Elliptic equations and QC maps- I Distribution, Sobolev spaces and Partial equation

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This is an addition materials to B. Jovanovic book [6](at this moment it is a rough version); see also [8]. The material covered by the author in the course Partial equation (elliptic of the second order) is roughly first 62 pages (including Morrey's inequality). In some parts we consider materials which reflects author's interest. (For example in Section 5 we consider Beltrami equation and Absolute Continuity on Lines). We denote by $*$ the parts which can be omitted in first reading.

The basic idea in distribution theory is to reinterpret functions as linear functionals acting on a space of test functions. Standard functions act by integration against a test function, but many other linear functionals do not arise in this way, and these are the "generalized functions". There are different possible choices for the space of test functions, leading to different spaces of distributions. The basic space of test function consists of smooth functions with compact support, leading to standard distributions. Use of the space of smooth, rapidly (faster than any polynomial increases) decreasing test functions (these functions are called Schwartz functions) gives instead the tempered distributions, which are important because they have a well-defined distributional Fourier transform. Every tempered distribution is a distribution in the normal sense, but the converse is not true: in general the larger the space of test functions, the more restrictive the notion of distribution. On the other hand, the use of spaces of analytic test functions leads to Sato's theory of hyperfunctions; this theory has a different character from the previous ones because there are no analytic functions with non-empty compact support.

1 harmonic in $\mathbb{R}^n$

1.1 Stokes' theorem, Green's theorem, the divergence theorem-Gauss's theorem or Ostrogradsky's theorem

Green's theorem

Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. If L and M are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then

$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx\,dy$$

where the path of integration along C is anticlockwise.

In physics, Green's theorem is mostly used to solve two-dimensional flow integrals, stating that the sum of fluid outflows from a volume is equal to the total outflow summed about an enclosing area. In plane geometry, and in particular, area surveying, Green's theorem can be used to determine the area and centroid of plane figures solely by integrating over the perimeter. In vector calculus, the divergence theorem, also known as Gauss's theorem or Ostrogradsky's theorem, is a result that relates the flow (that is, flux) of a vector field through a surface to the behavior of the vector field inside the surface.

More precisely, the divergence theorem states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface. Intuitively, it states that the sum of all sources (with sinks regarded as negative sources) gives the net flux out of a region.

The divergence theorem is an important result for the mathematics of physics and engineering, in particular in electrostatics and fluid dynamics.

In physics and engineering, the divergence theorem is usually applied in three dimensions. However, it generalizes to any number of dimensions. In one dimension, it is equivalent to the fundamental theorem of calculus. In two dimensions, it is equivalent to Green's theorem.

The theorem is a special case of the more general Stokes' theorem

Suppose V is a subset of $\mathbb{R}^n$ (in the case of $n = 3$, V represents a volume in 3D space) which is compact and has a piecewise smooth boundary S (also indicated with $\partial V = S$). If F is a continuously differentiable vector field defined on a neighborhood of V, then we have:

$$\iiint_V (\nabla \cdot \mathbf{F})\; dV = \oint \oint_S (\mathbf{F} \cdot \mathbf{n})\, dS.$$

The left side is a volume integral over the volume V, the right side is the surface integral over the boundary of the volume V. The closed manifold ∂V is quite generally the boundary of V oriented by outward-pointing normals, and n is the outward pointing unit normal field of the boundary ∂V. (dS may be used as a shorthand for n dS.) The symbol within the two integrals stresses once more that ∂V is a closed surface. In terms of the intuitive description above, the left-hand side of the equation represents the total of the sources in the volume V, and the right-hand side represents the total flow across the boundary S.

For admissible domains see [3].

Also we will use the notation $D[\phi, \psi] = \int\int_D (\sum \partial_k \phi\ \partial_k \psi\,)\,dx$ and $D[\phi] = D[\phi, \phi]$.

Suppose that D is simple (more generally admissible) in R^n and S boundary of D, $u \in C^1$ and $v \in C^2$ in $\overline{D}$ and $\partial/\partial n = D_n$ denotes differentation with respect to the outer unit normal n.

Let $dx_k \wedge d\sigma_k = dx$, that is $d\sigma_k = (-1)^{k-1} dx_1 \wedge \cdots \hat{dx_k} \cdots \wedge dx_n$.

Then $n_k d\sigma = d\sigma_k$,

$$\int_S u \frac{\partial v}{\partial n} d\sigma = \sum \int_S u \frac{\partial v}{\partial x_k} d\sigma_k = \tag{1}$$

$$\sum \int_D D_k \left(u \frac{\partial v}{\partial x_k}\right) dx \tag{2}$$

Thus

$$\int_S u \frac{\partial v}{\partial n} d\sigma = D[u, v] + \int_D u \Delta v dx. \tag{3}$$

Green's indentity (which we need mainly in the special case when D is a ball)

If $u, v \in C^2$ in $\overline{D}$, the second Green's indentity

$$\int_S \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) d\sigma = \int_D (u \Delta v - v \Delta u)\, dx, \tag{4}$$

A twice differentiable mapping $u = (u_1, \ldots, u_m) : \Omega \to \mathbb{R}^m$ defined in the open set Ω of the Euclidean space $\mathbb{R}^n$ is called harmonic if the real functions u_i, $i = 1, \ldots, m$, are harmonic. By $B = B^n$ we denote the unit ball in $\mathbb{R}^n$ and by S^{n-1} the unit $n-1$ dimensional sphere. for $a \in R^n$ by $B(a, R)$, $\overline{B}(a, R)$ and $S(a, r)$ we denote the ball, the closed ball and the sphere of radius r with center at a.

For $n = 2$ we frequently write $\mathbb{U}$, $D(a, R)$ and $K(a, r)$ instead of $\mathbb{B}^2$, $B(a, R)$ and $S(a, r)$.

?? Let $m = m_n$ denote the usual the Lebesgue measure on $\mathbb{R}^n$. Sometimes we use notation $dx = dx_1 \ldots dx_n$ and $|A|$ instead of dm and $m(A)$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and A is a Lebesgue measurable set in $\mathbb{R}^n$, respectively. By $d\sigma$ we denote positive Borel measure on S^{n-1} invariant w.r. to orthogonal group $O(n)$ normalized such that $\sigma(S^{n-1}) = 1$.

For a function h, we use notation $\partial h = \frac{1}{2}(h'_x - i h'_y)$ and $\overline{\partial} h = \frac{1}{2}(h'_x + i h'_y)$; we also use notations $Dh = h_z$ and $\overline{D} h = h_{\overline{z}}$ instead of ∂h and $\overline{\partial} h$ respectively when it seems convenient.

Let f be of class C^1 on $D(z_0, R)$ and let f and $g = f_{\overline{z}}$ be continuous in $\overline{D}(z_0, R)$. Then we have the representation

$$f(z) = \frac{1}{2\pi i} \oint_{K(z_0, R)} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \int_{\overline{D}(z_0, R)} \frac{g(\zeta)}{\zeta - z} d\xi d\eta \,. \tag{5}$$

We also use notation

$$Tg(z) = T_D g(z) = -\frac{1}{\pi} \int_D \frac{g(\zeta)}{\zeta - z} d\xi d\eta$$

if this integral exists in some sense.

Green's formula

Suppose we want to find the solution u of the Poisson equation in a domain $D \subset \mathbb{R}^n$:$\Delta u(x) = f(x), x \in D$ subject to some homogeneous boundary condition. Imagine f is the heat source and u is the temperature. The idea of Greens function is that if we know the temperature responding to an impulsive heat source at any point $x_0 \in D$, then we can just sum up the result with the weight function $f(x_0)$ (it specifies the strength of the heat at point x_0) to obtain the temperature responding to the heat source $f(x)$ in D. Mathematically, one may express this idea by defining the Greens function as the following: Let $u = u(x)$, $x = (x_1, ...x_n)$, be the solution of the following problem: $\Delta u(x) = f(x), x \in D$ satisfies some homogeneous boundary condition along the boundary ∂D.

Set $E(x) = \hat{E}_2(x) = \frac{\ln|x|}{2\pi}$, $x \in \mathbb{R}^2 \setminus \{0\}$; $E_n(x) = |x|^{2-n}$, and $\hat{E}_n = \frac{E_n}{(2-n)\sigma_n}$, $x \in \mathbb{R}^n \setminus \{0\}$, $n \geq 3$. We also use Γ instead of $\hat{E}_n$ and $\Gamma_y(x) = \Gamma(x - y)$.

In order to get Greens representation formula [16], it is convenient to introduce Greens function.

We define the Greens function G on a domain Ω with Dirichlet BC by
(i) $G(x, x_0) = 0$, $x \in \partial\Omega$ and
(ii) $G(x, x_0) = \hat{E}_n(x - x_0) + h(x, x_0)$, where $h(x, x_0)$ is a harmonic function in x.

We also use notation $G_{x_0}(x) = G(x, x_0)$. For given $x_0 \in \Omega$, let h_{x_0} be harmonic in Ω and $-\Gamma_{x_0}$ on $\partial\Omega$. Then $G_{x_0} = \Gamma_{x_0} +$.

In the distribution notation (ii) means $\Delta G(x, x_0) = \delta_{x_0}$.

GREENS FUNCTION FOR LAPLACIAN XX. Greens function for the upper half plane $H = \{y > 0\}$. We first construct the Greens function in the upper half plane with the Dirichlet boundary condition:

$G(z, z_0) = \Gamma(z - z_0) - \Gamma(z - \overline{z_0}) = \frac{1}{4\pi} \ln \frac{|z-z_0|^2}{|z-\overline{z_0}|^2}$

The outward unit normal to the boundary of the upper half plane is in the $-y$ direction. So we calculate

$$D_y(z, z_0) = \frac{1}{4\pi}[\frac{2(y - y_0)}{|z - z_0|^2} - \frac{2(y + y_0)}{|z - \overline{z_0}|^2}]$$

Since $|z - z_0| = |z - \overline{z_0}|$ for $z = (x, 0$, $D_y(x, 0, z_0) = -\frac{1}{\pi}\frac{y_0}{|x-z_0|^2}$

At point $(x, 0)$, $n = n(x) = (0, -1)$ and therefore $D_n(x, 0, z_0) = -D_y(x, 0, z_0) = \frac{1}{\pi}\frac{y_0}{|x-z_0|^2}$.

Greens function for the unit disk

$z^* = Jz = z/|z|^2$.

If $z_0 = r_0 e^{i\alpha}$ then $z_0^* = e^{i\alpha}/r_0$. Hence, for $z = e^{it}$, $r_0|z - z_0^*| = r_0|e^{it} - e^{i\alpha}/r_0| = |r_0 e^{it} - e^{i\alpha}| = |e^{it} - r_0 e^{i\alpha}|$. Thus $|z - z_0| = |z_0||z - z_0^*|$.

$G(z, z_0) = \frac{1}{2\pi}[\ln|z - z_0| - \ln|z - z_0^*| - \ln|z_0|]$

For $z = e^{i\theta}$, $n = e^{i\theta}$ and $D_n G(z, z_0) = D_r G(z, z_0)$

$|z - z_0|^2 = r^2 - 2rr_0\cos(\theta - \theta_0) + r_0^2$

$D_r|z - z_0|^2 = 2(r - r_0\cos(\theta - \theta_0))$ and $D_r|z - z_0| = \frac{r-r_0\cos(\theta-\theta_0)}{|z-z_0|}$

and $D_r \ln|z - z_0| = \frac{r-r_0\cos(\theta-\theta_0)}{|z-z_0|^2}$

$D_r \ln|z-z_0^*| = \frac{r-\frac{1}{r_0}\cos(\theta-\theta_0)}{|z-z_0^*|^2}$. Hence for $z=e^{i\theta}$

$$(D_r \ln|z-z_0|)_{z=e^{i\theta}} = \frac{1-r_0\cos(\theta-\theta_0)}{|e^{i\theta}-z_0|^2},$$

$$(D_r \ln|z-z_0^*|)_{z=e^{i\theta}} = \frac{r_0^2-r_0\cos(\theta-\theta_0)}{|e^{i\theta}-z_0|^2},$$

$$D_r G(z,z_0)|_{z=e^{i\theta}} = \frac{1-r_0^2}{|e^{i\theta}-z_0|^2}.$$

Hence for boundary data f we have solution of Dirichlet's problem $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} P(z_0,e^{i\theta})f(e^{i\theta})d\theta$.

$T_a(z) = T(a,z) = \frac{a-z}{1-\overline{a}z}$ is aut $\mathbb{U}$.

Using Green's formula under some conditions we show that $f = T(\overline{D}f)$. Then $G(a,z) = -\frac{1}{2\pi i}\ln|T(a,z)|$ is Green's function for $\mathbb{U}$.

Suppose that D is simply-connected domain in $\mathbb{C}$ different then $\mathbb{C}$ and ϕ conformal mapping of D onto $\mathbb{U}$. Then by

$G_D(b,w) = -\frac{1}{2\pi}\ln|T_D(b,w)|$, $b,w\in D$, where $T_D(b,w) = T(\phi(b),\phi(w))$, is given Green's function for D. In particular $A(w)=w/R$ maps $D(0,R)$ onto $\mathbb{U}$ and $T(b,w,R) = T(b/R,w/R) = \frac{R(b-w)}{R^2-\overline{b}w}$ and

$G(b,w,R) = -\frac{1}{2\pi}\ln|T(b,w,R)| = \frac{1}{2\pi i}\ln|\frac{R^2-\overline{b}w}{R(b-w)}|$, $b,w\in D(0,R)$, is Green's function for $D(0,R)$.

$B(w) = w-w_0$ maps $D(w_0,R)$ onto $D(0,R)$ and

$$G(b,w;w_0,R) = \frac{1}{2\pi}\ln|\frac{R^2-\overline{b-w_0}(w-w_0)}{R(b-w)}|, \quad b,w\in D(w_0,R),$$

is given Green's function for $D(w_0,R)$.

In particular, for $b=w_0$, we have

(1) $G(w_0,w;w_0,R) = G(w_0,w) = \frac{1}{2\pi}\ln|\frac{R}{(w_0-w)}|$.

Fix $b\in D(0,R)$ and set $A(b,\epsilon) = A(b,0;\epsilon,R)$, $P = R^2-\overline{b}w$ and $Q = R(b-w)$. Then $G(b,w,R) = \frac{1}{2\pi}\ln|X|$, where $X = P/Q$.

Apply Green's formula $u=f$ and $v = G(b,w,R)$ on $A(b,\epsilon)$. Compute $D_n v(w)$ for $w\in K(0,R)$ and $w\in K(b,\epsilon)$. For a $w\in K(0,R)$ locally v is real part of function $\frac{1}{2\pi}F$, where F is a branch $\ln X$ and $X=(P/Q)$. $Y = F' = \frac{Q}{P}X'$, $X' = \frac{R^2-|b|^2}{(b-w)^2}$, $Y = \frac{1}{R}\frac{R^2-|b|^2}{(b-w)(R^2-\overline{b}w)}$.

For $w\in K(0,R)$ we write $w=Re^{it}$; then $ds = R\frac{dw}{iw}$ and $(b-w)(R^2-\overline{b}w) = Re^{it}(b-w)(\overline{w}-\overline{b}) = Re^{it}|b-w|^2$ Since $n=n(w)=e^{it}$ is the outer unit normal at w on the circle $K(0,R)$,

$D_n g = Re(e^{it}F') = \frac{R^2-|b|^2}{|b-w|^2} = Re\frac{w+b}{w-b}$, and therefore

$D_n g ds = Re\frac{w+b}{w-b}\frac{dw}{iw}$.

Set $J(\epsilon) = \int_{S_\epsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds$, $J_1(\epsilon) = \int_{S_\epsilon} u\frac{\partial v}{\partial n} ds$, $J_2(\epsilon) = \int_{S_\epsilon} v\frac{\partial u}{\partial n} ds$ and $I(R) = \int_{K_R} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds$. Then
(2) $I(R) = \int_{K_R} u\frac{\partial v}{\partial n} ds$.

(3) $J_2(\epsilon) \to 0$ if $\epsilon \to 0_+$,

Using polar coordinate $z - b = \rho e^{it}$, we have

$v = -\ln|z-b| + h(z) = -\ln\rho + h(z)$, where h is a harmonic function.

If $n = n(w) = -e^{it}$ is the outer unit normal at w on the circle $K(0,\epsilon)$, $D_n v = -1/\rho + c + o(1)$ when $\rho \to 0$. Hence, since $ds = 2\pi\rho dt$, we first $u\frac{\partial v}{\partial n} ds = 2\pi\rho(-1/\rho + c + o(1))$ when $\rho \to 0$ and then
(4) $J_1(\epsilon) \to -u(a)$ if $\epsilon \to 0_+$.

Now by Green's formula

$$I(R) - J(\epsilon) = \iint_{D(0,R)} (u\Delta v - v\Delta u)\, dx\, dy = -\iint_{D(0,R)} v(z)\Delta u(z)\, dx\, dy$$

If $\epsilon \to 0_+$, using (2), (3) and (4), we get

$f(b) = \int_{K_R} D_n G(b,z) f(z) ds - \iint_{D(0,R)} G(b,z)\Delta u(z)\, dx\, dy$.

An immediate corollary is if $\varphi \in C_0^2(R^2)$ and $supp\varphi \subset D(0,R)$, then

$$\iint_{D(0,R)} \ln|z|\Delta\varphi(z)\, dx\, dy = -\iint_{D(0,R)} G(0,z)\Delta\varphi(z)\, dx\, dy = 2\pi\varphi(0). \quad (6)$$

In the distribution notation $\langle \ln|z| \rangle = 2\pi\delta$.

solutions of equation $\Delta f = g$

Green's function

$$G(\zeta, z, R) = \frac{1}{2\pi} \ln|\frac{R^2 - \overline{z}\zeta}{R(\zeta - z)}|.$$

The representation of function and partial derivatives by Green's function and Laplacian: We have

$$G_w = \frac{1}{4\pi}(\frac{1}{\zeta - w} - \frac{\overline{\zeta}}{R^2 - w\overline{\zeta}}), G_{ww} = \frac{1}{4\pi}(\frac{1}{(\zeta - w)^2} - \frac{\overline{\zeta}^2}{(R^2 - w\overline{\zeta})^2}) \quad (7)$$

$$|G_w| \le \frac{1}{2\pi}(\frac{1}{|\zeta - w|}, |G_{ww}| = \frac{1}{2\pi}\frac{1}{|\zeta - w|^2}. \quad (8)$$

Let f be continuous on $\overline{D}(0,R)$ and let f be of class C^2 on $D(0,R)$ and let $g = \Delta f = 4f_{z\overline{z}}$ be continuous on $\overline{D}(0,R)$. Then for $z \in D(0,R)$,

$$f(z) = \frac{1}{2\pi i}\oint_{K(0,R)} Re(\frac{\zeta + z}{\zeta - z})\frac{f(\zeta)}{\zeta} d\zeta - \frac{1}{\pi}\int_{\overline{D}(0,R)} G(\zeta, z, R) g(\zeta) d\xi d\eta, (9)$$

$$f_z(z) = \frac{1}{2\pi i}\oint_{K(0,R)} \frac{f(\zeta)}{(\zeta - w)^2} d\zeta - \frac{1}{\pi}\int_{\overline{D}(0,R)} G_z(\zeta, z, R) g(\zeta) d\xi d\eta\,. \quad (10)$$

Set $f_1(z) = \frac{1}{2\pi i}\oint_{K(0,R)} Re(\frac{\zeta+z}{\zeta-z})\frac{f(\zeta)}{\zeta} d\zeta$ and $V(z) = \frac{1}{\pi}\int_{\overline{D}(0,R)} G(\zeta, z, R) g(\zeta)$ $d\xi d\eta\,$.

