, then $\sigma \geq \rho$.*

This theorem remains valid if ρ is ultrahyperbolic metric and σ superhyperbolic metric on $\mathbb{D}$. Also, we can get further generalizations if $\mathbb{D}$ is replaced by a Riemann surface.

Suppose that Ω is a hyperbolic domain and
(a) $H_0 : \Omega \to (0,\infty)$ is continuous (more general, lower semicontinuous) on Ω,
(b) The generalized Gaussian curvature of H_0, $K_{H_0} \geq -1$ on Ω.
Then $\lambda_\Omega \leq H$.

For convenient of the reader we recall the definition of the curvature. Let D be a domain in $z = x + iy$-plane and a Riemannian metric be given by the fundamental form

$$ds^2 = \sigma |dz|^2 = \sigma(dx^2 + dy^2)$$

which is conformal with euclidian metric. If $M = (D, \sigma|dz|^2$, then the Gaussian curvature of M is

$$K_M = -\frac{1}{2\,\sigma}\triangle \ln \sigma\,.$$

Instead of K_M it is also convenient to use notation K_σ and call σ shortly metric coefficient.

Often in the literature a Riemannian metric is given by $ds = \rho|dz|$, $\rho > 0$, that is by the fundamental form

$$ds^2 = \rho^2(dx^2 + dy^2)\,.$$

In some situations it is convenient to to call ρ shortly metric density.

If $\rho > 0$ is a C^2 function on D and $M = (D, \rho|dz|)$, the Gaussian curvature of M is expressed by the formula

$$K_M = \bar{K}_\rho := -\rho^{-2}\triangle \ln \rho.$$

We also call the above term the Gaussian curvature of a Riemannian metric density ρ on D. Also we write $K(\rho)$ and $\bar{K}_\rho$ instead of K_ρ and $\bar{K}(\rho)$ respectively. It is clear that $\bar{K}_\rho = K(\rho^2)$.

Proposition 5.2 ([74])**.** If $\rho, \rho_0, \tilde{\rho}$ metric density on $B_0 = B(z_0; r_0)$. Suppose that
(i) $\eta := \frac{\tilde{\rho}}{\rho}$ has a local minimum at z_0
$\bar{K}_{\tilde{\rho}} \geq \bar{K}_{\rho_0}$ at z_0, and
$\rho_0 \geq \rho$ and $\rho(z_0) = \rho_0(z_0)$.
Then $\rho \leq \tilde{\rho}$ on B_0.

Proof. If $\bar{K}_{\tilde{\rho}} \geq \bar{K}_{\rho_0}$ then

$$I = \frac{\triangle \ln \tilde{\rho}}{\triangle \ln \rho_0} \leq \chi := \frac{\tilde{\rho}^2}{\rho_0^2}$$

If $\frac{\tilde{\rho}}{\rho}$ has a local minimum at z_0 on $B_0 = B(z_0; r_0)$, then $\triangle \ln \tilde{\rho} \geq \triangle \ln \rho$. If $\triangle \ln \rho >$ 0, then $I \geq 1$, and therefore $\chi(z_0) \geq 1$. Hence $\rho_0 \leq \tilde{\rho}$ on B_0. $\rho_0 \geq \rho$ $\rho(z_0) = \rho_0(z_0)$. Set $\eta := \frac{\tilde{\rho}}{\rho}$ and $\eta_0 := \frac{\tilde{\rho}}{\rho_0}$. If η has a local minimum at z_0 on $B_0 = B(z_0; r_0)$, then η_0 does. Since $\eta_0(z_0) = 1$, we conclude $\rho \leq \tilde{\rho}$ on B_0. □

A metric ρ is said to be ultrahyperbolic in a region Ω if it has the following properties :
(a) ρ is upper semicontinuous; and
(b) at every z_0 with $\rho(z_0) > 0$ there exits a supporting metric ρ_0,defined and class C^2 in a neighborhood V of z_0 , such that $\rho_0 \leq \rho$ and $\bar{K}_{\rho_0} \leq -1$ in V, while $\rho_0(z_0) = \rho(z_0)$.
Set $I_u(a, r) := \int_0^{2\pi} \big(u(a + re^{it}) - u(a)\big)dt$. If u is a C^2 function in a neighborhood V, then $u(a + re^{it}) - u(a) = Ar\cos t + Br\sin t + Dr^2\cos^2 t + Er^2\cos t\sin t + Fr^2\cos^2 + o(r^2)$, where $D = u_{xx}(a)/2$ and $F = u_{yy}(a)/2$. Hence $I_u(a, r) = \frac{\pi}{2}r^2\triangle u(a) + o(r^2)$ and therefore
(i) $\frac{2}{\pi}r^{-2}I_u(a, r)$ tends to $\triangle u(a)$ if r tends to 0.

Definition 5.13. If u is an upper semicontinuous function, the *lower generalized Laplacian* of u is defined by ([2], see also [25])

$$\triangle_L\, u(\omega) = 4\liminf_{r\to 0}\frac{1}{r^2}\Big[\frac{1}{2\pi}\int_0^{2\pi}\big(u(\omega + re^{it}) - u(\omega)\big)dt\Big].$$

When u is a C^2 function, then by (i) we conclude that the lower generalized Laplacian of u reduces to the usual Laplacian

$$\triangle u = u_{xx} + u_{yy}\ .$$

Definition 5.14. A region Ω is hyperbolic if $\mathbb{C}\backslash\Omega$ contains at least two points. The hyperbolic metric λ_Ω on Ω is the unique metric on Ω such that $\lambda_D(z) = \lambda_\Omega(z)|f'(z)|$, where $f : D \to \Omega$ is any holomorphic universal covering projection. The hyperbolic metric has constant curvature -1.

Theorem 5.15 (Ahlfors Lemma 1)**.** *Suppose ρ is an ultrahyperbolic metric on $\mathbb{D}$. Then* (1) $\rho \le \lambda$.

Definition 5.16. A conformai metric $\rho(z)|dz|$ on a region Ω is called an SK metric provided $\rho : \Omega \to [0, +\infty)$ is upper semicontinuous and $\triangle_L \log\rho(a) \ge \rho^2(a)$ at each point $a \in \Omega$ such that $\rho(a) > 0$.

Thus, an SK metric is a conformai metric with generalized curvature at most -1 at each point where it does not vanish.

Here $\lambda_D|dz|$ is the hyperbolic metric on normalized to have curvature -1. (In some references and some parts of this paper the curvature is taken to be -4; the reader can translate all such results to the context of curvature -1.) Ahlfors did not show that equality in (1) at a single point implied $\rho = \lambda_D$ which would be the analog of the equality statement in Schwarz 's lemma. Heins [2] [2] introduced the class of SK metrics, which includes ultrahyperbolic metrics, and verified that (1) remains valid for SK metrics. In addition, he showed that equality at a single point implied $\rho = \lambda_D$. However, his proof of the equality statement is not as elementary as the proof of Ahlfors' lemma since it relies on an integral representation for a solution of the nonlinear partial differential equation $\Delta u = \exp(2u)$.

In [53] D. Minda also considered the strong form of Ahlfors' lemma and present a relatively elementary proof of the equality statement for Ahlfors' lemma for SK metrics; it relies on the fact that the Laplacian of a real-valued function is nonpositive at any point where the function has a relative maximum. His proof is in the spirit of Ahlfors' derivation of (1) and is a modification of a method introduced by Hopf [3] for linear partial differential equations. A related proof was given by Jorgensen Jorgensen [4] in the special case of metrics with constant curvature -1.

Theorem 5.17. *Let Ω be a hyperbolic region in C and λ_Ω the hyperbolic metric on Ω. If $\rho(z)|dz|$ is an SK metric on Ω, then either $\rho(z) < \lambda_\Omega(z)$ for all $z \in \Omega$ or else $\rho(z) = \lambda_\Omega(z)$ for all $z \in \Omega$.*

Proposition 5.3 ([53])**.** Suppose G is a region in $\mathbb{C}$, $u : G \to [-\infty, \infty)$ is upper semicontinuous and there is a positive constant K such that $\Delta u(z) \ge Ku(z)$ at any point $z \in G$ with $u(z) > -\infty$. If $\limsup_{z\to\zeta} u(z) \le 0$ for all $\zeta \in \partial G$, then either $u(z) < 0$ for all $z \in G$ or else $u(z) = 0$ for all $z \in G$.

Outline. Fix $a \in G$ and take $r > 0$ such that $B = B(a, r)$. There exists $M > 0$ such that $\rho \le \lambda \le M$ on B. now $u = \ln(\rho/\lambda)$, is upper semicontinuous on B, $u(z) \le 0$ for $z \in B$ and at any point $z \in B$ where $u(z) > -\infty$; that is, where $\rho(z) > 0$, we have

$\underline{\triangle} u \ge \rho^2 - \lambda^2 \ge 2M(\rho - \lambda)$. Hence $\underline{\triangle} u \ge 2M^2 u$. Theorem 1 implies that either $\rho(z) < \lambda_G(z$ for all $z \in B$ or else $\rho(z) = \lambda_G(z$ for all $z \in B$.

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[2]See D. Minda [53] for papers Heins [2], Hopf[3],Jorgensen [4].

Note that H.L Royden [The Ahlfors-Schwarz lemma: the case of equality, J. Analyse Math. 46 (1986), 261-270] also established the sharp form of Ahlfors' lemma by a different method.

Theorem 5.18 ([74])**.** *If ρ and σ are two metrics (density) on $\mathbb{D}$, σ complete and $0 > \bar{K}_\sigma \geq \bar{K}_\rho$ on $\mathbb{D}$ (where K_σ and K_ρ are the generalized curvatures), then $\sigma \geq \rho$.*

The method of sub-solutions and super-solutions have been used in study harmonic maps between surfaces cf. [38].

6. HYPERBOLIC GEOMETRY, MÖBIUS TRANSFORMATIONS AND CAYLEY-KLEIN MODEL IN SEVERAL VERIABLES

The unit sphere in three-dimensional space $\mathbb{R}^3$ is the set of points (x, y, z) such that $x^2 + y^2 + z^2 = 1$. Let $N = (0, 0, 1)$ be the "north pole", and let M be the rest of the sphere. The plane $z = 0$ contains the center of the sphere; the "equator" is the intersection of the sphere with this plane.

For any point P on M, there is a unique line through N and P, and this line intersects the plane $z = 0$ in exactly one point P'. Define the stereographic projection of P to be this point P' in the plane.

In Cartesian coordinates (x, y, z) on the sphere and (X, Y) on the plane, the projection and its inverse are given by the formulas

$$(X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right),$$

$$(x, y, z) = \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2}\right).$$

If M is a surface in $\mathbb{R}^3$ space and suppose that f is a difeomorphism of $\mathbb{U}$ onto M we can transfer the Poincaré disk model onto M. For example, L is line on M if $f^{-1}(L)$ is a U-line on $\mathbb{U}$. We define $d_{hyp,M}(p, q) = d_{hyp,\mathbb{U}}(f^{-1}(p), f^{-1}(q))$, $p, q \in M$. The disk model and M- model are isomorphic under f.

The inverse of the stereographic projection S maps the unit disk onto the hemisphere $\mathbb{S}^2_-$ and defines a $\mathbb{S}^2_-$- hyperbolic model and an orthogonal (orthographic) projection of this model on xy-plane defines the Klein model on $\mathbb{U}$.

Thus, the two models are related through a projection on or from the hemisphere model.

Shortly, the Klein model is an orthographic projection of the hemisphere model, while the Poincaré disk model is a stereographic projection.

Given two distinct points U and V in the open unit ball of the model in Euclidean space, the unique straight line connecting them intersects the unit sphere at two ideal points A and B, labeled so that the points are, in order along the line, A, U, V, B. Taking the centre of the unit ball of the model as the origin, and assigning position vectors $\mathbf{u}, \mathbf{v}, \mathbf{a}, \mathbf{b}$ respectively to the points U, V, A, B, we have that that $|a - v| > |a - u|$ and $|u - b| > |v - b|$, where $|\cdot|$ denotes the Euclidean norm. Then the distance between U and V in the modelled hyperbolic space is expressed as

$$d(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \log \frac{\|\mathbf{v} - \mathbf{a}\| \, \|\mathbf{b} - \mathbf{u}\|}{\|\mathbf{u} - \mathbf{a}\| \, \|\mathbf{b} - \mathbf{v}\|},$$

where the factor of one half is needed to make the curvature -1.

We will prove below that on the unit ball in $\mathbb{R}^n$ the associated metric tensor is given by the formula: if $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n$, then

$$ds^2(\mathbf{v}) = \mathrm{Kle}(\mathbf{x}, \mathbf{v}) = \frac{\|d\mathbf{v}\|^2}{1-\|\mathbf{x}\|^2} + \frac{(\sum_{k=1}^n x_k v_k)^2}{\left(1-\|\mathbf{x}\|^2\right)^2}\,. \tag{6.1}$$

It is supposedly classical and can be found in the literature that the restriction of the Beltrami-Klein metric on the ball of $\mathbb{R}^n$ to any minimal surface (minimal with respect tot he flat metric) has curvature ≤ -1. Unfortunately, the B-K metric is not conformally equivalent to the Euclidean one. Hence, a conformal minimal disk is not isothermal with respect to the B-K metric, and the pull-back is not a hermitian metric on the disk. Probably it is not even quasiconformal.

6.1. **The Cayley-Klein model of hyperbolic geometry.** The Poincaré disk model also called the conformal disk model, is a model of 2-dimensional hyperbolic geometry in which the points of the geometry are inside the unit disk, and the straight lines consist of all segments of circles contained within that disk that are orthogonal to the boundary of the disk, plus all diameters of the disk. Hyperbolic straight lines consist of all arcs of Euclidean circles contained within the disk that are orthogonal to the boundary of the disk, plus all diameters of the disk.

By arcosh and arsinh we denote inverses of hyperbolic functions:

$\operatorname{arsinh} x = \ln\left(x+\sqrt{x^2+1}\right)$, $\operatorname{arcosh} x = \ln\left(x+\sqrt{x^2-1}\right); x \geq 1$. By (14.1), we find

$$\cosh d = \frac{1}{2}(e^d + e^{-d}) = \frac{1}{2}(\frac{1+\sigma}{1-\sigma} + \frac{1-\sigma}{1+\sigma}) = \frac{1+\sigma^2}{1-\sigma^2} = 1 + \frac{2\sigma^2}{1-\sigma^2},$$

where $d = d_{\mathrm{hyp},\mathbb{H}}$ and $\sigma = \delta_{\mathbb{H}}$. Hence,

$$\cosh d = 1 + 2\frac{|z_1 - z_2|^2}{|z_1 - \overline{z_2}|^2 - |z_1 - z_2|^2},$$

and

since $|z_1 - \overline{z_2}|^2 - |z_1 - z_2|^2 = 4y_1y_2$, we find

$$\cosh d = 1 + \frac{|z_1 - z_2|^2}{2y_1y_2}\,.$$

Thus, in general, the distance between two points in $\mathbb{H}$ measured in hyperbolic metric along such a hyperbolic geodesic is:

$$\mathrm{dist}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = \operatorname{arcosh}\left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1y_2}\right).$$

The Cayley-Klein model of hyperbolic geometry

Distances in this model are Cayley-Klein metrics. Given two distinct points p and q inside the disk, the unique hyperbolic line connecting them intersects the boundary at two ideal points, a and b, label them so that the points are, in order, a, p, q, b and $|aq| > |ap|$ and $|pb| > |qb|$.

The hyperbolic distance between p and q is then

$d(p, q) = \log \dfrac{|aq|\ |pb|}{|ap|\ |qb|}$.

Set $\{p,q\} = \frac{|aq|\,|pb|}{|ap|\,|qb|}$. If the ideal points, a and b, label them so that the points are, in order, a, p, q, r, b, then $\{p,q\}\{q,r\} = \{p,r\}$ and therefore $d(p,r) = d(p,q)+d(q,r)$.

The vertical bars indicate Euclidean length of the line segment connecting the points between them in the model (not along the circle arc), log is the natural logarithm. Both the Poincaré disk model and the Klein disk model are models of the hyperbolic plane. An advantage of the Poincaré disk model is that it is conformal (circles and angles are not distorted); a disadvantage is that lines of the geometry are circular arcs orthogonal to the boundary circle of the disk. This section focuses on the projection of the unit sphere from the north pole onto the plane through the equator. Other formulations are treated in later sections.

The unit sphere in three-dimensional space $\mathbb{R}^3$ is the set of points (x, y, z) such that $x^2 + y^2 + z^2 = 1$. Let $N = (0, 0, 1)$ be the "north pole", and let M be the rest of the sphere. The plane $z = 0$ contains the center of the sphere; the "equator" is the intersection of the sphere with this plane.

For any point P on M, there is a unique line through N and P, and this line intersects the plane $z = 0$ in exactly one point P'. Define the stereographic projection of P to be this point P' in the plane.

In Cartesian coordinates (x, y, z) on the sphere and (X, Y) on the plane, the projection and its inverse are given by the formulas

$$(X,Y) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right),$$
$$(x,y,z) = \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2}\right).$$

If M is a surface in $\mathbb{R}^3$ space and suppose that f is a difeomorphism of $\mathbb{U}$ onto M we can transfer the Poincaré disk model onto M. For example, L is line on M if $f^{-1}(L)$ is a U-line on $\mathbb{U}$. We define $d_{hyp,M}(p,q) = d_{hyp,U}(f^{-1}(p), f^{-1}(q))$, $p, q \in M$. The disk model and M- model are isomorphic under f.

The inverse of the stereographic projection S maps the unit disk onto the hemisphere $\mathbb{S}^2_-$ and defines a $\mathbb{S}^2_-$- hyperbolic model and an orthogonal (orthographic) projection of this model on xy-plane defines the Klein model on $\mathbb{U}$.

Thus, the two models are related through a projection on or from the hemisphere model.

Shortly, the Klein model is an orthographic projection of the hemisphere model, while the Poincaré disk model is a stereographic projection.

Let o be an orthographic projection defined by $o(y_1, y_2, y_3) = (y_1, y_2, 0)$ and denote by S the inverse of the stereographic projection. Then S maps the unit disk onto S^2_- and $S_1 = o \circ S$ the unit disk onto itself. S maps circles K orthogonal to $\mathbb{T}$ onto circles $S(K)$ in S^2_- orthogonal to T and every $S(K)$ belongs to a plane parallel to e_3. Let L be a plane parallel to e_3 and let the half- circle K be the intersection of L and S^2_-. If $a, b \in S(K)$, c and d are ideal point on K, and $a' = o(a)$ and $b' = o(b)$, then by similarity $|a-d|^2 = 2R|a'-d|$ and $|a-c|^2 = 2R|a'-c|$, where R is the radius of $S(K)$. Hence

(i) $|a', b', c, d| = |a, b, c, d|^2$. Now let $z, w \in U$ be points on the circle K and let points z^*, w^* be the intersection of the unit circle by the circle K.

Since $S(z^*) = z^*$, $S(w^*) = w^*$ and the absolute cross ratio is invariant under Möbius, (ii) $|z, w; z^*, w^*| = |Sz, Sw; z^*, w^*|$.

Let K_l (in honor of Klein) denote the inverse of S_1. Note that K_l fixes the points on the unit circle $\mathbb{T}$.

Proposition 6.1. The distance in Klein model is $d_{kle}(z,w) = d_{hyp}(K_l(z), K_l(w))$ an it equals $\frac{1}{2}\ln|z,w;\hat{z},\hat{w}|$, where $\hat{z},\hat{w}$ are the intersection of the unit circle by line zw.

When projecting the same lines in both models on one disk both lines go through the same two ideal points(the ideal points remain on the same spot) also the pole of the chord is the centre of the circle that contains the arc.

6.2. **The Hyperbolic Metric and Möbius transformations.** For Möbius transformations in several dimensions see [6]. By $e_1,\dots,e_n$ we denote the coordinate unit vectors of $\mathbb{R}^n$. For example, $e_1 = (1,0,\dots,0)$, $e_2 = (0,1,\dots,0)$ and $e_3 = (0,0,1,\dots,0)$. We denote by $x_1,\dots,x_n$ the coordinates of a point $x \in \mathbb{R}^n$. Thus $x = (x_1,\dots,x_n)$ and $x = x_1e_1 + \cdots + x_ne_n$.

We denote by $\mathbb{R}^n_\infty = \overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ the one point compactification of $\mathbb{R}^n$.

By $B^n(a;r)$ we denote ball $\{x \in \mathbb{R}^n : |x-a| < r\}$ and by $S^{n-1}(a;r)$ sphere $\{x \in \mathbb{R}^n : |x-a| = r\}$.

Möbius transformation is a mapping which is composition of a finte number of the following:

(1) Translation: $f(x) = x + a$
(2) Stretching $f(x) = rx$, $r > 0$
(3) Orthogonal: f is linear and $|f(x)| = |x|$ for all $x \in \mathbb{R}^n$.
(4) Inversion in a sphere $S = S(a;r)$: $J(x) = a + r^2\frac{x-a}{|x-a|^2}$.

Every isometry of R^n can be uniquely written as the composition $t \circ k$ where t is a translation and k is an isometry xing the origin.

An $n\times n$ matrix A is called orthogonal if $A^TA = I_n$, or equivalently if $AA^T = I_n$. The geometric meaning of the condition $A^TA = I_n$ is that the columns of A are mutually perpendicular unit vectors (check!). Let $O(n) = O_n(R)$ denote the set of $n \times n$ orthogonal matrices.

The group of similarities consists of all mappings $x \mapsto mx + b$ where $b \in \mathbb{R}^n$ and m is a conformal matrix, i.e. $m = \lambda k$ with $\lambda > 0$ and $k \in O(n)$. Every Möbius can be expressed as a composite of inversions.

The reflection with respect to the unit sphere in R^n is defined by

$$x \mapsto x^* = Jx = x/|x|^2, \quad J_0 = \infty,\ J_\infty = 0\,.$$

The matrix $J'(x)$ has components $J'(x)_{ij} = \frac{1}{|x|^2}(\delta_{ij} - \frac{2x_ix_j}{|x|^2})$. We adapt a special notation for the matrix $Q(x)$ with entries $Q(x)_{ij} = \frac{x_ix_j}{|x|^2}$. This enables us to write $J'(x) = \frac{1}{|x|^2}(I - 2Q(x))$. This an important formula. From $Q^2 = Q$ we obtain $(I - 2Q)^2 = I$. In higher dimensions, a Möbius transformation is a homeomorphism of $\overline{\mathbb{R}^n}$, the one-point compactification of $\mathbb{R}^n$, which is a finite composition of inversions in spheres and reflections in hyperplanes. Lioville's theorem in conformal geometry states that in dimension at least three, all conformal transformations are Möbius transformations. Every Möbius transformation can be put in the form

$$f(x) = b + \frac{\alpha A(x-a)}{|x-a|^\epsilon}$$

where $a, b \in \mathbb{R}^n$, $\alpha \in \mathbb{R}, A$ is an orthogonal matrix, and ϵ is 0 or 2. The group of Möbius transformations is also called the full Möbius group and denote by $\hat{M}(\mathbb{R}^n)$.

The orientation-preserving Möbius transformations form The sub-group of $\hat{M}(\mathbb{R}^n)$ which we denote by $M(\mathbb{R}^n)$ and which is the connected component of the identity in the Möbius group. For any $\gamma \in \hat{M}(\mathbb{R}^n)$ we denote by $|\gamma'(x)|$ the positive number such that $\gamma'(x)/|\gamma'(x)| \in O(n)$. In other words, $|\gamma'(x)|$ is the linear change of scale at x which is the same in all directions. In higher dimensions, we define absolute cross ratio $|a, b, c, d| = \frac{|a-c|}{|a-d|} : \frac{|b-c|}{|b-d|}$, which is invariant $|\gamma a, \gamma b, \gamma c, \gamma d| = |a, b, c, d|$, $\gamma \in \hat{M}(\mathbb{R}^n)$. This is clear when γ is a similarity, and for J we obtain $|Jx - Jy|^2 = |J'(x)||J'(y)||x - y|^2$.

For $a \in B^n$ $(a \neq 0)$, $R = R(a) = (|a^*|^2 - 1)^{1/2} = \sqrt{1 - |a|^2}/|a|$. Then $S^{n-1}(a^*, R(a))$ is orthogonal to the unit sphere S.

The reflection (inversion) with respect to this sphere is given by

$$\sigma_a x = a^* + R(a)^2 (x - a^*)^* . \tag{6.2}$$

Define canonical mapping

$$T_a(x) = (I - 2Q(a))\sigma_a x . \tag{6.3}$$

The explicit expression for $T_a(x)$ is

$$T_a(x) = -a + (1 - |a|^2)(x^* - a)^* = \frac{(1 - |a|^2)(x - a) - |x - a|a}{[x, a]^2}, \tag{6.4}$$

where $[x, a] = |x||x^* - a| = |a||x - a^*|$.
If $\gamma \in \hat{M}(\mathbb{R}^n)$ maps a in 0, then $\gamma = kv$, where $k \in O(n)$.

Let $x = (x_1, \ldots, x_n)$ be the coordinates on $\mathbb{R}^n$. The Poincaré metric on the unit $\mathbb{B} \subset \mathbb{R}^n$ is given by

$$ds^2_{\mathbb{B}} = \frac{4|dx|^2}{(1 - |x|^2)^2}.$$

It is conformally equivalent to the Euclidean metric. The 2-dimensional case $n = 2$ is the standard Poincaré metric on the unit disk $\mathbb{D} \subset \mathbb{R}^2 \cong \mathbb{C}$.

The Hyperbolic Metric.

Let $\mathbb{B}^3$ be the unit ball $\{x \in \mathbb{R}^3 : ||x|| < 1\}$ in Euclidean 3-space. Using analogy with the planar unit disk

the hyperbolic density on $\mathbb{B}^3$ is defined by

$$\lambda(x) = \frac{2}{1 - ||x||^2}.$$

The hyperbolic length of a smooth curve $\gamma : [a, b] \to \mathbb{B}^3$ is then

$$L(\gamma) = \int_a^b \lambda(\gamma(t))||\gamma'(t)||dt = \int_a^b \frac{2||\gamma'(t)||}{1 - ||\gamma(t)||^2} dt.$$

The hyperbolic metric λ on $\mathbb{B}^3$ is defined by $\lambda(x_0, x_1) = \inf\{L(\gamma)$, where the infimum is taken over all γ smooth curves in $\mathbb{B}^3$ from x_0 to x_1.

A curve that attains this infimum is a hyperbolic geodesic from x_0 to x_1. The arguments used for the hyperbolic metric on the unit disc (Lemma XX and Theorem XX) show that:

Proposition 6.2 (Hyperbolic metric on $\mathbb{B}^3$)**.** The hyperbolic metric is a metric on the unit ball $\mathbb{B}^3$.

Moreover, the hyperbolic geodesic from the origin 0 to any point $x \in \mathbb{B}^3$ is a radial path with hyperbolic length $\log \frac{1+||x||}{1-||x||}$.

Hyperbolic distance between arbitrary point $x, y \in \mathbb{B}^3$ is

$$\lambda(x,y) = \log|y,x,\xi,\eta| = \log \frac{1+||T_y(x)||}{1-||T_y(x)||},$$

where ξ and η are ends of geodesics through x and y, and T_y is defined by (6.4).

We can use the arguments above for any ball in $\mathbb{R}^3_\infty$ and obtain a hyperbolic metric on the ball for which the orientation preserving isometries are the Möbius transformations. The most important example is when the ball is the upper half-space:$\mathbb{R}^3_+ = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 > 0\}$. The boundary of this is the extended complex plane $\mathbb{C}_\infty = \mathbb{R}^2_\infty$. We can show that any Möbius transformation acting on this boundary extends to an orientation preserving isometry of the upper half-space for the hyperbolic metric with density: $\lambda(x) = \frac{1}{x_3}$. We can also deduce the results for the upper half-space directly from those for the ball $\mathbb{B}^3$ for inversion in the sphere.

We have seen how to put a hyperbolic metric on the unit ball $\mathbb{B}^3$ in $\mathbb{R}^3$ or the upper half-space $\mathbb{R}^3_+$. We will denote both of these by $\mathbb{H}^3$ and call them hyperbolic 3-space. The orientation preserving isometries for hyperbolic 3-space have been identified with the group of Möbius transformations acting on the boundary $\partial\mathbb{H}^3$.

6.3. **Klein model.** We can show that (A) the group $M(\mathbb{H}^n)$ is isomorphic with the group $M(\mathbb{B}^n)$, using a Mobius transformation of $\mathbb{H}^n$ onto $\mathbb{B}^n$. We choose so that $0, e_n, \infty$ correspond by $y = \sigma x$ to $-e_n, 0, e_n$, where e_n is the last coordinate vector. The restriction of σ on R^{n-1} is the usual stereographic projection.

The correspondence is given by

$$y = \sigma x = (e_n + 2(x-e_n)^*)^*$$
$$x = \sigma^{-1}(y) = e_n + 2(y^* - e_n)^*$$

When $x_n = 0$, one verifies that $|y|^2 = 1$, $y^* = y$ and (X) reduces to

$$y_i = \frac{2x_i}{1+|x|^2}, y_n = \frac{|x|^2-1}{1+|x|^2} \tag{6.5}$$

and for $|y| = 1$, we find

$$x_i = \frac{y_i}{1-y_n}, x_n = 0 \tag{6.6}$$

the stereographic projection (6.5) maps ball $B^{n-1} = \{(x_1, x_2, ..., x_{n-1}, 0) : x_1^2 + x_2^2 + \cdots + x_{n-1}^2 < 1\}$ on the lower hemi-sphere S^{n-1}. The composition of the stereographic projection with the mapping $(y_1, y_2, ..., y_{n-1}, y_n) \mapsto (y_1, y_2, ..., y_{n-1})$

$$y = \frac{2x}{1+|x|^2}, (x \in \mathbb{R}^{n-1}, |x| < 1), \tag{6.7}$$

and it maps $\mathbb{B}^{n-1}$ onto itself. The inverse mapping is L is given by

$$x = Ly = \frac{y}{1-y_n}, (y \in \mathbb{R}^{n-1}, |y| < 1), \tag{6.8}$$

where $-y_n = (1-|y|^2)^{1/2}$. Note that here $y = (y_1, y_2, ..., y_{n-1})$.

The equation of an orthogonal circle is of the form $|x-a|^2 = |a|^2 - 1$, $|a| > 1$ or $|x|^2 + 1 = 2xa$ and (6.7) is equivalent to $ay = 1$, the equation of a straight line. This can be used to construct the Klein model of hyperbolic space. In this model the noneuclidean lines are the lines segments in B^{n-1}.

Proof of (6.1) (Klein-Finsler norm) (see also 14.5).

Set $\omega_{hyp} = 2(1-|X|^2)^{-1}|dX|$, $\omega^2_{hyp} = 4(1-|X|^2)^{-2}|dX|^2$ and $\omega_{\rm kle} = L^*\omega_{\rm hyp}$. Note that

(1) $1 - y_n = \frac{1}{1+|X|^2}$

(2) $1 - |X|^2 = \frac{2y_n}{1-y_n}$

(3) $y_n dy_n = -\omega$, where $\omega = y_1 dy_1 + y_2 dy_2 + \cdots + y_{n-1} dy_{n-1} = y \cdot dy$.

Hence

$$dX = \frac{dY}{1-y_n} + Y\frac{dy_n}{(1-y_n)^2} = \frac{dY}{1-y_n} + Y\frac{\omega}{y_n(1-y_n)^2}$$

and

$(1-|X|^2)^{-1}|dX| = \frac{1}{2y_n}|\omega_1|$, where $\omega_1 = dY - Y\frac{\omega}{y_n(1-y_n)}$. Set $Pv = P_Y v = (Y,v)Y/|Y|$ and $Qv = v - Pv$. For $v \in T_Y\mathbb{R}^{n-1}$, we find

$$\omega_1(v) = Pv + Qv - \frac{|Y|^2}{y_n(1-y_n)}Pv = -\frac{1}{y_n}Pv + Qv,$$

and therefore

$$\omega^2_{\rm kle}(v) = \frac{|v|^2}{y_n^2} + \frac{|(v,Y)|^2}{y_n^4} = \frac{|v|^2}{1-|Y|^2} + \frac{|(v,Y)|^2}{(1-|Y|^2)^2}. \tag{6.9}$$

6.4. **Conformal minimal immersion.** Let $x = (x_1, \ldots, x_n)$ be the coordinates on $\mathbb{R}^n$. The Poincaré metric on the unit $\mathbb{B} \subset \mathbb{R}^n$ is given by

$$ds^2_{\mathbb{B}} = \frac{4|dx|^2}{(1-|x|^2)^2}.$$

It is conformally equivalent to the Euclidean metric. The 2-dimensional case $n = 2$ is the standard Poincaré metric on the unit disk $\mathbb{D} \subset \mathbb{R}^2 \cong \mathbb{C}$. Let $S \subset \mathbb{B}$ be a minimal surface (with respect to the Euclidean metric). Let $h = ds^2_{hyp|S}$ be the metric on S obtained by restricting the Poincaré metric form $ds^2_{\mathbb{B}}$ on S (inherited from form $ds^2_{\mathbb{B}}$) . Since ds^2_P is conformally equivalent to the Euclidean metric, it introduces the same conformal structure on S as the Euclidean metric.