If f is C^2 on $D(w',r]$, since $G(w',w;w',r)=0$ for $w\in K(w',r)$, an application of the above formula and (1) yield

$$I_f(w') = \frac{1}{2\pi i}\oint_0^{2\pi}[f(w'+re^{it})-f(w')]dt = \int_{D(w',r)} \ln\frac{r}{|w-w'|}\Delta f dudv\,. \quad (11)$$

Example 1. *If $\Delta^2 u = 0$, $u=0$ and $\frac{\partial u}{\partial n}=0$ on S, then u is harmonic.*
If we set $v=\Delta u$, then

$$0=\int_S\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)d\sigma=\int_D(v\Delta u-u\Delta v)\,dx=\int_D v^2dx=0\,. \quad (12)$$

Hence $v=0$, ie. u is harmonic.

Green's indentity follows easily from the familiar divergence theorem of advanced calculus.

$$\Gamma(x)=\Gamma_n(x)=\Gamma(|x|)=\begin{cases}\frac{1}{n(2-n)\Omega_n}|x|^{2-n} & \text{if } n>2\\ \frac{1}{2\pi}\ln|x| & \text{if } n=2.\end{cases}$$

$$D_i\Gamma(x-y)=\frac{1}{n\omega_n}(x_i-y_i)|x-y|^{-n} \quad (13)$$

$$|D_i\Gamma(x-y)|=\frac{1}{n\omega_n}|x-y|^{1-n}\,. \quad (14)$$

It is convenient to use notation $f^y(x)=f(x,y)$. Suppose that D is admissible domain and $u\in C^2(\overline{D})$. For $y\in D$ and $v=\Gamma^y$ we wish to apply the second Green's identity, but the function Γ^y has a singularity for $x=y$. If $B_\varepsilon=B_\varepsilon(y)$ is the ball $B(y,\varepsilon)$, $S_\varepsilon=S(y;\varepsilon)$ and $D_\varepsilon=D\setminus B(y,\varepsilon]$, then an application of the second Green identity by $v=\Gamma^y$ on D_ε yields

$$\int_{D_\varepsilon}\Gamma\Delta u dx=\int_S\left(\Gamma\frac{\partial u}{\partial n}-u\frac{\partial\Gamma}{\partial n}\right)d\sigma+\int_{S_\varepsilon}\left(\Gamma\frac{\partial u}{\partial n}-u\frac{\partial\Gamma}{\partial n}\right)d\sigma. \quad (15)$$

Next $\Gamma^y(x)=\Gamma(\varepsilon)$, for $x\in S_\varepsilon$, and therefore

$$I_1(\varepsilon):=\int_{S_\varepsilon}\Gamma\frac{\partial u}{\partial n}d\sigma=\Gamma(\varepsilon)\int_{S_\varepsilon}\frac{\partial u}{\partial n}d\sigma$$

Hence $|I_1(\varepsilon)|\le n\omega_n\varepsilon^{n-1}|\Gamma(\varepsilon)|\sup_{B_\varepsilon}|Du|\to 0$ when $\varepsilon\to 0$.
Since $\Gamma'(r)=\frac{1}{\sigma_n}r^{1-n}$ and $D_n\Gamma^y(x)=-\frac{1}{\sigma_n}\varepsilon^{1-n}$, for $x\in S_\varepsilon$, we have

$$I_2(\varepsilon):=\int_{S_\varepsilon}u\frac{\partial\Gamma}{\partial n}d\sigma=-\Gamma'(\varepsilon)\int_{S_\varepsilon}uds= \quad (16)$$

$$-\frac{1}{n\omega_n\varepsilon^{n-1}}\int_{S_\varepsilon}uds\to -u(y) \quad (17)$$

when $\varepsilon\to 0$.

Set $I(\varepsilon) := \int_{S_\varepsilon} \left(\Gamma\frac{\partial u}{\partial n} - u\frac{\partial \Gamma}{\partial n}\right) d\sigma$. Then $I(\varepsilon) = I_1(\varepsilon) - I_2(\varepsilon)$ and therefore $I(\varepsilon) \to u(y)$ if $\varepsilon \to 0$. Hence, when $\varepsilon \to 0$ in (15), we get

$$u(y) = \int_S \left(u\frac{\partial \Gamma}{\partial n} - \Gamma\frac{\partial u}{\partial n}\right) d\sigma + \int_D \Gamma \Delta u dx, \quad y \in D. \tag{18}$$

Thus $u = V + V^0 + V^1$, where

V is the Newtonian potential of a integrable function $f = \Delta u$, which is defined as the convolution

$$V(x) = \Gamma * u(x) = (\Gamma * u)_D(x) = \int_D \Gamma(x-y) f(y)\, dy,$$

V^0 potential of simple layer:

$$V^0(x) = -(\Gamma * \frac{\partial u}{\partial n})_S(x) = -\int_S \Gamma(x-y)\frac{\partial u}{\partial n} d\sigma(y) \tag{19}$$

and V^1 - Double-layer potential:

$$V^1(x) = (\frac{\partial \Gamma}{\partial n_y} * u)_D(x) = \int_S u(y)\frac{\partial \Gamma(x-y)}{\partial n_y} d\sigma(y) \tag{20}$$

In particular, if $u \in C_0^1(D)$, then $u(y) = \int_D \Gamma_y \Delta u dx$. Thus $\langle \Delta \Gamma_y \rangle = \delta_y$.

Repeat

If D is admissible domain, S the boundary of G and $u \in C^1(\overline{D}) \cap C^2(D)$, then for $x \in D$

$$u(x) = \int_D \Gamma(x-y)\Delta u(y)\, dy + \int_S \Gamma(x-y)\frac{\partial u}{\partial n} d\sigma(y) + \int_S u(y)\frac{\partial \Gamma(x-y)}{\partial n_y} d\sigma(y). \tag{21}$$

Suppose that $h \in C^1(\overline{D}) \cap C^2(D)$ and $g = \Gamma + h$

$$\int_S \left(u\frac{\partial h}{\partial n} - h\frac{\partial u}{\partial n}\right) d\sigma = -\int_D h\Delta u$$

$$u(y) = \int_S \left(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n}\right) d\sigma + \int_D g\Delta u\,.$$

If $g = 0$ on S, then

$$u(y) = \int_S u\frac{\partial g}{\partial n}\, d\sigma + \int_D g\Delta u\,. \tag{22}$$

In particular if $u \in C_0^2(D)$, then $u(y) = \int_D g^y \Delta u dx = \langle G^y, \Delta u\rangle$. Since $\langle \Delta G^y, u\rangle = \langle G^y, \Delta u\rangle = u(y)$ we find $\langle \Delta G^y \rangle = \delta_y$.

Representation of harmonic function by boundary values. If u is harmonic in $C^1(\overline{D}) \cap C^1(D)$, then

$$u(y) = \int_S u \frac{\partial g}{\partial n}\, d\sigma\,. \tag{23}$$

For $\eta \in S$, $\frac{\partial g}{\partial n}(x,\eta)$ is har in x as a limit of har functions ??
Embedding theorems have important role, cf. $ 2.7 Theorem 3 [6].
$W_0^{1,p}(G) \subset C^0(\overline{G})$, for $p > n$, [2], p.154.
Let $m = [n/2] + 1$. If $f \in C_0^m(G)$, then

$$|f|_{C(\overline{G})} \le c|f|_{H^m(G)}. \tag{24}$$

By passing to the limit we conclude that this estimate holds for $f \in H_0^m(G)$. Hence $H_0^m(G) \subset \overline{G}$.

Theorem 1.1 ([4], p.155,$ 2.7 Theorem 3 [6]). *Let $k = [n/2] + l + 1$ ($l = k - [n/2] - 1 \ge 0$).*

If G be of class C^k, then $H^k(G) \subset C^l(\overline{G})$. For $n = 2, 3$, $[n/2] = 1$, and we have $H^{2+l}(G) \subset C^l(\overline{G})$. In particular, $H^2(G) \subset C^0(\overline{G})$ and $H^3(G) \subset C^1(\overline{G})$.

Dokaz. For $f \in H^k(G)$, $\overline{G} \subset G'$ there is F which is an extension of f and finite in $\subset G'$ such that $F \in H^k(G')$. □

$$\tag{25}$$

The Newtonian potential w of f is a solution of the Poisson equation
$\Delta w = f$,
which is to say that the operation of taking the Newtonian potential of a function is a partial inverse to the Laplace operator. The solution is not unique, since addition of any harmonic function to w will not affect the equation. This fact can be used to prove existence and uniqueness of solutions to the Dirichlet problem for the Poisson equation in suitably regular domains, and for suitably well-behaved functions : one first applies a Newtonian potential to obtain a solution, and then adjusts by adding a harmonic function to get the correct boundary data.

A Green's function $g(x,\xi)$ in D is a function which is
harmonic in x in D except for $x = \xi$,
continuous in $\overline{D}$ except for $x = \xi$, $g = 0$ on ∂D
and
$g - |x - \xi|^{m-2}$ harmonic for $x = \xi$.

The Mean -Value Property

If u is harmonic on $\overline{B}(a,r)$, then $u(a)$ equals the average of u over $S(a,r)$

$$u(a) = \int_S u(a + r\xi)\, d\sigma(\xi).$$

we applied Green's identity with $A = \{x : \varepsilon < |x| < 1\}$ and $v(y) = |y|^{2-n}$

$$\int_S uds = \varepsilon^{1-n} \int_{\varepsilon S} uds$$

$$\int_S ud\sigma = \int_S u(\varepsilon\xi)\, d\sigma(\xi)$$

The Mean -Value Property, Volume Version If u is harmonic on $\overline{B}(a,r)$, then $u(a)$ equals the average of u over $B(a,r)$

$$u(a) = \frac{1}{\Omega_n r^n} \int_{B(a,r)} u(x) dV.$$

Let Ω_n be volume of unit ball in R^n and σ_n of unit sphere, then $\sigma_n = n\Omega_n$.

?? we applied Green's identity with $v(y) = |y|^{2-n}$, $\Gamma(x) = \frac{1}{n(2-n)\Omega_n}|x|^{2-n}$, $n > 2$

$$u(x) = \frac{1}{2-n} \int_S uD_n v d\sigma$$

A computation of $D_n v$ yields

$$P(x,\eta) = P_B(x,\eta) = \frac{1-|x|^2}{|x-\eta|^n} \tag{26}$$

For $\eta \in S$, $P(x,\eta)$ is harmonic in x as a limit of harmonic functions ??

the maximum principle Let G be connected, and let $u : G \to \mathbb{R}$ be harmonic. If u has either a maximum or a minimum in G, then u is constant.

Symmetry lemma

$$\big||y|^{-1}y - |y|x\big| = \big||x|^{-1}x - |x|y\big|$$

illustration $A = |x|^{-1}x, B = |y|^{-1}y, A_1 = |y|x, B_1 = |x|y$ triangles OAB_1 and OBA_1 are equilateral.

For $y \in \mathbb{S}$, $\big|y - x\big| = \big||x|^{-1}x - |x|y\big| = |x|\big|Jx - y\big|$ and $L(y) = R(y)$, where $L(y) = |y-x|^{2-n}$, and $R(y) = |x|^{2-n}\big|Jx - y\big|^{2-n}$.

Denote ξ with A, $J\xi$ with A' and arbitrary point x with Q,

$OQ = r, OA = \rho$ and θ angle QOA. Then

$$AQ^2 = r^2 + \rho^2 - 2r\rho\cos\theta, A'Q^2 = r^2 + \frac{R^4}{\rho^2} - 2r\frac{R^2}{\rho}\cos\theta \tag{27}$$

$$A'Q\partial_r A'Q = r - \frac{R^2}{\rho}\cos\theta \tag{28}$$

for $Q \in S(0,R)$, $A'Q = AQR/\rho$

for $m > 2$

$$D_n g = [\frac{1}{AQ^{m-2}} - \frac{R}{\rho AQ^{m-2}}]_r = \frac{(m-2)(R^2-\rho^2)}{R(AQ)^m} \tag{29}$$

Hence

$$D_n g = \frac{((R^2-\rho^2)}{R(R^2+\rho^2-2R\rho\cos\theta)}, \quad m=2 \tag{30}$$

$$D_n g = \frac{(m-2)(R^2-\rho^2)}{R(R^2+\rho^2-2R\rho\cos\theta)^{m/2}}, \quad m>2. \tag{31}$$

$\varepsilon \to 0$ on $S(\xi, \varepsilon)$

$$u(x) = u(\xi) + O(\varepsilon), D_n u = O(1), \tag{32}$$

$$g(x) = o(\varepsilon^{1-m}), D_n u = -\varepsilon^{-1} + O(1), m = 2; \tag{33}$$

$$D_n u = -(m-2)\varepsilon^{1-m} + O(1), m > 2, \tag{34}$$

$$\int_{S(\xi,\varepsilon)} ds = c_m \varepsilon^{m-1} \tag{35}$$

Hence

$$\int_{S(\xi,\varepsilon)} \left(u\frac{\partial g}{\partial n} - g\frac{\partial u}{\partial n} \right) d\sigma = \begin{cases} -c_2 u(\xi) & \text{if } m=2 \\ -(m-2)c_m u(\xi) + o(1) & \text{if } m>2. \end{cases}$$

and

$$u(\xi) = \frac{1}{c_m} \int_{S(0,R)} \frac{(R^2-\rho^2)u(x)}{R(R^2+\rho^2-2R\rho\cos\theta)^{m/2}} ds_x \tag{36}$$

Let $t \in S_R$, $n = t/|t|$, $L(t) = |x-t|^{2-n}$, and $M = t$. Then
$K(x,t) = D_k v(M) = (2-n)|x-t|^{-n}(x_k - t_k)$,
$|t| D_n L(M) = (2-n)|x-t|^{-n} \sum_{k=1}^n (x_k - t_k) t_k$ and
for fixed t function $|x-t|^{-n}(x-t,t)$ is harmonic in x ??.
Since $|x|^2 = |x-t|^2 + 2(x-t,t) + |t|^2$, hence $K(x,t)$ is harmonic in x.
If $v(t) = L(t) - R(t)$, then $K(x,t) = D_n v(M)$.
Poisson kernel for ball $B(a,R)$

$$K(x,\eta) = K_B(x,\eta) = \frac{R^2 - |x-a|^2}{n\omega_n R|x-\eta|^n}$$

and in particular Poisson kernel for ball $B(0,R)$ is

$$K(x,\eta) = K_B(x,\eta) = \frac{R^2 - |x|^2}{n\omega_n R|x-\eta|^n}\ .$$

If u is harmonic on $\overline{B(a,R)}$, then

$$u(x) = \int_{S(a,R)} K(x,\eta)u(\eta)d\sigma(\eta) \tag{37}$$

for every $x \in B(a, R)$.

Solution of Dirichlet problem for the ball:

Suppose f is continuous and bounded on S. Define u on $\overline{B}$ by

$$u(x) = \begin{cases} P_B[f](x) & \text{if } x \in B \\ f(x) & \text{if } x \in \mathbb{S}^{n-1}. \end{cases}$$

Then u is continuous on $\overline{B}$ and harmonic on B.

Check that for $t \in S$, $P(\cdot, t)$ is harmonic on $R^n \setminus \{t\}$. Namely,

$|x - t|^2 = 1 - 2x \cdot t + |x|^2$ and since $t \in S$, $1 - x \cdot t = (t - x) \cdot t$, $1 - ||x||^2 = 2(t - x) \cdot t - |x - t|^2$, and therefore $P(x, t) = 2(1 - x \cdot t)|x - t|^{-n} - |x - t|^{2-n}$. Since functions $P_1(x, t) = (t - x) \cdot t|x - t|^{-n} = \sum(t_i - x_i)t_i|x - t|^{-n}$ and $|x - t|^{2-n}$ are harmonic in x, $P(x, t)$ is harmonic.

Let $f : S^{n-1} \to \mathbb{R}^m$ be a bounded integrable mapping defined on the unit sphere S^{n-1}. Then the function u defined by

$$u(x) = P[f](x) = \int_{S^{n-1}} \frac{1 - ||x||^2}{||x - \eta||^n} f(\eta) d\sigma(\eta), \tag{38}$$

is a bounded harmonic mapping on the unit ball B^n.

It is known that σ satisfies the condition:

$$\int_{S^{n-1}} \frac{1 - ||x||^2}{||x - \eta||^n} d\sigma(\eta) = 1, \quad x \in B. \tag{39}$$

If u is harmonic in Ω, then u is real analytic in Ω. It follows from

$$P(x, \eta) = P_B(x, \eta) = \frac{1 - |x|^2}{|x - \eta|^n} = (1 - |x|^2) \sum c_i(|x|^2 - 2x \cdot \eta)^i \tag{40}$$

and $t^{n/2} = \sum c_i(t - 1)^i$ on $(0, 2)$.

Example 2. *Laplacian in polar and spherical coordinates*

(a) $\Delta u = \frac{1}{r} U_r(r U_r) + \frac{1}{r^2} U''_{\theta\theta}$

(b) $\Delta u = \frac{1}{r^2} U_r(r^2 U_r) + \frac{1}{r^2 \sin\varphi} U_\varphi(\sin\varphi U_\varphi) + \frac{1}{r^2 \sin^2\varphi} U''_{\theta\theta}$

Interior estimate

Suppose that $f_0 : S^{n-1}(a, r) \to \mathbb{R}^n$ is a continuous vector-valued function, and let $f = P[f_0]$ and

$$M_a^* = \sup\{|f_0(y) - f_0(a)| : y \in S^{n-1}(a, r)\}.$$

Then

$$r|f'(a)| \le n M_a^* . \tag{41}$$

2 Distribution

2.1 AC

Definicija 1. *Neka je $I \subset \mathbb{R}$ interval. Funkcija $f : I \to \mathbb{R}$ je apsolutno neprekidna na I ako za svako $\varepsilon > 0$ postoji $\delta > 0$ takvo da za svaki konačan niz $(x_1, y_1), \ldots, (x_n, y_n)$ međusobno disjunktnih podintervala intervala I takvih da je*

$$\sum_{j=1}^{n} (y_j - x_j) < \delta$$

važi

$$\sum_{j=1}^{n} |f(y_j) - f(x_j)| < \varepsilon.$$

Stav 1. *Neka je $f \in L_1[a, b]$ i neka je $F(x) = \int_a^x f(t)dt$, $x \in [a, b\,]$. Tada F ima konačan izvod skoro svuda na $[a, b\,]$ i važi $F' = f$ skoro svuda na $[a, b\,]$.*

Stav 2. *Neka su $f, g : I \to \mathbb{R}$ apsolutno neprekidne funkcije. Tada su $f + g$ i $-f$ apsolutno neprekidne funkcije na I.*

Stav 3. *Neka je I zatvoren interval i neka su $f, g : I \to \mathbb{R}$ apsolutno neprekidne funkcije. Tada je $f \cdot g$ apsolutno neprekidna funkcija na I.*

Napomena 1. *Sa f' obeležavamo klasičan izvod funkcije f a sa Df uopšteni izvod funkcije f.*

Stav 4. *Neka je $f \in L_1[a, b]$ i neka je $F(x) = \int\limits_a^x f(t)dt$, $x \in [a, b]$. Tada*

(1) Funkcija F ima konačan izvod skoro svuda na $[a, b]$;

(2) $F' = f$ skoro svuda na $[a, b]$;

(3) Funkcija F je apsolutno neprekidna na $[a, b]$.

Stav 5. *Neka je $f : [a, b] \to \mathbb{R}$ apsolutno neprekidna na $[a, b]$. Tada*

(1) f' postoji skoro svuda na $[a, b]$;

(2) $f' \in L_1[a, b]$;

(3) Za svako $x \in [a, b]$ važi $\int\limits_a^x f'(t)dt = f(x) - f(a)$.

Zadatak 1. *Neka ja $f \in L_1[a, b\,]$. Funkcija f ima uopšteni izvod $Df \in L_1[a, b\,]$ (izvod u smislu distribucija, slab izvod) ako i samo ako $f \in AC[a, b\,]$.*

Rešenje.

Smer ($\Leftarrow$). Neka je $\varphi \in C_c^1(a,b)$ proizvoljna. Tada postoji funkcija $\widetilde{\varphi} \in C^1[a,b]$ takva da je $\widetilde{\varphi}(x) = \varphi(x)$ za svako $x \in (a,b)$ i $\widetilde{\varphi}(a) = \widetilde{\varphi}(b) = 0$. Zbog lakšeg obeležavanja u nastavku rešenja umesto $\widetilde{\varphi}$ pišemo φ.

Kako je $\varphi \in C^1[a,b]$ sledi da $\varphi \in AC[a,b]$. Otuda je $f\varphi \in AC[a,b]$.

Kako su $f\varphi, f, \varphi \in AC[a,b]$ sledi da $(f\varphi)'$,f' i φ' postoje skoro svuda na $[a,b]$ i važi $(f\varphi)', f', \varphi' \in L_1[a,b]$. Otuda

$$(f\varphi)' = f'\varphi + f\varphi'$$

skoro svuda na $[a,b]$.

Dalje, kako je $\varphi \in C^1[a,b]$ sledi da $(f\varphi)', f'\varphi, f\varphi' \in L_1[a,b]$.

Kako je $f\varphi \in AC[a,b]$ i kako je $\varphi(a) = \varphi(b) = 0$ sledi

$$\int_a^b (f\varphi)'(t)dt = (f\varphi)(b) - (f\varphi)(a) = 0 - 0 = 0.$$

Otuda je

$$\int_a^b (f'\varphi + f\varphi')(t)dt = 0,$$

odnosno

$$\int_a^b f'(t)\varphi(t)dt = -\int_a^b f(t)\varphi'(t)dt.$$

Kako je $\varphi \in C_c^1(a,b)$ proizvoljna iz poslednje formule, a na osnovu definicije uopštenog izvoda, sledi da $Df = f'$ i $Df \in L_1[a,b]$.

Smer ($\Rightarrow$). Neka je $g = Df$. Kako je $g \in L_1[a,b]$ korektno je definisana funkcija

$$F(x) = \int_a^x g(t)dt, \qquad x \in [a,b].$$

Štaviše $F \in AC[a,b]$ i $F' = g$ skoro svuda na $[a,b]$. Na osnovu dokazanog smera funkcija F ima uopšteni izvod i važi $DF = g$. Otuda funkcije f i F imaju jednake uopštene izvode, pa postoji konstanta c tako da važi $f = F + c$. Kako je konstantna funkcija apsolutno neprekidna i kako je zbir dve apsolutno neprekidne funkcije apsolutno neprekidna funkcija sledi da $f \in AC[a,b]$. $\triangle$

Potrebno je dokazati da ako za $f, g \in L_1[a,b]$ važi $Df = Dg$ onda postoji konstanta c takva da je $f = g + c$.

2.2 Distribution

Let U be an open subset of $\mathbb{R}^n$. The space $\mathcal{D}(U)$ of test functions on U is defined as follows. A function $\varphi : U \to R$ is said to have compact support if there exists a compact subset K of U such that $\varphi(x) = 0$ for all x in $U \setminus K$. The elements of $\mathcal{D}(U)$ are the infinitely differentiable functions $\varphi : U \to R$ with compact support also known as bump functions. This is a real vector space. It can be given a topology by defining the limit of a sequence of elements of D(U). A sequence (φ_k) in $\mathcal{D}(U)$ is said to converge to $\varphi \in \mathcal{D}(U)$ if the following two conditions hold: There is a compact set $K \subset U$ containing the supports of all φ_k:

$\bigcup_k \operatorname{supp}(\varphi_k) \subset K$.

For each multi-index α, the sequence of partial derivatives $\partial^\alpha \varphi_k$ tends uniformly to $\partial^\alpha \varphi$

With this definition, $\mathcal{D}(U)$ becomes a complete locally convex topological vector space satisfying the Heine-Borel property.

Distributions

A distribution on U is a continuous linear functional $T : \mathcal{D}(U) \to R$ (or $T : \mathcal{D}(U) \to C$). That is, a distribution T assigns to each test function f a real (or complex) scalar $T(f)$ such that

$T(c_1\varphi_1 + c_2\varphi_2) = c_1 T(\varphi_1) + c_2 T(\varphi_2)$ for all test functions φ_1, φ_2 and scalars c_1, c_2. Moreover, T is continuous if and only if

$\lim_{k\to\infty} T(\varphi_k) = T\left(\lim_{k\to\infty} \varphi_k\right)$ for every convergent sequence φ_k in $\mathcal{D}(U)$. (Even though the topology of $\mathcal{D}(U)$ is not metrizable, a linear functional on $\mathcal{D}(U)$ is continuous if and only if it is sequentially continuous.) Equivalently, T is continuous if and only if for every compact subset K of U there exists a positive constant C_K and a non-negative integer n_K such that

$|T(\varphi)| \leq C_K \sup_K |\partial^\alpha \varphi|$ for all test functions φ with support contained in K and all multi-indices a with $|\alpha| \leq n_K$.

Why absolute values of Jacobians in change of variables for multiple integrals but not single integrals?

Let $I \subset \mathbb{R}$ be an interval and $\varphi : [a, b] \to I$ be a differentiable function with integrable derivative. Suppose that $f : I \to \mathbb{R}$ is a continuous function. Then

$$\int_{\varphi(a)}^{\varphi(b)} f(x)\,dx = \int_a^b f(\varphi(t))\varphi'(t)\,dt.$$

One may also use substitution when integrating functions of several variables. Here the substitution function $(v_1, ..., v_n) = f(u_1, ..., u_n)$ needs to be injective and continuously differentiable, and the differentials transform as

$$dv = dv_1 \cdots dv_n = |\det(\mathrm{D}\,\varphi)(u_1, \ldots, u_n)|\, du_1 \cdots du_n$$

where $det(Df)(u_1, ..., u_n)$ denotes the determinant of the Jacobian matrix containing the partial derivatives of f. This formula expresses the fact that the absolute value of the determinant of a matrix equals the volume of the parallelotope spanned by its columns or rows.

More precisely, the change of variables formula is stated in the next theorem:

Theorem. Let U be an open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ an injective differentiable function with continuous partial derivatives, the Jacobian of which is nonzero for every x in U. Then for any real-valued, compactly supported, continuous function f, with support contained in $f(U)$,

$$\int_{\varphi(U)} f(\mathbf{v})\, d\mathbf{v} = \int_U f(\varphi(\mathbf{u}))\, |\det(\mathrm{D}\, \varphi)(\mathbf{u})|\; d\mathbf{u}.$$

The conditions on the theorem can be weakened in various ways. First, the requirement that f be continuously differentiable can be replaced by the weaker assumption that f be merely differentiable and have a continuous inverse (Rudin 1987, Theorem 7.26). This is guaranteed to hold if f is continuously differentiable by the inverse function theorem. Alternatively, the requirement that $Det(Df) \neq 0$ can be eliminated by applying Sard's theorem (Spivak 1965).

For Lebesgue measurable functions, the theorem can be stated in the following form (Fremlin 2010, Theorem 263D):

A weak solution (also called a generalized solution) to an ordinary or partial differential equation is a function for which the derivatives may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense. There are many different definitions of weak solution, appropriate for different classes of equations. One of the most important is based on the notion of distributions.

Avoiding the language of distributions, one starts with a differential equation and rewrites it in such a way that no derivatives of the solution of the equation show up (the new form is called the weak formulation, and the solutions to it are called weak solutions). Somewhat surprisingly, a differential equation may have solutions which are not differentiable; and the weak formulation allows one to find such solutions.

Weak solutions are important because a great many differential equations encountered in modelling real world phenomena do not admit sufficiently smooth solutions and then the only way of solving such equations is using the weak formulation. Even in situations where an equation does have differentiable solutions, it is often convenient to first prove the existence of weak solutions and only later show that those solutions are in fact smooth enough.

2.2.1 Absolutely continuous

However, the function $u(x) := x$, $x \in \mathbb{R}$, is absolutely continuous, but it is unbounded and $u'(x) = 1$, which is not Lebesgue integrable. Also the function $u(x) := \sin x$, $x \in \mathbb{R}$, is absolutely continuous, bounded, but u' is not Lebesgue integrable. These simple examples show that an absolutely continuous function may not have bounded pointwise variation. Proposition XX below will show that this can happen only on unbounded intervals.

Example 3. *Let $u : I \to \mathbb{R}$ be differentable with bounded derivative, then $u \in AC(I)$.*

If $u, v \in AC([a,b])$, then
(i) $u \pm v \in AC([a,b])$
(ii) $uv \in AC([a,b])$

Example 4. *Let $I \subset \mathbb{R}$ be an interval and let $u : I \to \mathbb{R}$ be uniformly continuous.*

i) Prove that u may be extended uniquely to $\overline{I}$ in such a way that the extended function is still uniformly continuous.
(ii) Prove that if u belongs to $AC(I)$, then its extension belongs to $AC(\overline{I})$ and, u' is Lebesgue integrable on bounded subintervals of I.
(iii) Prove that there exist $A, B > 0$ such that $|u(x)| \leq A + B|x|$ for all $x \in I$.

Hint. For $\varepsilon = 1$, there is $\delta > 0$ such that $|u(x) - u(y)| \leq 1$ for $|x - y| \leq \delta$. Set $k = 1/\delta$. Then $|u(n\delta)| \leq |u(0)| + n$. If $n\delta \leq x < (n+1)\delta$, then $|u(x)| \leq |u(n\delta)| + 1 \leq |u(0)| + n + 1 \leq |u(0)| + 1 + k|x|$.

A real-valued function f on the interval $[a,b]$ is said to be singular if it has the following properties:

f is continuous on $[a,b]$. (**)
there exists a set N of measure 0 such that for all x outside of N the derivative $f'(x)$ exists and is zero, that is, the derivative of f vanishes almost everywhere.
f is nondecreasing on $[a,b]$.
$f(a) < f(b)$.

A continuous monotone function f is said to be singular with respect to Lebesgue measure (written $f \perp m$) provided that f is non-constant yet $f'(x) = 0$ almost everywhere. A standard example of a singular function is the Cantor function, which is sometimes called the devil's staircase (a term also used for singular functions in general). There are, however, other functions that have been given that name. The Cantor function can be expressed in a ternary expansion.

Absolute continuity (AC) is a smoothness property of functions that is stronger than continuity and uniform continuity. The notion of absolute continuity allows one to obtain generalizations of the relationship between the two central operations of calculus, differentiation and integration, expressed by the fundamental theorem of calculus in the framework of Riemann integration. Such generalizations are often formulated in terms of Lebesgue integration The following conditions on a real-valued function f on a compact interval $[a,b]$ are equivalent:

(1) f is absolutely continuous;

(2) f has a derivative f' almost everywhere, the derivative is Lebesgue integrable, and

$f(x) = f(a) + \int_a^x f'(t)\,dt$
for all x on $[a,b]$;

(3) there exists a Lebesgue integrable function g on $[a,b]$ such that

$f(x) = f(a) + \int_a^x g(t)\,dt$
for all x on $[a, b]$.

If these equivalent conditions are satisfied then necessarily $g = f'$ almost everywhere.

Equivalence between (1) and (3) is known as the fundamental theorem of Lebesgue integral calculus, due to Lebesgue.

For an equivalent definition in terms of measures see the section Relation between the two notions of absolute continuity.

A $f \in BV[a, b]$ is singular if $f'(x) = 0$ almost everywhere.

If $f \in BV[a, b]$, then $f = f_{ac} + f_s$ where $f_{ac} \in AC[a, b]$ and f_s is singular.

Darboux's theorem

Let I be an open interval, $f\colon I \to \mathbb{R}$ a real-valued differentiable function. Then f' has the intermediate value property: If a and b are points in I with $a < b$, then for every y between $f'(a)$ and $f'(b)$, there exists an x in $[a, b]$ such that $f'(x) = y$. Consider the equation

$$y'(x) = \operatorname{sgn} x, \qquad x \in [-1, 1], \tag{42}$$

gde je $y : [-1, 1\,] \to \mathbb{R}$ nepoznata funkcija. Primetimo da funkcija sgn ima prekid prve vrste u tački 0.

A Darboux function is a real-valued function f which has the "intermediate value property": for any two values a and b in the domain of f, and any y between $f(a)$ and $f(b)$, there is some c between a and b with $f(c) = y$. By the intermediate value theorem, every continuous function is a Darboux function. Darboux's contribution was to show that there are discontinuous Darboux functions.

Every discontinuity of a Darboux function is essential, that is, at any point of discontinuity, at least one of the left hand and right hand limits does not exist.

By Darboux's theorem, the derivative of any differentiable function is a Darboux function. By Darboux's theorem, sgn is not a Darboux function in any nbgh $V = (-a, a)$, $a > 0$, of 0. Therefore there is no differentiable function u such that $u'(x) = sgnx$, $x \in V$.

Hence equation (42) there is no solutions in class differentiable function on $[-1, 1\,]$ ie. there is no function $y : [-1, 1\,] \to \mathbb{R}$ such that $y'(x) = \operatorname{sgn} x$ for every $x \in [-1, 1\,]$.

Instead of the equation (42) we can consider

$$\int_{-1}^{1} y'(x)\varphi(x)dx = \int_{-1}^{1} -y(x)\varphi'(x)dx = \int_{-1}^{1} \operatorname{sgn} x\varphi(x)dx := l(\varphi), \tag{43}$$

where φ arbitrary function of class $C^1[-1, 1\,]$ such that $\varphi(-1) = \varphi(1) = 0$. ?? Pri tome prirodno je rešenje jednačine (43) tražiti u klasi $C[-1, 1]$. We can check that $u(x) = |x| + c$ (c real constant) solutions of the equation (145) in class $C[-1, 1]$. Hence it solution we call *weak solution* of the equation (42).

By partial integration,

$l_u(\varphi) := \int_{-1}^{1} -u(x)\varphi'(x)dx = \int_{-1}^{1} -(|x|+c)\varphi'(x)dx = I_1 + I_2$, where

$I_1 = \int_0^1 -(|x|+c)\varphi'(x)dx = c\varphi(0) + \int_0^1 \operatorname{sgn} x\, \varphi(x)dx$ and $I_2 = \int_{-1}^{0} -(|x| + c)\, \varphi'(x)dx = -c\varphi(0) + \int_{-1}^{0} \operatorname{sgn} x\, \varphi(x)dx$, and therefore

$l_u(\varphi) = l(\varphi)$.

Moral. By definition we express integral of $u'\varphi$ by means of integral of $u\varphi'$, and then by partial integraratio we return again to the integral of $u'\varphi$ plus the jump of $u\varphi$ at discontinuity point of u.

If $f, \varphi \in C[a,b]$ and $f, \varphi \in C^1(a,b)$, then

$$\int_a^b y'(x)\varphi(x)dx = f\varphi|_a^b - \int_a^b y(x)\varphi'(x)dx, \tag{44}$$

Show $f \in L^{1,2}$ if $f, f' \in L^2$; $[f,g] = \int fgdx + \int f'g'dx$ is a scalar product.

If M is subset of $L^2(-\pi,\pi)$ such that $f \in M$ if $f' \in L^2(-\pi,\pi)$ and $|f'|_2 \leq 1$. Whether M is compact in $L^2(-\pi,\pi)$? A subspace of $L^{1,2}$ consisting of AC functions we denote by $W^{1,2}$.

What is difference between $L^{1,2}$ and $W^{1,2}$? ACL property Whether $W^{1,2} \subset L^{1,2}$? Cantor function

Solve $u'' = f_0$ on $I = [-1,1]$.

$\int_{-1}^{1} u''vdx = -\int_{-1}^{1} u'v'dx$, $l(v) = \int_{-1}^{1} vf_0dx = (v, f_0)_2$.

If $f_0 \in L_2$, l is bounded linear functional on $L^{1,2}$ and there is $u \in L^{1,2}$ such that $[u,v] = -l(v)$.

$u' = 0$ a.e. then u is a singular function and therefore $L^{1,2}$ is not right space.

To justify partial integration we need to suppose that v is $AC[-1,1]$. A subspace of $L^{1,2}$ consisting of AC functions we denote by $W^{1,2}$.

If we suppose that functions are AC then we have Sobolev space $W^{1,2}$.