Problem 1. Does the Gaussian curvature of (S, h) satisfy $K_h \le -1$?

If S is euclidean disk then the Gaussian curvature of (S, h) equals -1.

Under "hyperbolic" we mean the Poincare metric, then the answer is no, the curvature of minimal submanifolds need not decrease; we get this informaion via Forstneric [66].

Proposition 6.3. Let $f\colon \mathbb{D} \to \mathbb{B} \subset \mathbb{R}^n$ be a conformal immersion, $S = f(\mathbb{U})$ and $K_h \le -1$. Then $f^*ds_P \le ds_P$. That is, the pullback of the Poincaré metric on $\mathbb{B}$ to the disk $\mathbb{D}$ is bounded above by the Poincaré metric on $\mathbb{D}$. By integration we get

$$\operatorname*{dist}_{\mathbb{B}}(f(z), f(w)) \le \operatorname*{dist}_{\mathbb{D}}(z, w), \quad z, w \in \mathbb{D}.$$

Proof. The conformal surface (S, h) has a unique Riemann surface structure. In any isothermal local complex coordinate z on S we have $h = \lambda(z)|dz|^2$, so h is a Kähler metric. Furthermore, a conformal parametrization $f\colon \mathbb{D} \to S$ is a holomorphic map (up to a correct choice of orientation). If $K_h \leq -1$ were, we could apply the Ahlfors lemma which tells us that

$$f^*(ds^2_{\mathbb{B}}) = f^*(h) \leq ds^2_{\mathbb{D}}.$$

□

Via Forstneric [66], we get the following information.

It is supposedly classical and can be found in the literature that the restriction of the Beltrami-Klein metric on the ball of $\mathbb{R}^n$ to any minimal surface (minimal with respect tot he flat metric) has curvature ≤ -1. This is what we need. Unfortunately, the B-K metric is not conformally equivalent to the Euclidean one. Hence, a conformal minimal disk is not isothermal with respect to the B-K metric, and the pull-back is not a hermitian metric on the disk. Probably it is not even quasiconformal.

There is a related results related to the estimate of the Gaussian curvature of analytic disks and more generally for complex submanifolds of Hermitian manifolds. For the following result, see [35]:

Theorem 10. If M' is a complex submanifold of a Hermitian manifold M, then the holomorphic bisectional (sectional) curvature of M' does not exceed that of M.

It is also interesting fact that The Bergman metric and the Beltrami-Klein metric are tightly related.

The Bergman metric is on the unit ball in $\mathbb{C}^n$ is given by

$$ds^2 = (n+1)\Big(\frac{|d\mathbf{z}|^2}{1-|\mathbf{z}|^2} + \sum_{\mu,\nu=1} \frac{\overline{z}_\mu z_\nu dz_\mu d\overline{z}_\nu}{(1-|z|^2)^2}\Big)\,.$$

More precisely, if $\mathbf{v} \in T_{\mathbf{z}}\mathbb{C}^n$, then

$$ds^2(\mathbf{v}) = \mathrm{Ber}(\mathbf{z},\mathbf{v}) = (n+1)\Big(\frac{\|d\mathbf{v}\|^2}{1-\|\mathbf{z}\|^2} + \frac{(\sum_{k=1}^n z_k \overline{v_k})^2}{\big(1-\|\mathbf{z}\|^2\big)^2}\Big)\,. \tag{6.10}$$

The restriction of this metric on the unit ball in $\mathbb{R}^n$ is up to the constant the Klein metric. More precisely, if $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n$, then $\mathrm{Ber}(\mathbf{z},\mathbf{v}) = (n+1)\mathrm{Kle}(\mathbf{x},\mathbf{v})$.

7. Schwarz lemma in the unit ball

In this section we follow [48].For further result see [49]. If f is a function on a set X and $x \in X$ sometimes we write fx instead of $f(x)$. We write $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$.

On $\mathbb{C}^n$ we define the standard Hermitian inner product by $< z, w >= \sum_{k=1}^n z_k \overline{w_k}$ for $z, w \in \mathbb{C}^n$ and by $|z| = \sqrt{< z, z >}$ we denote the norm of vector z. We also use notation (z, w) instead of $< z, w >$ on some places. By $\mathbb{B}_n$ we denote the unit ball in $\mathbb{C}^n$. In particular we use also notation $\mathbb{U}$ and $\mathbb{D}$ for the unit disk in complex plane.

For planar domains G and D we denote by $\mathrm{Hol}(G, D)$ the class of all holomorphic mapping from G into D. For complex Banach manifold X and Y we denote by $\mathcal{O}(X, Y)$ the class of all holomorphic mapping from X into Y.

We need some properties of bi -holomorphic automorphisms of unit ball (see [69] for more details). For a fixed z, $B_z = \{w : (w - z, z) = 0, |w|^2 < 1\}$ and denote by $R(z)$ radius of ball B_z. Denote by $P_a(z)$ the orthogonal projection onto the subspace $[a]$ generated by a and let $Q_a = I - P_a$ be the projection on the orthogonal complement. For $z, a \in \mathbb{B}^n$ we define

$$\tilde{z} = \varphi_a(z) = \frac{a - Pz - s_a Qz}{1 - (z, a)}, \tag{7.1}$$

where $P_a(z) = \dfrac{< z, a >}{< a, a >} a$ and $s_a = (1 - |a|^2)^{1/2}$. Set $U^a = [a] \cap \mathbb{B}$, $Q^b = b + [a]^\perp \cap \mathbb{B}_n$,

$$\varphi_a^1(z) = \frac{a - Pz}{1 - (z, a)}, \quad \varphi_a^2(z) = \frac{-s_a Qz}{1 - (z, a)}$$

and $\delta(a, z) = |\varphi_a(z)|$.

Then one can check that

(I1) The restriction of φ_a onto U^a is automorphisam of U^a and the restriction onto B_z maps it bi-holomorphically mapping onto $B_{\tilde{z}}$.

A domain U is called complete circular if whenever $z \in U$ and $|\lambda| \leq 1$ then $\lambda z \in U$. Note in passing that a complete circular domain automatically contains 0.

We need a few results from Rudin [69].

For a we define $s = s_a = \sqrt{1 - |a|^2}$.

Theorem 11 (2.2.2 [69])**.** For every $a \in \mathbb{B}$, φ_a has the following properties:
(i) $\varphi_a(0) = a$ and $\varphi_a(a) = 0$
(ii) $\varphi_a'(0) = -s^2 P - sQ$, $\varphi_a'(a) = -P/s^2 - Q/s$
(iii) the identity

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - (z, a)|^2},$$

(iv) φ_a is an involution: $\varphi_a(\varphi_a(z)) = z$
(v) φ_a is a homeomorphism of $\overline{\mathbb{B}}$ onto $\overline{\mathbb{B}}$, and $\varphi_a \in \mathrm{Aut}(\mathbb{B})$.
(vi) $\mathrm{Aut}(\mathbb{B})$ acts transitively on $\mathbb{B}$.

We only outline a proof. Since $(1 - (z, a))^{-1} = 1 + < z, a > + O(|z|^2)$ and $|a|^2 Pz = a < z, a >$, $\varphi_a(z) = a - (P + sQ)z + a < z, a > + O(|z|^2)$. Hence

$$\varphi_a(z) - \varphi_a(0) = -s^2 Pz - sQz + O(|z|^2)$$

and therefore the first formula in (ii) follows; the second one follows from

$$\varphi_a(a + h) = \frac{-Ph - sQh}{s^2 - < h, a >}.$$

From (iv), it follows that φ_a is one-to-one of $\overline{\mathbb{B}}$ onto $\overline{\mathbb{B}}$, and that $\varphi_a^{-1} = \varphi_a$. If $a, b \in \mathbb{B}$, $\varphi_b \circ \varphi_a$ is an automorphism of B that takes a to b.

If $f \in Aut(\mathbb{B})$, $a = f^{-1}(0)$, $J_{\mathbb{R}} f$ denotes real Jacobian, then

$$J_{\mathbb{R}} f(z) = \left(\frac{(1 - |a|^2)}{|1 - (z, a)|^2}\right)^{n+1}. \tag{7.2}$$

Proposition 7.1 (Theorem 8.1.2)**.** Suppose that (i) G and G' are complete circular domains in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively,
(ii) G' convex and bounded
(iii) $F : G \to G'$ holomorphic
Then
(a) $F'(0)$ maps G into G'
(b) $F(rG) \subset rG'$ $(0 < r \leq 1)$ if $F(0) = 0$.

The following is an immediate corollary of Proposition 7.1:

Corollary 3. *Suppose that $f \in \mathcal{O}(\mathbb{B}_n, \mathbb{B}_m)$. If $f(0) = 0$, then*
(A1) $|f'(0)| \leq 1$.

We give another proof which is more in spirit of this paper.

Proof. For $z^* = z/|z|$ define $D_z = \{\zeta z^* : \zeta \in \mathbb{U}\}$ and $F(\zeta) = f(\zeta z^*)$, $\zeta \in \mathbb{U}$. Let p be projection of $\mathbb{B}_m$ on the slice $D_{f(z)}$. By one dim version of Schwarz lemma $|F(\zeta)| \leq |\zeta|$ and in particular for $\zeta = |z|$, $|f(z)| \leq |z|$. Hence (A1) $|f'(0)| \leq 1$. □

Proposition 7.2 (Theorem 8.1.4 [69])**.** Suppose that $f : \mathbb{B}_n \to \mathbb{B}_m$ holomorphic, $a \in \mathbb{B}_n$ and $b = f(a)$.
Then
$|\varphi_b(f(z))| \leq |\varphi_a(z)|$, $z \in \mathbb{B}_n$
or equivalently

$$\frac{|1-(fz,fa)|^2}{(1-|fa|^2)(1-|fz|^2)} \leq \frac{|1-(z,a)|^2}{(1-|a|^2)(1-|z|^2)}. \tag{7.3}$$

Set

$$\sigma_n(z,a) := \frac{|1-(z,a)|^2}{(1-|a|^2)(1-|z|^2)}.$$

For $z, w \in \mathbb{C}^n$, $|1- <z,w>|^2 = 1 + |<z,w>|^2 - (|z|^2 + |w|^2) + |z-w|^2$ and therefore
(A1) $|1- <z,w>|^2 \leq (s_z s_w)^2 + |z-w|^2$ and $|1- <z,w>|^2 = (s_z s_w)^2 + |z-w|^2$, $z, w \in \mathbb{C}$, that is

$$\sigma_n(z,w) \leq 1 + \frac{|z-w|^2}{(s_z s_w)^2}, \quad \sigma_1(z,w) = 1 + \frac{|z-w|^2}{(s_z s_w)^2}, z, w \in \mathbb{C}. \tag{$B1$}$$

Theorem 12 ([32, 48])**.** Suppose that $f \in \mathcal{O}(\mathbb{B}_n, \mathbb{B}_m)$, $a \in \mathbb{B}_n$ and $b = f(a)$.
(i) Then $s_a^2 |f'(a)| \leq s_b$, i.e. $(1-|a|^2)|f'(a)| \leq \sqrt{1-|f(a)|^2}$.
(ii) If $m = 1$, then $s_a^2 |f'(a)| \leq s_b^2$, and
(iii) If $m > 1$, the inequality (a) $\sigma_m(fz, fw) \leq \sigma_n(z,w)$, $z, w \in \mathbb{B}_n$, does not hold in general, but if $f \in \mathcal{O}(\mathbb{B}_n, \mathbb{B}_1)$ then $\sigma_1(fz, fw) \leq \sigma_n(z,w)$, that is the following inequality holds:

$$\sigma_1(fz,fa) = \frac{|fz-fa|^2}{(1-|fa|^2)(1-|fz|^2)} \leq \frac{|z-a)|^2}{(1-|a|^2)(1-|z|^2)}, \quad z \in \mathbb{B}_n. \tag{7.4}$$

Proof. (i) Suppose first that $f(0) = 0$ and take $z \in \mathbb{B}_n$. Hence (A1) $|f'(0)| \leq 1$.

For $u \in T_a C^m$, by Theorem 11(ii), $v = \varphi_a'(a)u = -Pu/s^2 - Qu/s$ and, by Pitagora's theorem,

$$|u| = \sqrt{|Pu|^2 + |Qu|^2}, |v|^2 = |Pu|^2/s^4 + |Qu|^2/s^2$$

and therefore we find

$$\frac{|u|}{s} \leq |\varphi_a'(a)u| \leq \frac{|u|}{s^2}. \tag{$B1$}$$

If $f(a) = b$, set $h = \varphi_b \circ f \circ \varphi_a$. By the chain rule $h'(0) = \varphi_b'(b) \circ f'(a) \circ \varphi_a'(0)$.

Set $u \in T_aC^n$, $v = f'(a)u \in T_aC^m$, $u' = \varphi_a'(a)u$ and $v' = \varphi_b'(b)v$. By (A1), $|v'| \le |u'|$. Since, by (B1),

$$\frac{|v|}{s_b} \le |v'| \quad \text{and} \quad |u'| \le \frac{|u|}{s_a^2},$$

hence $s_a^2|f'(a)| \le s_b$, i.e. $(1-|a|^2)|f'(a)| \le \sqrt{1-|f(a)|^2}$ and therefore (i) is proved.

(ii) If $m = 1$, then $s_b^2|v'| = |v|$ and (ii) follows.
(iii) By (B1) and (7.3),

$$\sigma_1(fz, fa) = 1 + \frac{|fz - fa|^2}{(s_{fz}s_{fa})^2} \le \sigma_n(z,a) = \frac{|1-(z,w)|^2}{(s_zs_a)^2} \le 1 + \frac{|z-a|^2}{(s_zs_a)^2}$$

and therefore (7.4). If z tends a, (ii) also follows from (7.4). If (a1) holds, then (b1) $s_a^2|f'(a)| \le s_b^2$. For function $f_0 = \varphi_b \circ \varphi_a$ we have $(1-|a|^2)|f_0'(a)| = (1-|b|^2)$ which yields a contradiction with (b1). □

8. Contraction properties of holomorphic functions with respect to Kobayashi distances

The author also published a paper [42] about holomorphic fixed point theorem on Riemann surfaces.

Definition 8.1. Let G be bounded connected open subset of complex Banach space, $p \in G$ and $\mathbf{v} \in T_pG$. We define $k_G(p,\mathbf{v}) = \inf\{|\mathbf{h}|\}$, where infimum is taking over all $\mathbf{h} \in T_0\mathbb{C}$ for which (i): there exists a holomorphic function $\phi : \mathbb{U} \to G$ such that $\phi(0) = p$ and $d\phi_0(\mathbf{h}) = \mathbf{v}$.

Let $H = H(p,v)$ be the set of functions ϕ for which (i) holds $I = I(p,v)$ be the set of $h > 0$ for which (i) holds and let $J = J(p,v)$ be the set of $\lambda > 0$ for which there exists a holomorphic function $\phi : \mathbb{U} \to G$ such that $\phi(0) = p$ and $d\phi_0(1) = \lambda\mathbf{v}$, and $\lambda_0 = \sup\{\lambda \in J\}$. Since $\lambda \in I$ iff $(\lambda)^{-1} \in J$,then $k_G(p,\mathbf{v}) = \inf\{h : h \in I\} = \inf\{h : h^{-1} \in J\} = \inf\{h^{-1} : h \in J\} = 1/\lambda_0$.

If ϕ is a holomorphic map of $\mathbb{U}$ into G, we define $L_Gu(p,v) = \sup\{\lambda : \phi(0) = p,\ d\phi_0(1) = \lambda v\}$, and $L_G(p,v) = \sup L_Gu(p,v)$, where the supremum is taken over all maps $\phi : \mathbb{U} \to G$ which are analytic in $\mathbb{U}$ with $\phi(0) = p$. Note that $L_G(p,v)k_G(p,v) = 1$. By Definition 8.1,

$$\mathrm{Kob}_G(p,v) = \frac{1}{L_G(p,v)}. \tag{8.1}$$

If G is the unit ball, we write $L_\phi(p,v)$ instead of $L_G\phi(p,v)$.

We define the distance function on G by integrating the pseudometric k_G: for $z, z_1 \in G$

$$\mathrm{Kob}_G(z,z_1) = \inf_\gamma \int_0^1 k_G(\gamma(t),\dot\gamma(t))\,dt \tag{8.2}$$

where the infimum is over all piecewise paths γ: $[0,1] \to G$ with $\gamma(0) = z$ and $\gamma(1) = z_1$. For complex Banach manifold X and Y we denote by $\mathcal{O}(X,Y)$ the class of all holomorphic mapping from X into Y. If $\phi \in \mathcal{O}(\mathbb{U},X)$ and $f \in \mathcal{O}(X,Y)$, then $\phi \circ f \in \mathcal{O}(\mathbb{U},Y)$.

We can express Kobayashi-Schwarz lemma in geometric form:

Theorem 8.2. *If $a \in X$ and $b = f(a)$, $u \in T_a X$ and $u_* = f'(a)u$, then*

$$\mathrm{Kob}(b, u_*) \le \mathrm{Kob}(a, u)\,. \tag{8.3}$$

The proof is based on the fact that if $\phi \in H(a,u)$, then $f \circ \phi \in H(b, u_*)$.
Using Theorem 8.2, one can prove:

Theorem 8.3. *Suppose that G and G_1 are bounded connected open subset of complex Banach space and $f : G \to G_1$ is holomorphic. Then*

$$\mathrm{Kob}_{G_1}(fz, fz_1) \le \mathrm{Kob}_G(z, z_1) \tag{8.4}$$

for all $z, z_1 \in G$.

Let $A = \{1 < |x| < 4\}$, $A^* = \{2 < |x| < 3\}$, $l(t) = 2 + \frac{1}{3}(t-1)$ and$f(x) = -l(|x|)x$. l maps the interval $(1,4)$ onto the interval $(2,3)$ and therefore f maps A onto $A^* \subset A$, but f has no fixed point (there is no point $x \in A$ such that $f(x) = x$. Hence this example shows that there is no a metric d on G such that f is a contraction wrt d. The situation is completely different for analytic functions.

Theorem 8.4. *Suppose that G is bounded connected open subset of complex Banach space and $G_* \subset G$, $s_0 = dist(G_*, G^c)$, $d_0 = diam(G)$ and $q_0 = \frac{d_0}{d_0 + s_0}$. Then*
(i) $\mathrm{Kob}_G \le q_0 \mathrm{Kob}_{G_*}$ *on G_*.*
(ii) *In addition if $f : G \to G_*$ is holomorphic, then*

$$\mathrm{Kob}_{G_*}(fz, fz_1) \le q_0 \mathrm{Kob}_{G_*}(z, z_1) \tag{8.5}$$

for $z, z_1 \in G_*$.

$$\mathrm{Kob}_G(fz, fz_1) \le q_0 \mathrm{Kob}_G(z, z_1) \tag{8.6}$$

for $z, z_1 \in G$.

Proof. Suppose that $p \in G_*$, $v \in T_p G_*$ and $\phi : \mathbb{U} \to G$ is a holomorphic function such that $\phi(0) = p$ and $d\phi(h) = v$. Set $R_s = \frac{d_0+s}{d_0}$ and $q_s = \frac{d_0}{d_0+s}$. For $h \in \mathbb{U}$ define $\phi_s(h) = p + R_s(\phi(h) - p)$. Then $\phi_s(h) - \phi(h) = (R_s - 1)(\phi(h) - p)$ and therefore $|\phi_s(h) - \phi(h)| \le s$. For $s < s_0$, ϕ_s maps $\mathbb{U}$ into G and $d\phi_s(h) = R_s v$. Hence $k_G(p,v) \le q_s k_{G_*}(p,v)$ and if s approaches s_0 we first get (i) $k_G(p,v) \le q_0 k_{G_*}(p,v)$ and by a standard procedure $\mathrm{Kob}_G \le q_0 \mathrm{Kob}_{G_*}$. Now, by (8.4), we have (ii) $\mathrm{Kob}_{G_*}(fz, fz_1) \le \mathrm{Kob}_G(z, z_1)$. Combining (i) and (ii) we get (8.5) and (8.6). □

If $d_0 = diam(G)$ is not finite, elementary example: $H_a = \{z : Imz > a\}$ with $f(z) = z + ia$ which maps H onto H_a, shows that the theorem does not hold.

Theorem 8.5 (Carthéodory)**.** *Let $D \subset C^n$ domain for which Kobayshi pseudodistance is distance and $f : D \to D$ holomorphic mapping such that $\overline{f(D)}$ is a compact subset of D. Then f is contraction with respect to Kobayshi (Carthéodory) metric on D. In particular f has fixed points in D.*

It is a corollary of Theorem 8.4. A version of Theorems 8.3-8.4 was proved in 1968 by Clifford Earle and Richard Hamilton [21] (see subsections 8.2 for further comments).

8.1. **Addition to the proof of Theorem 12**(iii) **and and Theorems 8.3-8.4.** The Schwarz-Pick lemma states that every holomorphic function from the unit disk $\mathbb{U}$ to itself, or from the upper half-plane $\mathbb{H}$ to itself, will not increase the Poincaré distance between points.

It is convenient to introduce a pseudo-distance

$$\delta(z,\omega)=|\varphi_z(\omega)|=\left|\frac{z-\omega}{1-\overline{\omega}\,z}\right|,\quad z,\omega\in\mathbb{U} \tag{8.7}$$

which is a *conformal invariant.*

Shwarz-Pick lemma: If f holomorphic function from the unit disk $\mathbb{U}$ to itself, then

$$\delta(f(z),f(\omega))\le\delta(z,\omega),\quad z,\omega\in\mathbb{U} \tag{8.8}$$

with equality only if f is a Möbius transformation of $\mathbb{D}$ onto itself.

For $z,w\in\mathbb{C}$, set $a=(1-|z|^2)(1-|w|^2)$, $b=|z-w|^2$, $A=(1-|fz|^2)(1-|fw|^2)$, and $B=|fz-fw|^2$. By this notation,
(A2) $|1-<z,w>|^2=1+|<z,w>|^2-(|z|^2+|w|^2)+|z-w|^2=a+b$,
(B2) $|1-<fz,fw>|^2=A+B$.

If $f\in\mathcal{O}(\mathbb{B}_1,\mathbb{B}_1)$, using (A2) and (B2) Shwarz-Pick lemma can be rewritten in the form $\frac{B}{b}\le\frac{A+B}{a+b}$ and therefore $Ba\le Ab$, that is

(I) $\quad |fz-fw|\sqrt{(1-|z|^2)}\sqrt{(1-|w|^2)}\le\sqrt{(1-|fz|^2)}\sqrt{(1-|fw|^2)}|z-w|.$[3]

We can rewrite (I) as

(I1) $\quad I_f(z,w):=\dfrac{|fz-fw|}{|z-w|}\sqrt{(1-|z|^2)}\sqrt{(1-|w|^2)}\le\sqrt{(1-|fz|^2)}\sqrt{(1-|fw|^2)}.$

Note if $w\to z$, then $I_f(z,w)\to(1-|z|^2)|f'(z)|$.

By $\mathcal{B}$ we denote the Bloch space of holomorphic function on $\mathbb{U}$ with the "norm" $|f|_{\mathcal{B}}:=\sup\{(1-|z|^2)|f'(z)|:z\in\mathbb{U}\}$. XX Since $g:=\frac{fz-fw}{z-w}$ is holomorphic in two variables z,w, show that max $M^g(r)$ of $|g|$ on U_r^2 is attained on T_r^2.

For $z,w\in T_r$ let $K(z,w)$ be circle arc joins z and w. Since $fz-fw=\int_{K(z,w)}f'(\zeta)d\zeta$, we have $(1-r^2)M^g(r)\le|f|_{\mathcal{B}}$. D. Jocić has mentioned the following question:

Question. Whether $\sup\{I_f(z,w):z,w\in\mathbb{U},z\ne w\}=\sup\{(1-|z|^2)|f'(z)|:z\in\mathbb{U}\}=|f|_{\mathcal{B}}$?

Set

(I2) $\quad I_f^2(z,w):=\dfrac{|fz-fw-f'(w)(z-w)|}{|z-w|^2}\sqrt{(1-|z|^2)}\sqrt{(1-|w|^2)}.$

Question. Determine F such that $I_f^2(z,w)\le\sqrt{(1-|fz|^2)}\sqrt{F(w)}$.

Question 1 (D. Jocić). If $f\in\mathcal{O}(\mathbb{B}_n,\mathbb{B}_m)$ whether (I) holds?

For $z,w\in\mathbb{C}^n$, $|z-w|^2=|z|^2+|w|^2-2\mathrm{Re}<z,w>$, and $|1-<z,w>|^2=1-2Re<z,w>+|<z,w>|^2$.
Hence
$|1-<z,w>|^2=1+|<z,w>|^2-(|z|^2+|w|^2)+|z-w|^2$ and
$|1-<fz,fw>|^2=1+|<fz,fw>|^2-(|fz|^2+|fw|^2)+|fz-fw|^2$ and

[3] D. Jocić turns my attantion on this form and after communication with him we have added the proof of (7.4))

By Cauchy-Shwarz inequality $| < z, w > |^2 \leq |z||w|$ and therefore

(C2) $|1- < z, w > |^2 \leq a_n + b_n$, where $a_n = (1-|z|^2)(1-|w|^2)$, and $b_n = |z-w|^2$. Set $A_m = (1-|fz|^2)(1-|fw|^2)$ and $B_m = |fz - fw|^2$. By (C2) and (7.3),

$$\sigma_1(fz, fw) = \frac{A_1 + B_1}{A_1} \leq \sigma_n(z, w) = \frac{|1-(z,w)|^2}{a_n} \leq \frac{a_n + b_n}{a_n}$$

and therefore (7.4).

We show that (I) does not hold in general. Contrary suppose that (I) holds and that $f \in \mathcal{O}(\mathbb{B}_n, \mathbb{B}_m)$, $a \in \mathbb{B}_n$ and $b = f(a)$.
Recall if $m > 1$ we proved,
(II) $(1-|a|^2)|f'(a)| \leq \sqrt{1-|f(a)|^2}$.
Note that for function $f_0 = \varphi_b \circ \varphi_a$ we have equality in (II).
If (I) holds and z tends a then we have,
(III) $(1-|a|^2)|f'(a)| \leq (1-|b|^2)$.
An application of (II) and (III) tof_0 shows that $s_b \leq s_b^2$ and consequently $s_b \geq 1$. Since $s_b < 1$ for $b \neq 0$, we have a contradiction.

8.2. **Further comments related to Theorems 8.3-8.4.** We have worked on the subject from time to time between 1980 -1990 and in that time we proved Theorems 8.3-8.4([4]). But we realized these days that it is a version of the Earle-Hamilton (1968) fixed point theorem, which may be viewed as a holomorphic formulation of Banach's contraction mapping theorem. A version of this result was proved in 1968 (when I enroled Math Faculty) by Clifford Earle and Richard Hamilton [21] by showing that, with respect to the Carathéodory metric on the domain, the holomorphic mapping becomes a contraction mapping to which the Banach fixed-point theorem can be applied. Perhaps there are applications of this result in the Teichmüller theory.

9. Curvature of Kobayashi and Carathéodory metric

9.1. **On the Hessian of the Carathéodory metric.** Here we collect some materials from Burbea's paper [13].

In [13], the generalized lower Hessian of an upper semi-continuous function near a point z in $\mathbb{C}^n$ is introduced (for $n = 1$ see Heins[31]). With this Burbea introduces a "sectional curvature" and he proves that the sectional curvature of the Carathéodory-Reiffen metric is always ≤ -4. This generalizes a result of Suita [60] in the one dimensional case. The sectional curvatures of the ball and polydisk are always -4. A few other properties of the Hessian of the above metric are shown.

Now we give more details.

For ζ in D we write $H_\zeta(D, \mathbb{U}) = \{f \in \mathrm{Hol}(D, \mathbb{U}) : f(\zeta) = 0\}$. For each ζ in D, $C_D(\zeta,)$ is the function defined on the complex tangent space of D at ζ by
$C_D(\zeta, \mathbf{v}) = \sup\{| < \partial f(\zeta), \mathbf{v} > | : f \in \mathrm{Hol}(D, \mathbb{U})\}$.

Exercise 7. For example, in the polydisk $C_{\mathbb{U}^n}(0, z) = \max_k \ln \frac{1+|z_k|}{1-|z_k|}$;and in the ball $C_{\mathbb{B}^n}(0, z) = \ln \frac{1+|z|}{1-|z|}$.

[4]we found a my hand written manuscript 1990 and did not pay much attention to it at that time

Let (z, w) and (z', w') be points in $\mathbb{U}^2$ and define $A(z, w) = (T_{z'}(z), T_{w'}(w))$. Then $A \in Aut(\mathbb{U}^2)$ and $A(z', w') = (0, 0)$ and if $|T_{z'}(z)| \geq |T_{w'}(w)|$, then

$$C_{\mathbb{U}^2}((z, w), (z', w')) = \ln \frac{1 + |T_{z'}(z)|}{1 - |T_{z'}(z)|}.$$

Since $\mathrm{Hol}(D, \mathbb{U})$ is a normal family, the supremum in the definition of $C_D(\zeta; \mathbf{v})$ is attained by some $F \in H_\zeta(D, \mathbb{U})$. Here $F(z) = F(z; \zeta, \mathbf{v})$. By a normal family argument $C_D(\zeta; \mathbf{v})$ is continuous in $(\zeta, \mathbf{v})$.

The Hessian.

Let f be upper semi-continuous near $z \in \mathbb{C}^n$ and let $\mathbf{u} \in \mathbb{C}^n \backslash \{0\}$. The generalized lower Hessian (or "Laplacian") of at z along the direction u is defined by

$$\triangle_{\mathbf{u}} f(z) = 4 \liminf_{r \to 0} \frac{1}{r^2} \Big[\frac{1}{2\pi} \int_0^{2\pi} \big(f(z + re^{it}\mathbf{u}) - f(z) \big) dt \Big].$$

Note that, if f is a C^2 function near z, then $\triangle_{\mathbf{u}} f(z)$ reduces to four times the usual Hessian of at z along u, that is

$$\Delta_{\mathbf{u}} f(z) = 4 \sum D^2_{z_i \overline{z_j}} f(z) u_i \overline{u_j} = 4 H_z(f, \mathbf{u}).$$

If u is the restriction of f on the complex line $z = l(z^0 + \zeta \mathbf{u})$, that is $u(\zeta) = f \circ l(\zeta)$, then using the chain rule we have

$$D^2_{\zeta\overline{\zeta}} u = H_z(f, \mathbf{u}) = \sum D^2_{z_i \overline{z_j}} f(z) u_i \overline{u_j}.$$

Hence, since $\Delta u = 4 D^2_{\zeta\overline{\zeta}} u$, $\Delta u = \Delta_{\mathbf{u}} f(z) = 4 H_z(f, \mathbf{u})$.

Especially, if f is a C^1 function near the point z, and $\mathbf{v} = (v_1, v_2, ..., v_n) \in \mathbb{C}^n$, then $< \partial f, v > = \sum_{j=1}^n D^c_j f \, v_j$. Let $v \in \mathbb{C}^n \setminus \{0\}$ and consider $F(z) = F(z, \zeta, \mathbf{v})$ as before. Define

$$\lambda(z; \mathbf{u}) = \frac{| < \partial F(z), \mathbf{u} > |}{1 - |F(z)|^2}.$$

Therefore, $\ln \lambda(z; \mathbf{u}) = \ln | < D^c F(z), \mathbf{u} > | - \ln(1 - |F(z)|^2)$. The first term on the right is pluriharmonic and hence its Hessian along any direction (independently of $\mathbf{u}$) is zero. Consequently,

$\triangle_{\mathbf{w}} \ln \lambda(z; \mathbf{u}) = 4\lambda(z; \mathbf{w})^2$, for each direction $\mathbf{w} \in \mathbb{C}^n$. Especially,

$\triangle_{\mathbf{w}} \ln \lambda(\zeta; \mathbf{v}) = 4\lambda(\zeta; \mathbf{w})^2$.

Note that $\lambda(\zeta; \mathbf{v}) = C_D(\zeta; \mathbf{v})$.

Theorem 13. Let $\zeta \in D$ and $\mathbf{v} \in \mathbb{C}^n \setminus \{0\}$ be fixed. Then

$\triangle_{\mathbf{u}} \ln C_D(\zeta; \mathbf{v}) \geq 4\lambda(\zeta; \mathbf{u})^2$ for each direction $\mathbf{u} \in \mathbb{C}^n$ and thus again $\log C_D(\zeta; \mathbf{v})$ is plurisubharmonic.