If u is AC and $u' = 0$ then $u = c$.

Solve $u'' = \delta_0$ on $I = [-1,1]$. If v is $AC[-1,1]$ and $v(-1) = v(1) = 0$, then $\delta_0(v) = v(0)$ and therefore $|\delta_0(v)| \leq \int_{-1}^{1} |v'(x)|$.

Hence δ_0 is a is bounded linear functional on $W^{1,2}$.

Example 5. *If f is of bounded variation on $[a,b]$ and $\mu = \mu_f$ measure defined by $\mu([x,y]) = f(y) - f(x)$, whether $\langle f' \rangle = \mu_f$?*

Hint. If $\varphi \in C_0^1([a,b])$, by partial integration $\int_a^b f\varphi'dx = -\int_a^b \varphi df$.

Example 6. *As an illustration of the concept, consider the first-order wave equation*

$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad (1)$

(see partial derivative for the notation) where $u = u(t,x)$ is a function of two real variables.

The function $u_0(x,t) = |x-t|$ has partial derivatives in classical sense on $G = \mathbb{R}^2 \setminus D$, where $D = \{(x,t) : x = t\}$ and satisfies this equation on D. We will see that u_0 is so called a weak solution of equation (1).

Assume that u is continuously differentiable on the Euclidean space $\mathbb{R}^2$, multiply this equation (1) by a smooth function φ of compact support, and integrate. One obtains

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial u(t,x)}{\partial t}\varphi(t,x)\,\mathrm{d}t\mathrm{d}x+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial u(t,x)}{\partial x}\varphi(t,x)\,\mathrm{d}t\mathrm{d}x=0.$$

Using Fubini's theorem which allows one to interchange the order of integration, as well as integration by parts (in t for the first term and in x for the second term) this equation becomes

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(t,x)\frac{\partial\varphi(t,x)}{\partial t}\,\mathrm{d}t\mathrm{d}x-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(t,x)\frac{\partial\varphi(t,x)}{\partial x}\,\mathrm{d}t\mathrm{d}x=0. \qquad (2)$$

(Notice that while the integrals go from $-\infty$ to ∞, the integrals are essentially over a finite box because φ has compact support, and it is this observation which also allows for integration by parts without the introduction of boundary terms.)

We have shown that equation (1) implies equation (2) as long as u is continuously differentiable. The key to the concept of weak solution is that there exist functions u which satisfy equation (2) for any φ, and such u may not be differentiable and thus, they do not satisfy equation (1). A simple example of such function is $u(t,x)=|t-x|$ for all t and x. (That u defined in this way satisfies equation (2) is easy enough to check, one needs to integrate separately on the regions above and below the line $x=t$ and use integration by parts.) A solution u of equation (2) is called a weak solution of equation (1).

Distributions are a class of linear functionals that map a set of test functions (conventional and well-behaved functions) into the set of real numbers. In the simplest case, the set of test functions considered is $\mathcal{D}(\mathbb{R})$, which is the set of functions $f:\mathbb{R}\to\mathbb{R}$ having two properties:

f is smooth (infinitely differentiable); f has compact support (is identically zero outside some bounded interval).

A distribution T is a linear mapping $T:\mathcal{D}(\mathbb{R})\to\mathbb{R}$. Instead of writing $T(f)$, it is conventional to write $\langle T,\varphi\rangle$ for the value of T acting on a test function f. A simple example of a distribution is the Dirac delta δ, defined by

$\langle\delta,\varphi\rangle=\varphi(0)$,

meaning that δ evaluates a test function at 0. Its physical interpretation is as the density of a point source.

As described next, there are straightforward mappings from both locally integrable functions and Radon measures to corresponding distributions, but not all distributions can be formed in this manner.

2.2.2 Differentiation of distributions

Differentiation

Suppose $A:D(U)\to D(U)$ is the partial derivative operator

$A\varphi=\frac{\partial\varphi}{\partial x_k}$.

If f and φ are in $D(U)$, then an integration by parts gives

$\int_U\frac{\partial\varphi}{\partial x_k}\psi\,dx=-\int_U\varphi\frac{\partial\psi}{\partial x_k}\,dx$, so that $A^t = -A$. This operator is a continuous linear transformation on D(U). So, if $T \in D'(U)$ is a distribution, then the partial derivative of T with respect to the coordinate x_k is defined by the formula

$\left\langle \frac{\partial T}{\partial x_k}, \varphi \right\rangle = -\left\langle T, \frac{\partial \varphi}{\partial x_k} \right\rangle$ for all $\varphi \in D(U)$.

With this definition, every distribution is infinitely differentiable, and the derivative in the direction x_k is a linear operator on $D'(U)$.

More generally, if $\alpha = (\alpha_1, ..., \alpha_n)$ is an arbitrary multi-index and ∂^α is the associated partial derivative operator, then the partial derivative $\partial^\alpha T$ of the distribution $T \in D'(U)$ is defined by

$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle$ for all$\varphi \in$ D(U).

Differentiation of distributions is a continuous operator on $D'(U)$; this is an important and desirable property that is not shared by most other notions of differentiation.

For a locally integrable function f defined on a domain G in $\mathbb{R}^n$,we define $I_f(\varphi) = \int_G f(x)\varphi(x)dx$, $\varphi \in \mathcal{D}(G)$. It is also convenient to use notation $\langle f \rangle$ (or simply f if it is not confusing) instead of I_f. For example, if $G = \mathbb{R}$, we define $I_f(\varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x)dx$. For a constant c, define $I_c(\varphi) = c\int_{-\infty}^{\infty} \varphi(x)dx$. It is also convenient to use notation $\langle c \rangle$ (or simply c if it is not confusing) instead of I_c. We call distribution I_c constant distribution.

?? $f \in \mathcal{D}'$ $\langle D^\alpha f \rangle$ distributional derivative

If for some function f defined on a domain G in $\mathbb{R}^n$, $D^\alpha f$ is locally integrable on G, for classical derivative at x we use notation $\{D^\alpha f\}(x)$ and for distribution defined by this function notation $\{D^\alpha f\}$.

Then $\{D^\alpha f\}$

is different from $\langle D^\alpha f \rangle$ in general. Example H', where H is the Heaviside function.

Bois-Raymond theorem. A distribution induced by a locally integrable function f is 0 at domain G iff f is is 0 a.e, G.

Example 7. *For $\varepsilon > 0$ define, $\omega_\varepsilon(x) = c_\varepsilon \exp(-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2})$, $|x| < \varepsilon$, and 0 for $|x| \geq \varepsilon$.*

Check $\omega_\varepsilon \in \mathcal{D}$ and $\omega_\varepsilon(x) = \varepsilon^{-n}\omega_1(x/\varepsilon)$.

Proposition 1. *$\omega_\varepsilon \to \delta$ as $\varepsilon \to +0$ in distribution sense.*

$I(\varepsilon, \varphi) = \int_{R^n} \omega_\varepsilon(x)\varphi(x)dx - \varphi(0) = \int_{R^n} \omega_\varepsilon(x)(\varphi(x) - \varphi(0))dx = \int_{K_\varepsilon} \omega_\varepsilon(x)(\varphi(x) - \varphi(0))dx$. Hence

$|I(\varepsilon, \varphi)| \leq M(\varepsilon)\int_{K_\varepsilon} \omega_\varepsilon(x)dx = M(\varepsilon)$, where $M(\varepsilon) = \max_{x \in K_\varepsilon} |\varphi(x) - \varphi(0)|$. Since $M(\varepsilon) \to 0$ as $\varepsilon \to +0$,

Sohocki formula.

Distribution $P_0 = P\frac{1}{x}$ or principal vale of function $\frac{1}{x}$ is defined by

$I(\varphi) = \langle P\frac{1}{x}, \varphi \rangle = vp\int_{-\infty}^{\infty} \frac{\varphi(x)}{x}dx$, $\varphi \in D(\mathbb{R})$.

Prove that $xP_0 = 1$

Set $I(\varepsilon) = I(\varphi, \varepsilon) = \int_{-\infty}^{\infty} \frac{\varphi(x)}{x + i\varepsilon}dx$.

Distribution $P_+ = \frac{1}{x + i0}$ is defined as $\lim_{\varepsilon \to +0} I(\varphi, \varepsilon)$.

Prove that $P_+ = -i\pi\delta + P_0$.

Then

$R(\varepsilon) = I(\varepsilon) - I = \int_{-a}^{a} f_\varepsilon(x)dx$, where $f_\varepsilon(x) = -\frac{i\varepsilon\varphi(x)}{x(x+i\varepsilon)}$ and $supp\varphi \subset [-a, a]$.

If $\varphi(0) = 0$, then $|\varphi(x)| \leq M_0|x|$ and therefore $|f_\varepsilon(x)| \leq M_0$, $x \in \mathbb{R}$.

Since $|f_\varepsilon(x)| \leq M_0$, by Lebesgue dominated convergence thm $R(\varepsilon)$ tends to 0 if ε tends to 0. If $\psi(x) = \varphi(x) - \varphi(0)$, then $I(\varphi, \varepsilon) = I(\psi, \varepsilon) + \varphi(0)J(\varepsilon)$, where $J(\varepsilon) = \int_{-a}^{a} \frac{dx}{x+i\varepsilon}$. Since $J(\varepsilon)$ tends to $-\pi i$ if ε tends to 0, we find $vpI(\varphi) = I(\psi) - i\pi\varphi(0)$. Using $I(\psi) = vp.I(\varphi)$, we get

$P_+ = -i\pi\delta + P_0$.

Hint. $J(\varepsilon) = \int_{-a}^{a} \frac{dx}{x+i\varepsilon} = \ln(x + i\varepsilon)|_{-a}^{a} = arg(a + i\varepsilon) - arg(a - i\varepsilon)$, where arg is branch of Arg defined by $0 < arg < \pi$.

Proposition 2. *Denote by $\{D^\alpha f\}$ ordinary (classical) derivative, and set $[f]_{x_0} = f(x_0 + 0) - f(x_0 - 0)$.*

Suppose that $f \in C^1(-\infty, x_0] \cap C^1[x_0, \infty)$ and that f has discontinuity of first order at x_0, then

$f'(x) = \{f'(x)\} + [f]_{x_0}\delta_{x_0}$, *that is*

$(f', \varphi) = -(f, \varphi') = [f]_{x_0}\varphi(x_0) + L_{\{f'\}}(\varphi)$.

Rešenje. $(f', \varphi) = -(f, \varphi') = I_1 + I_2$, where

$I_1 = -\int_{-\infty}^{x_0} f(x)\varphi'(x)dx$ and $I_2 = -\int_{x_0}^{\infty} f(x)\varphi'(x)dx$.

By partial integration, $I_1 = -f(x)\varphi(x)|_{-\infty}^{x_0} + \int_{-\infty}^{x_0}\{f'(x)\}\varphi(x)dx$ and

$I_2 = -f(x)\varphi(x)|_{x_0}^{\infty} + \int_{x_0}^{\infty}\{f'(x)\}\varphi(x)dx$

Hence $(f', \varphi) = -f(x)\varphi(x)|_{-\infty}^{x_0} + \int_{-\infty}^{x_0}\{f'(x)\}\varphi(x)dx - f(x)\varphi(x)|_{x_0}^{\infty} + \int_{x_0}^{\infty}\{f'(x)\}\varphi(x)dx = [f]_{x_0}\varphi(x_0) + I_{\{f'\}}(\varphi)$. △

Sometimes, in the literature by $\{D^\alpha f\}$ is denoted generalized derivative, see [11]; by this notation $\{f'(x)\} = f'(x) + [f]_{x_0}\delta_{x_0}$.

If $f \in AC([a, b])$, then the classical derivative $f'(x)$ exist a.e. and $\{f'\} = I_{f'}$ ($\{f'(x)\} = f'(x) = I_{f'}$).

Proposition 3. *More generally, if $f \in C^1(-\infty, x_0) \cap C^1(x_0, \infty)$ and if f satisfy regularity condition with limit A at x_0:*

$f(x)(x - x_0)$ tends 0 if x tends to x_0, $f(x_0 + \varepsilon) - f(x_0 - \varepsilon)$ tends to A if ε tends to 0

and there is $P\{f'(x)\}|_{x_0}$, then

$f'(x) = P\{f'(x)\} + A\delta_{x_0}$

if f has discontinuity of first order at isolated points a_j then

$f' = \{f'\} + \sum[f]_{a_j}\delta a_j$

$f'(x) = \{f'(x)\} + \sum[f]_{a_j}\delta_{a_j}$

?? Suppose that $A = \{a_j\}$ is discrete and that $f \in C^1(\mathbb{R} \setminus A)$. Suppose further that f satisfies regularity condition at every point a_j with limit A_j. If $P\{f'\}|_A$ exists, then $f'(x) = \sum A_j\delta_{a_j} + P\{f'\}|_A$.

Example 8. 1. *If $f(x) = |x|$, then $f'(x) = \{f'(x)\} = sgnx$, $x \neq 0$, in $\mathcal{D}'$.*

Solution of equation $u' = sgn$ in $\mathcal{D}'$ is $u = |x| + I_c$

2. $(\ln|x|)' = P\frac{1}{x}$

3. $f_a(x) = f(a,x) = \frac{\sin(ax)}{x}$ *limit in distribution sense if* $a \to 0$ *and* $a \to \infty$
   Set $v = (\varphi(x) - \varphi(0))/x$. *Since* $v \in L^1$, $l(\varphi) = (f_a, \varphi) = (\sin ax, v) + \varphi(0)\pi/2$ *then by RiemannLebesgue lemma* $l(\varphi)$ *tends to* $\delta_0\pi/2$.

4. *Solve* $xu' = 1$; $u = c_1 + c_2H(x) + \ln|x|$

5. $f_0(x) = \frac{1}{2} - \frac{x}{2\pi}$, $x \in [0, 2\pi)$, 2π*-periodic*

6. *Generlized solutions of* $x^m u = 0$, *are given by* $u = \sum_{k=0}^{m-1} c_k\delta^{(k)}$.

$$\langle f_0' \rangle = -\frac{1}{2\pi} + \sum_{k=-\infty}^{\infty} \delta_{2\pi k}. \tag{1}$$

7. *Check that*

$$\sum_{n=-\infty}^{\infty} e^{inx} = 2\pi \sum_{k=-\infty}^{\infty} \delta(x + 2\pi k) = 2\pi \sum_{k=-\infty}^{\infty} \delta_{2\pi k} \tag{2}$$

   as distributions.
   Hint. $F(x) = \int_0^x f_0(t)dt = \frac{x}{2} - \frac{x^2}{4\pi}$, $x \in [0, 2\pi)$, *where* f_0 *is given in the item 5;*

$$F(x) = \frac{\pi}{6} - \frac{1}{2\pi} \sum_{k=-\infty, k\neq 0}^{\infty} \frac{1}{k^2} e^{ikx}. \tag{3}$$

   Differentiating (3) two times and using (1), we find (2).

8. $\triangle \ln|z| = 2\pi\delta_z$

9. $\triangle|x|^{2-n} = (2-n)\sigma_n\delta_x$

10. *Show that the function*

$$E(x,t) = \frac{1}{2a}H(at - |x|),$$

   where $a > 0$, H *is the Heaviside step function, satisfies the equation*

$$\Box_a E = \delta,$$

   where $\Box_a := \frac{\partial}{\partial t^2} - a^2\frac{\partial}{\partial x^2}$ *and* $\delta(x,t)$ *is* δ*-function in* $\mathcal{D}'(\mathbb{R}^2)$.

2.3 The Heaviside step function

The Heaviside step function, or the unit step function, usually denoted by H, is a discontinuous function whose value is zero for negative argument and one for positive argument. It is an example of the general class of step functions, all of which can be represented as linear combinations of translations of this one.

The function was originally developed in operational calculus for the solution of differential equations, where it represents a signal that switches on at a specified time and stays switched on indefinitely. Oliver Heaviside, who developed the operational calculus as a tool in the analysis of telegraphic communications, represented the function as 1. The distributional derivative of the Heaviside step function is the Dirac delta function: $\frac{dH(x)}{dx} = \delta(x)$. Since H is usually used in integration, and the value of a function at a single point does not affect its integral, it rarely matters what particular value is chosen of $H(0)$. Indeed when H is considered as a distribution or an element of L^∞ (see L_p space) it does not even make sense to talk of a value at zero, since such objects are only defined almost everywhere. If using some analytic approximation (as in the examples above) then often whatever happens to be the relevant limit at zero is used.

There exist various reasons for choosing a particular value.

$H(0) =$ is often used since the graph then has rotational symmetry; put another way, H_- is then an odd function. In this case the following relation with the sign function holds for all x:

$H(x) = \frac{1}{2}(1 + \mathrm{sgn}(x))$.

$H(0) = 1$ is used when H needs to be right-continuous. For instance cumulative distribution functions are usually taken to be right continuous, as are functions integrated against in LebesgueStieltjes integration. In this case H is the indicator function of a closed semi-infinite interval:

$H(x) = \mathbf{1}_{[0,\infty)}(x)$. The corresponding probability distribution is the degenerate distribution.

$H(0) = 0$ is used when H needs to be left-continuous. In this case H is an indicator function of an open semi-infinite interval:

$H(x) = \mathbf{1}_{(0,\infty)}(x)$.

In functional-analysis contexts from optimization and game theory, it is often useful to define the Heaviside function as a set-valued function to preserve the continuity of the limiting functions and ensure the existence of certain solutions. In these cases, the Heaviside function returns a whole interval of possible solutions, $H(0) = [0, 1]$.

2.4 Primitive distribution

If $\varphi \in \mathcal{D}(\mathbb{R})$, then $\int_{-\infty}^{x} \varphi(t)dt$ is not distribution in general. There is a distrubution ψ such that (i) $\varphi = \psi' + A\omega_\varepsilon$.

If (i) holds then $\psi' = \varphi - A\omega_\varepsilon$. Hence $I_1(\psi') = I_1(\varphi) - A$. For $A = I_1(\varphi)$, we have $I_1(\psi') = 0$.

Set $\mathcal{D}_1(\mathbb{R}) = \{\varphi' : \varphi \in \mathcal{D}(\mathbb{R})\}$. $f^{(-1)} \in \mathcal{D}'(\mathbb{R})$ is primitive distribution of $f \in \mathcal{D}'(\mathbb{R})$ if

$\langle f^{(-1)}, \varphi' \rangle = -\langle f, \varphi \rangle$ for every $\varphi \in \mathcal{D}(\mathbb{R})$. $f^{(-1)}$ is not defined on all $\mathcal{D}(\mathbb{R})$, but we show that $f^{(-1)}$ can be extended on $\mathcal{D}(\mathbb{R})$.

$\hat{\varphi} = \varphi - A\omega_\varepsilon$, where $A = I_1(\varphi)$, and $\psi(x) = \int_{-\infty}^{x} \hat{\varphi}(t)dt$. Since $\psi' = \hat{\varphi}$, $\hat{\varphi} \in D_1(\mathbb{R})$.

If $f^{(-1)}$ exists (can be extended on $\mathcal{D}(\mathbb{R})$) we choose $c =< \langle f^{(-1)}, \omega_\varepsilon >$ for a fixed ε. Note ω_ε is not in $\mathcal{D}_1(\mathbb{R})$.

By additivity, $\langle f^{(-1)}, \varphi \rangle =< \langle f^{(-1)}, \psi' > +A < \langle f^{(-1)}, \omega_\varepsilon >= -\langle f, \psi \rangle + I_c(\varphi)$, where $c =< \langle f^{(-1)}, \omega_\varepsilon >$.

The converse holds.

In particular, $\langle 0^{(-1)}, \varphi \rangle = I_c(\varphi)$

(I) If $u' = 0$ in distribution sense then $u = I_c$.

$0 = (u', \varphi) = (u, \varphi')$. Hence $0 = (u, \hat{\varphi}) = (u, \varphi) - I_1(\varphi)(u, \omega_\varepsilon)$.

We can choose a fixed $\varepsilon > 0$ and set $c = (u, \omega_\varepsilon)$. Hence $u = I_c$.

If F_1 and F_2 are primitive distributions of f, then $F_1 = F_2 + I_c$. Usually in the literature it is written c instead of I_c.

General solution of equation $u' = f$ in distribution sense is $u = f^{(-1)} + I_c$. If f is continuous, we define $F_0(x) = \int_0^x f(t)dt$. Here F_0 is primitive function and $u = F_0 + I_c$.

In this setting, is it correct to say that that every solution is reduced to classical solution.

Since $H' = \delta_0$, $\delta_0^{(-1)} = H + I_c = H + c$.

Fourier analysis

Superposition of sinusoidal wave basis functions to form a sawtooth wave Spherical harmonics, an orthonormal basis for the Hilbert space of square-integrable functions on the sphere, shown graphed along the radial direction

One of the basic goals of Fourier analysis is to decompose a function into a (possibly infinite) linear combination of given basis functions: the associated Fourier series. The classical Fourier series associated to a function f defined on the interval $[0, 1]$ is a series of the form

$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\theta}$

where

$a_n = \int_0^1 f(\theta) e^{-2\pi i n\theta}\, d\theta$.

The example of adding up the first few terms in a Fourier series for a sawtooth function is shown in the figure. The basis functions are sine waves with wavelengths λ/n (n=integer) shorter than the wavelength λ of the sawtooth itself (except for n=1, the fundamental wave). All basis functions have nodes at the nodes of the sawtooth, but all but the fundamental have additional nodes. The oscillation of the summed terms about the sawtooth is called the Gibbs phenomenon.

A significant problem in classical Fourier series asks in what sense the Fourier series converges, if at all, to the function f. Hilbert space methods provide one possible answer to this question. The functions $e_n(\theta) = e^{2\pi i n\theta}$ form an orthogonal basis of the Hilbert space $L^2([0, 1])$. Consequently, any square-integrable function can be expressed as a series

$f(\theta) = \sum_n a_n e_n(\theta), \quad a_n = \langle f, e_n \rangle$

and, moreover, this series converges in the Hilbert space sense (that is, in the L^2 mean).

The problem can also be studied from the abstract point of view: every Hilbert space has an orthonormal basis, and every element of the Hilbert space can be written in a unique way as a sum of multiples of these basis elements. The coefficients appearing on these basis elements are sometimes known abstractly as the Fourier coefficients of the element of the space. The abstraction is especially useful when it is more natural to use different basis functions for a space such as $L^2([0,1])$. In many circumstances, it is desirable not to decompose a function into trigonometric functions, but rather into orthogonal polynomials or wavelets for instance, and in higher dimensions into spherical harmonics. Fourier series on a square

We can also define the Fourier series for functions of two variables x and y in the square $[-\pi,\pi]\times[-\pi,\pi]$:

$$f(x,y) = \sum_{j,k\in\mathbf{Z}\text{ (integers)}} c_{j,k} e^{ijx} e^{iky}, c_{j,k} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y) e^{-ijx} e^{-iky}\,dx\,dy.$$

Aside from being useful for solving partial differential equations such as the heat equation, one notable application of Fourier series on the square is in image compression. In particular, the jpeg image compression standard uses the two-dimensional discrete cosine transform, which is a Fourier transform using the cosine basis functions.

Example 9. *1. For $u \in C^2(G)$ and $v \in C_0^1(G)$,*
$\int_G \Delta uvdx = -\int_G Du\cdot Dvdx$.

2. *Check that $div(fA) = \sum D_k(fA_k) = f\sum D_k(A_k) + \sum D_k f A_k = f divA + \nabla f\cdot A$, where $A = (A_1,\cdots,A_n)$ vector field.*

3. *Set $E(x) = E_2(x) = \frac{\ln|x|}{2\pi}$, $x\in\mathbb{R}^2\setminus\{0\}$; $A_3(x) = \frac{x}{|x|^3}$ and $E(x) = E_3(x) = -\frac{1}{4\pi|x|}$, $x\in\mathbb{R}^3\setminus\{0\}$; $E_n(x) = |x|^{2-n}$, $x\in\mathbb{R}^n\setminus\{0\}$, $n\geq 3$.Set $A(x) = A_n(x) = \frac{x}{|x|^n}$. We will prove, see Lemma 2.1 below, that $\Delta E_2 = \delta$ and $\Delta E_n = c_n\delta$ in distribution sense, where $c_n = -(n-2)\sigma_n$, $divA(x) = \sigma_n\delta_0$. If we introduce $\hat{E}_n = \frac{E_n}{(2-n)\sigma_n}$, then $\langle\Delta\hat{E}_n\rangle = \delta$. Example 1*

4. *The negative of the Laplacian in $\mathbb{R}^d$ given by*

$$-\Delta u = -\sum_{i=1}^{d}\partial_i^2 u$$

is a uniformly elliptic operator. The Laplace operator occurs frequently in electrostatics. If ρ is the charge density within some region Ω in $\mathbb{R}^3$, the potential Φ must satisfy the equation

$$-\Delta\Phi = 4\pi\rho.$$

If the unit charge is concentrated at 0*, then* $\Phi = \frac{1}{|x|}$.

2.5 Differentation of Distrubutions

Suppose u(x) and v(x) are two continuously differentiable functions. The product rule states (in Leibniz's notation): $(uv)' = u'v + uv'$ then applying the definition of indefinite integral,

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx$$

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

gives the formula for integration by parts. Suppose u(x) and v(x) are two continuously differentiable functions on $[a,b]$ and $v(a) = v(b) = 0$. Then $\int_a^b u'(x)v(x)\,dx = -\int_a^b u(x)v'(x)\,dx$

$x = (x_1, x')$, $x' = (x_2, x_3, ..., x_n)$ If $\varphi \in \mathcal{D}$ and $supp\varphi \subset I_n$, $I_n = [-a, a]^n$. If $f \in C^1(I_n)$, then $\int_{I_n} D_1 f\varphi dx = \int_{I_{n-1}} dx' \int_{-a}^{a} D_1 f(x_1, x')\varphi(x_1, x')dx_1$

Since $\int_{-a}^{a} D_1 f(x_1, x')\varphi(x_1, x')dx_1 = -\int_{-a}^{a} f(x_1, x')D_1\varphi(x_1, x')dx_1$, by Fubini's theorem $\int_{I_n} D_1 f\varphi dx = -\int_{I_n} fD_1\varphi dx$

$\int_{I_n} D^\alpha f\varphi dx = (-1)^{|\alpha|}\int_{I_n} fD^\alpha\varphi dx$

Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. By $D^\alpha f$ we denote functional on $\mathcal{D}(\mathbb{R}^n$ defined by

$$< D^\alpha f, \varphi >= (-1)^{|\alpha|} < f, D^\alpha\varphi >, \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{1}$$

?? Suppose that $\alpha \in \mathbb{N}_0^n$ is a multi-index. A function $f \in L_{1,loc}(G)$ has weak derivative $D^\alpha f \in L_{1,loc}(G)$ if (1) holds.

Example 10. 1. *Check* $(\ln|x|)' = P\frac{1}{x}$.

Set $E(x) = E_2(x) = \frac{\ln|x|}{2\pi}$, $x \in \mathbb{R}^2\setminus\{0\}$; $A_3(x) = \frac{x}{|x|^3}$ *and* $E(x) = E_3(x) = -\frac{1}{4\pi|x|}$, $E_n(x) = |x|^{2-n}$, $x \in \mathbb{R}^3 \setminus \{0\}$.

Check that $\Delta E_2 = \delta$ *and* $\Delta E_n = c_n\delta$ *in distribution sense, where* $c_n = -(n-2)\sigma_n$.

2. *Recall*

$$f' = \{f'\} + (\rfloor f) \cdot \delta, \tag{45}$$

where $\rfloor f$ *is jump of* f.

3. *the Dirac delta function* δ *on surface.*

For a smooth hyper surface S*, we define*

$$(\delta_S, \varphi) := \int_S \varphi d\sigma.$$

δ_S *is not regular distribution.*

Suppose that S smooth hyper surface, f is C^∞ on $R^n \setminus S$ and for every $x \in S$ there is limit from arbitrary side (locally) for all partial derivatives. Difference between these limits we call jump we denote by $\rfloor \partial_i f$. We associate to $\partial_i f$ distribution $\{\partial_i f\}$

$$< \{\partial_i f\}, \varphi >:= - < f, \partial_i \varphi >= - \int_{R^n} (f \cdot \partial_i \varphi) dx.$$

In sense of differentiation of distribution

$$\partial_i f = \{\partial_i f\} + (\rfloor f)_S \cos \alpha_i \, \delta_S. \tag{46}$$

Consider the case $i = 1$. For fixed $(x^2, \cdots x^n)$, by (45), we find
$J(x^2, \cdots x^n) = - \int_{-\infty}^{\infty} f \, \partial_i \varphi dx^1 = [f]_S \varphi + \int_{-\infty}^{\infty} \partial_i f \, \varphi dx^1$
Since $dx^2, \cdots dx^n = \cos \alpha_i d\sigma$,

$$I_1 = \int_{x^2} \cdots \int_{x^n} [f]_S \, \varphi \, dx^2, \cdots dx^n = \tag{47}$$

$$\int_S [f]_S \, \varphi \, \cos \alpha_1 \, d\sigma = ([f]_S \cos \alpha_1 \delta_S)(\varphi) \tag{48}$$

Hence

$$(\partial_i f, \varphi) := -(f, \partial_i \varphi) = \int_{x^2} \cdots \int_{x^n} J(x^2, \cdots x^n) dx^2, \cdots dx^n = \tag{49}$$

$$I_1 + \int_{R^n} \partial_i f \, \varphi dx. \tag{50}$$

4. *Let D be a bounded domain in $\mathbb{R}^m$ with smooth boundary S. Suppose that $A = (A^1, \cdots A^n)$ vector field continuous in $\overline{D}$, and $divA$ integrable in D. Set that vector field A is 0 out of $\overline{D}$. Applying (46) to each componet A^i of the field A and summing, we find*

$$divA = \{divA\} + (A \cdot n) \, \delta_S \tag{51}$$

If $\psi \in C_0^\infty$ equals 1 on $\overline{D}$, then

$$0 = (divA, \psi) = \{divA\} + \big((A \cdot n) \, \delta_S, \psi\big)$$

and in classical notation it yields Gauss-Ostrogradski formula

$$0 = \int_D divA - \int_S (A \cdot n) d\sigma.$$

5. *$E(x) = \frac{1}{4\pi |x|}$ is fundamental solution of Laplace equation in $\mathbb{R}^3$.*

2.5.1 ACL-property & Sobolev spaces

If $f \in \mathcal{D}'(G)$ is induced by a locally integrable functions g_1 and g_2 on$G$, then by Raymond lemma g_1 is equivalent with g_2 on G.

Definicija 2. *For $k \in \mathbb{N}_0$ we denote*

$$H^k_{loc}(G) = \{f \in L_{2,loc}(G) : \langle D^\alpha f\rangle \in L_{2,loc}(G), \text{for every } \alpha : |\alpha| \le k\}\,,$$

$$H^k(G) = \{f \in L_2(G) : \langle D^\alpha f\rangle \in L_2(G), \text{for every } \alpha : |\alpha| \le k\}\,.$$

If $f \in L_{2,loc}(G)$, then $\langle D^\alpha f\rangle$ is a distribution; $f \in H^k_{loc}(G)$ (res $f \in H^k(G)$) means that for every $|\alpha| \le k$ the distribution $\langle D^\alpha f\rangle$ can be identified with a function $g \in H^k_{loc}(G)$ (res $g \in H^k(G)$).

$H^k(G)$ is a complex Hilbert space with Hermitian scalar product:

$$\langle f, g\rangle_{H^k(G)} = \sum_{|\alpha|\le k} \int_G D^\alpha f(x)\overline{D^\alpha g(x)}\, dx, \tag{52}$$

and the norm

$$|f|_{H^k(G)} = (\sum_{|\alpha|\le k} \int_G |D^\alpha f|^2 dx)^{1/2}. \tag{53}$$

For $f \in L_2(G)$, we define $f_\varepsilon = \int_G \omega_\varepsilon(x-y)f(y)dy$.

If $\tilde{f}(x) = f(x)$ for $x \in G$ and 0 for $x \in R^n \setminus G$, then $f_\varepsilon = \tilde{f} * \omega_\varepsilon$.

If $f \in H^k(G)$, then $|f - f_\varepsilon|_{H^k(G')} \to 0$ if $\varepsilon \to 0$, for every subdomain G' in G. If f is finite in G, then $|f - f_\varepsilon|_{H^k(G)} \to 0$ if $\varepsilon \to 0$.

We denote $R^{n-1}_k = \{x \in \mathbb{R}^n : x_k = 0\}$. The projection P_k, given by $P_k x = x - x_k\, e_k$, is the orthogonal projection of $\mathbb{R}^n$ onto R^{n-1}_k.

Let $I = \{x \in \mathbb{R}^n : a_k \le x_k \le b_k\}$ be a closed n-interval.

A mapping $f : I \to \mathbb{R}^m$ is said to be absolutely continuous on lines (ACL) if f is absolutely continuous on almost every line segment in I, parallel to the coordinate axes.

More precisely, if E_k is the set of all $x \in P_k I$ such that the functions $t \to u(x + te_k)$ is not absolutely continuous on $[a_k, b_k]$, then $m_{n-1}(E_k) = 0$ for $1 \le k \le n$.

If Ω is an open set in $\mathbb{R}^n$, a mapping $f : \Omega \to \mathbb{R}^m$ is ACL (absolutely continuous on lines) if $f|I$ is ACL for every closed interval $I \subset \Omega$.

In general, if $f : \Omega \to \mathbb{R}^m$ we say that $f \in W^{1,p}(\Omega)$ if f is ACL on Ω and $D_k f \in L^p(\Omega)$.

Some authors consider only continuous mapping. For example in qc theory:

?? If $f : \Omega \to R$ is continuous we say that $f \in W^{1,p}$ if f is ACL and $D_k f \in L^p$.

Note that there is unbounded function $f \in W^{1,n}(\mathbb{R}^n)$ so in general we can not consider only continuous function.

If $\Omega \subset \mathbb{R}^n$, we say that $f : \Omega \to \mathbb{R}^n$ is K-qc if f is continuous, $f \in W^{1,n}(G)$ and $|f'(x)|^n \le J_f(x)$ for a.e. $x \in G$.

Proposition 4. *Let* $Q = \{x \in \mathbb{R}^n : a_k < x_k < b_k\}$ *be an open n-interval. If* $f \in ACL(Q)$ *and* $\varphi \in \mathcal{D}(Q)$*, then*

$$\int_Q D_k f \varphi dx = \int_Q f D_k \varphi dx.$$

Absolutely Continuous on Lines (ACL) characterization of Sobolev functions

Theorem 2.1 (see for example Theorem 4.2 [8])**.** *Let* Ω *be an open set in* $\mathbb{R}^n$ *and* $1 \leq p \leq \infty$*. If a function is in* $W^{1,p}(\Omega)$*, then, possibly after modifying the function on a set of measure zero, the restriction to almost every line parallel to the coordinate directions in* $\mathbb{R}^n$ *is absolutely continuous (Note in general it does not mean that there is a modifying function which is continuous function); what's more, the classical derivative along the lines that are parallel to the coordinate directions are in* $L^p(\Omega)$*. Conversely, if the restriction of* f *to almost every line parallel to the coordinate directions is absolutely continuous, then the pointwise gradient* ∇f *exists almost everywhere, and* f *is in* $W^{1,p}(\Omega)$ *provided* f *and* $|\nabla f|$ *are both in* $L^p(\Omega)$*. In particular, in this case the weak partial derivatives of f and pointwise partial derivatives of f agree almost everywhere. The ACL characterization of the Sobolev spaces was established by Otto M. Nikodym (1933); see (Maz'ya 1985, 1.1.3).*

Note that in general for a function in $W^{1,p}(\Omega)$ there is no a modifying function which is continuous function on Ω.

Theorem 2.2. *Let* $u \in W^{1,1}(G)$*, where* $G \subset \mathbb{R}^n$ *is an open set having the form* $G = \{x = (x_1, x') : x' := (x_2, ..., x_n) \in G', \alpha(x') < x_1 < \beta(x')\}$*. Then there exists a function* $\tilde{u}$ *with* $\tilde{u} = u$ *a.e.,such that the following holds: the functions* $U^{x'}$ *defined by* $U^{x'}(x_1) = \tilde{u}(x_1, x')$*, are AC on* $I(x') := [\alpha(x'), \beta(x')]$ *for a.e.* $x' := (x_2, ..., x_n) \in G'$ *(w.r.t. the (n-1)-dimensional Lebesgue measure). Its derivative coincides a.e. with the weak derivative* $D_1 u$*.*

Rešenje. There is $u_k \in C^\infty(G)$, $|u_k - u|_{W^{1,1}(G)} < 2^{-k}$, $u_k(x) \to u(x)$ and $D_1 u_k(x) \to D_1 u(x)$ a.e. G

$f(x) = |u_1(x)| + \sum_{k=1}^\infty |u_{k+1}(x) - u_k(x)|$,

$g(x) = |D_1 u_1(x)| + \sum_{k=1}^\infty |D_1 u_{k+1}(x) - D_1 u_k(x)|$

Choose a point $y_1 \in (\alpha(x'), \beta(x'))$ where the pointwise convergence For every $\alpha(x') < x_1 < \beta(x')$, since u_k is smooth we

$$u_k(x_1, x') = u_k(y_1, x') + \int_{y_1}^{x_1} D_1 u_k(t, x') dt \tag{54}$$

hence $f, g \in L_1(G)$ and the series in (4.3) are absolutely convergent for a.e. $x \in G$. Therefore, they converge pointwise almost everywhere. Moreover, we have the bounds $|u_k(x)| \leq f(x)$, $|D_1 u_k(x)| \leq g(x)$ or all $k \geq 1$, $x \in G$. By the dominated convergence theorem, the right hand of XX thus converges to

$$\tilde{u}(x_1, x') := u(y_1, x') + \int_{y_1}^{x_1} D_1 u(t, x') dt \tag{55}$$

Clearly, the right hand side of (4.7) is an absolutely continuous function of the scalar variable x_1. On the other hand, the left hand side satisfies

$\tilde{u}(x_1, x') := \lim_{k\to\infty} u_k(x_1, x') = u(x_1, x')$ for a.e. $x_1 \in I(x')$

This achieves the proof. △

A stronger result holds in the case $p > n$. A function in $W^{1,p}(\Omega)$ is, after modifying on a set of measure zero, Hölder continuous of exponent $\gamma = 1 - n/p$, by Morrey's inequality. In particular, if $p = +\infty$, then the function is Lipschitz continuous.