Let $\mathbf{v} \in \mathbb{C}^n \setminus \{0\}$ and assume that the metric density $\mu(z; \mathbf{v})$ is a positive upper semi-continuous function at z. The "curvature" of $\mu(z; \mathbf{v})$ at z in the direction $\mathbf{v}$ is given by

$$K(\mu; z, \mathbf{v}) = -\frac{1}{\mu(z; \mathbf{v})^2} \triangle_v \ln \mu(z; \mathbf{v}).$$

If $\rho = \mu_{\mathbf{v}}$ is the restriction of μ on the complex line $z = l(z^0 + \omega \mathbf{v})$, then $\rho = \mu_{\mathbf{v}}$ is function of one complex variable ω. If we consider ρ as a metric density then $K_\rho = K(\mu; z, \mathbf{v})$.

The "curvature" of $\lambda(z; \mathbf{v})$ is -4 at $z = \zeta$.

Theorem 14. The curvature of $C_D(\zeta;\mathbf{v})$ is always ≤ -4.

Proof. By Theorem 13, $\triangle_{\mathbf{v}} \ln C_D(\zeta;\mathbf{v}) \ge 4\lambda(\zeta;\mathbf{v})^2$ and, since $\lambda(\zeta;\mathbf{v}) = C_D(\zeta;\mathbf{v})$, the assertion follows. $\square$

We also note that the Carathéodory metric has the "distance-decreasing" property that is, if $f : D \to D^*$ is a holomorphic mapping, then $C_D(f(z); f^*(v)) \le C_D(z;v)$. The Carathéodory metric may be defined on arbitrary complex manifolds, although it may be zero in some directions v. Clearly the Carathéodory metric for the unit disk $\mathbb{U}$ is given by (B1) $C_{\mathbb{U}}(z;\mathbf{v}) = \frac{|\mathbf{v}|}{1-|z|^2}$.

Proposition 9.1. $C_D(z;\mathbf{v})||t;||/d(z)$, where $d(z)$ is the distance of $z \in D$ from the boundary of D.

Proposition 9.2. Let $v \in \mathbb{C}^n\setminus\{0\}$ be fixed. Then $\log C_D(z;\mathbf{v})$ is plurisubharmonic in $z \in D$.

9.2. **Kobayashi and Carathéodory metric.** For complex Banach manifold X and Y we denote by $\mathcal{O}(X,Y)$ the class of all holomorphic mapping from X into Y.

A complex Finsler metric F on a complex (Banach) manifold M is an upper semicontinuous function $F : T^{1,0}M \to \mathbb{R}^+$ satisfying
(i) $F(p\,;\mathbf{v}) > 0$ for all $p \in M$ and $\mathbf{v} \in T^{1,0}_pM$ with $v \ne 0$;
(ii) $F(p\,;\lambda\mathbf{v}) = |\lambda|F(p\,;\mathbf{v})$ for all $p \in M$, $\mathbf{v} \in T^{1,0}_pM$ and $\lambda \in \mathbb{C}$.

B. Wong proved the following interesting result, see also [13]:

Theorem 15 ([64])**.** (A) If G is a hyperbolic manifold in the sense of Kobayashi and the differential Kobayashi metric K_G is of class C^2, then the holomorphic curvature of K_G is greater than or equal to -4.
(B)If G is Carathéodory-hyperbolic and the differential Caratheodory metric C_G is of class C^2, then the holomorphic curvature of C_G is less than or equal to -4.

Here we shortly outline Wong approach [64].

Lemma 3. Let M, N be complex manifolds, and N complete hyperbolic in the sense of Kobayashi, suppose that we fix two points x_1 and x_2 in M and N respectively. Then $S = \{f \in \mathcal{O}(M,N), f(x_1) = x_2\}$ is compact in $\mathcal{O}(M,N)$ with respect to the compact open topology.

Definition 9.1. (a)Suppose that F is a C^2 hermitian Finsler metric on a complex one dimensional manifold. It is obvious that in this case F is just a C^2 hermitian metric in the usual sense of differential geometry. Then the holomorphic curvature of F is given by the following formula:

$$K(F) = -\frac{D^2_{z\overline{z}} \ln F}{F}\,. \tag{9.1}$$

(b) Let G be a complex manifold as before and $M_p(\mathbf{v})$ any complex one dimensional submanifold through the point p and whose tangent space at p is spanned by $\{\mathbf{v}, J\mathbf{v}\}$. In the following, $G(\mathbf{v})_p$ is the set of all $M_p(v)$. The holomorphic curvature $k_F(p,\mathbf{v})$ of a C^2 hermitian Finsler metric F at $(p,\mathbf{v}) \in T(G)$ is defined to be the following number:

$$k_F(p,\mathbf{v}) = \sup_{G(\mathbf{v})_p} \{\text{the holomorphic curvature of the restriction of } F \text{ to } M_p(\mathbf{v})\}. \tag{9.2}$$

If X and Y are complex Banach manifolds by $\mathcal{O}(X,Y)$(the notation $\mathrm{Hol}(X,Y)$ is also used in the literature) we denote the family of holomorphic mappings of X into Y. Let G be a complex manifold and $T(G)$ the tangent bundle; we define the differential Caratheodory metric as follows: $C_G : T(G) \to \mathbb{R}^+ \cup \{0\}$,

$$C_G(z,v) = \sup\{|df_z(v)| : f \in \mathcal{O}(G,\mathbb{U}),\ f(z) = 0\} \tag{9.3}$$

one can obtain a mapping f belonging to $\mathcal{O}(G,\mathbb{U})$ satisfying the following conditions:
(1) $f(p) = 0$ and
(2) $C_G(p,v) = |df_p(v)|$.

We observe that $df_p \neq 0$. For any $M_p(v)$ one can choose a neighborhood U of the origin 0 in $\mathbb{U}$ such that $f : M_p(v) \to U$ is a biholomorphism (for sufficiently small choice of $M_p(v)$).

Proof. (A) Let us fix a tangent $(p,\mathbf{v})$ at p. It is clear from Definition 9.1(b) that it suffices to prove the holomorphic curvature of the restriction of C_G to any $M_p(\mathbf{v})$ of $G(\mathbf{v})_p$ is less than -4.

With respect to the local coordinates $\{z,\overline{z}\}$ of $M_p(\mathbf{v})$, $f^*(B_{\mathbb{U}})$ (the pullback of $B_{\mathbb{U}}$ by f restricted to $M_p(\mathbf{v})$) and the restriction of C_G to $M_p(\mathbf{v})$ can be written as follows:

The restriction of $f^*(B_{\mathbb{U}})$ to $M_p(\mathbf{v})$ is $hdzd\overline{z}$; The restriction of C_G to $M_p(\mathbf{v})$ is $gdzd\overline{z}$, where h and $g = g_C$ are smooth functions on $M_p(\mathbf{v})$.

It is important to point out here that the Caratheodory metrics enjoy the distance decreasing property under holomorphic mappings. Therefore we have the following inequality: $f^*(B_{\mathbb{U}}) \le C_G$ (i.e. $h \le g$), where $B_{\mathbb{U}}$ is Finsler form of Poincare metric $\mathbb{U}$.

We let $u = h/g$. From (2) we have $h(p) = g(p)$ (f realizes C_G at the point p). Together with the above inequality ($h \le g$ on $M_p(\mathbf{v})$) and the definition of u, one can obtain the following two conditions of u:
(a) $u(p) = 1$ (i.e. log u(P) = 0),
(b) $u \le 1$ on $M_p(\mathbf{v})$ (i.e. $logu \le 0$).

This means $\log u$ attains a maximum at p. Therefore we have $D^c_{z\overline{z}} \ln u(p) \le 0$. From the fact that $\log u = \log h - \log g$, we have the following inequality:

$$D^c_{z\overline{z}} \ln h(p) \le D^c_{z\overline{z}} \ln g(p).$$

However, since $h(P) = g(P)$ and $1/h \ge 1/g$, we easily get

$$-\frac{1}{h} D^c_{z\overline{z}} \ln h(p) \ge -\frac{1}{g} D^c_{z\overline{z}} \ln g(p),$$

that is $-4 = K_h(p) \ge K_g(p)$.

The left-hand side is just the holomorphic curvature of the Poincaré metric on $\mathbb{U}$, which is equal to -4. The right-hand side is the holomorphic curvature of the restriction of C_G to $M_p(v)$. This completes the proof.

(B) From the definition of the differential Kobayashi metric there exists a sequence of holomorphic functions $\{f_i\}$ in $\mathcal{O}(\mathbb{U},G)$, such that $f_i(0) = p$, $K(P,v) = \lim df_i(w_i)|$, where $df_i(w_i) = v$, w_i is a tangent at the origin of $\mathbb{U}$, and $|w_i|$ is taken with respect to the Poincaré metric in the unit disc $\mathbb{U}$.

G is assumed to be hyperbolic in the sense of Kobayashi, so that it is a tight manifold in the sense of Wu (see [65]). Therefore there exist neighborhoods U_1, U_2

of 0 and p, respectively, such that $f_i(U_1) \subset U_2$ for all i. Furthermore, U_2 can be chosen to be complete hyperbolic (for example, biholomorphic to the unit ball).

Applying the lemma in part (A) again, we obtain a holomorphic mapping $f : U_1 \to U_2$ satisfying the following conditions: $f : U_1 \to U_2$ satisfying the following conditions:

(i) $f(0) = p$, $(df)_0(w_0) = \upsilon$ and $|w_0| = K_G(p, \upsilon)$.

Since f does not increase the corresponding distances we have

(ii) $\mathrm{Hyp}_{\mathbb{U}}(z, w) \geq K_G(f(z), (df)_0(w))$ for all $(z, w) \in T(D)$, $z \in U_1$.

Let $gdzd\overline{z}$ ($g = g_K$) be the pullback of the restriction of K_G to $M_p(v)$ by f, and let $hdzd\overline{z}$ be the Poincaré metric of the unit disc $\mathbb{U}$. We let $u = h/g$. Clearly, by (i), $\log u$ attains a minimum at 0 in $\mathbb{U}$. Hence we have the following inequalities:

Therefore we have $D^2_{z\overline{z}} \ln u(0) \geq 0$ and

$K_h(0) \leq K_g(0)$. One observes that the left-hand side of the above second inequality is the holomorphic curvature of the Poincaré metric, which is identically equal to -4 ($K_h(0) = -4$). The right-hand side is equal to the holomorphic curvature of the restriction of K_G to $M_p(v)$ at the point p. Our proof is therefore completed. □

Exercise 8. (a)Check that in unit ball $K_{\mathbb{B}}(0, z) = C_{\mathbb{B}}(0, z) = \ln \frac{1+|z|}{1-|z|}$

(b) Fill the details for the proofs

$D^c_{z\overline{z}} \ln h(p) \leq D^c_{z\overline{z}} \ln g_C(p)$ in the case (A) and

$-4 = K_h(0) \leq K_g(0)$, where $g = g_K$.

The following question is fundamental in hyperbolic complex analysis. If G is complete hyperbolic, does K_G satisfy the maximum modulus principle in $T(G)$.

A Schwarz-Pick system is a functor, denoted by $X \mapsto d_X$, that assigns to each complex Banach manifold X a pseudometric d_X so that the following conditions hold: (a) The pseudometric assigned to D is the Poincare metric (b) If X and Y are complex Banach manifolds then (2.2) $d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2)$ if $x_1 \in X$, $x_2 \in X$ and $f \in \mathcal{O}(X, Y)$.

Because of conditions (a) and (b) the sets $\mathcal{O}(\mathbb{D}, X)$ and $\mathcal{O}(X, \mathbb{D})$ provide upper and lower bounds for d_X. These upper and lower bounds lead to the definitions of the Kobayashi and Caratheodory pseudometrics, which we shall study in the remainder of this paper.

In this paper $d_{\mathbb{D}}$ will always be the Poincaré metric (2.1) on the unit disk D.

Definition 9.2. A Schwarz-Pick pseudometric on the complex Banach manifold X is a pseudometric d such that (3.1) $d(f(z), f(w)) \leq d_{\mathbb{D}}(z, w)$ for all z and w in $\mathbb{D}$ and f in $\mathcal{O}(\mathbb{D}, X)$. If $X \mapsto d_X$ is a Schwarz-Pick system, then d_X is obviously a Schwarz-Pick pseudometric on X for every complex Banach manifold X.

The Carathéodory length of a piecewise C^1 curve $\gamma : [a, b] \to X$ in X is $\tilde{L}_X(\gamma) = \int_a^b c_X(\gamma(t), \gamma'(t))dt$ and the distance $\tilde{C}_X(x, y)$ is the infimum of the lengths of all piecewise C^1 curves joining x to y. Observe that the integrand in (4.6) is piecewise continuous. The functor assigning $\tilde{C}_X$ to each complex Banach manifold X is a SchwarzPick system. In particular, if x and y are points in X, then $d(fx, fy) \leq \tilde{C}_X(x, y)$ for all f in $\mathcal{O}(X, \mathbb{D})$. Definition (4.3) therefore implies that $C_X(x, y) \leq \tilde{C}_X(x, y)$ for all x and y in X.

Complex geodesics. Since $X \mapsto \tilde{C}_X$ is a Schwarz-Pick system, $\tilde{C}_X$ is a Schwarz-Pick pseudometric on X for every complex Banach manifold X. Therefore $\tilde{C}_X \leq K_X$ for every X. Combining that inequality with (4.7) we obtain

$C_X(f(z), f(w)) \leq \tilde{C}_X(f(z), f(w)) \leq K_X(f(z), f(w)) \leq d_{\mathbb{D}}(z, w)$ whenever X is a complex Banach manifold, $f \in \mathcal{O}(\mathbb{D}, X)$, and z and w are points of $\mathbb{D}$.

Following Vesentini [Ves81], we call f in $\mathcal{O}(\mathbb{D}, X)$ a complex geodesic (more precisely a complex C_X- geodesic) if there is a pair of distinct points z and w in $\mathbb{D}$ with

$$C_X(f(z), f(w)) = d_{\mathbb{D}}(z, w) \tag{9.4}$$

so that none of the inequalities in (5.1) is strict.

Definition 9.3. A holomorphic map $\varphi : \mathbb{U} \to X$ in a complex manifold X is a complex geodesic if it is an isometry between the Poincaré distance d_{hyp} and the Kobayashi distance k_X.

It is a well-known result of Lempert [40] that on convex domains the Kobayashi and Carathodory distances(resp metrics) coincide.

In his famous 1981 paper [40], Lempert proved that given a point in a strongly convex domain the complex geodesics (i.e., the extremal disks) for the Kobayashi metric passing through that point provide a very useful fibration of the domain.

In communication with Forstneric and the author the following question has been mentioned:

Question. Whether, in the ball or a bounded convex domains of R^n, there exist minimal geodesics, i.e. conformal minimal (=harmonic) disks which are extremal at every point. This holds for holomorphic disks in any bounded convex domain in C^n by a famous theorem of Lempert (1981).

9.3. **Calculation of the curvature.** Let G be a bounded domain and $K(z; w)$ be the Bergman Kernel on G. Write $\phi(z) = log K(z, z)$. The Bergman metric

$$ds^2 = \sum_{\mu,\nu=1}^{n} D^2_{z_\mu \overline{z_\nu}} \phi \, dz_\mu d\overline{z_\nu} \tag{9.5}$$

is the Kahler metric with Kahler form $i\partial\overline{\partial}\phi$. We use notation b_D or $Berg_D$ for the Bergman distance on D. Note that the distance in the Bergman metric from the origin in the unit ball $\mathbb{B} \subset \mathbb{C}^n$ is

$$b_B(0, z) = \sqrt{n+1} \ln \frac{1 + |z|}{1 - |z|} = \sqrt{n+1} b_B(0, z) \,. \tag{9.6}$$

In the polydisk $b_{\mathbb{U}}(0, z) = \frac{1}{\sqrt{2}} \sqrt{\sum_{k=1}^n \ln \frac{1+|z_k|}{1-|z_k|}}$ and

$C_{\mathbb{U}}(0, z) = \max_k \ln \frac{1+|z_k|}{1-|z_k|}$.

Let (z, w) and (z', w') be points in $\mathbb{U}^2$ and define $A(z, w) = (T_{z'}(z), T_{w'}(w))$. Then $A \in Aut(\mathbb{U}^2)$ and $A(z', w') = (0, 0)$ and if $|T_{z'}(z)| \geq |T_{w'}(w)|$, then $C^2_{\mathbb{U}}((z, w), (z', w')) = \ln \frac{1+|T_{z'}(z)|}{1-|T_{z'}(z)|}$.

For $u > 0$:

$$\Delta \ln u = \frac{u \Delta u - |\nabla u|^2}{u^2}.$$

Let $g, f \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$ and let f map $\mathbb{D}$ into the unit ball in $\mathbb{C}^n$.

$$|g|\Delta|g| = 2|g'(z)|^2 - \frac{| < g, g' > |^2}{|g|^2}.$$

$$|\nabla|g||^2 = \frac{| < g, g' > |^2}{|g|^2} := A.$$

$$\Delta|f^2| = 4|f'|^2$$

$$|\nabla|f|^2|^2 = 4| < f', f > |^2.$$

Set $u = |g|$, $g = f'$, $\lambda_1 := \ln u$, $v := 1 - |f|^2$, $\lambda_2 := \ln v$, $I_1 = \Delta\lambda_1$ and $I_2 = -\Delta\lambda_2$. Then

$$I_1 = u^{-2}2(|g'|^2 - A) \quad \text{and} \quad I_2 = -\Delta\lambda_2 = 4v^{-2}(v|f'|^2 + | < f', f > |^2).$$

Set

$$\rho = |f'|(1 - |f|^2)^{-1}.$$

Hence $I = \Delta \ln \rho = I_1 + I_2$. Since

$$I_2 = 4v^{-2}(|f'|^2 - |f|^2|f'|^2 + | < f', f > |^2,$$

it seems that for $n = 1$, $I = \Delta \ln \rho \geq 4\rho^2$. But, for $n > 1$, we have some difficulties. Note that $I_1 \geq 0$. Perhaps, we can try to apply Schwarz's

$$(1 - |z|^2)^{-1}|f'(z)| \leq v = 1 - |f|^2$$

to estimate

$$R = I_1 + 4v^{-2}(-|f|^2|f'|^2 + | < f', f > |^2)..$$

Wikipedia says that the metric

$$h = \frac{4|dx|^2}{(1 - |x|^2)^2} \tag{9.7}$$

on the ball $\mathbb{B} = \{|x|^2 < 1\} \subset \mathbb{R}^n$ (for any $n \in \mathbb{N}$) is the Poincare model of the hyperbolic space; presumably it has constant (Gaussian?) curvature -1. However, it seems that this metric might not the most suitable for our purposes.

There is another model of a hyperbolic n-space, the so called *Beltrami-Klein model*, which is represented by the ball $\mathbb{B} \subset \mathbb{R}^n$ by the metric

$$g = 4\frac{|dx|^2 + (x\cdot dx)^2}{(1 - |x|^2)^2}; \quad x\cdot dx = \sum_{j=1}^{n} x_j dx_j. \tag{9.8}$$

In the complex case, replacing $\mathbb{R}^n$ by $\mathbb{C}^n$, the metric h is obviously not Kähler. The natural standard Kähler metric on the ball is the *Bergman metric*. Up to a normalizing constant, the Bergman kernel for the ball $\mathbb{B}_n \subset \mathbb{C}^n$ on the diagonal $z = w$ equals

$$K_{\mathbb{B}}(z) = \frac{1}{(1 - |z|^2)^{n+1}} \tag{9.9}$$

The Bergman metric on $\mathbb{B} = \mathbb{B}_n$ is defined by the Kähler $(1, 1)$-form

$$k_{\mathbb{B}} = -i(n + 1)\partial\overline{\partial}\log(1 - |z|^2) = \frac{i(n + 1)}{(1 - |z|^2)^2}\sum_{j,k=1}^{n} \bar{z}_j z_k \, dz_j \wedge d\bar{z}_k. \tag{9.10}$$

That is, up to a constant we have

$$ds^2_{\mathbb{B}} = \frac{n+1}{(1-|z|^2)^2} \sum_{j,k=1}^{n} \bar{z}_j z_k \, dz_j \otimes d\bar{z}_k. \tag{9.11}$$

This is similar to the expression for g (9.8), while h (9.7) amounts to the diagonal terms.

Now, if $F = (F_1, \dots, F_n) \colon \mathbb{D} \to \mathbb{B}^n$ is a holomorphic map the the pull-back F^* commutes with the ∂ and $\overline{\partial}$ operators. Hence, the induced metric on $\mathbb{D}$ is given by the Kähler form

$$F^*(k_B) = -i(n+1)\partial\overline{\partial}\log(1-|F|^2).$$

Up to a constant, the metric is therefore given by

$$F^*(ds^2_{\mathbb{B}}) = \frac{n+1}{2} \triangle \log(1-|F(\zeta)|^2)|d\zeta|^2.$$

It might now be possible to complete the calculation of the curvature and use the comparison principle.

Concerning the conformal minimal disks, there might be another problem with using the Beltrami-Klein metric (9.8): this metric is not conformally equivalent to the Poincaré metric (9.7). Hence, disks which are conformal harmonic in the standard flat (Euclidean) metric (and hence in the Poincaré metric) are no longer conformal in the Beltrami-Klein metric. Maybe this is not essential since in calculations we get various expressions of standard inner products which may anyhow simplify if the the disk is conformal in the Euclidean metric.

10. boundary Schwarz lemma and harmonic functions

For the next results see [74, 75]. We outline short proofs of some results related to Kalaj's communication [33].

Theorem 10.1. *A) Let $f = g + \overline{h}$ be complex valued continuous on $\overline{\mathbb{U}}$ and harmonic on $\mathbb{U}$ and $\gamma(t) = f(e^{it})$, $t \in [0, 2\pi]$. If γ is a rectifiable curve and $L = |\gamma|$ is length of γ, then*
a) $2\pi|g'(0)| \leq L$, $2\pi \max\{|g'(0)|, |h'(0)|\} \leq L$ *with equality iff* $f = L$, *where* $L(z) := az + b$.

b) $2\pi(1-|z|^2)|g'(z)| \leq L$, $z \in \overline{\mathbb{U}}$ *with equality iff* $f = L \circ \varphi_z$.

As a corollary we get, $\pi(|g'(0)| + |h'(0)|) \leq L$.

Proof. a)Since $f_t = ig'e^{it} + \overline{ih'e^{it}}$, $2\pi i g'(0) = \int_0^{2\pi} f_t e^{-it} dt$. Hence
$2\pi|g'(0)| \leq \int_0^{2\pi} |f_t e^{-it}| dt = L$.

Set $X(t) = g' - \overline{h'e^{2it}}$. If the equality holds in a) then there is c such that $cX = A^+$, where A^+ is a nonnegative function Set $u = P[X]$ and $H = h'z^2$. Then $u = g' - \overline{H}$ and cu is a nonnegative function. Hence $\mathrm{Im}(cg') = \mathrm{Im}(c\overline{H})$ and therefore $cg' = -\overline{c}H + c_1$, ie $g' = c_2 H + c_3$. Thus $X = c_2 H + c_3 - \overline{H}$ on T, $u = c_2 H + c_3 - \overline{H}$ and $\mathrm{Im} c(c_2 H + c_3) = \mathrm{Im} c\overline{H}$, ie $c_4 H + c_5 = c_6 H$. Hence $H = 0$ and therefore $g' = a$, ie $g = az + b$.

b) For $z \in \overline{\mathbb{U}}$ apply a) on $f \circ \varphi_z$. □

Theorem 10.2. *B) Suppose that $f : \overline{\mathbb{U}} \to \mathbb{R}^3$ conformal and harmonic and $\gamma(t) = f(e^{it})$, $t \in [0, 2\pi]$. If γ is a rectifiable curve and $L = |\gamma|$ is length of γ, then*
a) $2\pi|f'_x(0)| \le L$.
b) $2\pi(1-|z|^2)|f'_x(z)| \le L$, $z \in \overline{\mathbb{U}}$.
b1) *The equality holds in* b) *for some $z \in \mathbb{U}$ iff $f(\mathbb{U})$ is in a plane say X and $f = f(z) + R$, where $R : \mathbb{U} \to X$ is a composition of a rotation in X around z and homotety wrt z. After rotation we can suppose that X is y_1y_2 which we can identify with $\mathbb{C}$-plane. Then $f = L \circ \varphi_z$.*

Proof. a) Let $S = f(\overline{\mathbb{U}})$, $M_0 = f(0)$ and X tangent plane of S at M_0. Further suppose that P is the projection of S into X, $f^1 = P \circ f$, and $\gamma_1(t) = F(e^{it})$. Then $f^1 = g^1 + \overline{h^1}$. If $L_1 = |\gamma_1|$ is length of γ_1, then by Theorem 10.1a)
$2\pi|(g^1)'(0)| \le L_1$.
Since $L_1 \le L$ and $|(g^1)'(0)| = f^1_x(0) = f_x(0)$, we get the part a) of B).
b) Apply a) on $f \circ \varphi_z$. Note that $L_1 \le L$ with equality iff $f(\mathbb{U})$ is in a plane. If for some $z \in \mathbb{U}$ the equality holds in b), then $L_1 = L$. □

Check that
C) Let $f : \mathbb{U} \to \mathbb{R}^m$ be continuous on $\overline{\mathbb{U}}$ and harmonic on $\mathbb{U}$, and $\gamma(t) = f(e^{it})$. Then $f = g + \overline{h}$, where $g, h : \mathbb{U} \to \mathbb{C}^m$ are holomorphic on on $\mathbb{U}$ and $2\pi|g'(0)| \le L$, where $L = |\gamma|$ is length of γ.

Define the harmonic density

$$\mathrm{Har}(w) = \frac{1}{\sqrt{2}}\frac{1}{\hat{R}(w)} \tag{10.1}$$

on $\mathbb{U}$, where $\hat{R}(w) = \sqrt{1-|w|^2}$ and denote by d_{har} the corresponding distance.

Theorem 10.3. *If f is a harmonic mapping from the unit disk $\mathbb{U}$ into self, then*

$$d_{\mathrm{har}}(fz, fz') \le d_{\mathrm{hyp}}(z, z'), \quad z, z' \in \mathbb{U}.$$

For a function h, we use notation $\partial h = \frac{1}{2}(h'_x - ih'_y)$ and $\overline{\partial} h = \frac{1}{2}(h'_x + ih'_y)$; we also use notations Dh and $\overline{D}h$ instead of ∂h and $\overline{\partial} h$ respectively when it seems convenient. We use the notation $\lambda_f(z) = \big||\partial f(z)| - |\bar{\partial} f(z)|\big|$ and $\Lambda_f(z) = |\partial f(z)| + |\bar{\partial} f(z)|$, if $\partial f(z)$ and $\bar{\partial} f(z)$ exist.

Theorem 10.4 ([74])**.** *Let $f : \mathbb{U} \to \mathbb{U}$ be harmonic. Assume that $f(0) = 0$. Further assume that there is a point $b \in \mathbb{T}$ so that f extends continuously to b, $|f(b)| = 1$ (say that $f(b) = b'$), and and f is $\mathbb{R}$- differentiable at b. Then $|\Lambda_f(b)| \ge 2/\pi$.*

Define $A(z) = (1+z)(1-z)^{-1}$, $A_g := \underline{A}_g = (1+g)(1-g)^{-1}$, $B_g = R_g = A_g - A$ and $h = Re(A_g - A)$. Then $A' = 2(1-z)^{-2}$, and if g is a holomorphic function we have $A'_g = 2(1-g)^{-2}g'$, and $R'_g = 2(1-g)^{-2}g' - 2(1-z)^{-2}$.

XXX Ornek-Akyel [54, 55] use the following form of maximum principle:

Proposition 10.1. If u is harmonic on the unit disk and for every $w \in \mathbb{T}$, $\liminf_{z \to w} u(z) \ge 0$, then $u \ge 0$.

Theorem 10.5. *Let $B : \mathbb{U} \to \mathbb{U}$ be a finite Blaschke product which equal w_0 on a finite set $X = X(w_0) \subset \mathbb{T}$ and let f be a holomorphic function in the unit disc and $|f(z) - 1| < 1$ for $|z| < 1$. Suppose the following condition is satisfied*
(ii.2) *For all $a \in A$, $f(z) = 1 + B(z) + o(z-a)^2$, $z \in \mathbb{T}$, $z \to a$.*
Then $u(z) = Re\underline{A}_F - Re\underline{A}_B$ is continuous on $U \cup A$ and satisfies the condition

(ii.1) $\liminf_{z\to w} u(z) \geq 0$ *for every* $w \in \mathbb{T}$, *and*
it is non-negative on $\mathbb{U}$.
For every $a \in X$, $u(z) = o(1)$ *if* z *tends* a.

Theorem 10.6 ([74])**.** *Let* $B : \mathbb{U} \to \mathbb{U}$ *be a finite Blaschke product which equal* w_0 *on a finite set* $A \subset \mathbb{T}$ *and let* f *be a holomorphic function in the unit disc and* $|f(z) - 1| < 1$ *for* $|z| < 1$. *Suppose in addition to* (ii.2) *the following conditions is satisfied*
(ii.3) *there is a* $a_0 \in A$, *such that* $f(z) = 1 + B(z) + o(z - a_0)^3$, $z \in \mathbb{T}$, $z \to a_0$.
Then $u(z) = o(z - 1)$ *if* z *tends* 1 *and*
$f = 1 + B$.

In joint work with M. Knežević and the author, it is proved:

Proposition 10.2 (the unit disk euclidean-qch version,[38])**.** Let f be a k - quasiconformal euclidean harmonic mapping from the unit disc $\mathbb{U}$ into itself. Then for all $z \in \mathbb{U}$ we have

$$|f_z(z)| \leq \frac{1}{1-k}\,\frac{1-|f(z)|^2}{1-|z|^2}.$$

Using $L_f(z) \leq (1+k)|f_z(z)|$, we get
(A) $L_f(z) \leq K\frac{1-|f(z)|^2}{1-|z|^2}$ and therefore
(A1) $\lambda(fz', fz) \leq K\lambda(z', z)$, $z', z \in \mathbb{U}$.

In proof we use the metric density $\sigma_f(z) = (1-k)^2\lambda(f(z))|f_z(z)|^2$ and check that the curvature $\mathbf{K}(\sigma)(z) \leq -1$.

In communication with Pavlović appears the following question:

Question 2. Whether (A1) holds if f is k-qr? We announce a positive answer to this question in [48]:

Theorem 16. (i): Let f be a k - quasiregular euclidean harmonic mapping from the unit disc $\mathbb{U}$ into itself.

Then for any two points z_1 and z_2 in $\mathbb{U}$ we have

$$\lambda(f(z_1), f(z_2)) \leq \frac{1+k}{1-k}\,\lambda(z_1, z_2)\,.$$

11. Further comments

Because of limited space we mention only a few papers related to Schwarz lemma holomorphic maps.We first recall the definition of normality. Let X, Y be two complex manifolds; a family F of holomorphic maps from X to Y is normal if every sequence in F admits either a convergent subsequence or a compactly divergent subsequence. A complex manifold X is taut if $\mathrm{Hol}(\mathbb{U}, X)$ is a normal family. Let X be a taut complex manifold. Then $\mathrm{Hol}(Y, X)$ is a normal family for every complex manifold Y. A connected complex manifold X is (Kobayashi) hyperbolic if k_X is a true distance. Every complete hyperbolic manifold is taut.

(1) For a review about Schwarz lemma holomorphic maps between Kahler manifolds see Jianbo Chen [15], Abstract. In Section 1, we introduce some background knowledge of complex geometry. In Section 2, classical Schwarz lemma and its interpretation is discussed. In Section 3, we study the Ahlfors-Schwarz's lemma and its generalization to holomorphic maps between the unit disk and Kahler manifolds with holomorphic sectional curvature bounded from above by a negative constant. In Section

4, we focus on the case when equality holds at a certain point is discussed for holomorphic maps between the unit disk and classical bounded symmetric domains of type I, II and III. In Section 5, two higher-dimensional generalizations of the Ahlfors-Schwarz lemma for holomorphic maps from a compact Kahler manifold to another Kahler manifold, both of which satisfy respective conditions on curvature, are studied. In Section 6, we investigate two applications of various versions of Ahlfors-Schwarz lemma.

(2) James J. Faran [23], Abstract. The problem of local equivalence of Hermitian Finsler metrics under holomorphic changes of coordinates is solved. On such a Finsler metric we find some differential conditions which imply that the Finsler metric is the Kobayashi metric of the underlying manifold (these conditions are satisfied if the metric is the Kobayashi metric on a bounded, strictly convex domain in $\mathbb{C}^n$ with smooth boundary).