Let Ω be an open set in $\mathbb{R}^n$. The Sobolev space$W^{1,2}(\Omega)$ is also denoted by $H^1(\Omega)$. It is a Hilbert space, with an important subspace $H_0^1(\Omega)$ defined to be the closure in $H^1(\Omega)$ of the infinitely differentiable functions compactly supported in Ω. The Sobolev norm defined above reduces here to

$$\|f\|_{H^1} = \left(\int_\Omega \left(|f|^2 + |\nabla f|^2 \right) \right)^{\frac{1}{2}}.$$

When Ω is bounded, the Poincar inequality states that there is a constant $C = C(\Omega)$ such that

$$\int_\Omega |f|^2 \leq C^2 \int_\Omega |\nabla f|^2, \quad f \in H_0^1(\Omega).$$

When Ω is bounded, the injection from $H_0^1(\Omega)$ to $L^2(\Omega)$ is compact. This fact plays a role in the study of the Dirichlet problem, and in the fact that there exists an orthonormal basis of $L^2(\Omega)$ consisting of eigenvectors of the Laplace operator (with Dirichlet boundary condition).

When Ω has a regular boundary, $H_0^1(\Omega)$ can be described as the space of functions in $H^1(\Omega)$ that vanish at the boundary, in the sense of traces (see below). When $n = 1$, if $\Omega = (a, b)$ is a bounded interval, then $H_0^1(a, b)$ consists of continuous functions on $[a, b]$ of the form

$$f(x) = \int_a^x f'(t)\,\mathrm{d}t, \qquad x \in [a, b]$$

where the generalized derivative f' is in $L^2(a, b)$ and has 0 integral, so that $f(b) = f(a) = 0$. If for a function f exists f''. Check $f'' = (f')'$ in distribution sense.

Example 11 (Example 20, Tao)**.** *The function* $|\sin x|$ *lies in* $W^{1,\infty}(\mathbf{R})$*, but is not everywhere differentiable in the classical sense; nevertheless, it has a bounded weak derivative of* $\cos x \, sgn(\sin(x))$*. On the other hand, the Cantor function (aka the Devils staircase) is not in* $W^{1,\infty}(\mathbf{R})$*, despite having a classical derivative of zero at almost every point; the weak derivative is a Cantor measure, which does not lie in any* L^p *space. Thus one really does need to work with weak derivatives rather than classical derivatives to define Sobolev spaces properly (in contrast to the* $C^{k,\alpha}$ *spaces).*

Example 12. *If* $f \in D'$*, then* $f'' = (f')'$ *in distribution sense.*

If $f \in D'$, then $(f'', \varphi) = (f, \varphi'')$, and $((f')', \varphi) = -((f'), \varphi') = (f, \varphi'')$

Proposition 5. *If f is continuous on $I = (a, b)$ and ?? $f \in W^{1,2}(I)$, then f' is AC and $f \in C^1(I)$.*

Proposition 6. *(i) If f is continuous on R and $f \in W^{1,1}(R)$, then $f(x)$ tends 0 if x tends ∞.*
(ii) If f is continuous on R, $g = f^2$ and $f \in W^{1,2}(R)$, then $g \in W^{1,1}(R)$. In particular, $f(x)$ tends 0 if x tends ∞.
If f is continuous on $I = (a, b)$ and $f \in W^{1,1}(I)$, then f has continuous extension on $[a, b]$.

Set $A = \int_0^\infty f'(t)dt$. Since $f(x) - f(0) = \int_0^x f'(t)dt$, $f(x)$ tends $f(0) + A$ if x tends ∞.

Proposition 7. *Suppose that (a, b) is bounded interval and M a bounded set in $H^1(a, b)$. Then M is compact in $C[a, b]$ and in particular in $L_2(a, b)$.*

Rešenje. Let M be a bounded set of functions in $H^1(a, b)$, ie. there is a constant c such that $|f|_{H^1} \le c$.

If $x_0, x \in (a, b)$, then $f(x) - f(x_0) = \int_{x_0}^x f'(t)dt$, and therefore $|f(x) - f(x_0)| \le (|x - x_0|)^{1/2}|f'|_2 \le c(|x - x_0|)^{1/2}$.
If (a, b) is bounded and m_0 minimum $|f|$, then $m_0 \le c(|b - a|)^{1/2}$ and $|f(x)| \le m_0 + c(|b - a|)^{1/2} \le 2c(|b - a|)^{1/2}$.

Let $a_1 < a < b < b_1$. Then we can extend every function $f \in H^1(a, b)$ by $\tilde{f} = f$ on (a, b), $\tilde{f}(x) = f(b-)$ for $x \in [b, b_1)$ and $\tilde{f}(x) = f(a+)$ for $x \in (a_1, a]$. The set $M_1 = \{\tilde{f} : f \in M\}$ is bounded in $H^1(a_1, b_1)$ by a constant c_1. By Arzela-Ascoli, M_1 is normal in $C(a_1, b_1)$.

What happens if the interval $[a, b]$ is replaced by an arbitrary interval $I \subset R$ (possible unbounded)?

For a C^1 domain in space (even for a rectangle in the plane) the proof is much more difficult (way ?). △

Proposition 8. *Let $R = I \times (0, c)$, $I = (a, b)$, be an open rectangle in the upper half-plane. Suppose that*
f is continuous on $\mathbb{R}$ and $f \in W^{1,1}(\mathbb{R})$,
then, for almost every $y \in (0, c)$,
$f(\cdot, y)$ belongs to $f \in W^{1,1}(I)$, and
limits $f(a+, y)$ and $f(b-, y)$ of $f(x, y)$ exist if x tends $a+$ (respectively $b-$).

f is ACL on R and in particular $\int_0^c I(y)dy < +\infty$, where $I(y) = \int_a^b |D_1 f(x, y)|dx < +\infty$ for almost every $y \in (0, c)$.

2.6 Newton potential and embedding theorems

.

Recall: Let Ω_n be volume of unit ball in $\mathbb{R}^n$ and σ_n of unit sphere, then $\sigma_n = n\Omega_n$ and $\Gamma(x) = \frac{1}{n(2-n)\Omega_n}|x|^{2-n}$. If D is admissible domain, S the boundary of G and $u \in C^1(\overline{D}) \cap C^2(D)$, then for $x \in D$

$$u(x) = \int_D \Gamma(x-y)\Delta u(y)\,dy + \int_S \Gamma(x-y)\frac{\partial u}{\partial n}d\sigma(y) + \int_S u(y)\frac{\partial \Gamma(x-y)}{\partial n_y}d\sigma(y). \tag{56}$$

If $divA = 0$, then $div(fA) = \nabla f \cdot A$.

Recall

Set $E(x) = E_2(x) = \ln|x|$ and $E(x) = \hat{E}_2(x) = \frac{\ln|x|}{2\pi}$, $x \in \mathbb{R}^2 \setminus \{0\}$; $A_3(x) = \frac{x}{|x|^3}$ and $E(x) = E_3(x) = -\frac{1}{|x|}$, $E(x) = \hat{E}_3(x) = -\frac{1}{4\pi|x|}$, $x \in \mathbb{R}^3 \setminus \{0\}$; $E_n(x) = |x|^{2-n}$, $x \in \mathbb{R}^n \setminus \{0\}$, $n \geq 3$.

Check that $\Delta E_2 = 2\pi\delta$ and $\Delta E_n = c_n\delta$ in distribution sense, where $c_n = -(n-2)\sigma_n$.

If we introduce $\hat{E}_n = \frac{E_n}{(2-n)\sigma_n}$, then $\langle \Delta \hat{E}_n \rangle = \delta$.

Lemma 2.1. *Set $A(x) = A_n(x) = \frac{x}{|x|^n}$. Prove that $divA(x) = \sigma_n\delta_0$.*

An immediate corollary is $\langle \Delta \hat{E}_n \rangle = \delta$.

Rešenje. Let $\varphi \in D(\mathbb{R}^n)$. For $\varepsilon > 0$, set $A(\varepsilon) = \{x : \varepsilon \leq |x| \leq \frac{1}{\varepsilon}\}$ and $I(\varepsilon) = -\int_{A(\varepsilon)} A \cdot \nabla\varphi dx$. Then $I = \langle divA(x), \varphi \rangle = -\lim_{\varepsilon\to 0}\int_{A(\varepsilon)} A \cdot \nabla\varphi dx = \lim_{\varepsilon\to 0} I(\varepsilon)$. Since $divA = 0$, we have $\int_{A(\varepsilon)} A \cdot \nabla\varphi dx = \int_{A(\varepsilon)} div(\varphi A)dx$ and by the divergence theorem

$$I(\varepsilon) = \int_{A(\varepsilon)} div(\varphi A)dx = \int_{S_\varepsilon} \varphi A \cdot n dx\,.$$

For $x \in S_\varepsilon$, $x \cdot n = -|x|$ and therefore $A \cdot n = \frac{1}{\varepsilon^{n-1}}$. Hence $I(\varepsilon) = \frac{1}{\varepsilon^{n-1}}\int_{S_\varepsilon} \varphi dx = \sigma_n\varphi(x_\varepsilon)$, where $x_\varepsilon \in S_\varepsilon$, and $I = \langle divA(x), \varphi\rangle = -\lim_{\varepsilon\to 0}\int_{A(\varepsilon)} A \cdot \nabla\varphi dx = \lim_{\varepsilon\to 0} I(\varepsilon) = \sigma_n\varphi(0)$.

△

Example 13. 1. *If $f \in C^2$ and $g \in C^2$, then*
$\Delta(f \circ g) = f"(g(x))|\nabla g(x)|^2 + f'(g(x))\Delta g(x)$.

2. *Check*
   (a) $D_k|x|^2 = 2|x|D_k|x|$.
   Since $|x|^2 = \sum_{k=1}^n x_k^2$, we find
   $D_k|x|^2 = 2x_k$, and therefore $D_k|x| = x_k/|x|$. Thus
   (b) $D_k|x|^p = p|x|^{p-2}x_k$.
   Hence (c) $\nabla|x|^p = p|x|^{p-2}x$.
   Using $D_{kk}|x|^p = D_k(D_k|x|^p) = p\big((p-2)|x|^{p-4}x_k^2 + |x|^{p-2}\big)$, we have
   (d) $\Delta|x|^p = \sum_{k=1}^n D_{kk}|x|^p = p(p+n-2)|x|^{p-2}$.
   In particular, for $p+n-2 = 0$, i.e. $p = 2-n$, $\Delta|x|^{2-n} = 0$.

3. *Set* $A(x) = x/|x|^{-n}$, *we have* $\nabla |x|^{2-n} = (2-n)A(x)$ *and therefore* $divA = 0$.

   Hence $|x|^{2-n}$ *is harmonic and* $divA = 0$

   For $n = 3$ *and* $p = -1$, $\frac{1}{|x|}$ *and* $D_k \frac{1}{|x|} = -\frac{x_k}{|x|^3}$ *are harmonic.*

   $\nabla \frac{1}{|x|} = -\frac{x}{|x|^3}$. *We will see later that this is related to potential of electrostatic and gravity field.*

4. *Set* $E_2(z) = \ln|z|$. *Then* $\langle \Delta E_2 \rangle \langle = \Delta \ln|z| \rangle = 2\pi\delta$

   Let $\varphi \in \mathcal{D}$ *and supp*$\varphi \subset U_R$.

   Set $I(\epsilon) = \int_{A(\epsilon,r)} \ln|z| \Delta\varphi dz$ *and* $J(\epsilon) = \int_{S_\epsilon^-} (\ln|z| D_n\varphi - \varphi D_n \ln|z|) ds$.

   Then by definition $A = \langle \Delta E_2, \varphi \rangle = \int_{U_R} \ln|z| \Delta\varphi dz$ *and* A *is limit of* $I(\epsilon)$ *when* $\epsilon \to 0$. *By Green formula,*

   $I(\epsilon) = \int_{A(\epsilon,r)} \ln|z| \Delta\varphi dz = J(\epsilon) = \int_{S_\epsilon} (\varphi D_n \ln|z| - \ln|z| D_n\varphi) ds$. *Then* $J(\epsilon) = I_1(\epsilon) + I_2(\epsilon)$, *where* $I_1(\epsilon) = \int_{S_\epsilon} (\varphi D_n \ln|z|) ds$ *and* $I_1(\epsilon) = -\int_{S_\epsilon} (\ln|z| D_n\varphi) ds$. *Since* $I_1(\epsilon) = \int_{S_\epsilon} (\varphi(z)\frac{1}{r}) ds = \frac{1}{\epsilon}\int_{S_\epsilon} \varphi(z)$, *where* $r = |z|$, $I_1(\epsilon) \to 2\pi\varphi(0)$ *when* $\epsilon \to 0$ *and* $I_1(\epsilon) \to 0$, *we conclude that* $J(\epsilon) \to 2\pi\varphi(0)$.

5. $\Delta(fg) = g\Delta f + 2\sum D_k f D_k g + f\Delta g$. *Set* $f = \ln|x|$ *and* $g = E_n$. *Then* $\Delta g = 0$, $\Delta f = (n-2)|x|^{-2}$, $D_k g = (2-n)x_k|x|^{-n}$, $\nabla f \cdot \nabla g = (2-n)|x|^{-n}$ *and therefore* $\Delta(fg) = g\Delta f + 2\sum D_k f D_k g = (n-2)E_n|x|^{-2} + 2(2-n)|x|^{-n} = (2-n)|x|^{-n}$.

   $\Delta(\ln|x| E_n(x) = (2-n)|x|^{-n}$.

2.7 Convolution

distribution

Set $P_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}$.

If $f : \mathbb{R} \to \mathbb{R}$ continuous, bounded and $u(x,y) = f * P_y$, show that $\Delta u = f * (\Delta P_y)$, $y > 0$.

Suppose that (i) $f, g \in L_{1,loc}(\mathbb{R}^n)$ and define
$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$.
Set $h = |f| * |g|$ and suppose that (ii) $h \in L_{1,loc}(\mathbb{R}^n)$.

We say that a sequence $\eta_j \in \mathcal{D}(\mathbb{R}^n)$ converges to 1 in $\mathbb{R}^n$ if

a) For every compact K there is index n_0 such that $\eta_j = 1$ on K for $j \geq n_0$.

b) η_j together with $D^\alpha \eta_j$ are uniformly bounded in $\mathbb{R}^n$.

If (i) and (ii) hold, check that

$$\langle f * g, \varphi \rangle = \int_{\mathbb{R}^n} f(x)g(y)\varphi(x+y)dxdy \tag{1}$$

and

$$(2) \qquad \langle f * g, \varphi\rangle = \lim_{j\to\infty} \langle f(x)g(y), \eta_j(x,y)\varphi(x+y)\rangle$$

for $\varphi \in D(\mathbb{R}^n)$, where $\eta_j(x,y) \in \mathcal{D}(\mathbb{R}^{2n})$ is an arbitrary sequence which converges to 1 in $\mathbb{R}^{2n}$.

We take the formula (2) for the definition of convolution.

For $f \in \mathcal{D}'(\mathbb{R}^n)$ and $g \in \mathcal{D}'(\mathbb{R}^m)$ we define the direct product $f \cdot g$ by $\langle f \cdot g, \varphi\rangle = \langle f(x)g(y), \varphi(x+y)\rangle = \langle f, \langle g, \varphi_x\rangle\rangle$ for $\varphi(x,y) \in \mathcal{D}(\mathbb{R}^{n+m})$, where $\varphi_x(y) = \varphi(x,y)$.

Lemma 2.2. *Set $\psi(x) = \langle g(y), \varphi(x+y)\rangle$. If $g \in D'(R^m)$ and $\varphi(x,y) \in \mathcal{D}(\mathbb{R}^{n+m})$, then $\psi \in D(R^n)$ and $D^\alpha\psi = \langle g(y), D_x^\alpha \varphi(x+y)\rangle$. In particular, $\triangle\psi(x) = \langle g(y), \triangle_x \varphi(x+y)\rangle$.*

Theorem 2.3. *For $f, g \in \mathcal{D}'(\mathbb{R}^n)$, if g has finite support, the convolution $f * g$ exist and*

$$\langle f * g, \varphi\rangle = \langle f \cdot g, \varphi\rangle = \langle f(x)g(y), \eta(y)\varphi(x+y)\rangle,$$

where η is an arbitray test function equals 1 on neighborhood (nbg) supp g.

Set $\psi(x) = \langle f(y), \eta(y)\varphi(x+y)\rangle$, where $\eta \in \mathcal{D}(\mathbb{R}^n)$ equals 1 on nbg *supp* f and note that

(I) $\psi(0) = \langle f(y), \eta(y)\varphi(y)\rangle = \langle f, \varphi\rangle$.

Set $V_f = E_n * f$. Formally, $V_f(x) = \int_{R^n} E_n(x-y)f(y)dy = \int_{R^n} E_n^y(x)f(y)dy$.

$\Delta V_f(x) = \int_{R^n} \Delta E_n^y(x)f(y)dy = c_n \int_{R^n} \delta_x(y)f(y)dy = c_n f(x)$.

Theorem 2.4. *Suppose that $f \in D'(\mathbb{R}^n)$ has finite support. Then $\langle \Delta V_f, \varphi\rangle = -(n-2)\sigma_n f = \langle E_n, \Delta\psi\rangle = -(n-2)\sigma_n\psi(0) = -(n-2)\sigma_n\langle f, \varphi\rangle$*

Rešenje. Let $\varphi \in D(\mathbb{R}^n)$. By Theorem 2.3, we can express the convolution by the direct product

$I = \langle \triangle V_f, \varphi\rangle = \langle V_f, \triangle\varphi\rangle = \langle E_n * f, \triangle\varphi\rangle = \langle E_n(x)f(y), \eta(y)\triangle\varphi(x+y)\rangle$.

Further, by definition of the direct product and by Lemma 2.2, we have

$I = \big\langle E_n(x), \langle f(y), \eta(y)\triangle\varphi(x+y)\rangle\big\rangle = \langle E_n, \Delta\psi\rangle$,

where $\psi(x) = \langle f(y), \eta(y)\varphi(x+y)\rangle$. Now an application of Lemma 2.1 and (I) yields $I = I(\varphi) = -(n-2)\sigma_n\psi(0) = -(n-2)\sigma_n$. $\triangle$

For $f \in \mathcal{D}'(\mathbb{R}^n)$, $f * \delta = \delta * f = f$.

For examples see [11].

Example 14. *Let G be bounded domain in $\mathbb{R}^n$ bounded by piecewise smooth surface S . Define*

δ_S by $< \delta_S, \varphi >= \int_S \varphi dA$ and for

$\mu \in \mathcal{D}$, $< \mu\delta_S, \varphi >= \int_S \mu\varphi dA$.

Let $A = (A^1, ..., A^n)$ be vector field continuous in $\overline{G}$, such that $divA$ integrable in G. If A is zero out of $\overline{G}$

then $divA = \{divA\} - (A \cdot n)\delta_S$.

Example 15. *Here we will use physical notation and give physical interpretation of a three-dimensional delta function. In electromagnetism, charge density is a measure of electric charge per unit volume of space, in one, two or three dimensions.*

Set $A(x) = A_n(x) = \frac{x}{|x|^n}$. *Recall that (1)* $divA(x) = \sigma_n \delta_0$. *The easiest way to define a three-dimensional delta function is just to take the product of three one-dimensional functions:*

$$\delta_3(\mathbf{r}) \equiv \delta(x)\delta(y)\delta(z) \qquad (1).$$

The integral of this function over any volume V *containing the origin is again 1, and the integral of any function of* $\mathbf{r}$ *is a simple extension of the one-dimensional case:*

$$\int_G f(\mathbf{r})\delta_3(\mathbf{r}-\mathbf{a})d^3\mathbf{r} = f(\mathbf{a}) \qquad (2).$$

In electrostatics, there is one situation where the delta function is needed to explain an apparent inconsistency involving the divergence theorem. If we have a point charge q *at the origin, the electric field of that charge is*

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \qquad (3),$$

where $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$.

By (1), $\nabla \cdot \mathbf{E} = \sigma_3\delta_0 = 4\pi\delta_0$.

Suppose that smooth surface is boundary of V*.According to the divergence theorem, the surface integral of the field is equal to the volume integral of the divergence of that field:*

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{E} d^3\mathbf{r} \qquad (4),$$

where the integral on the left is over some closed surface, and that on the right is over the volume enclosed by the surface. In electrostatics, the integral on the right evaluates to the total charge contained in the volume divided by ϵ_0

$$\int_V \nabla \cdot \mathbf{E} d^3\mathbf{r} = \int_V \frac{q}{\epsilon_0}\delta_0(\mathbf{r})d^3\mathbf{r} = \frac{q}{\epsilon_0} \qquad (5).$$

We only state,[6]:

$F[\varphi](y) = (2\pi)^{-n/2}\int_{R^n}\varphi(x)e^{iy\cdot x}dx$, $F^{-1}[\varphi](y) = (2\pi)^{-n/2}\int_{R^n}\varphi(x)e^{-iy\cdot x}dx$

$D^\alpha F[\varphi](y) = (2\pi)^{-n/2}\int_{R^n}(ix)^\alpha\varphi(x)e^{iy\cdot x}dx$

Fourier transform is injective of S onto itself.

$F[f * g] = F[f]F[g]$.

2.8 The Newtonian potential

the Newtonian potential or Newton potential is an operator in vector calculus that acts as the inverse to the negative Laplacian, on functions that are smooth and decay rapidly enough at infinity. As such, it is a fundamental object of study in potential theory. In its general nature, it is a singular integral operator, defined by convolution with a function having a mathematical singularity at the origin, the Newtonian kernel Γ which is the fundamental solution of the Laplace equation. It is named for Isaac Newton, who first discovered it and proved that it was a harmonic function in the special case of three variables, where it served as the fundamental gravitational potential in Newton's law of universal gravitation. In modern potential theory, the Newtonian potential is instead thought of as an electrostatic potential.

The Newtonian potential of a compactly supported integrable function f is defined as the convolution

$$u(x) = \Gamma * f(x) = \int_{\mathbb{R}^d} \Gamma(x-y) f(y)\, dy$$

where the Newtonian kernel Γ in dimension d is defined by XX

The Newtonian potential w of f is a solution of the Poisson equation $\Delta w = f$, which is to say that the operation of taking the Newtonian potential of a function is a partial inverse to the Laplace operator. The solution is not unique, since addition of any harmonic function to w will not affect the equation. This fact can be used to prove existence and uniqueness of solutions to the Dirichlet problem for the Poisson equation in suitably regular domains, and for suitably well-behaved functions f: one first applies a Newtonian potential to obtain a solution, and then adjusts by adding a harmonic function to get the correct boundary data.

The Newtonian potential is defined more broadly as the convolution

$$\Gamma * \mu(x) = \int_{\mathbb{R}^d} \Gamma(x-y) d\mu(y)$$

when μ is a compactly supported Radon measure. It satisfies the Poisson equation

$$\Delta w = \mu$$

in the sense of distributions. Moreover, when the measure is positive, the Newtonian potential is subharmonic on R^d.

If f is a compactly supported continuous function (or, more generally, a finite measure) that is rotationally invariant, then the convolution of f with Γ satisfies for x outside the support of f

$$f * \Gamma(x) = \lambda \Gamma(x), \quad \lambda = \int_{\mathbb{R}^d} f(y)\, dy.$$

In dimension $d = 3$, this reduces to Newton's theorem that the potential energy of a small mass outside a much larger spherically symmetric mass distribution is the same as if all of the mass of the larger object were concentrated at its center.

2.9 Extension of functions

$f \in L_2(\Omega)$, $f_\varepsilon = \int_\Omega \omega_\varepsilon(x-y)f(y)dy$. Let $\tilde{f}$ be an extension of f such that $\tilde{f}(x) = 0$, $x \in R^n \setminus \Omega$. Then $f_\varepsilon = \tilde{f} * \omega_\varepsilon$.

Lemma 2.3. *If $f \in H^k(\Omega)$ and Ω' subdomain of Ω, then $|f_\varepsilon - f|_{H^k(\Omega')} \to 0$, $\varepsilon \to 0$.*

Lemma 2.4. *If Q is a paralepiped then the set $C^\infty(\overline{Q})$ (in particular $C^k(\overline{Q})$) is dense in $H^k(Q)$.*

For hypersurface S in R^n we say that of class C^m if for every $x_0 \in S$ there is ball $B = B(x_0, r)$ and function $\varphi = \varphi_{x_0}$ in $C^m(B)$ such that $\nabla\varphi(x_0) \neq 0$ and $\{\varphi(x) = 0\} = S \cap B$.

$\varphi(x, y) = x^2 - y^2$

We say that a bounded domain Ω in $\mathbb{R}^n$ and its boundary belong to class $C^{k,\alpha}$, $0 \leq \alpha \leq 1$ if for every point $x_0 \in \partial\Omega$ there exists a ball $B = B(x_0)$ and mapping ψ from B onto D such that ([2], p. 95)

$\psi(B \cap \Omega) \subset R^n_+$,

$\psi(B \cap \partial\Omega) \subset \partial R^n_+$,

$\psi \in C^{k,\alpha}(B)$, $\psi^{-1} \in C^{k,\alpha}(D)$.

We say that domain is with Lipschitz boundary if ψ and ψ^{-1} are Lipschitz functions.

the Schwarz reflection principle is a way to extend the domain of definition of an analytic function of a complex variable, F, which is defined on the upper half-plane and has well-defined and real number boundary values on the real axis. In that case, the putative extension of F to the rest of the complex plane is

$\overline{F(\bar{z})}$ or $F(\bar{z}) = \overline{F(z)}$. That is, we make the definition that agrees along the real axis.

If $f \in C[0, 1)$, define $f(x) = f(0) + f'(0)x + f'(0)x^2/2$, $x \in (-1, 0)$. See

A Vandermonde matrix, named after Alexandre-Thophile Vandermonde, is a matrix with the terms of a geometric progression in each row, i.e., an $m \times n$ matrix V: $V_{i,j} = \alpha_i^{j-1}$ for all indices i and j. (Some authors use the transpose of the above matrix.) The determinant of a square Vandermonde matrix (where $m = n$) can be expressed as: $\det(V) = \prod_{1 \leq i < j \leq n}(\alpha_j - \alpha_i)$.

See [6] $ 2.3 Lema 1.

Lemma 2.5 (Extension 1)**.** *Let $Q_a = \{x \in \mathbb{R}^n : |x_i| \leq a\}$ and $Q_a^+ = Q_a \cap \{x \in \mathbb{R}^n : x_n > 0\}$. Let $f \in C^k(\overline{Q}_a{}^+)$ (respectively $f \in H^k(\overline{Q}_a{}^+)$), then f can be extended to $F \in C^k(\overline{Q}_a)$ (respectively $F \in H^k(\overline{Q}_a)$).*

(i)If we set $f(x', x_n) = f(x', -x_n)$, we have $D_n f(x', x_n) = -D_n f(x', -x_n)$, and $D_n f(x', x_n)$ is not continuous at points $(x', 0)$. Set $g(x', x_n) = g_j(x', x_n) = f(x', -x_n/j)$, and $F := Ef = \sum_{j=1}^{k+1} A_j g_j(x', x_n)$. Then

$D_n^s g(x', x_n) = (-1/j)^s D_n^s f(x', -x_n/j)$, $D_n^s F = \sum_{j=1}^{k+1} A_j D_n^s g_j(x', x_n) = \sum A_j (-1/j)^s D_n^s f(x', -x_n/j)$

and therefore $D_n^s F(x', 0) = \sum A_j (-1/j)^s D_n^s f(x', 0) = (\sum_{j=1}^{k+1} A_j (-1/j)^s) D_n^s f(x', 0)$, $s = 0, 1, ..., k$. Hence we get the system (1) $\sum_{j=1}^{k+1} A_j (-1/j)^s = 1$, $s = 0, 1, ..., k$. Since the corresponding Vandermonde matrix is not singular, we can choose $A = (A_1, A_2, ..., A_n)$ to be the solution of system $\sum_{j=1}^{k+1} A_j (-1/j)^s = 1$, $s = 0, 1, ..., k$.

Check that

$$||F||_{H^k(Q_a)} \leq c||f||_{H^k(Q_a^+)}. \tag{57}$$

(ii) If $f \in H^k(\overline{Q}_a{}^+)$, there is a sequence $f_p \in C^k(\overline{Q}_a{}^+)$ which converges to f in $H^k(\overline{Q}_a{}^+)$. Denote by $F_p = Ef_p$ extension of f_p. By (57),

$$||F_p - F_q||_{H^k(Q_a)} \leq c||f_p - f_q||_{H^k(Q_a^+)}.$$

F_p is Cauchy sequence in $H^k(Q_a)$ and since $H^k(Q_a)$ is complete, it converges to a $F \in H^k(Q_a)$.

$F_p(x) = f_p(x)$ and $f_p(x) \to f(x)$, a.e. $x \in Q_a^+$, $p \to \infty$. Therefore F is extension of f in $H^k(Q_a)$.

The inequality (57) holds in this case.

Theorem 2.5 (Extension 2)**.** *Let Ω and Ω' be domain in R^n and $\Omega \subset \Omega'$. Then for every $f \in C^k(\overline{\Omega})$ (resp $f \in H^k(\Omega)$) there is an extension $F \in C^k(\overline{\Omega'})$ (resp $F \in H^k(\overline{\Omega'})$) finite in Ω' such that*

$|F|_{H^k(\Omega')} \leq C|f|_{H^k(\Omega)}$,

where the constant C depends only on Ω and Ω' .

In general, for $f \in H^k(G)$ there is no equivalent function which is continuous on G. There we can not use this result to define trace of f on $\partial\Omega$.

2.10 Trace

We say that $u \in C^k(\overline{\Omega})$ if all derivatives of order $\leq k$ have continuous extension to $\overline{\Omega}$.

Let $C^k(\overline{\Omega})$ denote family of functions (mappings) which belong $C^k(\Omega)$ and all derivatives of order $\leq k$ have continuous extension to $\overline{\Omega}$.

A bounded domain Ω in $\mathbb{R}^n$ and its boundary belong to class $C^{k,\alpha}$, $0 \leq \alpha \leq 1$ if for every point $x_0 \in \partial\Omega$ there exists a ball $B = B(x_0)$ and mapping ψ from B onto D such that ([2], p. 95):

(i) $\psi(B \cap \Omega) \subset R_+^n$;

(ii) $\psi(B \cap \partial\Omega) \subset \partial R_+^n$;

(iii) $\psi \in C^{k,\alpha}(B)$, $\psi^{-1} \in C^{k,\alpha}(D)$.

Chapter 3 Introduction to Sobolev Spaces

Orane Jecker, Sobolev spaces, Trace theorems and Green's functions. Boundary Element Methods for Waves Scattering Numerical Analysis Seminar. October 21, 2010, https://www2.math.ethz.ch/education/bachelor/seminars/hs2010/wave/Orane
A bounded domain Ω in $\mathbb{R}^n$ is a Lipschitz domain (or domain with Lipschitz boundary) if ψ and ψ^{-1} are both Lipschitz continuous functions. Many of the Sobolev embedding theorems require that the domain of study be a Lipschitz domain. Consequently, many partial differential equations and variational problems are defined on Lipschitz domains. Domains with Lipschitz boundary are, for example, balls or polygonal domains in two dimensions where the domain is always on one side of the boundary.

Intuitively, a Lipschitz continuous function is limited in how fast it can change: there exists a definite real number such that, for every pair of points on the graph of this function, the absolute value of the slope of the line connecting them is not greater than this real number; this bound is called a Lipschitz constant of the function (or modulus of uniform continuity). For instance, every function that has bounded first derivatives is Lipschitz. In the theory of differential equations, Lipschitz continuity is the central condition of the Picard-Lindelöf theorem which guarantees the existence and uniqueness of the solution to an initial value problem. A special type of Lipschitz continuity, called contraction, is used in the Banach fixed point theorem.

We have the following chain of inclusions for functions over a closed and bounded [xx] subset of the real line

Continuously differentiable $\subseteq$ Lipschitz continuous $\subseteq$ a-Hölder continuous $\subseteq$ uniformly continuous $\subseteq$ continuous, where $0 < a \leq 1$. We also have

Lipschitz continuous $\subseteq$ absolutely continuous $\subseteq$ bounded variation $\subseteq$ differentiable almost everywhere

Given two metric spaces (X, d_X) and (Y, d_Y), where d_X denotes the metric on the set X and d_Y is the metric on set Y, a function $f : X \to Y$ is called Lipschitz continuous if there exists a real constant $K \geq 0$ such that, for all x_1 and x_2 in X,

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

Any such K is referred to as a Lipschitz constant for the function f. The smallest constant is sometimes called the (best) Lipschitz constant; however, in most cases, the latter notion is less relevant. If $K = 1$ the function is called a short map, and if $0 \leq K < 1$ the function is called a contraction. Remark 1. In general we can say that u Dirichlet Eigenfunction Ω if $u \in C(\overline{\Omega})$, $u = 0$ on $\partial\Omega$, $\Delta u = \lambda u$ on Ω. In this setting we do not suppose that $u \in W_0^{1,2}(\Omega)$.

If $u \in W^{1,p}(\Omega)$ and u has a continuous extension on $\overline{\Omega}$ which is 0 on $\partial\Omega$, then $u \in W_0^{1,p}(\Omega)$. The converse is not true in general.

Lamberti: In order to guaranteed the validity of the converse, you need more information on the boundary because if the boundary is "thick"then you cannot have pointwise control of the function "far from the interior". The condition Ω has a Lipschitz boundary guarantees the converse. To be honest, I have never read a counterexample, but I am confident that such counterexample exists and could probably be found either in the big book of Mazya (Sobolev spaces) or in the smaller book of Mazya and Poborchi (Differentiable functions on domains).

It is interesting that that the direct implication does not require boundary regularity. Simple proofs of these facts can be found in the book by Brezis "Functional Analysis".

Question (Lamberti). It would be interesting to see if it is true that given an eigenfunction u in the broad sense (C^2 inside and continuous up to the boundary) then u is in $W^{1,2}$. I am not sure this is an easy question.

$W_0^{1,p}(\Omega) \subset C^0(\overline{\Omega})$ for $p > n$; in planar case for $p > 2$. A Lipschitz domain (or domain with Lipschitz boundary) is a domain in Euclidean space whose boundary is "sufficiently regular" in the sense that it can be thought of as locally being the graph of a Lipschitz continuous function.

Lamberti: This is true if Ω has the extension property, for example if Ω has a Lipschitz boundary. Indeed, if $p > N$ then Morrey inequality holds in $\mathbb{R}^N$, hence you have even Holder continuity. A simple discussion of this issue can be found in the popular book in PDE's by L. Evans. If you do not have the extension property, then you can only conclude that if $p > N$ then u in continuous in the interior and is globally bounded.

Remark 2. It seems to me if we suppose that the Dirichlet eigenfunction solution $w \in W_0^{1,2}(\Omega)$ and Ω is convex, then we have that gradient of w is bounded.

So the question is what we mean when we say that u is the Dirichlet eigenfunction (in particular, whether we suppose that $w \in W_0^{1,2}(\Omega)$?

Let G be a Lipschitz open subset of R^d. There exists a unique continuous linear mapping $E : H^1(G) \to L_2(\partial G)$ such that for all $u \in C^1(\overline{G})$, we have

$$E(u) = u|_{\partial G} .$$

In other words, the trace is the unique reasonable way of defining a boundary value for H^1 functions, as the continuous extension of the restriction to the boundary for functions for which this restriction makes sense.

Outline

Theorem 2.6. *Let G be C^1 domain and S C^1 hyper-surface (compact) in $\overline{G}$. Let $f \in H^1(G)$ and let f_p be arbitrary sequence from $C^1(\overline{G})$ which converges to f in $H^1(G)$. Then $f_p|_S$ converges to f_0 in $L_2(S)$.*

Rešenje. We call f_0 the trace of f on S.