(3) Gunnar Pór Magnússon [41], Abstract. Lars Alhfors proved a differential geometric version of the classical Schwarz lemma in 1938. His version of the lemma gives an interesting connection between the existence of nonconstant entire functions with values in a given domain and metrics with negative curvature on such domain. We recall the classical Schwarz lemma and review the notions necessary to understand Ahlfors' lemma, before proving both the new form of the lemma and giving some applications.

(4) In the survey [22] (see Abstract), C. Frosini and F. Vlacci give geometric interpretations of some standard results on boundary behaviour of holomorphic selfmaps in the unit disc of $\mathbb{C}$ and generalize them for holomorphic selfmaps of some particular domains of $\mathbb{C}^n$.

(5) Marco Abate [3], Abstract.These are the notes of a short course I gave in the school "Aspects métriques et dynamiques en analyse complete", Lille, May 2015. The aim of this notes is to describe how to use a geometric structure (namely, the Kobayashi distance) to explore and encode analytic properties of holomorphic functions and maps defined on complex manifolds. We shall first describe the main properties of the Kobayashi distance, and then we shall present applications to holomorphic dynamics in taut manifolds, strongly pseudo convex domains and convex domains, and to operator theory in Bergman spaces (Carleson measures and Toeplitz operators).

Theorem 11.1 (Theorem 2.40 (Budzynska; Abate-Raissy, [3])). *Let $D \subset\subset \mathbb{C}^n$ be a bounded strictly convex domain, and take $f \in \mathrm{Hol}(D, D)$ without fixed points. Then the sequence of iterates $\{f^k\}$ converges to a constant map defined by $z_0 \in \partial D$.*

As often happens with objects introduced via a general definition, the Kobayashi pseudodistance can seldom be explicitly computed. "Besides the cases listed in Proposition 1.17 [3], as far as we know there are formulas only for some complex ellipsoids [39], bounded symmetric domains [38], the symmetrized bidisk [11] and a few other scattered examples. " Recall that in particular (see [35], p.47):

$$\mathrm{Kob}(z, w) = \max\{\mathrm{Kob}(z_k, w_k) : k = 1, \cdots, n\}.$$

On the other hand, it is possible and important to estimate the Kobayashi distance; see Subsection 1.5 [3].

(6) For Iteration theory of holomorphic maps on taut manifolds we refer the interested reader to M. Abate monograph[4], in which a Wolff-Denjoy theorem for hyperbolic Riemann surfaces is proved:

Theorem 11.2 (Theorem 1.3.12, [4]). *Let X be a hyperbolic Riemann surface, and let $f \in \mathrm{Hol}(X, X)$. Then either:*
(i) *f has an attractive fixed point in X, or*
(ii) *f is a periodic automorphism,or*
(iii) *f is a pseudoperiodic automorphism,or*
(iv) *the sequence $\{f^k\}$ is compactly divergent. Furthermore, the case* (iii) *can occur only if X is either simply connected* (*and f has a fixed point*) *or doubly connected* (*and f has no fixed points*).

If X is compact, Theorem 1.3.12 drastically simplifies, becoming: Corollary 1.3.13: Let X be a compact hyperbolic Riemann surface. Then every function $f \in Hol(X, X)$ is either constant or a periodic automorphism.

(7) Suzuki, Masaaki,[73]. Abstract. In this paper we study the intrinsic metrics for the circular domains in $\mathbb{C}^n$. We calculate the Kobayashi (pseudo-) metric at its center for pseudoconvex complete circular domain D using the result of Sadullaev. From this we have that such D is hyperbolic iff D is bounded. If a convex complete circular domain is complete hyperbolic, then the Caratheodory and Kobayashi metrics coincide at the center. Using this and the results of Hua we explicitly compute the intrinsic metrics of the classical domains. Furthermore we define the extremal function and extremal disc for intrinsic metrics and compute them in some special cases.
(8) In [1], the author(Simonic) introduces Ahlfors' generalization of the Schwarz lemma. With this powerful geometric tool of complex functions in one variable, he is able to prove some theorems concerning the size of images under holomorphic mappings, including the celebrated Picards theorems. The article concludes with a brief insight into the theory of Kobayashi hyperbolic complex manifolds.
(9) Filippo Bracci, John Erik Fornaess, Erlend Fornaess Wold, [11], Abstract. We prove that for a strongly pseudoconvex domain $D \subset \mathbb{C}^n$, the infinitesimal Carathéodory metric $C(z, v)$ and the infinitesimal Kobayashi metric $K(z, v)$ coincide if z is sufficiently close to bD and if v is sufficiently close to being tangential to bD. Also, we show that every two close points of D sufficiently close to the boundary and whose difference is almost tangential to bD can be joined by a (unique up to reparameterization) complex geodesic of D which is also a holomorphic retract of D. The same continues to hold if D is a worm domain, as long as the points are sufficiently close to a strongly pseudoconvex boundary point. We also show that a strongly pseudoconvex boundary point of a worm domain can be globally exposed, this has consequences for the behavior of the squeezing function.
(10) and let $f : U \to \overline{U}$ be a holomorphic function with $f(0) = 0$. It is easy to show that the inequality

$$|f(z) - f'(0)z| \leq \frac{2|z|^2}{1 - |z|}(1 - |f'(0)|). \tag{11.1}$$

Lawrence A. Harris,[30],Abstract. Our main result is an inequality which shows that a holomorphic function mapping the open unit ball of one normed linear space into the closed unit ball of another is close to being a linear map when the Frechet derivative of the function at 0 is close to being a surjective isometry. We deduce this result as a corollary of a kind of uniform rotundity at the identity of the sup norm on bounded holomorphic functions mapping the open unit ball of a normed linear space into the same space. In the following, a function h defined on an open subset of a complex normed linear space with range in another is called holomorphic if the Frechet derivative of h at x (denoted by $Dh(x)$ exists as a bounded complex-linear map for each x in the domain of definition of h.(See [7, Def. 3.16.4].) Denote the open (resp.,closed) unit ball of a normed linear space X by X_0 (resp.,X_1). Throughout, X and Y denote arbitrary complex normed linear spaces. Our main result is Let $h : X_0 \to Y_1$ be a holomorphic function with $h(0) = 0$.

Put $L = Dh(0)$ and let $\mathcal{U}$ he the set of all linear isometries of X onto Y. Suppose $\mathcal{U}$ is nonempty and let $d(L,\mathcal{U})$ denote the distance of L from $\mathcal{U}$ in the operator norm.

Theorem 11.3. *Then*

$$|h(x) - L(x) \leq \frac{8|x|^2}{(1-|x|)^2} d(L,\mathcal{U}), (x \in X_0). \tag{11.2}$$

I

(11) For Pluriharmonic Functions in Balls see Rudin [70]. Abstract. It is proved that a function is pluriharmonic in the open unit ball of $\mathbb{C}^n$ if and only if it is harmonic with respect to both the ordinary Laplacian and the invariant Laplace-Beltrami operator.

Theorem 11.4 ([74])**.** *Suppose that $G = G_n = (a,b) \times \mathbb{R}^{n-1}$, $-\infty < a < b \leq \infty$, $f : B_n \to G_n$. If f_1 is pluriharmonic on $\mathbb{B}_n$ and f is K-qc at $a \in \mathbb{B}_n$, then $k_G(fa)|f'(a)| \leq Kk_B(a)$, where k is quasi-hyperbolic density.*

12. SURFACES

See

By a standard argument in the calculus of variations, a minimal graph $z = f(x,y)$ can be shown to satisfy the nonlinear partial differential equation $(1 + f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1 + f_x^2)f_{yy} = 0$.

Let G be a planar domain enclosed by a Jordan curve C. $J[u] = \iint_G L(x,y,u,u_x,u_y)dxdy$

Set $h(t) = J[u + tv]$, $p = u_x$ and $q = u_y$. If $v = 0$ on bG, then $h'(0) = \iint_G (vL_u + v_xL_p + v_yL_q)dxdy$.

By partial integration, $h'(0) = \iint_G v\big(L_u - (L_p)_x - (L_q)_y\big)dxdy$. $L_u - (L_p)_x - (L_q)_y = 0$, $L_u - L_{xp} - L_{yq} - u_xL_{up} - u_yL_{uq} - u_{xx}L_{pp} - 2u_{xy}L_{pq} - u_{yy}L_{qq} = 0$

Let us derive the EulerLagrange equation for the minimal surface problem $L = \sqrt{1 + p^2 + q^2}$ Definition A smooth surface in $\mathbb{R}^3$ is a subset $S \subset \mathbb{R}^3$ such that each point has a neighbourhood $U \subset S$ and a map $X : V \to \mathbb{R}^3$ from an open set $V \subset \mathbb{R}^2$ such that

$X : V \to U$ is a homeomorphism

X is C^1

X_u and X_v are linearly independent. Definition (an abstract smooth surface) A smooth surface is a surface with a class of homeomorphisms φ_U such that each map $\varphi_{U'}\varphi_U^{-1}$ is a smoothly invertible homeomorphism.

Clearly, since a holomorphic function has partial derivatives of all orders in x, y, a Riemann surface is an example of an abstract smooth surface.

Definition. Two surfaces S, S' are isometric if there is a smooth homeomorphism $f : S \to S'$ which maps curves in S to curves in S' of the same length.

The cylinder $X(u,v) = (a\cos v, a\sin v, u)\ du^2 + a^2dv^2$

locally isometry $X(u,v) = (a\cos(v/a), a\sin(v/a), u)\ du^2 + dv^2$

The helicoid is homeomorphic to the plane $\mathbb{R}^2$. To see this, let α decrease continuously from its given value down to zero.

However, harmonic minimal immersions in $\mathbb{R}^3$ are not necessarily conformal as the following parameterization of the half helicoid shows $X : \mathbb{C} \to \mathbb{R}^3$, $X(z) = Re(e^z, ie^z, iz) = (e^x \cos y, -e^x \sin y, -y$.

http://www.michaelbeeson.com/research/papers/IntroMinimal.pdf Notes on Minimal Surfaces Michael Beeson Aug. 9, 2007 with revisions and additions fall 2010 file last touched November 11, 2015

Unlike Gauss curvature, the mean curvature is extrinsic and depends on the embedding, for instance, a cylinder and a plane are locally isometric but the mean curvature of a plane is zero while that of a cylinder is nonzero. A surface of least area bounded by γ would be a critical point of A, but not necessarily conversely. There could be relative minima of area which are not absolute minima of area. There can also be unstable critical points of area which are not even relative minima.

A parametric surface is the image of an open subset of the Euclidean plane $\mathbb{R}^2$ by a continuous function, in a topological space, generally an Euclidean space of dimension at least three. Usually the function is supposed to be continuously differentiable, and this will be always the case in this article.

Specifically, a parametric surface in $\mathbb{R}^3$ is given by three functions of two variables u and v, called parameters

$$\begin{aligned} x &= f_1(u,v) \\ y &= f_2(u,v) \\ z &= f_3(u,v)\,. \end{aligned}$$

As the image of such a function may be a curve (for example if the three functions are constant with respect to v), a further condition is required, generally that, for almost all values of the parameters, the Jacobian matrix

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} \end{bmatrix}$$

has rank two. Here "almost all" means that the values of the parameters where the rank is two contain a dense open subset of the range of the parametrization. For surfaces in a space of higher dimension, the condition is the same, except for the number of columns of the Jacobian matrix. Tangent plane and normal vector

A point p where the above Jacobian matrix has rank two is called regular, or, more properly, the parametrization is called regular at p.

The tangent plane at a regular point p is the unique plane passing through p and having a direction parallel to the two row vectors of the Jacobian matrix. The tangent plane is an affine concept, because its definition is independent of the choice of a metric. In other words, any affine transformation maps the tangent plane to the surface at a point to the tangent plane to the image of the surface at the image of the point.

The normal line, or simply normal at a point of a surface is the unique line passing through the point and perpendicular to the tangent plane. A normal vector is a vector which is parallel to the normal.

The Gaussian curvature at a point on an embedded smooth surface given locally by the equation

$z = F(x, y)$

in Euclidean space (E3), is defined to be the product of the principal curvatures at the point;[5] the mean curvature is defined to be their average. The principal curvatures are the maximum and minimum curvatures of the plane curves obtained by intersecting the surface with planes normal to the tangent plane at the point. If the point is $(0, 0, 0)$ with tangent plane z = 0, then, after a rotation about the z-axis setting the coefficient on xy to zero, F will have the Taylor series expansion

$$F(x, y) = \tfrac{1}{2}k_1x^2 + \tfrac{1}{2}k_2y^2 + \cdots$$

The principal curvatures are k1 and k2. In this case, the Gaussian curvature is given by

$K = k_1 \cdot k_2.K = k_1 \cdot k_2.$

and the mean curvature by

$K_m = \frac{1}{2}(k_1 + k_2).$

Since K and K_m are invariant under isometries of E3, in general

$$K = \frac{RT - S^2}{(1 + P^2 + Q^2)^2}$$

and

$$K_m = \frac{ET + GR - 2FS}{(1 + P^2 + Q^2)^2}$$

where the derivatives at the point are given by $P = F_x, Q = F_y, R = F_{xx}, S = F_{xy}$, and $T = F_{yy}$.

For every oriented embedded surface the Gauss map is the map into the unit sphere sending each point to the (outward pointing) unit normal vector to the oriented tangent plane at the point. In coordinates the map sends (x,y,z) to

$$N(x, y, z) = \frac{1}{\sqrt{1 + P^2 + Q^2}}(P, Q, -1).$$

Direct computation shows that: the Gaussian curvature is the Jacobian of the Gauss map.

Line and area elements

Taking a local chart, for example by projecting onto the xy-plane (z = 0), the line element ds and the area element dA can be written in terms of local coordinates as

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2$$

and

$dA = (EG - F2)^{1/2}dxdy.$

The expression $Edx^2 + 2Fdxdy + Gdy^2$ is called the first fundamental form.

The matrix

$$\begin{pmatrix} E(x,y) & F(x,y) \\ F(x,y) & G(x,y) \end{pmatrix}$$

is required to be positive-definite and to depend smoothly on x and y.

In a similar way line and area elements can be associated to any abstract Riemannian 2-manifold in a local chart.

Second fundamental form

Definition of second fundamental form

The extrinsic geometry of surfaces studies the properties of surfaces embedded into a Euclidean space, typically $\mathbb{E}^3$. In intrinsic geometry, two surfaces are "the same" if it is possible to unfold one surface onto the other without stretching it, i.e. a map of one surface onto the other preserving distance. Thus a cylinder is locally "the same" as the plane. In extrinsic geometry, two surfaces are "the same" if they are congruent in the ambient Euclidean space, i.e. there is an isometry of $\mathbb{E}^3$ carrying one surface onto the other. With this more rigid definition of similitude, the cylinder and the plane are obviously no longer the same.

Although the primary invariant in the study of the intrinsic geometry of surfaces is the metric (the first fundamental form) and the Gaussian curvature, certain properties of surfaces also depend on an embedding into $\mathbb{E}^3$ (or a higher dimensional Euclidean space). The most important example is the second fundamental form, defined classically as follows.

Take a point (x, y) on the surface in a local chart. The Euclidean distance from a nearby point $(x + dx, y + dy)$ to the tangent plane at (x, y), i.e. the length of the perpendicular dropped from the nearby point to the tangent plane, has the form

$edx^2 + 2fdxdy + gdy^2$

plus third and higher order corrections. The above expression, a symmetric bilinear form at each point, is the second fundamental form. It is described by a 2×2 symmetric matrix

$$\begin{pmatrix} e(x,y) & f(x,y) \\ f(x,y) & g(x,y) \end{pmatrix}$$

which depends smoothly on x and y. The Gaussian curvature can be calculated as the ratio of the determinants of the second and first fundamental forms:

$$K = \frac{eg - f^2}{EG - F^2}.$$

For example, a sphere of radius r has Gaussian curvature $1/r^2$ everywhere, and a flat plane and a cylinder have Gaussian curvature 0 everywhere. The Gaussian curvature can also be negative, as in the case of a hyperboloid or the inside of a torus.

Remarkably Gauss proved that it is an intrinsic invariant (see his Theorema Egregium below). One of the other extrinsic numerical invariants of a surface is the mean curvature K_m defined as the sum of the principal curvatures. It is given by the formula.

$$K_m = \frac{1}{2} \cdot \frac{eG + gE - 2fF}{EG - F^2}$$

The coefficients of the first and second fundamental forms satisfy certain compatibility conditions known as the Gauss-Codazzi equations; they involve the Christoffel symbols Γ^k_{ij} associated with the first fundamental form:

$$e_y - f_x = e\Gamma^1_{12} + f(\Gamma^2_{12} - \Gamma^1_{11}) - g\Gamma^2_{11}, \quad f_y - g_x = e\Gamma^1_{22} + f(\Gamma^2_{22} - \Gamma^1_{12}) - g\Gamma^2_{12}.$$

These equations can also be succinctly expressed and derived in the language of connection forms due to lie Cartan.[23] Pierre Bonnet proved that two quadratic forms satisfying the Gauss-Codazzi equations always uniquely determine an embedded surface locally.[24] For this reason the Gauss-Codazzi equations are often called the fundamental equations for embedded surfaces, precisely identifying where the intrinsic and extrinsic curvatures come from. They admit generalizations to surfaces embedded in more general Riemannian manifolds.

A curve C on the surface S is defined as an equivalence class of mappings $X = \Phi(U(t))$ from a real interval $a \leq t \leq b$ to G. If we set $\underline{C}(t) = U(t)$, then $C = X \circ \underline{C}$. The differential of the mapping $X = \Phi(U)$ is $dX = X_u du + X_v dv$. The arclength s of a smooth curve C on S is found by integrating the square root of the differential form $ds^2 = |dX|^2 = dX \cdot dX = Edu^2 + 2Fdudv + Gdv^2$, where $E = X_u \cdot X_u, F = X_u \cdot X_v, G = X_v \cdot X_v$. This is known as the first fundamental form of the surface. It is invariant under change of parameters although its individual coefficients are not. In particular, the arclength of C is well-defined, being independent of the choice of parameters.

A tangent vector to the curve C at a point $X_0 = X(t_0)$ is $X'(t_0) = X_u u'(t_0) + X_v v'(t_0)$.

The tangent plane of S at X_0 is the set of all tangent vectors to curves on the surface through X_0, the two-dimensional subspace spanned by the (independent) vectors X_u and X_v. The cross product $X_u \times X_v$ is orthogonal to the tangent plane. When normalized to have unit length, it becomes the unit normal vector

$$n = \frac{X_u \times X_v}{|X_u \times X_v|}$$

at the point X0. Thus, the orientation of parameters assigns a local orientation to the surface. However, the surface may not be (globally) orientable; it may be impossible to assign a normal direction in a continuous and consistent manner over the whole surface. The Mobius strip and the Klein bottle are familiar examples of nonorientable surfaces. The components of the cross product $X_u \times X_v$ are exactly the three Jacobians that appear in the formula for surface area. Thus,

Curvature is a second-order effect, requiring the assumption that the surface S has continuous second partial derivatives in its parametric representations. It will also be assumed that the curve C on S is regular ($U'(t) \neq 0$) and twice continuously differentiable. If C is parametrized in terms of arclength s, the tangent vector $T = X'(s)$ has unit length and is called the unit tangent vector. The curvature vector $T'(s) = dT/ds$ is orthogonal to T. Its normal projection $\kappa(T) = T'(s) \cdot n$

is called the normal curvature of S at X_0 in the direction T. The normal curvature will be shown to depend only on the tangent direction T of the curve C at X_0. Intuitively, it measures the rate at which the surface is rising out of its tangent plane in a specified direction. A more concrete way to define normal curvature is to consider only the normal sections of S. In other words, for each tangent direction T at X_0, let C be the curve of intersection of the surface S with the plane through

X_0 that contains both the normal vector n and the tangent vector T. Then dT/ds is parallel to n and $k(T) = |dT/ds|$, the sign depending on the choice of orientation of the surface. The principal curvatures k_1 and k_2 of S at X_0 are the maximum and minimum of $k(T)$ as T ranges over all directions in the tangent space. The mean curvature of S at X_0 is the average value $H = \frac{1}{2}(k_1 + k_2)$, whereas the Gauss curvature is the product $K = k_1 k_2$. The beautiful theorema egregium of Gauss asserts that K is a "bending invariant," unchanged whenever the surface is deformed without stretching. The mean curvature and the Gauss curvature can be computed in terms of surface invariants.

By the chain rule, the unit tangent vector of the curve C is
$k = T'(s) \cdot n = Lu'(s)^2 + 2Mu'(s)v'(s) + Nv'(s)^2$, where
$L = X_{uu} \cdot n, M = X_{uv} \cdot n, N = X_{vv} \cdot n.$

The notation e, f, g is also used instead of L, M, N respectively. For an orthogonal parametrization (i.e., F = 0), Gaussian curvature is:

$$K = -\frac{1}{2\sqrt{EG}} \left(\frac{\partial}{\partial u} \frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v} \frac{E_v}{\sqrt{EG}} \right).$$

For a surface described as graph of a function $z = F(x, y)$, Gaussian curvature is:

$$K = \frac{F_{xx} \cdot F_{yy} - F_{xy}^2}{(1 + F_x^2 + F_y^2)^2}$$

Isothermal Parameters

In studying the intrinsic properties of surfaces, it is advantageous to choose parameters that will reflect in some way the geometry of the surface. Isothermal parameters are those that preserve angles. In other words, the angle between a pair of curves in the parameter plane is equal to the angle between the corresponding pair of curves on the surface. Here the curves are oriented by their parametrizations, and the angle between them is understood to be the angle between their tangent vectors. As usual, this angle is chosen to lie between 0 and π.

The conclusion is that angles between curves are preserved everywhere if and only if the first fundamental form has the structure $ds^2 = \lambda^2(du^2 + dv^2), \lambda = \lambda(u, v) > 0$. In this case the surface is said to be represented in terms of isothermal parameters.

If a surface $X = \Phi(U)$ is expressed in terms of isothermal parameters, we show that the Laplacian $\Delta X = X_{uu} + X_{vv}$ is orthogonal to both X_u and X_v:

$$\Delta X \cdot X_u = \Delta X \cdot X_v = 0.$$

This means that ΔX is orthogonal to the tangent plane of the surface, so that $|\Delta X| = \pm \Delta X \cdot n = \pm(L + N)$. In isothermal parameters, $F = 0$ and $E = G = \lambda^2$, so the formula reduces to

$$H = \frac{1}{2\lambda^2}(L + N).$$

13. Weierstrass-Enneper parameterization of minimal surfaces

Heinz's inequality asserts that $|f_z(0)|^2 + |f_z(0)|^2 \geq c$ for all harmonic mappings f of the unit disk onto itself with $f(0) = 0$, where $c > 0$ is an absolute constant. the sharp form of the inequality with the constant $c = 27/4\pi^2 = 0.6839...$, established by Richard Hall [2] in 1982.

For the sharp estimate of curvature, a constrained form of Heinz's lemma is required. Specifically, it is required to find the sharp lower bound of $|h'(0)|^2+|g'(0)|^2$ among all harmonic self-mappings of the disk with $f(0)=0$ and dilatation $\omega = q^2$ for some analytic function q. It is reasonable to conjecture that the "extremal function" is now the canonical mapping of the disk onto an inscribed square (see Section 4.2), with dilatation $\omega(z)=z^2$. This mapping has $a_1 = h'(0) = 2\sqrt{2}/\pi$ and $b_1 = g'(0) = 0$, so the constrained form of Heinz's lemma can be expected to run as follows.

Conjecture 1. Let f be a harmonic mapping of D onto D with $f(0)=0$, whose dilatation $\omega = \overline{f_{\overline{z}}}/f_z$ is the square of an analytic function. Then $|f_z(0)|^2+|f_{\overline{z}}(0)|^2 \geq \frac{8}{\pi^2}$ and the bound is sharp. The canonical mapping onto the square lifts to Scherk's first surface, as was observed in Section 9.4. The Weierstrass-Enneper parameters for Scherk's surface, adjusted to lie above a square inscribed in the unit circle, are $p(z) = \frac{2\sqrt{2}}{\pi(1-z^4)}$ and $q(z)=iz$.

It seems that as an application of Heinz's inequality, we can get the following estimate. Define $P(x_1,x_2,x_3)=(x_1,x_2)$ and let let D be a domain in plane and $X : D \to \mathbb{R}^3$, the projecting mapping f of X is $P \circ X$.

Theorem 13.1. *Let $X : \mathbb{U} \to \mathbb{R}^3$ be harmonic such that the projecting mapping f is injective and $f(\mathbb{U}) \supset \mathbb{U}$. If $f(0)=0$, then $|X_z(0)|^2 + |X_{\overline{z}}(0)|^2 \geq c_0$, where $c_0 = 27/4\pi^2 = 0.6839...$.*

In particular, let S be a minimal graph (minimal surface) over a domain G which contains the unit disk $\mathbb{U}$ and $X : \mathbb{U} \to S$ isothermal parametrization. If $P(X(0)) = 0$, then the above estimate holds.

The only minimal graphs that extend over the entire plane are themselves planes, a theorem of S. Bernstein.

Let $S = \{(u,v,F(u,v)) : u+iv \in \mathbb{C}\}$ be a regular minimal graph over a simply connected domain $\mathbb{C}$, and let $X : \mathbb{U} \to S \subset \mathbb{R}^3$ be harmonic conformal. Then $??|X_z(0)|^2 + |X_z(0)|^2 \geq c_0 R^2$.

Finally, the upper bound for curvature has an interesting consequence. We have shown that if a minimal graph lies above the entire unit disk, then its Gauss curvature at the point above the origin is bounded by the absolute constant $C_0 = 16\pi^2/27$. From this it follows more generally that whenever a minimal graph covers a full disk of radius R, then its curvature at the point above the center of that disk satisfies $|K| \leq C/R^2$. Consequently, if a minimal graph actually lies above the entire plane, its Gauss curvature at every point is $K=0$. Since the mean curvature of a minimal surface also vanishes, it follows that any such minimal graph must have both principal curvatures equal to zero at every point. This proves a classical theorem of S. Bernstein.

Bernstein's Theorem. A minimal graph that lies above the entire plane must itself be a plane.

$$S(f) = S(h) + \frac{2\overline{q}}{1+|q|^2}\left(q'' - \frac{q'h''}{h'}\right) - 4\left(\frac{q'\overline{q}}{1+|q|^2}\right)^2.$$

Theorem 1. For a locally univalent sense-preserving harmonic function f with dilatation $\omega = q^2$, the following are equivalent: (i) S(f) is analytic.
(ii) The curvature K of the minimal surface locally associated with f is constant.
(iii) $K \equiv 0$ so that the corresponding minimal surface is a plane.

(iv) The dilatation of f is constant.
(v) $f = h + a\overline{h}$ for some analytic locally univalent function h and for some complex constant a with $|a| < 1$. A surface can be viewed informally as a two-dimensional set of points in three-dimensional Euclidean space $\mathbb{R}^3$. Formally, it is a two-dimensional manifold with a "smooth" structure. For most of our purposes it will suffice to regard a surface S as an equivalence class of differentiable mappings $X = \Phi(U)$ from a domain $D \subset \mathbb{R}^2$ onto a set $S \subset \mathbb{R}^3$. Here $U = (u, v)$ and $X = (x, y, z)$ denote points in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

The surface S is said to be regular at a point if the Jacobian matrix has rank 2 there or, equivalently, if the two row vectors X_u and X_v are linearly independent. This means it is possible to solve locally for one of the coordinates x, y, or z in terms of the other two. The surface S is regular if it is regular at every point. Regularity is an intrinsic property of S, independent of the choice of parameters. Henceforth we will assume that S is regular. A surface is said to be embedded in $\mathbb{R}^3$ if it has no self-intersections. A nonparametric surface is one with the special form $z = f(x, y)$ or with x or y expressed in terms of the other two coordinates. Thus, a regular surface is locally nonparametric but need not be (globally) nonparametric. A nonparametric surface is also called a graph.

Lagrange's identity in vector analysis shows that
$|X_u \cdot X_v|^2 = |X_u|^2|X_v|^2 - (X_u \cdot X_v)^2 = EG - F^2$.

The differential form $Ldu^2 + 2Mdudv + Ndv^2$ is known as the second fundamental form of the surface. Like the first fundamental form, it is intrinsic to the surface, invariant under sense-preserving change of parameters. Because the first fundamental form represents the arclength differential ds^2, the normal curvature may be expressed symbolically as a ratio of the two fundamental forms:

We now turn to minimal surfaces. A surface S is called a minimal surface if for each sufficiently small simple closed curve C on S the portion of S enclosed by C has the minimum area among all surfaces spanning C. Minimal surfaces can be constructed physically by dipping a loop of wire into soap solution. Because of surface tension, the resulting soap film will assume the shape that minimizes surface area.

A nonparametric minimal surface will be called a minimal graph.

By a standard argument in the calculus of variations, a minimal graph $z = f(x, y)$ can be shown to satisfy the nonlinear partial differential equation $(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0$. This equation has the elegant geometric interpretation that the mean curvature of the surface is everywhere equal to zero. Indeed, the coefficients of the first and second fundamental forms are easily computed in the nonparametric case, and the minimal surface equation is seen to reduce to $EN - 2FM + GL = 0$, or $H = 0$. It is convenient to take the identical vanishing of mean curvature as the definition of a minimal surface. As an immediate consequence, the Gauss curvature of a minimal surface is negative unless both principal curvatures vanish. The simplest example of a minimal surface is of course the plane. Two other classical examples are the catenoid $z = cosh^{-1}r$, where $r^2 = x^2 + y^2$, and the helicoid $z = tan^{-1}(y/x)$, both shown in Figure XX 9.1. Aside from the plane, the catenoid is the only minimal surface of revolution, and the helicoid is the only ruled minimal surface. The only minimal surface of translation $z = f(x) + g(y)$ is Scherk's first surface, $z = log\frac{cosy}{cosx}, |x| < \pi/2, |y| < \pi/2$.

$H = \frac{1}{2}\frac{EN-2FM+GL}{EG-F^2}$. In isothermal parameters, $F = 0$ and $E = G = \lambda^2$, so the formula reduces to $H = \frac{1}{2\lambda^2}(L + N)$.

In view of the relation $|\triangle X| = \pm(L+N)$, this shows that $\triangle X = 0$ if and only if $H = 0$. But minimal surfaces are characterized by the vanishing of mean curvature. We have therefore proved the following theorem.

Theorem 17. Let a regular surface S be expressed in terms of isothermal parameters. Then the position vector is a harmonic function of the parameters if and only if S is a minimal surface.

Corollary 4. *If a nonparametric minimal surface is expressed in terms of isothermal parameters, the projection onto the base plane defines a harmonic mapping.*

To be more specific, suppose a nonparametric minimal surface $t = F(u, v)$ lies over a region G in the uv- plane. Suppose it is represented by isothermal parameters (x, y) in a region D of the xy-plane. Then the three coordinate functions $u = u(x, y), v = v(x, y)$, and $t = t(x, y) = F(u(x, y), v(x, y))$ are all harmonic. Because the mapping from D to the surface is injective and the surface is nonparametric, the projection $w = u + iv = f(z)$, where $z = x + iy$, defines a (univalent) harmonic mapping of D onto G.

construct three complex-valued functions φ_k by the operation $2D_cX = (\varphi_1, \varphi_2, \varphi_3) = D_uX - iD_vX$. In other words, $\varphi_k = 2D_cx_k == D_uX_k - iD_vX_k$. Recalling the definitions of E, F, and G in the first fundamental form of S, one finds by direct calculation that $\varphi_k^2 = (D_ux_k)^2 - 2iD_ux_kD_vx_k - (D_vx_k)^2$ and $|\varphi_k|^2 = (D_ux_k)^2 + (D_vx_k)^2$ and therefore

$$\sum_{k=1}^{3}\varphi_k^2 = E - G - 2iF, \sum_{k=1}^{3}|\varphi_k|^2 = E + G\,. \tag{13.1}$$

Suppose now that the parametric representation of S is isothermal. Then $F = 0$ and $E = G > 0$, so

$$\sum_{k=1}^{3}\varphi_k^2 = 0, \quad \sum_{k=1}^{3}|\varphi_k|^2 > 0\,. \tag{13.2}$$

If S is a minimal surface, the coordinates x_k are harmonic and, hence, the functions φ_k are analytic. Consequently, to every regular minimal surface there correspond three analytic functions φ_k with the properties (13.2). Furthermore, the process is reversible and the converse is true. In other words, each triple of analytic functions satisfying (13.2) will generate a regular minimal surface. Let us formulate the result more precisely as a theorem.

Theorem 18. Let $X = \phi(U)$ be an isothermal parametrization of a regular minimal surface and let $w = u + iv$. Then the function $\varphi = 2D_wX$ is analytic and have the properties (13.2). Conversely, let $\{\varphi_1, \varphi_2, \varphi_3\}$ be an arbitrary triple of functions analytic in a simply connected region, satisfying (13.2). Then the function $X = \mathrm{Re}F$, where $F = \int\varphi$, gives an isothermal parametrization of a regular minimal surface.

Proof. We have already established the first statement. Only the converse remains to be proved. Set $F_k = \int\varphi_k(w)dw$ and $x_k = ReF_k = Re\int\varphi_k(w)dw$. Let S be defined by X.

Then $D_u x_k = ReF'_k$. Since $(F_k)_u = \varphi_k$, therefore $X_u = Re(\varphi_1, \varphi_2, \varphi_3)$ and $X_v = -Im\varphi$. Hence $E - G - 2iF = \sum_{k=1}^{3} \varphi_k^2 = 0$, ie. $E = G$ and $F = 0$. □

The following lemma describes the relevant triples of analytic functions.

Lemma 4. Let p be an analytic function and q a meromorphic function in some domain $D \subset \mathbb{C}$. Suppose that p has a zero of order at least $2m$ wherever q has a pole of order m. Then the functions

$$\varphi_1 = p(1+q^2), \varphi_2 = -ip(1-q^2), \varphi_3 = -2ipq \tag{13.3}$$

are analytic in D and have the property

$$\sum_{k=1}^{3} \varphi_k^2 = 0. \tag{13.4}$$

Conversely, every ordered triple of functions $\varphi_1, \varphi_2, \varphi_3$ analytic in D with the property (13.4) has the structure (13.3), unless $\varphi_2 = i\varphi_1$. The representation is unique.

p and q are called the Weierstrass-Enneper parameters associated to φ.

Outline. It is easy to verify that every triple with the structure (13.3) satisfies (13.4). The condition on the zeros of p ensures that the functions φ_k are analytic. Conversely, let φ_k be any analytic functions with the property (13.4). If $\varphi_2 \neq i\varphi_1$ we may define

$$p = \frac{1}{2}(\varphi_1 + i\varphi_2), q = \frac{i\varphi_3}{(\varphi_1 + i\varphi_2)}, \tag{13.5}$$

so that $i\varphi_3 = 2pq$. If $\varphi_2 = i\varphi_1$, then $\varphi_1^2 + \varphi_2^2 = 0$, and it follows that $\varphi_3 = 0$. The representation (13.3) then follows with the uniquely determined choices $p = \varphi_1$ and $q = 0$. In this degenerate case, the corresponding minimal surface is a horizontal plane.

The conclusion is that angles between curves are preserved everywhere if and only if the first fundamental form has the structure $ds^2 = \lambda^2(du^2 + dv^2), \lambda = \lambda(u,v) > 0$.

Let S be a regular minimal surface and $\phi : D \to S$ be an isothermal parametrization of S and let $z = x+iy$(we write also $X = \phi(z)$). Then the function $\varphi = 2D_z X$ is analytic and have the properties (13.2). We say that φ is associated to ϕ(or to S). p and q are called the Weierstrass-Enneper parameters associated to φ(or S). Let $f = (\phi_1, \phi_1)$ be the projection mapping and $f = h + \overline{g}$ be the canonical decomposition. Set $w = u + iv$ and $w = f(z)$. Then $D_z u = (\varphi_1)/2$ and $D_z v = (\varphi_2)/2$, $h' = D_z f$ and $g' = D_z\overline{f}$. Hence $h' = D_z f = D_z(u+iv) = D_z u + iD_z v = (\varphi_1 + i\varphi_2)/2$ and $g' = D_z\overline{f} = (\varphi_1 - i\varphi_2)/2$.

Theorem 19. the dilatation $\omega = g'/h'$ of f is given by $\omega = -\frac{\varphi_3^2}{4h'^2}$ and therefore it is the square of a meromorphic function.
If f is sense-preserving, then $\omega = q^2$. In particular, f is sense-preserving if and only if q is analytic and $|q(z)| < 1$ in $\mathbb{U}$

Then the dilatation $\omega = g'/h'$ of f is an analytic function with $|\omega(z)| < 1$ in $\mathbb{U}$ and with the further property that $\omega = q^2$ for some function q analytic in $\mathbb{U}$. The minimal surface S over $\mathbb{U}$ has the isothermal representation
$x_k = Re\int \varphi_k(w)dw$,$k = 1,2,3$,for $z \in \mathbb{U}$, with
$\varphi_1 = h' + g' = p(1+q^2), \varphi_2 = -i(h' - g') = -ip(1-q^2)$,
and $\varphi_3 = -2ipq$, where p and q are the Weierstrass-Enneper parameters. Thus

$\varphi_3^2 = -4\omega h'^2$ and $h' = p$. This shows that $\omega = -\frac{\varphi_3^2}{4h'^2}$ is the square of a meromorphic function. In other words, the harmonic mappings that result from projection of minimal graphs have dilatations with single valued square roots. If f is sense-preserving, this is equivalent to saying that its dilatation function ω has no zeros of odd order. The formula for ω is further illuminated when the Weierstrass-Enneper functions p and q are introduced. Then

and a similar calculation shows that $h' = p$ and $g' = pq^2$, which gives the elegant expression $\omega = q^2$ for the dilatation of the projected harmonic mapping f. In particular, f is sense-preserving if and only if q is analytic and $|q(z)| < 1$ in $\mathbb{U}$.

The first fundamental form of S is $ds^2 = \lambda^2 |dz|^2$, where

$$\lambda^2 = \frac{1}{2}\sum_{k=1}^{3} |\varphi_k|^2 > 0\,.$$

A direct calculation shows that $\lambda^2 == |h'|^2 + |g'|^2 + 2|h'||g'| = (|h'| + |g'|)^2$ so that $\lambda = |h'| + |g'| = |p|(1 + |q|^2)$.

We are now prepared to connect minimal surfaces with harmonic mappings. If a surface $X = \Phi(U)$ is expressed in terms of isothermal parameters, then, as we have just seen, $X_u \cdot X_v = 0$ and $X_u \cdot X_u = X_v \cdot X_v$. Further differentiations produce the relations

the Weierstrass-Enneper parameterization of minimal surfaces is a classical piece of differential geometry.

Alfred Enneper and Karl Weierstrass studied minimal surfaces as far back as 1863.

Let f and g be functions on either the entire complex plane or the unit disk, where g is meromorphic and f is analytic, such that wherever g has a pole of order m, f has a zero of order $2m$ (or equivalently, such that the product fg^2 is holomorphic), and let c_1, c_2, c_3 be constants. Then the surface with coordinates (x_1, x_2, x_3) is minimal, where the x_k are defined using the real part of a complex integral, as follows:

$$x_k(\zeta) = \Re\left\{\int_0^{\zeta} \varphi_k(z)\,dz\right\} + c_k, \qquad k = 1, 2, 3 \tag{13.6}$$

$$\varphi_1 = f(1 - g^2)/2 \tag{13.7}$$

$$\varphi_2 = \mathbf{i}f(1 + g^2)/2 \tag{13.8}$$

$$\varphi_3 = fg \tag{13.9}$$

The converse is also true: every nonplanar minimal surface defined over a simply connected domain can be given a parametrization of this type.[1]

For example, Enneper's surface has $f(z) = 1$, $g(z) = z$.

It was shown in the previous chapter that when a minimal surface is represented by isothermal parameters, its three coordinate functions are harmonic. As a consequence, the projection of a minimal graph to its base plane is a harmonic mapping. Our object is now to characterize the harmonic mappings obtained in this way and to show how they lift to minimal surfaces.

Consider a regular minimal graph $S = \{(u, v, F(u, v)) : u + iv \in G\}$ over a simply connected domain $G \subset \mathbb{C}$ containing the origin. Suppose that G is not the whole plane. (It will be shown later that the only minimal graphs that extend over the entire plane are themselves planes, a theorem of S. Bernstein.) In view of the Weierstrass-Enneper representation, as developed in Section 9.3, the surface has a reparametrization by isothermal parameters $z = x + iy$ in the unit disk $\mathbb{U}$.

There is no loss of generality in supposing that z ranges over the unit disk, because any other isothermal representation can be precomposed with a conformal map from the disk whose existence is guaranteed by the Riemann mapping theorem. The functions φ_k may be expressed in the form
$\varphi_1 = p(1+q^2), \varphi_2 = -ip(1-q^2), \varphi_3 = -2ipq,$
where p is analytic and q is meromorphic in $\mathbb{U}$, with p nonvanishing except for a zero of order $2m$ wherever q has a pole of order m.

In terms of p and q, the Weierstrass-Enneper representation is
$u = Re(\int_0^z p(1+q^2)d\zeta), v = Im(\int_0^z p(1-q^2)d\zeta), F(u,v) = 2Im(\int_0^z pqd\zeta), z \in \mathbb{U}.$

Now let $w = u + iv$ and let $w = f(z)$ denote the projection of S onto its base plane: The formula for ω is further illuminated when the Weierstrass-Enneper functions p and q are introduced. Then
$f(z) = Re(\int_0^z p(1+q^2)d\zeta) + iIm(\int_0^z p(1-q^2)d\zeta)$
and a similar calculation shows that $h' = p$ and $g' = pq^2$, which gives the elegant expression $\omega = q^2$ for the dilatation of the projected harmonic mapping f. In particular, f is sense-preserving if and only if q is analytic and $|q(z)| < 1$ in $\mathbb{U}$. In view of the remarks at the end of Section 9.3, the relation $\omega = q^2$ also identifies $-i/\sqrt{w}$ as the stereographic projection of the Gauss map of the corresponding minimal surface. The problem now arises to give a full description of the harmonic mappings that are projections of minimal surfaces. In other words, what properties of a harmonic mapping are necessary and sufficient for it to lift to a minimal graph expressed by isothermal parameters? A necessary condition, as we have just shown, is that the dilatation of the harmonic mapping is the square of a meromorphic function. Surprisingly, the condition is also sufficient. To verify this assertion, we may suppose without loss of generality that the mapping is sense-preserving.

Theorem 20. If a minimal graph $S = \{(u, v, F(u, v)) : u+iv \in G\}$ is parametrized by sense-preserving isothermal parameters $z = x + iy \in \mathbb{U}$, the projection onto its base plane defines a harmonic mapping $w = u + iv = f(z)$ of $\mathbb{U}$ onto G whose dilatation is the square of an analytic function. Conversely, if $f = h + \overline{g}$ is a sense-preserving harmonic mapping of $\mathbb{U}$ onto some domain G with dilatation $\omega = q^2$ for some function q analytic in $\mathbb{U}$, then the formulas
$u = Ref(z), v = Imf(z), t = 2Im(\int_0^z q(\zeta)h'(\zeta)d\zeta)$
define by isothermal parameters a minimal graph whose projection is f. Except for the choice of sign and an arbitrary additive constant in the third coordinate function, this is the only such surface.

Question

Let $Q = [-1, 1]^2$ and $p_0 = (u_0, v_0) \in Q$. Denote by $\hat{S}_{p_0}$ family of minimal graphs S with base Q, parametrized by $F : U \to S$, for which Q is tangent plane at p_0. Find $J(p_0) = \sup\{|F'(0)| : F(U) \in \hat{S}_{p_0}\}$. Whether $J(p_0) = \frac{4}{\pi}\cos(\frac{\pi}{2}u_0)$ if $v_0 \le u_0$.

Set $I_s = [-s, s]$. For s near 1 Q is not extremal disk in $Q \times I_s$.

Let $r < 1$, $A_0(z) = \frac{1+z}{1-z}$, and let $\phi = i\dfrac{2}{\pi}\ln A_0$;. that is $\phi = \phi_0 \circ A_0$, where $\phi_0 = i\frac{2}{\pi}\ln$. Let $\hat{\phi}$ be defined by $\hat{\phi}(z) = \phi(iz)$. Note that $\hat{\phi} = \frac{4}{\pi}$ arctan is the inverse of f_0.

For a second example, consider the function

$$f(z) = \frac{1+i}{\pi\sqrt{2}} \sum_{k=0}^{3} i^k \arg(\frac{z-i^k}{z-i^{k+1}})$$

which maps $\mathbb{U}$ harmonically onto the square region $Q = [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^2$. Check that $u = \frac{\sqrt{2}}{\pi}\arg A_0(iz), v = \frac{\sqrt{2}}{\pi}\arg A_0(z), t = \frac{\sqrt{2}}{\pi}\ln|A_0(z^2)|$ This can be recognized as Scherk's surface, as presented in Duren, Section 9.4, with the scale factor $\frac{\sqrt{2}}{\pi}$.

Example 16. $f(z) = Imf_1(z)$, where $f_1(z) = \frac{2}{\pi}\ln A(z)$ and $A(z) = \frac{1+z}{1-z}$.

Since $f'(0) = \frac{4}{\pi}$, this example shows that (ii) is sharp.

$g = h = -if_1/2, \omega = 1, t = 2Im(\int_0^z q(\zeta)h'(\zeta)d\zeta) = 2Im(h(z)) = -Im(if_1(z)) = -Ref_1(z) = \frac{2}{\pi}\ln|A(z)|$.

For example, we can consider the unit ball $\mathbb{B}$ in $\mathbb{R}^3$ and metric h on minimal surface S induced by hyperbolic metric on the unit ball B and try to compute the Gaussian curvature of (S, h).

Roughly speaking then we can try to apply theorem:

If ρ and σ are two metrics (density) on $\mathbb{U}$, σ complete and $0 > \bar{K}_\sigma \geq \bar{K}_\rho$ on $\mathbb{U}$, then $\sigma \geq \rho$.

$\bar{K}$ is Gaussian curvature.

$\mathbb{U}$ the unit disk

For hyperbolic density $\bar{K} = -4$ or -1 (it depends of normalization).

Two examples will now be given to illustrate the process of lifting a harmonic mapping to a minimal surface. First consider the function $f(z) = z - \frac{1}{3}\bar{z}^3$, which provides a harmonic mapping of the unit disk $\mathbb{U}$ onto the region G inside a hypocycloid of four cusps inscribed in the circle $|w| = 4/3$ (see Section 1.1).

$\omega = -z^2$. Because ω is the square of an analytic function, the theorem says that f lifts to the minimal surface defined by the equations
$u = Ref(z), v = Imf(z), t = 2Im(\int_0^z q(\zeta)h'(\zeta)d\zeta) = Rez^2$.
This is a nonparametric portion of Enneper's surface, as presented in Section 9.4. The complete surface is obtained as z ranges over the whole plane.

Therefore, in terms of the Weierstrass-Enneper parameters, the Gauss curvature is found to be $K = -\frac{4|q'|^2}{|p|^2(1+|q|^2)^4}$. Since $|q(z)| < 1$, the SchwarzPick lemma gives $(1-|z|^2)|q'(z)| \leq 1-|q|^2$.

Furthermore, the bound is sharp everywhere (but is attained only at the origin) for univalent harmonic mappings f of $\mathbb{U}$ onto itself with $f(0) = 0$. Extremal function $f_0(z) = \frac{2}{\pi}\arg\frac{1+z}{1-z}$. Since $\ln\frac{1+z}{1-z} = 2z + 0(z)$, when $z \to 0$,

$f_0(z) = \frac{4}{\pi}y$ and $L_{f_0}(0) = \frac{4}{\pi}$.

A metric defined by CH disks

pseudo Finsler norm

Let G be bounded connected open subset of R^n (in general,real or complex Banach space ??), $p \in G$ and $v \in T_pG$. We define $k_G(p, v) = \inf\{|h|\}$, where infimum is taking over all $h \in T_0\mathbb{C}$ for which there exists conformal harmonic mappings $\phi : \mathbb{U} \to G$ such that $\phi(0) = p$ and $d\phi(h) = v$. Hence $hd\phi(1) = v$ and $k_G(p, v)d\phi(1) = v$, $k_G(p, v)|d\phi(1)| = |v|$

Since $h = \frac{|v|}{|d\phi(1)|}$, $k_G(p, v) = \inf\{|h|\} = |v|\frac{|v|}{\sup|d\phi(1)|}$, where supremum is taking over all conformal harmonic mappings $\phi : \mathbb{U} \to G$ such that $\phi(0) = p$ and $d\phi(h) =$

sv, $s \in \mathbb{R}$. Hence we define pseudo Finsler density as $k_G(p) = \frac{1}{\sup|d\phi(1)|}$ and $D_g(p) = \sup|d\phi(1)|$. We call the corresponding distance CH-distance. Are there some interesting application of CH-distances (as Kobayashi, Caratheodory, etc)?

Your question related to the analogy famous theorem of Lempert (1981) seems to be interesting.

Today I have exams (so I am very busy) but as soon as possible I will give more details. Shortly, I will give only an idea.

There is a version of Koebe one quoter theorem for harmonic maps. We can try to prove it for minimal surface.

By e we denote euclidean distance in $\mathbb{R}^n$ space.

In some settings the following is true: If $f : \mathbb{U} \to R^3$ conformal-harmonic, $f(0) = 0$ and $S = f(\mathbb{U})$, then $e(0, \partial S) \geq \frac{|f'(0)|}{16}$.

Suppose that S is a minimal graph defined over a domain G in xy-plane, by $z = F(x, y)$, $(x, y) \in G$.

Let $p(x, y, z) = (x, y)$ and $\underline{f} = p \circ f$. Then $\underline{f}$ is univalent harmonic and an application

the version of Koebe one quoter theorem for harmonic maps, yields the result.

Let S be a minimal surface in the unit ball $\mathbb{B}$ in R^3 and consider the metric $d = d_S$ on S induced by the hyperbolic metric on the unit ball $\mathbb{B}$ and try to compute the Gaussian curvature K_S of (S, d).

Question: It will be nice if $K_S \leq -1$. Is it possible that $K_S \geq -1$? If S is the intersection a plane through the origin with $\mathbb{B}$, then $K_S = -1$.

Let f be isothermal parametrization of S defined on the unit disk U. If $K_S \leq -1$, then $d_{hyp,B}(fz, fw) \leq d(fz, fw) \leq d_{hyp,U}(z, w)$, $z, w \in U$.

It seems that this implies that CH-Finsler density for unit ball at 0 is $CH(0) = 1$ and extremal disks are euclidean disks.

If (S, d) is complete and $K_S \geq -1$, then $d(fz, fw) \geq d_{hyp}(z, w)$, $z, w \in U$.

If $f : \mathbb{U} \to \mathbb{B}_n$ is holomorphic (analytic disk), then the pull back of Hermitian-euclidean metric e on $\mathbb{C}^n$ is $ds^2 = \sum_{k=1}^n |dw_k|^2 = (\sum_{k=1}^n |f_k'(z)|^2)|dz|^2 = |f'(z)|^2|dz|^2$ and $\lambda = |f'(z)|$. By Schwarz's lemma $\lambda \leq \frac{1}{1-|z|^2}$. Hence, if $d = d_{e,S}$ metric on $S = f(\mathbb{U})$ inherited from $\mathbb{C}^n$, then $d(fz, fw) \geq d_{hyp}(z, w)$, $z, w \in U$.

As I understand, in the setting of $\mathbb{C}^n$ (identified by $\mathbb{R}^{2n}$ analytic disks are CH-disks.

Is it known something about the Gaussian curvature K_S of analytic disk S in the unit ball B with metric inherited by Kobayshi metric of B?

Question:

d_{hyp} hyperbolic metric on U. K_S the Gaussian curvature of (S, d).

conformal harmonic map of the unit disk U into the unit ball B in R^n.

best regards, Miodrag

Dear Miodrag,

I do not see a connection between this comparison result and the calculation/estimation of the CH-metric; I must be missing something.

I am really curious whether, in the ball or a bounded convex domains of R^n, there exist minimal geodesics, i.e. conformal minimal (=harmonic) disks which are extremal at every point. This holds for holomorphic disks in any bounded convex domain in $\mathbb{C}^n$ by a famous theorem of Lempert (1981).

Q. Let $f : \mathbb{D} \to \mathbb{B}^n$ be a harmonic univalent mapping with $l(f'(0) \geq 1$ and $f(0) = 0$. Is there an absolute constant $c > 0$, such that $\delta = dist(0, \partial h(\mathbb{D}) \geq c$??

Q. Let $f : \mathbb{D} \to \mathbb{B}^n$ be a K-qr harmonic mapping. Whether f is a quasi-isometry w.r. the hyperbolic distances ?

14. Appendix- Schwarz lemma 2

The model of the hyperbolic plane is the half-plane model. The underlying space of this model is the upper half-plane model $\mathbb{H}$ in the complex plane $\mathbb{C}$,defined to be $\mathbb{H} = \{z \in C : Im(z) > 0\}$. In coordinates (x, y) the line element is defined as $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$. The geodesics of this space are semicircles centered on the x-axis and vertical half-lines. The geometrical properties of the figures on the half-plane are studied by considering quantities invariant under an action of the general Möbius group, which consists of compositions of Möbius transformations and reflections [4]. The curvilinear triangle formed by circular arcs of three intersecting semicircles is one of the principal figures of the upper half-plane model $\mathbb{H}$. The hyperbolic laws of sines-cosines for that triangle are proved by using properties of the Möbius group and the upper half-plane $\mathbb{H}$.

In [67] Yamaleev suggests another way of construction of proofs of the sines-cosines theorems of the Poincaré model. The curvilinear triangle formed by circular arcs is the figure of the Euclidean plane; consequently, on the Euclidean plane we have to find relationships antecedent to the sines-cosines hyperbolic laws. Therefore, first of all, we establish these relationships by making use of axioms of the Euclidean plane, only. Secondly, we prove that these relationships can be formulated as the hyperbolic sine-cosine theorems. For that purpose we refer to the general complex calculus and within its framework establish a relationship between exponential function and the cross-ratio. In this way the hyperbolic trigonometry emerges on Euclidean plane in a natural way.

For the benefit of the reader we add some details concerning some parts of subsection 2.2. The cross-ratio of a 4-tuple of distinct points on the real line with coordinates z_1, z_2, z_3, z_4 is given by

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

Let γ be a circular arc (geodesic), orthogonal to T at the points w_1 and w_2, that contains the points z_1 and z_2 of the unit disk (suppose that the points w_1, z_1, z_2, w_2 occur in this order). Since $(r, 0, -1, 1) = (1 + r)/(1 - r)$, we find

$$\lambda(z_1, z_2) = \ln(z_2, z_1, w_1, w_2).$$

We leave to the interested reader to check that $\{z_1, z_2\} = (z_2, z_1, w_1, w_2) > 0$ if the points are in the order indicated above.

In this form we can consider λ as the oriented distance which changes the sign of the permutation z_1 and z_2. Additivity of the distance on geodesics follow from $(z_2, z_1, w_1, w_2) = (z_2, z_3, w_1, w_2)(z_3, z_1, w_1, w_2)$.

Let $K = K(z_1, z_2)$ circle orthogonal on $\mathbb{T}$ throughout points z_1 i z_2 and denote by a and b the intersection points K and $\mathbb{T}$. Usually we denote the intersection points such that z_1 is between a and z_2.

Recall $[z_1, z_2; a, b] = \frac{z_1-a}{z_1-b} : \frac{z_2-a}{z_2-b}$. For example if $a = -1$, $b = 1$, then according convention about the notation we write $z_1 = 0$ i $z_2 = r$, $0 < r < 1$.

Since $[0, r; -1, 1] = \frac{1-r}{1+r}$, it is $0 < [0, r; -1, 1] < 1$. Therefore it is convenient to define $\{z_1, z_2\} = [z_2, z_1; a, b]_2 = \frac{z_2-a}{z_2-b} : \frac{z_1-a}{z_1-b}$.

Check that according our convention on notation $\{z_1, z_2\} > 1$ and
$\{z_1, z_2\}{\cdot}\{z_2, z_3\} = \{z_1, z_3\}$. Define $d_{hyp}(z_1, z_2) = \ln\{z_1, z_2\}$. Check $\{0, r\} = \frac{1+r}{1-r}$.

Define $T_{z_1}(z) = \frac{z-z_1}{1-\overline{z_1}z}$, $\varphi_{z_1} = -T_{z_1}$ and
$\delta(z_1, z_2) = |T_{z_1}(z_2)| = |\frac{z-z_1}{1-\overline{z_1}z}|$. Schwarz's lema yields motivation to introduce the hyperbolic distance: If $f \in \mathrm{Hol}(\mathbb{U}, \mathbb{U})$, then $\delta(fz_1, fz_2) \leq \delta(z_1, z_2)$.

Consider $F = \varphi_{w_1} \circ f \circ \varphi_{z_1}$, $w_k = f(z_k)$. Then $F(0) = 0$ and $|\varphi_{w_1}(w_2)| \leq |\varphi_{z_1}(z_2)|$.

Hence

$$|f'(z)| \leq \frac{1 - |fz|^2}{1 - |z|^2}.$$

By notation $w = f(z)$ i $dw = f'(z)dz$,

$$\frac{|dw|}{1 - |w|^2} \leq \frac{|dz|}{1 - |z|^2}.$$

Define the density $\rho(z) = \frac{1}{1-|z|^2}$.

For $\mathbf{v} \in T_z\mathbb{C}$ vector def $|\mathbf{v}|_\rho = \rho(z)|\mathbf{v}|$ and set $\mathbf{v}^* = df_z(\mathbf{v})$. Then $|\mathbf{v}^*|_\rho \leq |\mathbf{v}|_\rho$.

If γ piecewise smooth then define $|\gamma|_\rho = \int_\gamma \rho(z)|dz|$ and $d(z_1, z_2) = \inf |\gamma|_\rho$, where the infimum is taken over all paths γ in $\mathbb{U}$ joining the points z_1 and z_2.

We summarize

$$\lambda_{\mathbb{U}} = \ln \frac{1 + \delta_{\mathbb{U}}}{1 - \delta_{\mathbb{U}}}, \quad \lambda_{\mathbb{H}} = \ln \frac{1 + \delta_{\mathbb{H}}}{1 - \delta_{\mathbb{H}}}\,. \tag{14.1}$$

For $\mathbf{v} \in T_z\mathbb{C}$ vector we define $|\mathbf{v}|_\rho = \rho(z)|\mathbf{v}|$ and set $\mathbf{v}^* = df_z(\mathbf{v})$.

Proposition 14.1. If $f \in \mathrm{Hol}(\mathbb{U}, \mathbb{U})$, then $|\mathbf{v}^*|_\rho \leq |\mathbf{v}|_\rho$.

If γ piecewise smooth then define $|\gamma|_\rho = \int_\gamma \rho(z)|dz|$ and $d(z_1, z_2) = \inf |\gamma|_\rho$, where the infimum is taken over all paths γ in $\mathbb{U}$ joining the points z_1 and z_2.

Let G be a simply connected domain different from $\mathbb{C}$ and let $\phi : G \to \mathbb{U}$ be a conformal isomorphism. Define the pseudo hyperbolic distance on G by $\varphi_a^G(z) = \varphi_b(\phi(z))$, where $b = \phi(a)$, and $\delta_G(a, z) = |\varphi_a^G(z)| = \delta_{\mathbb{U}}(\phi(a), \phi(z))$. Verify that the pseudo hyperbolic distance on G is independent of conformal mapping ϕ. In particular, using conformal isomorphism $A(w) = A_{w_0}(w) = \frac{w-w_0}{w-\overline{w_0}}$ of $\mathbb{H}$ onto $\mathbb{U}$, we find $\varphi_{H,w_0}(w) = A(w)$ and therefore $\delta_H(w, w_0) = |A(w)|$.

Proposition 14.2. The definition of δ_G is independent of conformal isomorphism.

Proof. Let $\phi, \phi_1 : G \to \mathbb{U}$ be two conformal isomorphism and let z_1, z_2 be two points in G and $w_k = \phi(z_k)$ and $w'_k = \phi_1(z_k)$, $k = 1, 2$. If $A = \phi_1^{-1} \circ \phi$, then $A \in \mathrm{Aut}(\mathbb{U})$ and $A(w_k) = w'_k$, $k = 1, 2$. Hence $\delta_{\mathbb{U}}(w_1, w_2) = \delta_{\mathbb{U}}(w'_1, w'_2)$. □

For a domain G in $\mathbb{C}$ and $z, z' \in G$ we define $\delta_G(z, z') = \sup \delta_{\mathbb{U}}(\phi(z'), \phi(z))$, where the suprimum is taken over all $\phi \in \mathrm{Hol}(G, \mathbb{U})$.

Proposition 14.3. (a) If G and D are conformally isomorphic to $\mathbb{U}$ and $f \in \mathrm{Hol}(G, D)$, then

$$\delta_D(fz, fz') \leq \delta_G(z, z'), \quad z, z' \in G.$$

(b) The result holds more generally if G and D are hyperbolic domains.

Proof. Let $\phi : G \to \mathbb{U}$ and $\phi_1 : D \to \mathbb{U}$ be two conformal isomorphisms and set $F = \phi_1 \circ f \circ \phi^{-1}$. Next let z_1, z_2 be two points in G and $w_k = f(z_k)$, $\zeta_k = \phi(z_k)$ and $\zeta'_k = \phi_1(w_k)$, $k = 1, 2$. Then $\delta_{\mathbb{U}}(\zeta'_1, \zeta'_2) \leq \delta_{\mathbb{U}}(\zeta_1, \zeta_2)$ and the result follows.
A proof of (b) can be based on the fact:
if $\phi \in \mathrm{Hol}(D, \mathbb{U})$, then $\phi \circ f \in \mathrm{Hol}(G, \mathbb{U})$. □

Set $d^0(z, w) = |\frac{z-w}{1-\overline{w}z}|$. In a response to nice post Principle of subordination [71] KKK(05/03/2012 at 8:27 pm) says: "In most textbooks, the hyperbolic metric g is given first and then it is deduced that the isometry group is exactly the group of all conformal self maps of the disk i.e. disk-preserving holomorphic and anti-holomorphic functions. Suppose we want to reverse this process and want to find a (hyperbolic) metric which is preserved by the conformal self maps. We observe from the equality case of the Pick's lemma that for $w = w(z)$ to be a conformal self map, $|\frac{dw}{dz}| = \frac{1-|w|^2}{1-|z|^2}$ i.e. $\frac{|dw|}{1-|w|^2} = \frac{|dz|}{1-|z|^2}$.

Thus we find that the hyperbolic metric is given by (up to a positive factor)
$g = \frac{|dz|^2}{(1-|z|^2)^2} = \frac{dx^2+dy^2}{(1-x^2-y^2)^2}$.

Interestingly, the other form of Picks lemma is given by $d^0(fz, fw) \leq d^0(z, w)$ with equality hold iff f is a conformal self map. It suggests (without integrating) that the hyperbolic distance is actually given by (up a scaling) $d^0(z, w) = |\frac{z-w}{1-\overline{w}z}|$. However, it is different from the usual definition $d(z, w) = \tanh^{-1}(|\frac{z-w}{1-\overline{w}z}|)$, I cant think if any way to "see" (except integrating) that the correct definition is the later one instead of the former one (actually I havent checked if the former one really satisfies the triangle inequality)."

Note that the former one really satisfies the triangle inequality and we call it the pseudo-hyperbolic distance. The pseudo-hyperbolic distance is not additive along hyperbolic geodesics.

14.1. **the hyperbolic law of cosines and sines.** A good reference for this subsection is Ahlfors book [6]. For a right triangle in hyperbolic geometry with sides a, b, c and with side c opposite a right angle, the relation between the sides takes the form:

$$\cosh c = \cosh a \, \cosh b$$

where cosh is the hyperbolic cosine. This formula is a special form of the hyperbolic law of cosines that applies to all hyperbolic triangles:

Proposition 14.4 (I. the hyperbolic law of cosines)**.**

$$\cosh c = \cosh a \; \cosh b - \sinh a \; \sinh b \; \cos \gamma$$

with γ the angle at the vertex opposite the side c.

By using the Maclaurin series for the hyperbolic cosine, $coshx = 1 + x^2/2 + o(x^2)$, it can be shown that as a hyperbolic triangle becomes very small (that is, as a, b, and c all approach zero), the hyperbolic relation for a right triangle approaches the form of Pythagoras' theorem.

In hyperbolic geometry when the curvature is -1, the law of sines becomes:

Proposition 14.5 (II. the hyperbolic law of sines)**.** For a hyperbolic triangle ABC,
$\dfrac{\sin A}{\sinh a} = \dfrac{\sin B}{\sinh b} = \dfrac{\sin C}{\sinh c}$.

In the special case when B is a right angle, one gets
$\sin C = \dfrac{\sinh c}{\sinh b}$,
which is the analog of the formula in Euclidean geometry expressing the sine of an angle as the opposite side divided by the hypotenuse.

Proof of I. By an abuse of notation, we use the same symbols for vertices and the measures of corresponding angles. Without loss of generality we can suppose $C = 0$, $0 < B < 1$ and $A = e^{iC}s$, where $0 < s < 1$. Then $B = \tanh(a/2)$, $A = \tanh(b/2)e^{iC}$ and $\sigma(A, B) = \tanh(c/2)$. As in euclidean trigonometry all trigonometric function we can express by tanh. For cos and sin (see also Proposition **??**),

$$\cosh x = \frac{1+\tanh^2(x/2)}{1-\tanh^2(x/2)}, \quad \sinh x = \frac{2\tanh(x/2)}{1-\tanh^2(x/2)}\,.$$

$$\cosh c = \tag{14.2}$$

$$\frac{(1+\tanh^2(a/2))(1+\tanh^2(b/2)) - 4\tanh(a/2)\tanh(b/2)\cos C}{(1-\tanh^2(a/2))(1-\tanh^2(b/2))} \tag{14.3}$$

$$= \cosh a\ \cosh b - \sinh a\ \sinh b\ \cos\gamma\,. \tag{14.4}$$

□

Proof of II. By the hyperbolic law of cosines,

$$\cos C = \frac{\mathrm{ch}a\,\mathrm{ch}b - \mathrm{ch}c}{\mathrm{sh}a\,\mathrm{sh}b}\,,$$

$$\sin^2 C = \frac{(\mathrm{ch}^2 a - 1)(\mathrm{ch}^2 b - 1) - (\mathrm{ch}a\,\mathrm{ch}b - \mathrm{ch}c)^2}{\mathrm{sh}^2 a\mathrm{sh}^2 b},$$

$$\frac{\sin^2 C}{\mathrm{sh}^2 c} = \frac{1 - \mathrm{ch}^2 a - \mathrm{ch}^2 b - \mathrm{ch}^2 c + 2\mathrm{ch}a\,\mathrm{ch}b\,\mathrm{ch}c}{\mathrm{sh}^2 a\mathrm{sh}^2 b\mathrm{sh}^2 c}\,.$$

Since the formula is symmetric, II follows. □

14.2. **Klein model.** Recall for given two distinct points U and V in the open unit ball of the model in Euclidean space, the unique straight line connecting them intersects the unit sphere at two ideal points A and B, labeled so that the points are, in order along the line, A, U, V, B. Taking the centre of the unit ball of the model as the origin, and assigning position vectors u, v, a, b respectively to the points U, V, A, B, we have that that $|a - v| > |a - u|$ and $|u - b| > |v - b|$, where $|\cdot|$ denotes the Euclidean norm. Then the distance between U and V in the modelled hyperbolic space is expressed as

$$d(\mathbf{u}, \mathbf{v}) = \frac{1}{2}\log\frac{\|\mathbf{v}-\mathbf{a}\|\ \|\mathbf{b}-\mathbf{u}\|}{\|\mathbf{u}-\mathbf{a}\|\ \|\mathbf{b}-\mathbf{v}\|},$$

where the factor of one half is needed to make the curvature -1.

XXX The associated metric tensor is given by

$$ds^2 = g(\mathbf{x}, d\mathbf{x}) = \frac{\|d\mathbf{x}\|^2}{1-\|\mathbf{x}\|^2} + \frac{(\mathbf{x}\cdot d\mathbf{x})^2}{\left(1-\|\mathbf{x}\|^2\right)^2}\,.$$

More precisely, if $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n$, then

$$ds^2(\mathbf{v}) = \mathrm{Kle}(\mathbf{x}, \mathbf{v}) = \frac{\|\mathbf{v}\|^2}{1 - \|\mathbf{x}\|^2} + \frac{(\sum_{k=1}^n x_k v_k)^2}{\left(1 - \|\mathbf{x}\|^2\right)^2}\,. \tag{14.5}$$

14.3. **Schwarz in R^n.** XX In terms of the so-called contact angle (the angle between the normal to the sphere at the bottom of the cap and the base plane)

Set $N_0 = (0, ..., 0, 1)$, $E_1 = (1, 0, ..., 0)$ and $Q(x, \xi) = |x|^2 + |\xi|^2 - 2\langle x, \xi\rangle$. Note that $P(x, \xi) = (1 - |x|^2)Q^{-n/2}$.

Check that
if $x = |x|N_0$, then $|x - \xi|^2 \le |x - E_1|^2 = 1 + |x|^2$, $\xi \in S^+$ and
$|x - E_1|^2 = 1 + |x|^2 \le |x - \xi|^2$, $\xi \in S^-$.
Set $M(x) = M_2(x) = (1 - |x|^2)(1 + |x|^2)^{-n/2}$. Then
$M(x) \le P(x, \xi)$ for $\xi \in S^+$, and $P(x, \xi) \le M(x)$ for $\xi \in S^-$. Define

$$U^0 = P[\chi_{S^+} - \chi_{S^-}]$$

Theorem 21 (Theorem 6.16 [ABR])**.** Suppose that u is complex valued harmonic on $\mathbb{B}$, $|u| < 1$ on $\mathbb{B}$, and $u(0) = 0$. Then

$$|u(x)| \le U_0(|x|N_0) \tag{14.6}$$

for every $x \in \mathbb{B}$.

After a rotation, we can assume that $x = |x|N_0 = (0, ..., 0, |x|)$.

First we consider the case in which u is real valued. There is $f \in L^\infty(\mathbb{S})$, such that $u = P[f]$. We claim that $u(x) \le U_0(x)$. This inequality is equivalent to $I_1 \le I_2$, where

$$I_1 := \int_{S^-} P(x, \xi)(1 + f)d\sigma(\xi) \quad \text{and} \quad I_2 := \int_{S^+} P(x, \xi)(1 - f)d\sigma(\xi).$$

It is convenient to introduce $J_1 := \int_{S^-} Q^{-n/2}(x, 0)(1 + f)d\sigma(\xi)$ and $J_2 := \int_{S^+} Q^{-n/2}(x, 0)(1 - f)d\sigma(\xi)$.

The condition $u(0) = 0$ implies $\int_{S^-} f d\sigma = -\int_{S^+} f d\sigma$ and therefore $J_1 = J_2$.

Since ξ_n is negative on S^- and positive on S^-, $Q^{-n/2}(x, \xi) \le Q^{-n/2}(x, 0)$, $\xi \in S^-$, and $Q^{-n/2}(x, 0) \le Q^{-n/2}(x, \xi)$, $\xi \in S^+$. Hence $I_1 \le J_1 = J_2 \le I_2$.

It is useful to present a small variation of the above proof. Set $A := \int_{S^+}(1 - f(\xi))d\sigma(\xi)$ and $B := \int_{S^-}(1 + f(\xi))d\sigma(\xi)$. The condition $u(0) = 0$ implies $A = B$. Then $J_1 = BM(x)$, $J_2 = AM(x)$ and $I_1 \le J_1 = J_2 \le I_2$.

Theorem 22. Suppose that u is complex valued harmonic on $\mathbb{B}$, $|u| < 1$ on $\mathbb{B}$. Then

$$|(\nabla u)(0)| \le \tau_n = 2\frac{V(B_{n-1})}{V(B_n)}. \tag{14.7}$$

Equality holds if and only if $u = U_0 \circ T$ for some orthogonal transformation.

Note that $B_1 = (-1, 1)$ and $V(B_1) = 2$.

Proof. For a fixed ξ, $D_n Q(x, \xi) = 2(x_n - \xi_n)$ and therefore
$D_n P(x, \xi) = -2x_n Q^{-n/2} - n(1 - |x|^2)Q^{-n/2-1}(x_n - \xi_n)$.

For $\xi \in S$, $Q(0,\xi) = 1$ and therefore $D_n P(0,\xi) = n\xi_n$. Hence $D_n u(0) = \int_S D_n P(0,\xi)\, f(\xi)\, d\sigma(\xi) = n\int_S \xi_n\, f(\xi)\, d\sigma(\xi) \le n\tau_n$, where $\tau_n := \int_S |\xi_n|\, d\sigma(\xi)$. □

Set $\omega_n = V(B_n)$.

Exercise 9. Check that $\omega_1 = 2$, $\omega_2 = \pi$, $\omega_3 = 4\pi/3$, $\tau_2 = 4/\pi$ and $\tau_3 = 3/2$.

When $n = 2$, $|(\nabla u)(0)| \le 4/\pi$. For $z = (x,y) \in \mathbb{B}_2$, $U_2^0(x,y) = \frac{2}{\pi}\arctan\frac{2y}{1-x^2-y^2}$ and $U_2^0(|z|N_0) = \frac{4}{\pi}\arctan|z|$.

Suppose w is a harmonic mapping of $\mathbb{U}$ into itself. Then, the following statement holds: $|w(z) - M(z)w(0)| \le \frac{4}{\pi}$, $z \in \mathbb{U}$.

Show that

$$U_3^0(|x|N_0) = \frac{1}{|x|}\left[1 - \frac{1-|x|^2}{\sqrt{1+|x|^2}}\right]$$

and

$$U_4^0(|x|N_0) = \frac{2}{\pi}\frac{(1+|x|^2)^2\arctan|x| - |x|(1-|x|^2)}{|x|^2(1+|x|^2)}.$$

Definition 14.1 (Har(p),Har$_c$(p))**.** For $p \in \mathbb{U}$, let Har(p) (respectively Har$_c$(p)) denote the family of all complex valued harmonics maps f from $\mathbb{U}$ into itself with $f(0) = p$ (which are conformal at 0 respectively).

Burgeth

$$H^c := \{\text{h is harmonic (hyperbolic-harmonic)} \quad \text{on } B^p : h(0) = c, 0 < h < 1\}$$

By $\mathbb{I}_A$ we denote the characteristic function of A. Set

$$M_c^p(|x|) = 2\int_{S^p} \mathbb{I}_{S(c,\hat{x})} P_x d\sigma - 1 \tag{14.8}$$

$$m_c^p(|x|) = 2\int_{S^p} \mathbb{I}_{S(c,-\hat{x})} P_x d\sigma - 1, \tag{14.9}$$

where $x \in B^p$ and $S(c,\hat{x})$ denotes the polar cap with center $\hat{x}$ and σ-measure c. Check that

$$M_c^p(|x|) = \frac{2}{\sqrt{\pi}}\frac{\Gamma(p/2)}{\Gamma(p/2)}(1-|x|^2))^\nu \int_0^{\alpha(c)} \frac{\sin^{p-2}\varphi}{(1-2|x|\cos\varphi + |x|^2)^\mu} d\varphi, \tag{14.10}$$

$$m_c^p(|x|) = \frac{2}{\sqrt{\pi}}\frac{\Gamma(p/2)}{\Gamma(p/2)}(1-|x|^2))^\nu \int_0^{\alpha(c)} \frac{\sin^{p-2}\varphi}{(1-2|x|\cos\varphi + |x|^2)^\mu} d\varphi, \tag{14.11}$$

where $(\nu,\mu) = (1, p/2)$ in the harmonic case resp. $(\nu,\mu) = (p-1, p-1)$ in the hyperbolicharmonic case and $\alpha(c)$ is the spherical angle of $S(c,\hat{x})$. For the sake of simplicity we consider $g = (1+h)/2 \in H^c$ $(g(0) = a)$. Our aim is to find the extreme values of the integral $\int_{S^p} P_x g^* d\sigma$ where g is varying in H^c.

Theorem 23. Let h be a harmonic or hyperbolic-harmonic function with $|h| \le 1$ and $h(0) = a$, $-1 < a < 1$. Then for XX $c = \sigma_n(a+1)/2$ and all $x \in \mathbb{B}^p$

$$m_c^p(|x|) \le h(x) \le M_c^p(|x|) \tag{14.12}$$

Equality on the right (resp.,left-) hand side for some $z \in \mathbb{B}^p \setminus \{0\}$ implies

$$h(x) = U_c^z(x) := 2\int_{S^p} \chi_{S(c,\hat{z})} P_x - 1 \tag{14.13}$$

$$h(x) = u_c^z(x) := 2\int_{S^p} \chi_{S(c,-\hat{z})} P_x - 1 \tag{14.14}$$

Set $A(t,x) = \{P_x > t\}$ and $\mu(t,x) = \sigma(A(t,x))$. First, note that the function defined by $t \mapsto \mu(t,x)$ ($t \in \mathbb{R}^+, x \in \mathbb{B}^p$) is continuous and strictly decreasing, so to each $c \in]0,1[$ there exists a unique number t_c such that $\mu(t_c,x) = c\sigma_p$. Further note that $\mathbb{I}_{S(c,\hat{x}} = \mathbb{I}_{A(t_c,x)} \in K^c$. Fixing $x \in \mathbb{B}^p$ we conclude that for every $g^* \in K^c$ with $g^* \neq \mathbb{I}_{S(c,\hat{x})}$

$$\begin{aligned}
&(14.15) \qquad \int_{S^p} P_x g^* d\sigma &&= \int (P_x - t_c) g^* d\sigma + c\sigma_p t_c \\
&(14.16) &&= \\
&(14.17) &&< \int (P_x - t_c)\mathbb{I}_{A(t_c,x)} d\sigma + c\sigma_p t_c \\
&(14.18) &&= \int \mathbb{I}_{S(c,\hat{x}} P_x d\sigma = \frac{1}{2}\big(M_c^p(|x|) + 1\big).
\end{aligned}$$

MM Set $M^p(r,c) = M_c^p(r)$.

$M^p(r,c)$ is increasing wrt c. Whether there is some kind of monotonicity with respect to r?

Let $g \in H^c$, $g = P[g^*]$ and $y_0 = g(x_0)$ there is c_0 such that $\int_{S_0} P_{x_0} d\sigma = y_0$, where $S_0 = S(\hat{x}_0, c_0)$; set $\sigma^0 = \sigma(S_0)$.

Then $P_{x_0} = t_0$ on bS_0.

$$\begin{aligned}
g(x) &= \int_S P_x g^* d\sigma = \int_{\mathbb{S}} (P_x - t_0) g^* d\sigma + t_0 \int_{\mathbb{S}} g^* d\sigma \\
&\leq \int_{S(\hat{x}_0,c_0)} (P_x - t_0) g^* d\sigma + t_0\sigma_p g(0) \leq \int_{S(\hat{x}_0,c_0)} (P_x - t_0) d\sigma + t_0\sigma_p g(0) \\
&= \int_{S(\hat{x}_0,c_0)} (P_x - t_0) d\sigma + t_0\sigma_p g(0) = \int_{S(\hat{x}_0,c_0)} P_x d\sigma - t_0\sigma^0 + t_0\sigma_p g(0) \\
&= M_{c_0}^p(|x|) + c_1(x)
\end{aligned}$$

?? $t_0 = t_0(x)$

Definition 14.2 (Har(p),Har$_c$(p))**.** For $p \in \mathbb{U}$, let Har(p) (respectively Har$_c$(p)) denote the family of all complex valued harmonics maps f from $\mathbb{U}$ into itself with $f(0) = p$ (which are conformal at 0 respectively). Set

$$\begin{aligned}
L_{\text{har}}(p) &= \sup\{|f'(0)| : f \in \text{Har}(p)\} \quad \text{and} \quad K_{\text{har}}(p) &&= \frac{L(p)}{\sqrt{1-|p|^2}}, \\
L_c(p) &= \sup\{|f'(0)| : f \in \text{Har}_c(p)\} \quad \text{and} \quad K_c(p) &&= \frac{L_c(p)}{1-|p|^2}.
\end{aligned}$$

For planar domains D and G and given points $z \in D$ and $p \in G$ denote by

$$L_{\text{har}}(z,p) = L_{\text{har}}(z,p;D,G) = \sup\{|f'(z)|\},$$

where the supremum is taken over all $f \in \text{Har}(D,G)$ with $f(z) = p$. If $I \subset \mathbb{R}$ is an interval, and $u_0 \in I$, we define $L_{\text{har}}(z,p;D,I)$ in a similar way.

If $D = \mathbb{U}$ we write Har(G) instead of Har($\mathbb{U}, G$) and if in addition $z = 0$, we write simply $L_{\text{har}}(p,G)$ (or shortly $L_h(p,G)$) and if in addition $G = \mathbb{U}$, we write $L_{\text{har}}(p)$.

For $a \in (-1,1)$, let Har^a denote the family of all real valued harmonics maps f from $\mathbb{U}$ into $(-1,1)$ with $f(0)=a$. Set $c=(a+1)/2$, $\overline{c}=2\pi c$, $\alpha=\alpha(c)=\overline{c}/2=(a+1)\pi/2$ and $X(r)=\frac{4}{\pi}\arctan\left(\frac{1+|z|}{1-|z|}\tan\frac{\alpha(c)}{2}\right)-1$, where $r=|z|$.

Theorem 24. If $h\in \mathrm{Har}^a$, then $h(z)\le X(|z|)$ and $|(dh)_0|\le X'(0)=\frac{4}{\pi}\sin\alpha$.

MM For $p=2$, $M^p_c(|z|)=X(r)=\frac{4}{\pi}\arctan\left(\frac{1+|z|}{1-|z|}\tan\frac{\alpha(c)}{2}\right)-1$, where $r=|z|$, $c=\sigma_2(a+1)/2=(a+1)\pi$ and $\alpha=\alpha(c)=\pi-c/2$. Then $M^2_c(0)=a$, $X'(0)=\frac{4}{\pi}\sin\alpha$ and therefore $L(a)=L_{\mathrm{har}}(a)=\frac{4}{\pi}\sin\alpha$, $0\le a<1$.

In general for $p\in\mathbb{U}$, $L(p)=L(|p|)$.

Set

$$
\begin{aligned}
L_{\mathrm{har}}(p) &= \sup\{|f'(0)| : f\in \mathrm{Har}(p)\} \quad \text{and} \quad K_{\mathrm{har}}(p) &= \frac{L(p)}{\sqrt{1-|p|^2}},\\
L_c(p) &= \sup\{|f'(0)| : f\in \mathrm{Har}_c(p)\} \quad \text{and} \quad K_c(p) &= \frac{L_c(p)}{1-|p|^2}.\\
U^p(z) &= \sup\{|f(z)| : f\in \mathrm{Har}(p)\}.
\end{aligned}
$$

$u(0)=a\in(0,1)$ $u=P[f]$

Set $A:=\int_{S^+}(1-f(\xi))d\sigma(\xi)$ and $B:=\int_{S^-}(1+f(\xi))d\sigma(\xi)$. The condition $u(0)=0$ implies $A=B$. Then $J_1=BM(x)$, $J_2=AM(x)$ and $I_1\le J_1=J_2\le I_2$.

Set $A(f):=\int_{A^+}(1-f(\xi))d\sigma(\xi)$ and $B(f):=\int_{A^-}(1+f(\xi))d\sigma(\xi)$.

$A^+=S(1,l_1)$ $A^-=S(1,l_2)$ $l(A^+)=l(A^-)+a$ $l_1=(1+a)/2$ $\alpha(a)=(1+a)\pi$ the central angle of cup A^+, $\zeta_0=e^{i\alpha/2}=b+i\sqrt{1-b^2}$

$|1-\zeta_0|^2=2(1+b)$ $U(z)=\int_{A^+}P_z d\sigma-\int_{A^-}P_z d\sigma$

Set $A(f):=\int_{A^+}(1-f(\xi))d\sigma(\xi)$ and $B(f):=\int_{A^-}(1+f(\xi))d\sigma(\xi)$. Since $U(0)=\int_{A^+}d\sigma(\xi)-\int_{A^-}d\sigma(\xi)=l(A^+)-l(A^-)=a$, $U(0)=u(0)$ and therefore $A(f)=B(f)$.

$I_1\le P_z(\zeta_0)A=P_z(\zeta_0)B\le I_2$

Let $\mathbb{C}^n$ be the complex Euclidean n-space. In this paper, we write a point $z\in\mathbb{C}^n$ as a column vector of the following $n\times 1$ matrix form.

Theorem B. ([3, Theorem 1.1']) Suppose f is a holomorphic self-mapping of $\mathbb{U}$ with $f(0)=0$, and, further, f is holomorphic at $z=a\in\mathbb{B}^n$ with $w(a)=b\in\mathbb{S}^1$. Then, the following two conclusions hold:

(1) $\bar{b}f'(a)a\ge 1$.

(2) $\bar{b}f'(a)a=1$ if and only if $f(z)=e^{i\alpha}z$, where $e^{i\alpha}$ and $\alpha\in\mathbb{R}$.

Theorem 1.4. Let f be a holomorphic self-mapping of $\mathbb{B}^n$. If f is holomorphic at $z=a\in\mathbb{B}^n$ with $w(a)=b\in\mathbb{S}^n$, then we have the following inequality:

$$
\overline{b}^T J_f(a)a \ge \frac{2|1-\overline{b}^T f(0)|^2}{1-|\overline{b}^T f(0)|^2+||J_f(0)||}. \tag{14.19}
$$

Set $g(\zeta)=\overline{b}^T f(a\zeta)$. Then g is a holomorphic self-mapping of $\mathbb{U}$ satisfying $g(0)=\overline{b}^T f(0)$, $g(1)=\overline{b}^T f(a)=1$, $g'(0)=\overline{b}^T J_f(0)a$ and $g'(1)=\overline{b}^T J_f(a)a$. Now apply complex 1-dim version, Theorem 3.8.

In 2016, Tang et al proved the following theorem which is an improvement of Theorem B.

Theorem F. ([17, Theorem 3.1]) Let $f:\mathbb{B}^n\to\mathbb{B}^n$ be a holomorphic mapping, and let $f(c)=c$ for some $c\in\mathbb{B}^n$. Then we have the following two conclusions: (1)

If f is holomorphic at $z = a \in \mathbb{B}^n$ with $w(a) = b \in \mathbb{S}^n$, then

$$\bar{b}^T J_f(a)a \geq \frac{|1-\bar{c}^T b|^2}{|1-\bar{c}^T a|^2} \tag{14.20}$$

Theorem 1.9. Let w be a pluriharmonic self-mapping of $\mathbb{B}^n$ satisfying $w(0) = 0$. If w(z) is differentiable at $z = a \in \mathbb{B}^n$ with $w(a) = b \in \mathbb{S}^n$, then

$$\mathrm{Re}\left[\overline{\mathrm{b}}^{\mathrm{T}}(\mathrm{w_z(a)a} + \mathrm{w_{\overline{z}}(\overline{a})a})\right] \geq \frac{4}{\pi}\frac{1}{1+\frac{\pi}{4}\Lambda_{\mathrm{w}}(0)}. \tag{14.21}$$

Using the authomorphism of $\mathbb{B}^n$ one can obtain version with $w(c) = 0$, where $c \in \mathbb{B}^n$.

See

Jian-Feng Zhu, Schwarz lemma and boundary Schwarz lemma for pluriharmonic mappings, manuscript December 2017

H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, Bull. Amer. Math. Soc., 42 (1936), 689692.

[3] T. Liu and X. Tang, A new boundary rigidity theorem for holomorphic self-mappings of the unit ball in C^n, Pure Appl. Math. Q., 11 (2015), 115130.

T. Liu, J. Wang and X. Tang, Schwarz lemma at the boundary of the unit ball in C^n and its applications, J. Geom. Anal., 25 (2015), 18901914.

T. Liu and X. Tang, Schwarz lemma at the boundary of strongly pseudoconvex domain in C^n, Math. Ann., 366 (2016), 655666.

In communication (4/8/2018) Jian-Feng Zhu asked questions:

Can we obtain similar results like Lemma 2.2 in the paper of "Liu Taishun JGA" and Lemma 2.3 in the paper of "Math Ann" for pluriharmonic mappings and harmonic K-q.c.?

14.4. **Hopf fibration.** We write $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$. On $\mathbb{C}^n$ we define the standard Hermitian inner product by

$$< z, w >= \sum_{k=1}^{n} z_k \overline{w_k}$$

for $z, w \in \mathbb{C}^n$ and by $|z| = \sqrt{< z, z >}$ we denote the norm of vector z. We also use notation (z, w) instead of $< z, w >$ on some places. By $\mathbb{B} = \mathbb{B}_n$ we denote the unit ball in $\mathbb{C}^n$. In particular we use also notation $\mathbb{U}$(and occasionally $\mathbb{D}$) and $\mathbb{H}$ for the unit disk and the upper half-plane in complex plane respectively. Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and define Euclidean scalar product on $\mathbb{C}^n$ with $< z, w >_e = \mathrm{Re} < z, w >$. If we wish to emphasize that $z \in \mathbb{C}^n$ we occasionally write z instead of $\d{z}$. Let $e_1 = (1, 0) \in C^2$. Set of vectors $w = (w_0, w_1)$ orthogonal on e_1 wrt Hermitian inner (res Euclidean) product is given by $e_1^\perp = \{w :< w, e_1 >= w_0 = 0\}$ (res $\d{e}_1^\perp = \{w :< w, e_1 >_e = \mathrm{Re} w_0 = 0\}$. The sets $e_1^\perp$ and $\d{e}_1^\perp$ have real dimensions 2 and 3 respectively.

Direct construction

Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$(where $\mathbb{C}$ denotes the complex numbers) by writing:

$$(x_1, x_2, x_3, x_4) \leftrightarrow (z_0, z_1) = (x_1 + ix_2, x_3 + ix_4)$$

and

$$(x_1, x_2, x_3) \leftrightarrow (z, x) = (x_1 + ix_2, x_3)(x_1, x_2, x_3) \leftrightarrow (z, x) = (x_1 + ix_2, x_3).$$

Thus S^3 is identified with the subset of all (z_0, z_1) in $\mathbb{C}^2$ such that $|z_0|^2+|z_1|^2 = 1$, and S^2 is identified with the subset of all (z, x) in $\mathbb{C} \times \mathbb{R}$ such that $|z|^2 + x^2 = 1$. (Here, for a complex number $z = x + iy, |z|^2 = zz^* = x^2 + y^2$, where the star denotes the complex conjugate.) Then the Hopf fibration p is defined by

$$p(z_0, z_1) = (2z_0z_1^*, |z_0|^2 - |z_1|^2).$$

By easy calculation $|p(z_0, z_1)|^2 = |2z_0z_1|^2 + (|z_0|^2 - |z_1|^2)^2 = (|z_0|^2 + |z_1|^2)^2$ and therefore p maps the 3-sphere into the 2-sphere. Furthermore, if two points on the 3-sphere map to the same point on the 2-sphere, i.e., if $p(z_0, z_1) = p(w_0, w_1)$, then (w_0, w_1) must equal $(\lambda z_0, \lambda z_1)$ for some complex number λ with $|\lambda|^2 = 1$. The converse is also true; any two points on the 3-sphere that differ by a common complex factor λ map to the same point on the 2-sphere. These conclusions follow, because the complex factor λ cancels with its complex conjugate λ^* in both parts of p: in the complex $2z_0z_1^*$ component and in the real component $|z_0|^2 - |z_1|^2$.

Since the set of complex numbers λ with $|\lambda|^2 = 1$ form the unit circle in the complex plane, it follows that for each point m in S^2, the inverse image $p^{-1}(m)$ is a circle, i.e., $p^{-1}m \cong S^1$. Thus the 3-sphere is realized as a disjoint union of these circular fibers.

A direct parametrization of the 3-sphere employing the Hopf map is as follows.

$$z_0 = e^{i\,\frac{\xi_1+\xi_2}{2}} \sin\eta, \quad z_1 = e^{i\,\frac{\xi_2-\xi_1}{2}} \cos\eta.$$

Where η runs over the range 0 to $\pi/2$, and ξ_1 and ξ_2 can take any values between 0 and 2π. Every value of η, except 0 and $\pi/2$ which specify circles, specifies a separate flat torus in the 3-sphere, and one round trip (0 to 2π) of either ξ_1 or ξ_2 causes you to make one full circle of both limbs of the torus. Geometric interpretation using rotations

Another geometric interpretation of the Hopf fibration can be obtained by considering rotations of the 2-sphere in ordinary 3-dimensional space. The rotation group SO(3) has a double cover, the spin group Spin(3), diffeomorphic to the 3-sphere. The spin group acts transitively on S^2 by rotations. The stabilizer of a point is isomorphic to the circle group. It follows easily that the 3-sphere is a principal circle bundle over the 2-sphere, and this is the Hopf fibration.

To make this more explicit, there are two approaches: the group Spin(3) can either be identified with the group Sp(1) of unit quaternions, or with the special unitary group SU(2).

In the first approach, a vector (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ is interpreted as a quaternion $q \in H$ by writing

$$q = x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4.$$

The 3-sphere is then identified with the versors, the quaternions of unit norm, those $q \in H$ for which $|q|^2 = 1$, where $|q|^2 = qq^*$, which is equal to $x_1^2 + x_2^2 + x_3^2 + x_4^2$ for q as above.

On the other hand, a vector (y_1, y_2, y_3) in $\mathbb{R}^3$ can be interpreted as an imaginary quaternion

$$p = \mathbf{i}y_1 + \mathbf{j}y_2 + \mathbf{k}y_3.$$

Then, as is well-known since Cayley (1845), the mapping

$$p \mapsto qpq^*$$

is a rotation in $\mathbb{R}^3$: indeed it is clearly an isometry, since $|qpq^*|2 = qpq^*qp^*q^* = qpp^*q^* = |p|^2$, and it is not hard to check that it preserves orientation. In fact, this identifies the group of versors with the group of rotations of $\mathbb{R}^3$, modulo the fact that the versors q and -q determine the same rotation. As noted above, the rotations act transitively on S^2, and the set of versors q which fix a given right versor p have the form $q = u + vp$, where u and v are real numbers with $u^2 + v^2 = 1$. This is a circle subgroup. For concreteness, one can take $p = k$, and then the Hopf fibration can be defined as the map sending a versor ω to $\omega k \omega^*$. All the quaternions ωq, where q is one of the circle of versors that fix k, get mapped to the same thing (which happens to be one of the two 180 rotations rotating k to the same place as ω does). Another way to look at this fibration is that every versor ω moves the plane spanned by $\{1, k\}$ to a new plane spanned by $\{\omega, \omega k\}$. Any quaternion ωq, where q is one of the circle of versors that fix k, will have the same effect. We put all these into one fibre, and the fibres can be mapped one-to-one to the 2-sphere of 180^o rotations which is the range of $\omega k \omega^*$. This approach is related to the direct construction by identifying a quaternion $q = x_1 + ix_2 + jx_3 + kx_4$ with the 2×2 matrix:

$$\begin{bmatrix} x_1 + \mathbf{i}x_2 & x_3 + \mathbf{i}x_4 \\ -x_3 + \mathbf{i}x_4 & x_1 - \mathbf{i}x_2 \end{bmatrix}.$$

This identifies the group of versors with SU(2), and the imaginary quaternions with the skew-hermitian 2×2 matrices (isomorphic to C R).

Fluid mechanics. If the Hopf fibration is treated as a vector field in 3 dimensional space then there is a solution to the (compressible, non-viscous) Navier-Stokes equations of fluid dynamics in which the fluid flows along the circles of the projection of the Hopf fibration in 3 dimensional space. The size of the velocities, the density and the pressure can be chosen at each point to satisfy the equations.

15. App

15.1. **several variables.** The definition of a holomorphic function generalizes to several complex variables in a straightforward way. Let D denote an open subset of C^n, and let $f : D \to C$. The function f is analytic at a point p in D if there exists an open neighborhood of p in which f is equal to a convergent power series in n complex variables.[14] Define f to be holomorphic if it is analytic at each point in its domain. Osgood's lemma shows (using the multivariate Cauchy integral formula) that, for a continuous function f, this is equivalent to f being holomorphic in each variable separately (meaning that if any n - 1 coordinates are fixed, then the restriction of f is a holomorphic function of the remaining coordinate). The much deeper Hartogs' theorem proves that the continuity hypothesis is unnecessary: f is holomorphic if and only if it is holomorphic in each variable separately.

More generally, a function of several complex variables that is square integrable over every compact subset of its domain is analytic if and only if it satisfies the CauchyRiemann equations in the sense of distributions.

Functions of several complex variables are in some basic ways more complicated than functions of a single complex variable. For example, the region of convergence of a power series is not necessarily an open ball; these regions are Reinhardt domains, the simplest example of which is a polydisk. However, they also come with some fundamental restrictions. Unlike functions of a single complex variable, the possible domains on which there are holomorphic functions that cannot be extended to larger domains are highly limited. Such a set is called a domain of holomorphy.

A function $f(z)$ defined on a domain $U \subset \mathbb{C}^n$ is called holomorphic if $f(z)$ satisfies one of the following two conditions.

(i) $a = (a^1, \ldots, a^n) \in U \subset \mathbb{C}^n$ $f(z)$ is expressed as a power series expansion that is convergent on U:

$$f(z) = \sum c_{k_1,\ldots,k_n}(z^1 - a^1)^{k_1} \cdots (z^n - a^n)^{k_n} \tag{1}$$

which was the origin of Weierstrass' analytic methods.

(ii) a) $f(z)$ is continuous on U, and b) for each variable z^λ, $f(z)$ is holomorphic, namely,

$$\frac{\partial f}{\partial \bar{z}^\lambda} = 0 \tag{2}$$

which is a generalization of the CauchyRiemann equations (using a partial Wirtinger derivative), and has the origin of Riemann's differential equation methods. (Using Hartogs'extension theorem, continuity in (ii) is not necessary.)

To show that above two conditions (i) and (ii) are equivalent, it is easy to prove (i) implies (ii). To prove (ii) implies (i) one uses Cauchy's integral formula on the n-multiple disc for several complex variables. Therefore, Liouville's theorem for entire functions, and the maximal principle hold for several variables. Also, the inverse function theorem and implicit function theorem hold as in the one variable case.

b) means the following: if $a \in U$, $1 \le i \le n$, and $g(\lambda) = f(a + \lambda e_i)$, then g is holomorphic in some neighborhood of 0 in $\mathbb{C}$.

$z = (z', z_n)$, $z' \in \mathbb{C}^{n-1}$ $K(z,w) = \prod_{j=1}^n (1 - \overline{w_j} z_j)^{-1}$ $f(z', \cdot)$

$$f(z', z_n) = \int_T f(z', w_n)(1 - \overline{w_n} z_n)^{-1} d\lambda_1(w_n)$$

$$f(z) = \int_{T^n} K(z,w) f(w) d\lambda_n(w)$$

$$K(z,w) = \Sigma_\alpha \overline{w}^\alpha z^\alpha$$

A biholomorphic function is a bijective holomorphic function whose inverse is also holomorphic.

If $n = 1$, every simply connected open set other than the whole complex plane is biholomorphic to the unit disc (this is the Riemann mapping theorem). The situation is very different in higher dimensions. For example, open unit balls and open unit polydiscs are not biholomorphically equivalent for $n > 1$. In fact, there does not exist even a proper holomorphic function from one to the other.

For every holomorphic $f : \mathbb{U}^n \to \mathbb{B}_m$ with $f(0) = 0$, the differential $A = Df|0$ maps $\mathbb{U}^n$ into $\mathbb{B}_m$, and conversely, for every holomorphic $g : \mathbb{B}_m \to \mathbb{U}^n$ with

$g(0) = 0$, the differential $Dg|0$ maps $\mathbb{B}_m$ into $\mathbb{U}^n$. Hence, if there exists any biholomorphism between the ball and the polydisk, then there exists a linear biholomorphism between them. Since a linear biholomorphism exists only in dimension one, where ball and polydisk are the same, the ball and the polydisk are not biholomorphically equivalent in complex dimensions $n \geq 2$.

For $n = 2$, set $a^k = A(e_k)$, $I = \{e_1 + \lambda e_2 : 0 \leq \lambda \leq 1\}$ and $I' = \{a^1 + \lambda a^2 : 0 \leq \lambda \leq 1\}$. Then I belongs $\partial \mathbb{U}^2$, $I' = A(I)$ and since I' does not belong $\partial \mathbb{B}_2$, we have a contradiction.

Note that a priori, we don't know that the considered biholomorphism is linear, but we know that there exists a linear biholomorphism by XX, and we look at that instead. A posteriori, theorem XX shows that the biholomorphism (that fixes 0) was indeed linear, and we looked at the original.

An analytic disc in $\mathbb{C}^n$ is a nonconstant holomorphic mapping $f : \mathbb{U} \to \mathbb{C}^n$. A closed analytic disc in $\mathbb{C}^n$ is a continuous mapping $g : \overline{\mathbb{U}} \to \mathbb{C}^n$ such that g is holomorphic on $\mathbb{U}$. In practice we may refer to either of these simply as an "analytic disc". The center of an analytic disc is $f(0)$ or $g(0)$.

In order to get a filling about the subject, we first construct bi-holomorphic automorphism of the unit ball in complex dimension 2. We will use notation (z, w) for points in $\mathbb{C}^2$. For a fixed w define $B^w = \{(z, w) : |z|^2 + |w|^2 < 1\}$ and for a fixed z, $B_z = \{(z, w) : |z|^2 + |w|^2 < 1\}$ and denote by $R(z)$ radius of ball B_z.

Check that the mapping $f(z, w) = (z, R(z)w)$ maps $\mathbb{U}^2$ onto $\mathbb{B}_2$.

Whether there is a holomorphic motion f of $\mathbb{U}$ such that $F(z, w) = (z, f(z, w))$ maps $\mathbb{U}^2$ onto $\mathbb{B}_2$.

Take $A = (a, 0) \in B^0$. We can identify points $(z, 0)$ with z and B^0 with $\mathbb{U}$, and consider automorphism $\tilde{z} = \varphi_a(z)$ of $\mathbb{U}$. Since $1 - |\tilde{z}|^2 = |\varphi_a'(z)|(1 - |z|^2)$, then

$$k(z) = \frac{R(\tilde{z})}{R(z)} = \frac{s_a}{|1 - \overline{a}z|},$$

where $s_a = (1 - |a|^2)^{1/2}$, and the mapping $Z = (z, w) \mapsto (\tilde{z}, k(z)w)$ maps B_z onto $B_{\tilde{z}}$. But this mapping is not holomorphic wrt z. We can modify this mapping to the bi -holomorphic automorphism of unit ball:

$$(z, w) \mapsto (\tilde{z}, \lambda(z)w) = \frac{(z - a, s_a w)}{1 - < z, a >}, \tag{1}$$

where $\lambda(z) = \frac{s_a}{1 - \overline{a}z}$. This mapping can be written in the form $\varphi = (\varphi^1, \varphi^2)$, where $\varphi^1(z, w) = \tilde{z}$ and $\varphi^1(z, w) = \lambda(z)w$. Since $R(\tilde{z}) = |\lambda(z)|R(z)$, φ^2 maps B_z onto $B_{\tilde{z}}$. In general, we consider the orthogonal projection $P_a(z)$ onto the subspace $[a]$ generated by a and let $Q_a = I - P_a$ be the projection on the orthogonal complement. If $A = (a, 0)$ then $P_A(z, w) = (z, 0)$ and $Q_A(z, w) = (0, w)$. (1) can be rewritten in the form

$$Z \to \frac{a - PZ - s_a QZ}{1 - (Z, A)}. \tag{2}$$

If we set $z^1 = Pz$ and $z^2 = Qz$, then $z = z^1 + z^2$, and $\varphi_a(z) = \varphi_a^1(z^1) + \varphi_a^2(z^2)$. Motivated by (2), for $z, a \in B^n$, we define the analytic disk $\phi_a = \hat{a}\varphi_{|a|}$,

$$\tilde{z} = \varphi_a^1(z) = \frac{a - Pz}{1 - (z, a)} \quad \text{and} \quad \varphi_a^2(z) = \frac{-s_a Qz}{1 - (z, a)},$$

and therefore

$$\varphi_a(z) = \frac{a - Pz - s_a Qz}{1 - (z, a)},$$

where $P_a(z) = \frac{<z,a>}{<a,a>}a$. Set $U^a = [a] \cap \mathbb{B}$, $Q^b = b + [a]^{\perp} \cap \mathbb{B}$. Note ϕ_a maps $\mathbb{U}$ onto U^a, and $\phi_a(\zeta) = \varphi_a^1(z)$, for $z = \hat{a}\zeta$ and $\zeta \in \mathbb{U}$. The restriction of φ_a onto U^a is automorphism of U^a and the restriction onto Q^z maps it bi-holomorphicaly onto $Q^{\tilde{z}}$.

φ_a maps a $n-1$-dimensional plane onto a $n-1$-dimensional plane.

Let $L = \{(z, b) = 0\}$ and $L' = \varphi_a(L)$. If $w \in L'$, then $\varphi_a(w) \in L$ and therefore $(\varphi_a(w), b) = 0$. Set $Sw = a - Pw - s_a Qw$.Hence

$(a - Pw - s_a Qw, b) = 0$ and $Sw = a - s_a w - (1 - s_a)Pw$, $(a, b) = s_a(w, a) + (1 - s_a)(Pw, b)$

Since $(Pw, b) = (a, b)(w, a)/|a|^2$, we conclude that L' is complex $n-1$-dimensional plane.

Let $u \in T_p\mathbb{C}^n$ and $p \in \mathbb{B}_n$. If $A = d\varphi_p$, set $|Au|_e = M^0(p, u)|u|_e$, ie.

$$M^0(p, u) = \frac{|Au|_e}{|u|_e}\,. \tag{15.1}$$

In general, if $\Omega \subset \mathbb{C}^n$, we define $M(p, u) = M_\Omega(p, u)$ by

$$\mathrm{Kob}(p, u) = M_\Omega(p, u)|u|_e.$$

We show below that on $\mathbb{B}_n$, $\mathrm{Kob}(0, u) = |u|_e$ and therefore $M(p, u) = M^0(p, u)$, $u \in T_p\mathbb{C}^n$.

Note that:

(V0) If $\varphi \in Aut(\Omega)$, $a \in \Omega$, $b = \varphi(a)$, $u \in T_p\mathbb{C}^n$ and $u_* = \varphi'(a)u$, then

(i) $\mathrm{Kob}(b, u_*) = M_\Omega(a, u)|u|_e$.

(V1) $M(a, u) = M(a, \hat{u})$.

(V2) In particular, if Ω is a planar hyperbolic domain then $\mathrm{Hyp}_\Omega(p) = 2M_\Omega(p, u)$, $p \in \Omega$. We first compute Kobayashi-Finsler norm at the origin 0 if Ω is the ball or the polydisk and use (V0) to compute Kobayashi-Finsler norm in these cases.

Proposition 15.1. If the measure of the angle between $u \in T_p\mathbb{C}^n$ and $p \in \mathbb{B}_n$ is $\alpha = \alpha(p, u)$, then

$$M^0(p, u) = M_B^0(p, u) = \sqrt{\frac{1}{s_p^4}\cos^2\alpha + \frac{1}{s_p^2}\sin^2\alpha}. \tag{15.2}$$

Proof. Set $A^k = d\varphi_p^k$ and $u = u_1 + u_2$, where $u_1 \in T_pU^p$ and $u_2 \in T_pQ^p$, and $u_k' = A^k(u_k)$, $k = 1, 2$. By the classical Schwarz lemma 2-the unit disk, Proposition **??** (Schwarz lemma 1-the unit ball) and (B1), $|u_1'|_e = |u_1|_e/s_p^2$ and $|u_2'|_e = |u_2|_e/s_p$. Then $u' = A(u) = u_1' + u_2'$ and u_1' and u_2' are orthogonal.

Hence, since $|u_1|_e = \cos\alpha|u|_e$, $|u_2|_e = (\sin\alpha)|u|_e$ and $|u'|_e = \sqrt{|u_1'|_e^2 + |u_2'|_e^2}$, we find (15.2). ☐

It is clear that

$$\frac{1}{s_p} \le M^0(p, u) \le \frac{1}{s_p^2}. \tag{15.3}$$

Suppose that (i) $f \in \mathcal{O}(\mathbb{B}_n, \mathbb{B}_m)$, $a \in \mathbb{B}_n$ and $b = f(a)$, $u \in T_p\mathbb{C}^n$ and $u_* = f'(a)u$.

Set $A = d\varphi_a$, $B = d\varphi_b$, $g = \varphi_b \circ f \circ \varphi_a$, $v = Au$ and $v_* = Bu_*$. We first conclude $(dg)_0 = B \circ (df)_a \circ A$ and therefore $v_* = (dg)_0(v)$. Then by Proposition 15.1, we find $|Au|_e = M(a,u)|u|_e$ and $|Bu_*|_e = M(b,u_*)|u_*|_e$. Finally, by Schwarz 1-unit ball, $|v_*|_e \le |v|_e$, ie. $|Bu_*|_e \le |Au|_e$. Hence

Theorem 15.1. *Suppose that $f \in \mathcal{O}(\mathbb{B}_n, \mathbb{B}_m)$, $a \in \mathbb{B}_n$ and $b = f(a)$, $u \in T_p\mathbb{C}^n$ and $u_* = f'(a)u$. Then*

$$M^0(b,u_*)|u_*|_e \le M^0(a,u)|u|_e\,. \tag{15.4}$$

In particular, we have

Theorem 15.2 (Schwarz lemma 2-unit ball,see [32, 49])**.** *Suppose that $f \in \mathcal{O}(\mathbb{B}_n, \mathbb{B}_m)$, $a \in \mathbb{B}_n$ and $b = f(a)$.*

Then $s_a^2|f'(a)| \le s_b$, i.e.

$$(1-|a|^2)|f'(a)| \le \sqrt{1-|f(a)|^2}.$$

Theorem 15.3. *Let $a \in \mathbb{B}_n$ and $v \in T_p\mathbb{C}^n$. For $\mathbb{B}_n$, $\mathrm{Kob}(a,v) = M^0(a,v)|v|_e$. In particular, $M(a,v) = M^0(a,v)$.*

Proof. Let ϕ be a holomorphic map of $\mathbb{U}$ into $\mathbb{B}_n$, $\phi(0) = a$, $v \in T_aC^n$, $|v|_e = 1$, $(d\phi)_0(1) = \lambda v = v'\}$.

1°. Consider first the case $a = 0$.

Let p be the projection on $[v]$. Then $(dp)_0 = p$, $\phi_1 = p \circ \phi$ is a holomorphic map of $\mathbb{U}$ into U^v and $(d\phi_1)_0(1) = (dp)_0(v') = \lambda v$. By classical Schwarz lemma $|\lambda| = |\phi_1'(0)| \le 1$ and therefore since $(d\phi^0)_0(1) = v$, where $\phi^0(\zeta) = v\zeta$, ϕ^0 is extremal. Hence $\mathrm{Kob}(0,v) = 1$.

2°. Let $a \ne 0$, $A = (d\varphi_a)_a$ and $v_* = A(v)$. Then by 1°, $\mathrm{Kob}(a,v) = \mathrm{Kob}(0,v_*) = |v_*|_e = M^0(a,v)|v|_e$.

We give another proof using analytic discs. If $a \ne 0$, in general the part of the projection of $\phi(\mathbb{U})$ on $a + [v]$ can be out of $\mathbb{B}_n$. So we can not use the procedure in 1° directly and we consider $g = \varphi_a \circ \phi$ and set $u = v_* = A(v)$. Since $(dg)_0(1) = A(v') = A(\lambda v) = \lambda v_*$, by classical Schwarz lemma $\lambda|v_*|_e \le 1$, ie. (i) $\lambda \le 1/|u|_e$. Further define ϕ^v by $\phi^v(\zeta) = \zeta\hat{u}$ and set $\phi^a = \varphi_a \circ \phi^v$. Then $(d\phi^a)_0(1) = A(u) = \frac{v}{|u|_e} = \lambda_0 v$, where $\lambda_0 = 1/|u|_e$. Hence by (i), the mapping ϕ^a is extremal and therefore $\mathrm{Kob}(a,v) = |u|_e$. □

Let $p \in \mathbb{B}_2$ and $v \in T_p\mathbb{B}^2$. Note that $df(v) = d\big(\mathrm{Re}\, f\big)(v) + id\big(\mathrm{Im}\, f\big)(v)$.

If the measure of the angle between $v \in T_p\mathbb{C}^2$ and x_1x_2-plane is $\beta = \beta(p,u)$, then

$|\big(d(\mathrm{Re}\, f)\big)_p(v)| = (\cos\beta')|df_p(v)|$, where $\beta' = \beta(p',v_*)$, $v_* = df_p(v)$ and $p' = f(p)$.

(h1) Let $h : \mathbb{B}_2 \to \mathbb{S}(a,b)$ be a pluriharmonic function and let $v' = (d\mathrm{Re}\, h)_p(v)$, $v_* = dh_p(v)$ and $p' = h(p)$.

We leave the reader to check that

(E1) there is an analytic function $f : \mathbb{B}_2 \to \mathbb{S}(a,b)$ such that $\mathrm{Re}\, f = \mathrm{Re}\, h$ on $\mathbb{B}_2$, and $\mathrm{Hyp}_{\mathbb{S}(a,b)}(w) = \mathrm{Hyp}_{\mathbb{S}(a,b)}(\mathrm{Re}\, w)$, $w \in \mathbb{S}_0$,

and then using (E1) to prove

(VI)Under the hypothesis (h1), $k_{\mathbb{S}(a,b)^2}(p',v_*) \le (\tan\beta')k_{\mathbb{B}^2}(p,v)\lambda(p_2')/\lambda(p_1')$, where λ is the hyperbolic density on $\mathbb{S}(a,b)$.

15.2. **stereographic projection.** Geometrically, a Möbius transformation can be obtained by first performing stereographic projection from the plane to the unit two-sphere, rotating and moving the sphere to a new location and orientation in space, and then performing stereographic projection (from the new position of the sphere) to the plane.

The inversion wrt the unit sphere is defined by $x \mapsto x^* = Jx = \frac{x}{|x|^2}$. J maps sphere $S(a,r)$ onto sphere. The $S(a,r)$ is defined by $|x-a|^2 = r^2$. Hence $|x|^2 - 2(x,a) + |a|^2 = r^2$. Suppose that $0 < r < |a|$ and set $s = |a|^2 - r^2$. Since $x = y^* = Jy$, then $\frac{1}{|y|^2} - \frac{(y,a)}{|y|^2} + s = 0$ and therefore $s|y|^2 - 2(y,a) + 1 = 0$. Hence $|y|^2 - 2(y, a/s) + 1/s = 0$, $|y - a/s|^2 = |a/s|^2 - 1/s = |r/s|^2$. If we set $b = a/s$ and $R = r/s$, then J maps sphere $S(a,r)$ onto $S(b,R)$. Check that J maps sphere $S(a,|a|)$ onto hyperplane.

The map A defined by $Ax = a + rx$ maps the unit sphere onto the sphere $S(a,r)$ the inversion wrt $S(a,r)$ is given by $J_a^r = A \circ J \circ A^{-1}$, $J_a^r(x) = a + r^2 \frac{x-a}{|x-a|^2}$.

Check that J_a^r maps sphere $S(b,r_1)$, $r_1 < |b-a|$ onto a sphere and a sphere S which contains a onto a hyperplane.

Set $E_3 = (0,0,1)$, $S_0 = S(E_3, 1)$ and $S_1 = S(E_3/2, 1/2)$.

The stereographic projection p wrt S_1 is restriction of the inversion J_1 wrt S_0.

Let K be a circle. K is the intersection of a hyperplane L and a sphere S. If E_3 does not belong to $L \cup S$, then $J_1(K)$ and $J_1(L)$ are respectively two sphere S' and L' and the intersection of S' and L' is a circle K'.

Lemma 15.1. *Neka je γ pozitivno orjentisana granica kruga B, funkcija f neprekidna na $\overline{B}$ i $\Gamma = f \circ \gamma$.*
Tada važi
a) Ako $b \notin f(\overline{B})$ tada je $Int_\Gamma b = 0$
b) ako je $Int_\Gamma b \neq 0$, onda $b \in f(B)$
c) ako je f jednolisno na $\overline{B}$, tada je Γ Jordan-ov put, i $f(B) = Int(\Gamma)$.

$\triangleright$ a) Ako $b \notin f(\overline{B})$ tada je sa $H(t,s) = f(s\gamma(t))$ definisana homotopija puta Γ u tačku, u $\mathbb{C} \setminus \{b\}$. Otuda je $Int_\Gamma b = 0$.
b) Iz a) sledi b).
c) Na osnovu b. $Int(\Gamma) \subset f(B)$. Ako $b \in Ext(\Gamma) \cap f(B)$ tada je, s obzirom da je f jednolisno na $\overline{B}$, prvo $\Gamma^* \cap f(B) = \emptyset$ i otuda $f(B) \subset Ext(\Gamma)$; što je kontradikcija.
□

16. QC

16.1. Definitions.

Example 17. Let $A(z) = az + b\overline{z}$.
1. If $|a| \neq |b|$, the mapping A is univalent; it maps lines onto lines, parallel lines onto parallel lines, squares onto rectangles .
Show that A maps
2. circles onto ellipses if $|a| \neq |b|$.
3. if $|a| > |b|$, positively oriented circles K_r of radius r with center at origin onto positively oriented ellipse E_r with major axis length $\mathbf{L}_r = \Lambda r$ and minor axis length $\mathbf{l}_r = \lambda r$, where $\Lambda = (|a| + |b|)$ i $\lambda = (|a| - |b|)$.

Outline for 2 i 3. Let $a = |a|e^{i\alpha}$, $b = |b|e^{i\beta}$ i $z = \rho e^{i\varphi}$. Then
$$A(z) = |a|e^{i\alpha}\rho e^{i\varphi} + |b|e^{i\beta}\rho e^{-i\varphi} = (|a|e^{i(\alpha+\varphi)} + |b|e^{i(\beta-\varphi)})\rho\,.$$

Let $\gamma = \frac{\alpha+\beta}{2}$ i $\gamma_0 = \frac{\alpha-\beta}{2}$. Hence

$$A(z) = (|a|e^{i(\gamma_0+\varphi)} + |b|e^{-i(\gamma_0+\varphi)})\rho e^{i\gamma}\,.$$

Define $u + iv = A_0(z) = (|a|e^{i\varphi} + |b|e^{-i\varphi})\rho$. We have $u = (|a| + |b|)\rho\cos\varphi$ i $v = (|a| - |b|)\rho\sin\varphi$ (for fixed ρ these parametric equation of ellipse). A is composition of two rotation and A_0. The mapping A_0 maps positively oriented circles onto positively oriented ellipse if $|a| > |b|$ and hence 3. follows. □

Let $f : \Omega \to f(\Omega)$ be a C^1-diffeomorphism and $z_0 \in \Omega$; and $z = x + iy$ is a coordinate on Ω at z_0 and $w = u+iv$ is a coordinate on $f(\Omega)$ at $w_0 = f(z_0)$. Then

$$du = u_x dx + u_y dy, dv = v_x dx + v_y dy,$$

$$f_x = u_x + iv_x, f_y = u_y + iv_y.$$

it is convenirt to use notation

$$p = Df = f_z = \frac{1}{2}(f_x - if_y),\ q = \overline{D}f = f_{\overline{z}} = \frac{1}{2}(f_x + if_y).$$

Hence $dw = df = f_z dz + f_{\overline{z}}\, d\overline{z}$ and

$$f_x = p + q,\ f_y = i(p - q). \tag{16.1}$$

It is convenient to use notation $A = df(z_0)$, $h = \rho e^{i\varphi} \in T_{z_0}$. Then $A(h) = ph + q\overline{h}$, i.e. $A(\rho e^{i\varphi}) = (p\, e^{i\varphi} + q\, e^{-i\varphi})\,\rho$. The maximum of $|pe^{i\varphi} + qe^{-i\varphi}|$ is attained when

$$\frac{qe^{-i\varphi}}{pe^{i\varphi}} = \frac{q}{p}\, e^{-2i\varphi} \tag{16.2}$$

is positive, the minimum when it is negative. If we introduce the complex dilatation

$$\mu_f = \frac{q}{p} \tag{16.3}$$

with $d_f = |\mu_f|$.

The maximum corresponds to the direction $\varphi = \alpha = \frac{1}{2}arg\mu$ and the minimum to the direction $\alpha \pm \pi/2$. In the w-plane the direction of the major axis is $\beta = \frac{1}{2}arg\,\nu$, where

$$\nu_f = \frac{q}{\overline{p}} = (\frac{p}{|p|})^2\,\mu_f \tag{16.4}$$

The quantity ν_f is called the second complex dilatation. Observe that $\beta - \alpha = argp$.

Recall that we suppose that we work with orientation preserving mapping. $\mu\, e^{-2i\,\alpha} = s > 0$; $arg\nu = argp^2 + arg\mu$.

Since f is a diffeomorphism, it is locally linear (in this case at z_0), ie. df is a linear map, which maps $\mathbb{D}$ onto an ellipse with major axis of length $?\Lambda = |p| + |q|$ and minor axis of length $\lambda = |p| - |q|$. The area of $\mathbb{D}$ is π, and the area of $df(\mathbb{D})$ is $\pi\Lambda\,\lambda$, so the Jacobian $J_f(z_0)$ is $\Lambda\,\lambda$.

Example 18. Show $J_f = |f_z|^2 - |f_{\overline{z}}|^2$.

Example 19. Let f be a diffeomorphism in a neighborhood U of a point z_0. Then f is orientation preserving mapping in U if and only if $J_f(z_0) > 0$.

The dilatation (or distortion) at z_0 is defined to be

$$D_f := \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \geq 1 \tag{16.5}$$

The complex dilatation at z_0 is

$$\mu_f = \frac{f_{\overline{z}}}{f_z} \tag{16.6}$$

It is often more convenient to consider

$$d_f = |\frac{f_{\overline{z}}}{f_z}|.$$

The dilatation and distortion are related by

$$D_f = \frac{1 + |\mu_f|}{1 - |\mu_f|}.$$

Proposition 16.1. Let $f \in C^1$, then f is conformal $\Rightarrow f_{\overline{z}} \equiv 0$ (Cauchy-Riemann equations).

If f is conformal, $D_f = 1$ and $\alpha = \beta$ above, so df maps circles to circles.

Definition 16.1 (Grotzsch analytic definition for regular mappings)**.** Let $f : \Omega \to \mathbb{C}$ be a diffeomorphism. We say that f is a *quasiconformal map* if $D_f(z)$ is bounded in Ω. We say f is a K-quasiconformal map if $D_f(z) \leq K$ for all $z \in \Omega$.

$K(f) = ess\sup_{z\in\Omega} D_f(z)$ is called the coefficient of quasi-conformality (or linear dilatation) of f in the domain Ω.

Proposition 16.2. A C^1 diffeomorphism is conformal iff it is 1-quasiconformal.

Definition 16.2. b1) For a C^1 mapping $u : \mathbb{U} \to \mathbb{R}^m$, set $S = u(\mathbb{U})$, $D[u] = \int_{\mathbb{U}}(|D_1 u|^2 + |D_2 u|^2)dxdy$, $E = |D_1 u|^2$, $G = |D_2 u|^2$, $F = D_1 u \cdot D_2 u$, $J_u = \sqrt{EG - F^2}$, and $A = A(S) == \int_{\mathbb{U}} J_u dxdy$.
b2) We say that u is $\underline{K}$-qc if $E + F \leq \underline{K} J_u$.
b3) Suppose that $u : \mathbb{U} \to \mathbb{R}^m$ is harmonic on $\mathbb{U}$. Then $u = \mathrm{Re}F$, where F is analytic. Set $\underline{D}[F] = \int_{\mathbb{U}} |F'(z)|^2 dxdy$.
b4) For a planar domain D and C^1 mapping $u : D \to \mathbb{R}^m$, set $S = u(D)$,

$$K_*(f, z) = \frac{E + G}{2J_u}$$

and $K_*(f) = ess\sup_{z\in D} K_*(f, z)$ which is called the coefficient of quasi-conformality (or linear dilatation) of f in the domain D.

If S is in a plane and K the standard coefficient of quasi-conformality, then $K_* = \frac{K^2+1}{2K}$,that is $K = K_* + \sqrt{K_*^2 - 1}$, where $K_* = K_*(f)$ and $K = K(f)$.

For a fixed integer k, $1 \leq k \leq n$ define P_k by $P_k x = x - x_k\, e_k$. It is readable that P_k is the orthogonal projection of R^n onto $R_k^{n-1} = \{x \in R^n : x_k = 0\}$.

Let $I = \{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k\}$ be a closed n-interval.

A mapping $f : I \to \mathbb{R}^m$ is said to be absolutely continuous on lines (ACL) if f is continuous and if f is absolutely continuous on almost every line segment in I, parallel to the coordinate axes.

More precisely, if E_k is the set of all $x \in P_k I$ such that the functions $t \to f(x+te_k)$ is not absolutely continuous on $[a_k, b_k]$, then $m_{n-1}(E_k) = 0$ for $1 \le k \le n$. Note that for fixed k and x the line $L(t) = x + te_k$ is parallel to e_k.

If Ω is an open set in $\mathbb{R}^n$, a mapping $f : \Omega \to \mathbb{R}^m$ is absolutely continuous if $f|I$ is ACL for every closed interval $I \subset \Omega$.
If Ω and Ω' are domain in $\overline{\mathbb{R}}^n$, a homeomorphisam $f : \Omega \to \Omega'$ is called ACL if $f|\Omega \setminus \{\infty, f^{-1}(\infty)\}$ is ACL.

If $f : \Omega \to \mathbb{R}^m$ is ACL, then the partial derivatives of f exist a.e. in Ω, and they are Borel functions.

Definition 16.3. ■ An ACL-mapping $f : \Omega \to \mathbb{R}^m$ is said to be ACL^p, $p \ge 1$, if the partial derivatives of f are locally L^p-integrable. A homeomorphism $f : D \to D'$ is called ACL^p if the restriction of f to $D \setminus \{\infty, f^{-1}\infty\}$ is ACL^p.

? Roughly speaking, Fuglede's theorem states that an ACL^p function is absolutely continuous on almost every path.

Theorem 16.4. (Fuglede's theorem) *Suppose that U is an open subset in R^n and that $f : U \to R^m$ is ACL^p. Let Γ be the family of all locally rectifiable path in U which have a closed subpath on which f is not absolutely continuous. Then $M_p(\Gamma) = 0$.*

Theorem 16.5. *Suppose that $f : \Omega \to \Omega^*$ is a homeomorphism such that $H(x, f)$ is bounded. Then*
1)f is ACL.
2) f is differentiable a.e. in Ω

1) Let Q be a closed n-interval in $\Omega \setminus \{\infty, f^{-1}(\infty)\}$. Consider the orthogonal projection $P : R^n \to R^{n-1}$. For each Borel set $A \subset intPQ$ define $E_A = Q \cap P^{-1}A$ and set $\varphi(A) = m(fE_A)$. By Lebesgue's theorem, φ has a finite derivative $\varphi'(y)$ a.e. $y \in PQ$. Fix such y. We shall prove that f is absolutely continuous on the segment $J = E_y$.

Let F be a compact subset of $J \cap intQ$. Prove that

$$\Lambda_1(fF)^n \le C\,\varphi'(y)\,m_1(F)^{n-1}.$$

2) f have partial derivatives a.e. and f has a.e. a finite volume derivative $\mu_f'(x)$. Consider x_0 such that $\mu_f'(x_0) < \infty$.

Since $H(x_0, f) < \infty$, there are positive numbers r_0 and H such that $L(x_0, f, r) \le H\,l(x_0, f, r)$ for $0 < r < r_0$. For all such r we have $\Omega_n\, L(x_0, f, r)^n \le H^n m(f\overline{B}(x_0, r))$. Hence

$$\frac{L(x_0, f, r)^n}{r^n} \le H^n \frac{m(f\overline{B}(x_0, r))}{m(\overline{B}(x_0, r))}.$$

Letting $r \to 0$ yields $L(x_0, f)^n \le H^n \mu_f'(x_0) < \infty$.

By the theorem of Rademacher-Stepanov, f is differentiable a.e.

Theorem 16.6. *Suppose that $f : \Omega \to \Omega^*$ is a homeomorphism. Then the following condition are equivalent*
1) $K_O(f) \le K$.
2)f is ACL, f is differentiable a.e. in Ω, and $|f'(x)|^n \le K|J(x, f)|$ a.e.

Suppose that $f : \Omega \to \Omega^*$ is a homeomorphism and $x \in \Omega$, $x \neq \infty$ and $f(x) \neq \infty$. For each $r > 0$ such that $S(x; r) \subset \Omega$ we set
$L(x, f, r) = \max_{|y-x|=r} |f(y) - f(x)$, $l(x, f, r) = \min_{|y-x|=r} |f(y) - f(x)$.

Definition 16.7. The linear dilatation of f at x is

$$H(x, f) = \limsup_{r\to 0} \frac{L(x, f, r)}{l(x, f, r)} .$$

Theorem 16.8 (The metric definition of qcty)**.** *A homeomorphism $f : \Omega \to \Omega^*$ is qc iff $H(x, f)$ is bounded.*

$H(f) = ess \sup_{x\in\Omega} H(x, f)$ is called the coefficient(metric) of quasi-conformality (or linear dilatation) of f in the domain Ω.

Let Ω be a domain in R^n and $f : \Omega \to \mathbb{R}^n$ be continuous. We say that f is quasiregular (shortly qr) if
(1) f belongs to Sobolev space $W^n_{1,\,loc}(\Omega)$
(2) there exists K, $1 \leq K < \infty$, such that

$$|f'(x)|^n \leq K\, J_f(x) \;\; a.e. \tag{16.7}$$

The smallest K in (16.7) is called the outer dilatation $K_O(f)$.
If f is qr, also

$$J_f(x) \leq K'\, l(f'(x))^n \;\; a.e. \tag{16.8}$$

for some K', $1 \leq K' < \infty$, where $l(f'(x)) = inf\{|f'(x)h| : |h| = 1\}$. The smallest K' in (16.8) is called the inner dilatation $K_I(f)$ and $K(f) = max(K_O(f), K_I(f))$ is called the maximal dilatation of f. If $K(f) \leq K$, f is called K-quasiregular.

Reshetnyak's main theorem: every nonconstant qr map is discrete and open.

Poleckii's inequality: If $f : M \to N$ nonconstant qr map and Γ a path family in M, then

$$M(f\Gamma) \leq K_I(f)M(\Gamma).$$

17. Holomorphic motions

Theorem 17.1. *Let $\mathcal{F}_K$ be the family of K-quasiconformal maps of S^2 fixing* 0,1 *and ∞. Then $\mathcal{F}_K$ is compact.*

17.1. **Cross-ratio.** Let z_2, z_3, z_4 be points of S^2. Define S such that $S(z_2) = 1$, $S(z_3) = 0$ $S(z_4) = \infty$; the explicit formula for S is given by

$$Sz = \frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)}.$$

Let z_1, z_2, z_3, z_4 be four points of S^2. Then the cross-ratio $\rho(z_1, z_2, z_3, z_4) = [z_1, z_2, z_3, z_4]$ of z_1, z_2, z_3, z_4 is $S(z_1)$.

We write $[z_1, z_2, z_3]$ instead $[z_1, z_2, z_3, \infty]$. Let $\mathcal{F}_K$ be the family of K-quasiconformal maps of S^2 fixing 0,1 and ∞. Then $\mathcal{F}_K$ is compact.

For given K and for every non-degenerate annulus A with center at origin, there exists non-degenerate annulus A_* with center at origin, such that every $f \in \mathcal{F}_K$ maps A in A_*. In particular, we find
a. there is a constant $C(K)$ such that

$$|f(z)| < C(K)$$

for every $f \in \mathcal{F}_K$ and for all $z \in \mathbf{T}$.
Using a. we can prove:
b. there is a constant $C(K)$ such that

$$|\rho(f(z_1), f(z_2), f(z_3), f(z_4))| < C(K)$$

for every K-qc mapping f of S^2(without normalization) and for all z_1, z_2, z_3, z_4 for which $\rho(z_1, z_2, z_3, z_4) = 1$.
and
Corollary 3.3 (see [86]): A K-qc mapping distors the cross-ratio of any 4 points by a bounded amount, as measured in the hyperbolic metric on $\mathbb{C}_{0,1} = \mathbb{C} \setminus 0, 1$; see also [5] p.53-61.

Proof. Let f a K-qc mapping of S^2, $w_k = f(z_k)$,

$$Sz = \frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)}, \quad S_* w = \frac{(w - w_3)(w_2 - w_4)}{(w - w_4)(w_2 - w_3)}.$$

and $\widetilde{f} = S_* \circ f \circ S^{-1}$. Since $\widetilde{f} \in \mathfrak{F}_K$,it follows that
c. $|\widetilde{f}(z)| < C(K)$ for all $z \in \mathbf{T}$.

Hence, by definition of cross-ratio (which is invariant under Mobius transformation), it follows that b. holds for all K-qc mappings of S^2 (without normalization). $\square$

We say that f (or family $\mathcal{F}_f$) satisfies condition C
if $f(\infty) = \infty$ and

$$|\frac{(w_1 - w_3)}{(w_2 - w_3)}| < C(K) \quad \text{for} \quad |\frac{(z_1 - z_3)}{(z_2 - z_3)}| = 1 \tag{17.1}$$

i.e. $|[z_1, z_2, z_3]| = 1$ implies $|[w_1, w_2, w_3]| \leq c$; hence, since $|[z_1, z_2, z_3]| = |[z_2, z_1, z_3]|^{-1}$, it follows that $|[z_1, z_2, z_3]| = 1$ implies $c^{-1} < |[w_1, w_2, w_3]| \leq c$

In particular, if $f(\infty) = \infty$, it follows from b. that f satisfies condition C.

Let $z = z_3$ be the center of a circle K_r and let $|f - f(z)|$ attains respectively maximum and minimum at points z_1, z_2 on K_r. If we apply condition (17.1), we get that circular dilation $H(z) < C(K)$ for every $z \in \mathbb{C}$. $\blacksquare$ Hence, it follows, known result:
d. If f is a K-qc mapping of S^2 (without normalization), then $H(z) < C(K)$ for every $z \in \mathbb{C} \setminus \{f^{-1}(\infty)\}$

For details about circular dilation (the the distortion function) $H = D_f$ see [**?**],pp.177-178.

Let $f : S^2 \to S^2$ be a homeomorphism and $\mathcal{F}_f = \{g = A \circ f \circ B : A, B \in$ Möb$, g$ fixes $0, 1, \infty\}$.
We say that f (or family $\mathcal{F}_f$) satisfies condition A:
if there is a constant $C = C(f)$ such that

$$|g(z)| < C(K)$$

for every $g \in \mathcal{F}_f$ and for all $z \in \mathbf{T}$.
In this setting we also say that the family $\mathcal{F}_f$ satisfies condition A.

We say that f (or family $\mathcal{F}_f$) satisfies condition B:
if there is a constant $C(f)$ such that

$$|\rho(g(z_1), g(z_2), g(z_3), g(z_4))| < C(f)$$

for every $g \in \mathcal{F}_f$ and for all z_1, z_2, z_3, z_4 for which $\rho(z_1, z_2, z_3, z_4) = 1$.

Theorem 17.2. *Let $f : S^2 \to S^2$ be a homeomorphism and $\mathcal{F}_f = \{g = A \circ f \circ B : A, B \in$ Möb$, g$ fixes $0, 1, \infty\}$. The following conditions are equivalent*
f is quasiconformal
$\mathcal{F}$ is compact
family $\mathcal{F}_f$ satisfies condition A
family $\mathcal{F}_f$ satisfies condition B
f satisfies the condition C

Proof. We will use the above notations. If $\mathcal{F}$ is compact, then $\mathcal{F}$ satisfies the condition a. Since, by $f \in \mathcal{F}$ and $\widetilde{f} = S_* \circ f \circ S^{-1}$ belongs $\mathcal{F}$, we get that for f holds condition b. and therefore d. Hence, it follows (see [**?**],pp.177-178) that f is qc.*
Converse follows from Theorem 17.1. □

Definition 17.3. If $E \subset S^2$ is any subset of the Riemann sphere S^2, the function $f : E \times \mathbb{D} \to S^2$ is called a *holomorphic motion* of E if
(a) $f(z, 0) = z$, for all $z \in E$
(b) $f(z, t)$ is $\overline{\mathbb{C}}$-valued holomorphic (meromorphic) in t for all $z \in E$ and
(c) $f(z, t)$ is injective univalent in z on E for each $t \in \mathbb{D}$.

To f we associate $F : E \times \mathbb{D} \to C^2$ defined by $F(z, \zeta) = (z, f(z, \zeta)$.
If $f(z, \zeta) = e^{it}(1 - r^2)^\alpha \zeta$, $z = re^{it}$ and $F = F^\alpha$ associated to f, describe $F(U^2)$. For $\alpha = 1/2$, whether $F(\mathbb{U}^2) = \mathbb{B}_2$.
Note that there is no assumption regarding the continuity of f as a function of z or the pair (z, t). That such continuity occurs is a consequence of the following remarkable λ-lemma of Mañé-Sullivan-Sad (extended by Slodkowski).

Example 20. a) If t has at least one rational coordinate, $f_t = e^t Id$, otherwise $f_t = -e^t Id$. Is f_t a *holomorphic motion* ? no.
b) Define

$$f_\zeta(z) = \begin{cases} z + \zeta/z & |z| \ge 1 \\ z + \zeta\overline{z} & |z| < 1. \end{cases}$$

f is a holomorphic motion of S^2.

If $f(z, t) : S^2 \times \mathbb{D}$ is a injective *holomorphic motion*, parameterized by the unit disk, then it is a qc *holomorphic motion*.
Suppose that z_1, z_2, z_3 are fixed for a moment; define $w_k = w_k(\zeta) = f^\zeta(z_k)$ and $\omega(\zeta) = [f^\zeta(z_1), f^\zeta(z_2), f^\zeta(z_3)]$.
Define $\mathbb{C}_{0,1} = \mathbb{C} \setminus \{0, 1\}$.

Note, since f^ζ is injective, if z_1, z_2, z_3 are different finite complex numbers, $\omega(\zeta) \in \mathbb{C}_{0,1}$ for every $\zeta \in \mathbb{D}$.
Denote by $\lambda_{0,1}$ hyperbolic metric on $\mathbb{C}_{0,1}$. Since f is a holomorphic motion of S^2, ω is a holomorphic function in $\mathbb{D}$; by $f^0 = Id$, $\omega(0) = [z_1, z_2, z_3]$ and then by

Schwarz lemma $\lambda_{0,1}(\omega(\zeta),\omega(0)) \leq \lambda(\zeta,0)$; in particular, if $r<1$, there is a constant $c_r=\lambda(r,0)$ such that $\lambda_{0,1}(\omega(\zeta),\omega(0)) \leq c_r$ for all $|[z_1,z_2,z_3]|=1$. Thus if $r<1$, there two constant $0<r_1<R_1$, which depends only on r, such that $\omega(\mathbb{D}_r)\subset A(r_1;R_1)$ for all $|[z_1,z_2,z_3]|=1$. Thus each f^ζ satisfies the condition (C) for every $\zeta\in\mathbb{D}$.
We also outline an alternative approach.
Consider $f^z(0)=z$, $|\frac{0-z}{0-1}|$, $\lambda(\omega)=\frac{0-f^z(\omega)}{0-1}$; λ maps holomorphic $\mathbb{C}_{0,1}$ into itself; in particular $\lambda(K_2)$ is a compact subset of $\mathbb{C}_{0,1}$, where K_2 is circle of radius 2.
Suppose that z_1,z_2,z_3 are fixed; define $w_k=w_k(\zeta)=f^\zeta(z_k)$ and $\phi(\zeta)=S_*\circ f^\zeta\circ S^{-1}$ and let $\widetilde{\phi}$ be lift of ϕ. Since f is a holomorphic motion of S^2, $\widetilde{\phi}$ is a holomorphic function; by $f^0=Id$, $\phi(0)=0$ and then by Schwarz lemma $|\widetilde{\phi}(\zeta)|\leq|\zeta|$; thus if $r<1$, $|Sz_1|=2$, then $|S_*w_1|\leq C$ for $|\zeta|\leq r$, where the constant C depends of r. Thus each f^ζ satisfies the condition (C).

XXX motion: using the analytic dependence $\Phi(\lambda,z)$ on λ, one can show that the Beltrami coefficient μ_λ is an analytic function of λ, valued in the unit ball of $L^\infty(\mathbb{C})$.

Let $l_\zeta(\upsilon)=\int_{\mathbb{D}}\mu_\zeta\upsilon$ and $||\upsilon||_1=1$; for fixed υ, l_ζ is holomorphic in ζ, and by classical (usual) Schwarz lemma

$|l_\zeta|\leq|\zeta|$; $|\mu_\zeta|_\infty=\sup|l_\zeta(\upsilon)|=\sup|\int_{\mathbb{D}}\mu_\zeta\upsilon|$ over $||\upsilon||_1=1$. Thus $|\mu_\zeta|_\infty\leq|\zeta|$ and $K(f^\lambda)\leq\frac{1+|\lambda|}{1-|\lambda|}$.
The Beltrami coefficients $\mu_\lambda(z)$ are an analytic function of λ for almost every $z\in\mathbb{C}$ and by the composition formula for Beltrami coefficients,

$$|\mu_\Psi(\Phi_{\lambda_2}(z))|=|\frac{\mu_{\lambda_1}(z)-\mu_{\lambda_2}(z)}{1-\mu_{\lambda_1}(z)\overline{\mu_{\lambda_2}(z)}}|\leq|\frac{\lambda_1-\lambda_2}{1-\lambda_1\overline{\lambda_2}}|$$

and $|\mu_\Psi|_\infty\leq|\frac{\lambda_1-\lambda_2}{1-\lambda_1\overline{\lambda_2}}|$.

18. App

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Erik Lundberg Allen Weitsman, Calculus of Variations and Partial Differential Equations

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We consider minimal graphs $u=u(x,y)>0$ over unbounded domains D with $u=0$ on bD. Assuming D contains a sector properly containing a halfplane, we show that u grows at most linearly. We also provide examples illustrating a range of growth.

Weitsman, A.: Growth of solutions to the minimal surface equation over domains in a half plane. Commun. Anal. Geom. 13, 10771087 (2005)

Weitsman, A.: On the growth of minimal graphs. Indiana Univ. Math. J. 54, 617625 (2005)

On the growth of solutions to the minimal surface equation over domains containing a halfplane For sharp Pointwise Estimates for Directional Derivatives and Khavinsons Type Extremal Problems for Harmonic Functions see Chapter 6, [37].

Sharp pointwise estimates for the gradient of harmonic functions are of use in problems relating electrostatics as well as hydrodynamics of ideal fluid, elasticity and hydrodynamics of the viscous incompressible fluid.

A certain estimate of such a type appeared in a D. Khavinson's problem for harmonic functions which will be describe now. In [Kh], Khavinson found the sharp constant $K(x)$ in the inequality for the radial derivative of a harmonic function u in the ball $B = \{x \in \mathbb{R}^3 : |x| < 1\}$

$$|D_r u(x)| \leq K(x) sup_{|y|<1} |u(y)|, x \in \mathbb{B}, \tag{18.1}$$

where $r = |x|$ and, in a conversation, made a conjecture that the same constant $K(x)$ should appear in the stronger sharp inequality

$$|\nabla u(x)| \leq \underline{K}(x) sup_{|y|<1} |u(y)|, x \in \mathbb{B}, \tag{18.2}$$

i.e. K(x) = K(x). Now, consider the sharp inequality

$$|D_\ell u(x)| \leq K(x; \ell) sup_{|y|<1} |u(y)|, x \in \mathbb{B}, \tag{18.3}$$

where u is a harmonic function in $\mathbb{B}$, $x \in \mathbb{B}$, and ℓ is an arbitrary unit vector in R3. By Khavinson's conjecture, given any $x \in \mathbb{B}$, the maximum value of $K(x; \ell)$ with respect to ℓ is attained at a radial direction. By Khavinson's extremal problem for harmonic functions we mean the problem of finding the direction of ℓ as well as the maximal value of $K(x; \ell)$. Note that at present no solution of this particular problem is known.

A PROOF OF KHAVINSONS CONJECTURE IN R4 DAVID KALAJ, arXiv:1601.03347v1 [math.CV] 13 Jan 2016 Abstract. The paper deals with an extremal problem for bounded harmonic functions in the unit ball of $\mathbb{R}^4$. We solve the generalized Khavinson problem in $\mathbb{R}^4$. This precise problem was formulated by G. Kresin and V. Maz'ya for harmonic functions in the unit ball and in the halfspace of $\mathbb{R}^4$. We find the optimal pointwise estimates for the norm of the gradient of bounded realvalued harmonic functions.

G. Kresin and V. Mazya, Sharp pointwise estimates for directional derivatives of harmonic functions in a multidimensional ball, J. Math. Sci. 169(2010), 167187.

[8] G. Kresin and V. Mazya, Optimal estimates for the gradient of harmonic functions in the multidimensional half-space, Discrete Contin. Dyn. Syst. 28(2010), 425440.

[9] G. Kresin and V. Mazya, Sharp real-part theorems. Springer, Berlin, Jan. 1, 2007. (Lecture Notes in Mathematics, 1903). ISBN: 978-3-540-69573-8.

[10] M. Marković, On harmonic functions and the hyperbolic metric, Indagationes Mathematicae 26(2015), 1923.

[11] M. Marković, Proof of the Khavinson conjecture near the boundary of the un it ball, arXiv:1508.00125v1. See Gehring-Osgood

1) F.W. Gehring and B.G. Osgood, *Uniform domains and the quasi-hyperbolic metric*, J. Anal. Math. 36(1979), 50-74.

2) THE VISUAL ANGLE METRIC AND QUASIREGULAR MAPS GENDI WANG AND MATTI VUORINEN, arXiv:1505.00607v3 [math.MG] 13 Jul 2016

Abstract. The distortion of distances between points under maps is studied. We first prove a Schwarz-type lemma for quasiregular maps of the unit disk involving the visual angle metric. Then we investigate conversely the quasiconformality of a bilipschitz map with respect to the visual angle metric on convex domains. For the unit ball or half space, we prove that a bilipschitz map with respect to the visual angle metric is also bilipschitz with respect to the hyperbolic metric. We also obtain various inequalities relating the visual angle metric to other metrics such as the distance ratio metric and the quasihyperbolic metric.

3) For K-qc maps in the plane $f : B \to B$, the best inequality of the form $d(f(z), f(w)) \leq c(K) max((d(z,w), d(z,w)^{1/K})$, where d is hyperbolic metric It was proved in the paper Wang-Vuorinen, PAMS 2016.

Wei DaiZhao LiuGuozhen Lu, Liouville Type Theorems for PDE and IE Systems Involving Fractional Laplacian on a Half Space, Sep 2016,Potential Analysis.

In this paper, let a be any real number between 0 and 2, we study the Dirichlet problem for semi-linear elliptic system involving the fractional Laplacian:

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = v^q(x), & x \in \mathbb{R}^n_+, \\ (-\Delta)^{\alpha/2}v(x) = u^p(x), & x \in \mathbb{R}^n_+, \\ u(x) = v(x) = 0, & x \notin \mathbb{R}^n_+. \end{cases}$$

(1) We will first establish the equivalence between PDE problem (1) and the corresponding integral equation (IE) system (Lemma 2). Then we use the moving planes method in integral forms to establish our main theorem, a Liouville type theorem for the integral system (Theorem 3). Then we conclude the Liouville type theorem for the above differential system involving the fractional Laplacian (Corollary 4).

(8) Potential Analysis — RG Impact & Description — Impact Rankings (2017 and 2018). Available from:

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For Theorema Egregium of Gauss see http://uregina.ca/ mareal/cs6.pdf

References

[1] A. Simonic, *The Ahlfors lemma and Picard's theorems*, arXiv:1506.07019v1 [math.CV] 22 Jun 2015

[2] Abate M., Patrizio G. , *Holomorphic curvature of Finsler metrics and complex geodesics*, Preprint, Max-Planck-Institut fur Matematik, Bonn (1992)

[3] Marco Abate, The Kobayashi distance in holomorphic dynamics and operator theory, http://www.dm.unipi.it/ abate/articoli/artric/files/AbateLille.pdf

[4] Abate,M.: Iteration theory of holomorphic maps on taut manifolds. Mediterranean Press, Cosenza(1989) [See also http://www.dm.unipi.it/ abate/libri/libriric/libriric.html

[5] Ahlfors, L., *Conformal invariants*, McGraw-Hill Book Company, 1973.

[6] Ahlfors,L.V.: Möbius transformations in several dimensions. -Lecture Notes, University of Minnesota, 1981

[ABR] S. Axler, P. Bourdon and W. Ramey: *Harmonic function theory*, Springer-Verlag, New York 1992.

[7] A. F. Beardon and D. Minda, *The hyperbolic Metric and Geometric Function Theory*, The Proceedings of IWQCMA05, Quasiconformal Mappings and their applications, Editors: S. Ponnusamy, T. Sugawa and M. Vuorinen, 2007, Narosa Publishing House, New Delhi, India, p. 9-56

[8] B. Burgeth: *A Schwarz lemma for harmonic and hyperbolic-harmonic functions in higher dimensions*, Manuscripta Math. **77** (1992), 283–291.

[9] F. Bracci, G. Patrizio, Monge-Ampere foliations with singularities at the boundary of strongly convex domains. Math. Ann. 332 (2005), 499522.

[10] F. Bracci, G. Patrizio, S. Trapani, The pluricomplex Poisson kernel for strongly convex domains. Trans. Amer. Math. Soc. 361 (2009), 979-1005.

[11] Filippo Bracci, John Erik Fornaess, Erlend Fornaess Wold, *Comparison of invariant metrics and distances on strongly pseudoconvex domains and worm domains*,arXiv:1710.04192v2 [math.CV] 15 Oct 2017.

[12] D.M. Burns and S.G. Krantz, Rigidity of holomorphic mappings and a new Schwarz Lemma at the boundary, J. Amer. Math. Soc. 7 (1994), 661-676.

[13] J. Burbea, *On the Hessian of the Caratheodory metric, Rocky Mountian Journal of Mathematics*, Volume 8, Number 3, Summer 1978

[14] J. W. Cannon, W. J. Floyd, R. Kenyon and W. R. Parry, Hyperbolic Geometry, Flavors of Geometry MSRI Publications Volume 31, 1997,http://library.msri.org/books/Book31/files/cannon.pdf

[15] Jianbo Chen, *Generalizations of Schwarz Lemma*, The final year report of Math 2999 under the guidance of Professor Ngaiming Mok,
https://www.stat.berkeley.edu/ jianbo/schwarz.pdf

[16] Cheng, S.Y. and Yau, S.,T., *Differential equations on Riemannian manifolds and their geometric applications*, Comm.Pure and Appl. Math. **28** (1975) 333-354

[17] V. N. Dubinin, The Schwarz inequality on the boundary for functions regular in the disc, J. Math. Sci. 122 (2004), 3623-3629.

[18] P. Duren, *Harmonic mappings in the plane*, Cambridge Univ. Press, 2004.

[19] L. Carleson, T. D. W. Gamelin, Complex dynamics, Universitext: Tracts in Mathematics, Springer-Verlag, ISBN 0-387-97942-5,(1993)

[20] C. C. Cowen, Fixed Points of Functions Analytic in the Unit Disk, Conference on Complex Analysis, University of Illinois, May 22, 2010,
https://www.math.iupui.edu/ ccowen/Talks/FixPts1005.pdf

[21] C. Earle and R. Hamilton, A fixed point theorem for holomorphic mappings, Proc. Symp. Pure Math., Vol. XVI, 1968, 61-65.

[22] C. Frosini, F. Vlacci, *A survey on geometric properties of holomorphic self-maps in some domains of* $\mathbb{C}^n$, http://web.math.unifi.it/users/frosini/SurveyFrosini-Vlacci.pdf

[23] James J. Faran, *Hermitian Finsler metrics and the Kobayashi metric.* J. Differential Geom. 31 (1990), no. 3, 601–625. doi:10.4310/jdg/1214444630.

[24] Gardiner,F.,P., *Teichmüller Theory and Quadratic Differentials*, New York: Awiley-Interscience Publication, 1987.

[25] Gentili G. and Visintin B., *Finsler Complex Geodesics and Holomorphic Curvature*, Rendinconti Accad. XL Memorie di Matematica bf 111 (1993) 153-170

[26] Graziano Gentili and Fabio Vlacci, *Rigidity for regular functions over Hamilton and Cayley numbers and a boundary Schwarz' Lemma*,Indag. Mathem., N.S., 19 (4), 535-545 December, 2008

[27] Robert E. Greene, Kang-Tae Kim, Steven G Krantz, Applications of Bergman Geometry, The Geometry of Complex Domains - Page 90 https://books.google.rs/books isbn=0817646221

[28] Clifford J. Earle, Lawrence A. Harris, John H. Hubbard, Sudeb Mitra, Schwarz's lemma and the Kobayashi and Carathéodory pseudometrics on complex Banach manifolds, Kleinian Groups and Hyperbolic 3-Manifolds: Proceedings of the Warwick Workshop September 2001, Lond. Math. Soc. Lec. Notes **299**, 363-384, edited by Y. Komori, V. Markovic, C. Series

[29] Lawrence A. Harris, Fixed Point Theorems for Infinite Dimensional Holomorphic Functions, J. Korean Math. Soc. 41(2004), No. 1, pp. 175192, http://www.ms.uky.edu/ larry/paper.dir/fixed_pt.pdf

[30] Lawrence A. Harris, *A continuous form of Schwarz's lemma in normed linear spaces*, Pacific J. Math. Volume 38, Number 3 (1971), 635-639.

[31] M. Heins, *On a class of conformal metrics*, Nagoya Math. J. Volume 21 (1962), 1-60.

[32] D. Kalaj, *Schwarz lemma for holomorphic mappings in the unit ball*, arXiv:1504.04823v2 [math.CV] 27 Apr 2015, Glasgow Mathematical Journal, https://doi.org/10.1017/S0017089517000052, Published online: 04 September 2017.

[33] D. Kalaj, *A sharp inequality for diffeomorphisms of the unit disk*, Communication at VIII Simpozijum Matematica i primene 2017, 17 i 18 nov 2017, Beograd.

[34] D. Khavinson, An extremal problem for harmonic functions in the ball, Canadian Math. Bulletin 35(2) (1992), 218-220

[35] S. Kobayashi, Hyperbolic manifolds and holomorphic mappings, Marcel-Dekker, New York, 1970.

[36] Steven G. Krantz, *The Carathéodory and Kobayashi Metrics and Applications in Complex Analysis*, arXiv:math/0608772v1 [math.CV] 31 Aug 200

[37] G. Kresin and V. Maz'ya, Maximum Principles and Sharp Constants for Solutions of Elliptic and Parabolic Systems, 2014

[38] M. Knežević, M. Mateljević, On the quasi-isometries of harmonic quasi-conformal mappings J. Math. Anal. Appl, 2007, 334(1), 404-413.

[39] M. Knezević, Harmoijska i kvazikonformna preslikavanja,kvazi - izometrije i krivina,2014, http://elibrary.matf.bg.ac.rs/bitstream/handle/123456789/4280/phdKNEZEVIC_MILJAN.pdf

[40] L. Lempert, *La métrique de Kobayashi et la répresentation des domaines sur la boule*, Bull. Soc. Math. France 109 (1981),427-474

[41] Gunnar Þór Magnússon,*A geometric version of the Schwarz lemma*, A talk given at KAUS 2010,https://notendur.hi.is/gthm1/files/kaus2010-schwarz.pdf

[42] Mateljević, M., *Holomorphic fixed point theorem on Riemann surfaces*, Math. Balkanica **12** (1-2) (1998), 1-4.

[43] M. Mateljević, *Ahlfors-Schwarz lemma and curvature,* Kragujevac Journal of Mathematics (Zbornik radova PMF), Vol. **25,** 2003, 155-164.

[44] M. Mateljević: Kompleksne Funkcije 1 & 2, Društvo matematičara Srbije, 2006.

[45] M. Mateljević, *Topics in Conformal, Quasiconformal and Harmonic maps*, Zavod za udžbenike, Beograd 2012.

[46] M. Mateljević,*Hyperbolic geometry and Schwarz lemma*, VI Simpozizijum Matematika i primene, p.1-17,Beograd, November 2016

[47] M. Mateljević, (a) *Schwarz lemma and Kobayashi Metrics for holomorphic and pluriharmonic functions*, arXiv:1704.06720v1[math.CV] 21 Apr 2017
(b) *Schwarz lemma, Kobayashi Metrics and FPT*, preprint November 2016.

[48] M. Mateljević, *Schwarz Lemma and Kobayashi Metrics for Holomorphic Functions*, Filomat Volume 31, Number 11, 2017, 3253-3262

[49] M. Mateljević, (a) *Schwarz lemma and Kobayashi Metrics for holomorphic and pluriharmonic functions*, arXiv:1704.06720v1 [math.CV] 21 Apr 2017
(b) *Schwarz lemma, Kobayashi Metrics and FPT*, preprint November 2016.

[50] M. Mateljević, Ahlfors-Schwarz lemma, hyperbolic geometry, the Caratheodory and Kobayashi metric, Zbornik radova sedmog Simpozijuma "Matematika i primene" (4 - 5. novembar 2016.), Univerzitet u Beogradu - Matematički fakultet, 2017,1-41.

[51] M. Mateljević, Schwarz lemma and Kobayashi metrics for harmonic and holomorphic functions, J. Math. Anal. Appl. 464 (2018) 78-100.

[52] M. Mateljević, Schwarz lemma and distortion for harmonic functions via length and area, arXiv:1805.02979v1 [math.CV] 8 May 2018

[53] D. Minda, *The strong form of Ahlfors' lemma.* Rocky Mountain J. Math. Volume 17, Number 3 (1987), 457-462.

[54] B. Örnek and T. Akyel, A Representation with majorant of the Schwarz lemma at the boundary, manuscript July 2015.

[55] T. Akyel and B. Örnek, *Some Remarks on Schwarz lemma at the boundary*, manuscript August 2015.

[56] B. Örnek, *Estimates for holomorphic functions concerned with Jack's lemma* (manuscript 15 156 Publ).

[57] R. Osserman, *A From Schwarz to Pick to Ahlfors and Beyond*, Notice AMS 46, 8, 868-873 (1999).

[58] R. Osserman, *A A new variant of the SchwarzPickAhlfors Lemma*, Manuscripta mathematica , V. 100, No 2, pp 123129, Oct. 1999

[59] R. Osserman, *A sharp Schwarz inequality on the boundary*, Proc. Amer. Math. Soc. 128 (2000), 3513-3517.

[60] N. Suita, *On a metric induced by analytic capacity. I.*, Kodai Math. Sem. Rep. 25 (1973), 215-218

[61] H. Unkelbach, *Uber die Randverzerrung bei konformer Abbildung*, Math. Z., 43 (1938), 739-742.

[62] B. V. Sabat, Vvedenie v kompleksnyi analiz,I, II 1976, Introduction to Complex Analysis, translated by American mathematical society, 1992.

[63] M. Vuorinen: *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math. 1319, Springer-Verlag, Berlin–New York, 1988.
[64] B. Wong, *On the Holomorphic Curvature of Some Intrinsic Metrics*, Proceedings of the American Mathematical Society Vol. 65, No. 1 (Jul., 1977), pp. 57-61
[65] H. Wu, *Normal families of holomorphic mappings*, Acta Math. 119 (1967), 193-233. MR 31# 468.
[66] Communication with Forstneric.
[67] Robert M. Yamaleev, *Hyperbolic Cosines and Sines Theorems for the Triangle Formed by Arcs of Intersecting Semicircles on Euclidean Plane*, Journal of Mathematics Volume 2013, Article ID 920528, 10 pages,http://dx.doi.org/10.1155/2013/920528
[68] Yau, S.,T., *Harmonic functions on complete Riemannian manifolds*, Comm.Pure and Appl. Math. **28** (1975) 201-228
[69] Rudin, W., *Function Theory in the Unit Ball of C^n*, Springer-Verlag, Berlin Heidelberg New York, 1980.
[70] W. Rudin, *Pluriharmonic Functions in Balls*, Proceedings of the American Mathematical Society Vol. 62, No. 1 (Jan., 1977), pp. 44-46
[71] https://cuhkmath.wordpress.com/2012/03/04/principle-of-subordination/, Principle of subordination Posted on 04/03/2012 by leonardwong
[72] Subordination principle in univalent functions theory.Available from: https://www.researchgate.net/publication/242278646_Subordination_principle_in_ univalent_functions_theory [accessed Oct 3, 2016].
[73] Suzuki, Masaaki. *The intrinsic metrics on the circular domains in $\mathbb{C}^n$*. Pacific J. Math. 112 (1984), no. 1, 249–256. http://projecteuclid.org/euclid.pjm/1102710113.
[74] M. Mateljević, Communications at Analysis Seminar, University of Belgrade.
[75] https://www.researchgate.net/post/What_are_the_most_recent_versions_of_The_Schwarz_Lemma [accessed Jul 31, 2017].
How to solve an exremal problems related to harmonic functions?. Available from: https://www.researchgate.net/post/How_to_solve_a_exremal_problems_related_to_harmonic_functions [accessed Aug 3, 2017].
[76] P. Melentijević, Invariant gradient in refinements of Schwarz lemma and Harnack inequalities, manuscript.
[77] M. Marković, On harmonic functions and the hyperbolic metric, Indag. Math., 26(1):19-23, 2015.
[78] K. Goebel, "Fixed points and invariant domains of holomorphic mappings of the Hilbert ball" Nonlin. Anal., 6 (1982) pp. 1327-1334
[79] M. Abate, "Horospheres and iterates of holomorphic maps" Math. Z., 198 (1988) pp. 225-238
[80] M. Abate, "Converging semigroups of holomorphic maps" Atti Accad. Naz. Lincei, 82 (1988) pp. 223-227
[81] Kapeluszny, J., T. Kuczumow, and S. Reich: *The DenjoyWolff theorem in the open unit ball of a strictly convex Banach space.* - Adv. Math. 143, 1999, 111123.
[82] Bas Lemmens, Brian Lins, Roger Nussbaum, and Marten Wortel, Denjoy-Wolff theorems for Hilbert's and Thompson's metric spaces, arXiv:1410.1056v3 [math.DS] 27 May 2015
[83] P. Yang, "Holomorphic curves and boundary regularity of biholomorphic maps of pseudo-convex domains" preprint (1978)
[84] Y. Kondratiev, Y. Kozitsky, D. Shoikhet, *Dynamical systems on sets of holomorphic functions*, Complex Analysis and Dynamical Systems IV, Contemp. Math. 553, 2011, 139-153.
[85] Xieping Wang, Guangbin Ren, Boundary Schwarz lemma for holomorphic self-mappings of strongly pseudoconvex domains, arXiv:1506.01569v3 [math.CV] 20 Mar 2016, Complex Analysis and Operator Theory 11(2) March 2016 DOI: 10.1007/s11785-016-0552-5
[86] C. McMullen, *Riemann surfaces, dynamics and geometry, Course Notes*, Harvard University, 2004

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