Suppose that domain G is in first quadrant and first that $f \in C_0^1(G)$.

Set $x' = (x_1, \cdots, x_{n-1})$. Let S' be simple part of S given by equation $x_n = \varphi(x')$, $\varphi \in C^1(\overline{D})$, where D is a domain in hyper-plane $x_n = 0$.

For $x \in S'$, $f(x) = f(x', \varphi(x')) = \int_0^{\varphi(x')} D_n f(x', x_n) dx_n$ and by Cauchy-Bunyakovsky-Schwarz Inequality $|f(x)|^2 \leq \varphi(x') A(x') \leq cA(x')$, where $A(x') = \int_0^{\varphi(x')} |D_n f(x', x_n)|^2 dx_n$. Hence
(A1) $|f(x', \varphi(x'))|^2 \leq cA(x')$, $x' \in D$.
Set $d\sigma = d_{S'}\sigma = B(x')dx'$, where $B(x') = \sqrt{1 + |D_1\varphi(x')|^2 + \cdots + |D_{n-1}\varphi(x')|^2}$.

Since $B(x') \leq c_1$ on D, we have $I = \int_{S'} |f|^2 d\sigma = \int_D |f(x', \varphi(x'))|^2 B(x')dx' \leq c_1 \int_D |f(x', \varphi(x'))|^2 dx'$, where $d\sigma = d_{S'}\sigma$. Hence, using (A1) we find
(A2) $I \leq c_2 \int_D A(x')dx'$, where $c_2 = cc_1$.
Set $V = \{x', x_n) : x' \in D, 0 \leq x_n \leq \varphi(x')\}$. By Fubini's $I_1 = \int_V D_n f(x', x_n)|^2 dx = \int_D (\int_0^{\varphi(x')} |D_n f(x', x_n)|^2 dx_n)dx'$ and therefore
$I_1 = \int_V D_n f(x', x_n)|^2 dx = \int_D A(x')dx'$.
Hence, $I \leq c_2 I_1$ and since $I_1 \leq |f|_{H^1(G)}$ we have $\sqrt{I} \leq c_2 |f|_{H^1(G)}$.

Since S can be covered by a finite number pieces of type S', we find by summing

$$|f|_{L_2(S)} \leq C|f|_{H^1(G)}, \tag{58}$$

where the constant C does not depend on the function f.

An application of Theorem 2.5, shows that this inequality holds for $f \in C^1(\overline{G})$.

For $f \in H^1(G)$, there is a sequence of functions f_p in $C^1(\overline{G})$ which converges to f in the norm $H^1(G)$.

The sequence of traces $f_p|S$ is Cauchy, and there is a function f_S to which the sequence of traces $f_p|S$ converges. △

For $f \in H^1(\mathbb{D}, \mathbb{R}^d)$, we define $E(f) = \int_{\mathbb{D}} |df|^2$.

Lemma 2.6 (Courant-Lebesgue lemma)**.** *(i) Let $f \in H^1(\mathbb{D}, \mathbb{R}^d)$, $E(f) \leq K$, $\delta < 1$, $p \in \mathbb{D}$.*
Then there exists $r \in (\delta, \sqrt{\delta})$ for which $f|T_r$ is AC, where $T_r = S(p, r) \cap \mathbb{D}$, and

$$|f(z_1) - f(z_2)| \leq (8\pi K)^{1/2} (\ln \frac{1}{\delta})^{1/2}, z_1, z_2 \in T_r. \tag{59}$$

Example of trace. The Dirichlet Space

The Definition. The Dirichlet space $\mathfrak{D}$ is the Hilbert space of the functions f analytic in the unit disc $\mathbb{D}$ for which the semi-norm

$$|f|_{\mathfrak{D},*} = \int_{\mathbb{D}} |f'(z)| dm_2 \tag{60}$$

is finite.

Proposition 9. *The Dirichlet space sits then inside the analytic Hardy space H^2. In particular, Dirichlet functions have nontangential limits f^* at a.e. point on the boundary of $\mathbb{D}$. If $f \in \mathbb{D}$, then*

$$|f|_{\mathfrak{D},*} = \sum_0^\infty |k||f_k|^2 < \infty\,,$$

and f^* is trace of f.
Hint. If r tends 1_-, f_r converges to f in $\mathbb{D}$ and to f^* in $L_2(T)$.
Construct a function f in $\mathbb{D}$ which has no continuous extension to $\overline{\mathbb{D}}$.

For a function h, we use notation $\partial h = h_z = \frac{1}{2}(h'_x - ih'_y)$ and $\overline{\partial} h = h_{\overline{z}} = \frac{1}{2}(h'_x + ih'_y)$; we also use notations Dh and $\overline{D}h$ instead of ∂h and $\overline{\partial} h$ respectively when it seems convenient.

Definition 2.1 (Quasiconformal mappings)**.** *A homeomorphism $f\colon D \mapsto G$, where D and G are subdomains of the complex plane $\mathbb{C}$, is said to be K-quasiconformal (K-q.c or k-q.c), $K \geq 1$, if f is absolutely continuous on a.e. horizontal and a.e. vertical line in D and there is $k \in [0,1)$ such that*

$$|f_{\bar{z}}| \leq k|f_z| \quad \text{a.e. on } D, \tag{61}$$

where $K = \frac{1+k}{1-k}$, YY i.e. $k = \frac{K-1}{K+1}$.

Note that the condition (61) can be written as

$$D_f := \frac{\Lambda}{\lambda} = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \leq K, \tag{62}$$

where $\Lambda = |f_z| + |f_{\overline{z}}|$, $\lambda = |f_z| - |f_{\overline{z}}|$ and $K = \frac{1+k}{1-k}$, i.e. $k = \frac{K-1}{K+1}$.

Example 16. *1. The Cantor function K defines the measure $\mu = \mu_K$ by $\mu[x,y] = K(y) - K(x)$. Whether $K' = \mu_K$ in $\mathcal{D}'$?*

2. Let $f(z) = \frac{1-|z|}{1-z}$, $z \in \mathbb{U}$). Show that $f \in W_0^{1,p}(\mathbb{U})$, $0 < p < 2$. Whether $f \in H^1(\mathbb{U})$? Hint. $I_p(r) = \int_0^{2\pi} \frac{1}{|1-re^{it}|^p}dt \approx (1-r^2)^{p-1}$, $p > 1$, and $I_1(r) =\approx \ln \frac{1}{1-r^2}$.

3. For $a \in \mathbb{R}$, define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \frac{1}{|x|^a}\,.$$

Then f is weakly differentiable if $a + 1 < n$ with weak derivative

$$D_k f == -\frac{a}{|x|^{a+1}}\frac{x_k}{|x|}\,.$$

4. If f is conformal mapping from the unit disc into $\mathbb{C}$ and f in The Dirichlet space, then f has nontangential limits f^* at a.e. point on the boundary of $\mathbb{D}$.

Define $l(f,z) := \int_0^1 |f'(tz)|dt$. If f conformal mapping from the unit disc into $\mathbb{C}$, then $A = m(f(U)) = \int_U |f'(z)|^2 dxdy$. If A is finite then $B = \int_U |f'(z)|dxdy = \int_0^{2\pi} l(f,e^{it})dt$ is finite and by Fubini's theorem $l(f,e^{it})$ is finite a.e. w.r. t.

Let $f \in H^1(\mathbb{U})$ and define curve $\gamma_\theta(\rho) = f(\rho e^{i\theta})$, $\rho \in [0,1)$.

If $L(\theta)$ is the length of curve γ_θ, then $L(\theta) \leq \int_0^1 |\nabla f(\rho e^{i\theta})|d\rho$. By Cauchy-Schwarz, $L^2(\theta) \leq S(\theta) := \int_0^1 |\nabla f(\rho e^{i\theta})|^2 d\rho$ and therefore

$\int_0^{2\pi} L^2(\theta)d\theta \leq \int_0^{2\pi} S(\theta)d\theta$.

If for a fixed ρ_0, we define curve $\gamma_\theta(\rho) = f(\rho e^{i\theta})$, $\rho \in [\rho_0, 1)$, then $\int_0^{2\pi} L^2(\theta)d\theta \leq \ln \frac{1}{\rho_0} \int_{A_{\rho_0}} |\nabla f(z)|^2 dxdy < \infty$, *where* $A_{\rho_0} = \{\rho_0 < |z| < 1\}$.

5. *Let $f = \sum f_k z^k$ be holomorphic function on $\mathbb{U}$ and $A(f) = \int_{\mathbb{U}} |f'(z)|^2 dxdy$. Then* $B(f) = \int_{\mathbb{U}} |f(z)|^2 dxdy \sum_{k=0}^{\infty} \frac{|f_k|^2}{k+1}$,
   $A(f) = \sum_{k=1}^{\infty} \frac{(k+1)^2 |f_{k+1}|^2}{k+1} = \sum_{k=1}^{\infty} k|f_k|^2$, *and* $\int_{-\pi}^{\pi} |f^*(e^{it})|^2 dt = 2\pi \sum_{k=0}^{\infty} |f_k|^2$.
   If $A(f) = \int_{\mathbb{U}} |f'(z)|^2 dxdy < \infty$, *then* $f \in W^{1,2}(\mathbb{U})$ *and* $f^* = Tf$ *on* $\mathbb{T}$.

6. *If f is a harmonic in $\mathbb{U}$, $f \in H^1(\mathbb{U})$ and trace of $f|T = 0$, then $f = 0$.*

7. *If $f : \mathbb{U} \xrightarrow{onto} G \subset \mathbb{C}$ is a K-qc mapping and $m_2(G) < \infty$, then $f \in H^1(\mathbb{U})$. If G is a Jordan domain, whether f has continuous extension to $\overline{\mathbb{U}}$.*

8. *If f is injective (homeomorphisam) and $G = f(\mathbb{D})$ a Jordan domain in $\mathbb{C}$ and $f \in H^1(\mathbb{D})$, whether f has continuous extension on $\overline{\mathbb{D}}$?*

9. *If $u \in W^{1,p}(\Omega)$ and u has a continuous extension on $\overline{\Omega}$ which is 0 on $\partial\Omega$, then $u \in W_0^{1,p}(\Omega)$. The converse is not true in general.*

   Lamberti: In order to guaranteed the validity of the converse, you need more information on the boundary because if the boundary is "thick"then you cannot have pointwise control of the function "far from the interior". The condition Ω has a Lipschitz boundary guarantees the converse. To be honest, I have never read a counterexample, but I am confident that such counterexample exists and could probably be found either in the big book of Mazya (Sobolev spaces) or in the smaller book of Mazya and Poborchi (Differentiable functions on domains).

   It is interesting that that the direct implication does not require boundary regularity. Simple proofs of these facts can be found in the book by Brezis "Functional Analysis".

10. *Question (Lamberti). It would be interesting to see if it is true that given an eigenfunction u in the broad sense (C^2 inside and continuous up to the boundary) then u is in $W^{1,2}$. I am not sure this is an easy question.*

11. *Whether there exists a function f such that $f \in W_0^{1,p}(\mathbb{U})$, for every $p < 2$, and f is injective and it has no continuous extension to $\overline{\mathbb{U}}$?*
    Yes. Hint. For $z = re^{it}$, define $f(z) = R(t)(1 + r)e^{it}$, where $R(t) = t$ for $0 \leq t \leq \pi/2$ and $R(t) = \pi - t$ for $\pi/2 \leq t \leq \pi$.

    Check that f is injective on $\mathbb{H}^+$ and that f has no continuous extension to 0.

    Set $g(z) = (1 - r)f(z)$. Then f is 2-to-1 and $g \in W_0^{1,p}(\mathbb{U}^+)$ for $p < 2$, and it has no continuous extension to 0.

12. *Let $G = \mathbb{B}$ be the open unit ball in $\mathbb{R}^n$, with $n \geq 2$. Prove that the unbounded function $f(x) = \ln\ln(1 + \frac{1}{|x|})$ is in $W^{1,n}(\mathbb{B})$.*

Whether $f \in W^{1,n}(\mathbb{R}^n)$*? Yes.*

Hint. If $A = \{x : r_1 \le |x| \le r_2\}$, $0 < r_1 < r_2$, *then* $f \in W_0^{1,\infty}(A)$.

Set $A(r) = 1+\frac{1}{r} = \frac{r+1}{r}$ *and* $g(r) = \ln\ln(A(r)$. *Then* $g'(r) = \frac{1}{\ln(A(r)}\frac{1}{A(r)}A'(r)$, *where* $A'(r) = -1/r^2$, *and therefore (i)* $g'(r) = \frac{1}{\ln(A(r)}\frac{1}{r(r+1)}$.

Then $f(x) = g(r)$, *where* $r = |x|$. *Since* $D_k r = x_k/r$, $|D_k r| = x_k|/r \le 1$ *and* $D_k f(x) = g'(r)x_k/r$, *we have* $|D_k f| \le c\frac{1}{\ln A(r)}\frac{1}{r}$. *Hence* $|\nabla f|^n \le cM(r)$, *where* $M(r) = \frac{1}{(\ln A(r)^n}\frac{1}{r^n(r+1)^n}$, *and therefore* $r^{n-1}M(r) \le \frac{1}{(\ln A(r)^n}\frac{1}{r(r+1)^n}$. *Set* $0 < r_0 < 1$. *Since* $r^{n-1}M(r) \le c\frac{1}{r(r+1)^n}$ *for* $r \ge r_0$, *we find*

(1) $I_1 = \int_{r_0}^{\infty} r^{n-1}M(r)dr < \infty$.

Since for $0 < r < 1$, $\ln A(r) \ge -lnr$, *we find* $|g'(r)| \le \frac{1}{-lnr}r^{-1}$, *and then*

$r^{n-1}M(r) \le \frac{1}{(\ln A(r)^n}\frac{1}{r} \le \frac{1}{(-lnr)^n}\frac{1}{r}$. *Set* $a_0 = -\ln r_0$ *and* $I_2 = \int_0^{r_0}\frac{1}{(-lnr)^n}\frac{r^{n-1}}{r^n}dr$. *Using the substitution* $t = -lnr$, *we find* $I_2 = \int_{a_0}^{\infty}\frac{dt}{t^n} < \infty$ *for* $n \ge 2$.

Hence $I = \int_{\mathbb{R}^n}|\nabla f|^n dx \le I_1 + I_2 < \infty$.

13. *Whether there exists a function* $f \in W_0^{1,n}(\mathbb{B})$ *which has no continuous extension to* $\overline{\mathbb{B}}$*?*

    Yes.Hint. Set $x_k = e_n/k$, $r_k(x) = |x - x_k|$, $f_k(x) = f(r_k x)$, $\omega_k(x) = \omega_{1/2k}(r_k(x))$, $g_k(x) = f_k(x)\omega_k(x)/k^2$, *and* $g(x) = \sum_{k=1}^{\infty} g_k(x)$. *Show that* $g \in W_0^{1,p}(\mathbb{B}^+)$ *and that the trace of* g *is* 0, *but* g *has no continuous extension at* 0.

    The same function works for $B(e_n;1)$*; take* $h(x) = g(x+e_n)$ *on* $\mathbb{B}$.

14. $W_0^{1,p}(\Omega) \subset C^0(\overline{\Omega})$ *for* $p > n$*; in planar case for* $p > 2$*,see for example [2]. A Lipschitz domain (or domain with Lipschitz boundary) is a domain in Euclidean space whose boundary is šufficiently regular"in the sense that it can be thought of as locally being the graph of a Lipschitz continuous function. Lamberti: This is true if* Ω *has the extension property, for example if* Ω *has a Lipschitz boundary. Indeed, if* $p > n$ *then Morrey inequality holds in* $\mathbb{R}^n$*, hence you have even Holder continuity. A simple discussion of this issue can be found in the popular book in PDE's by L. Evans. If you do not have the extension property, then you can only conclude that if* $p > n$ *then u in continuous in the interior and is globally bounded.*

    Remark 2. It seems to me if we suppose that the Dirichlet eigenfunction solution $w \in W_0^{1,2}(\Omega)$ *and* Ω *is convex, then we have that gradient of* w *is bounded.*

    So the question is what we mean when we say that u is the Dirichlet eigenfunction (in particular, whether we suppose that $w \in W_0^{1,2}(\Omega)$*)?*

Theorem 2.7 (see [4])**.** *For a function* $f \in L_2(T)$ *to be the trace on the circle* $\mathbb{T}$ *of the function belonging Sobolev space* $H^1(\mathbb{U})$ *it is necessary and sufficient* *that*

$$\sum_{-\infty}^{\infty} |k||f_k|^2 < \infty\,,$$

where f_k are Fourier coefficients of f on $\mathbb{T}$.

The function $f(\theta) = \sum_{k=1}^{\infty} \frac{\cos(k^3\theta)}{k^2}$ is continuous, but it is not the trace of any function belonging to $H^1(\mathbb{U})$.

Hint: If $u = P[f]$, then $u \in H^1(\mathbb{U})$.

For the next result see [10] and comments there.

Theorem 2.8. *Let $R = (a, b) \times (0, c)$ be an open rectangle in the upper half-plane. Suppose that*
(a) $F : R \to \mathbb{C}$ is bounded C^1 function, and
(b) $\overline{D}F \in L^p(R)$ for some $p > 1$.
Then $F_0(x) = \lim F(x + iy)$ if $y \searrow 0$ exist a.e. for $x \in (a, b)$.

Whether F_0 is a trace of F on (a, b)?

Note that (b) is significant generalization of classical assumption that $F \in H^\infty(R)$, ie. $\overline{D}F = 0$. It is not known whether the theorem is true if $p = 1$.

Whether the function belonging to Sobolev space $W^{1,p}(\mathbb{U})$ have trace?

Erich Miersemann, Linear Elliptic Equations of Second Order Lecture Notes,Version October, 2012 http://www.math.uni-leipzig.de/ miersemann/pde2book.pdf

2.11 Embeddings

We are reduced to evaluating a 1D integral, improper at 0, whose integrand behaves like $\rho^{n-1}g(\rho)$. By virtue of the assumptions about g, this is bounded above by a multiple of ρ^{n-1-s}. Provided $n - 1 - s > -1$, this converges (at a rate proportional to ρ^{n-s}). In the example, $n = 2$, $s = 1$, so convergence is assured.

In one space dimension, a solution of the ODE $-u'' = f$ is given in terms of the potential $u(x) = -\frac{1}{2}\int |x - y|f(y)dy$.

If $f \in C_c(\mathbb{R})$, then obviously $u \in C^2(\mathbb{R})$ and $\max|u''| = \max|f|$. In more than one space dimension, however, it is not possible estimate the maximum norm of the second derivative D^2u of the potential $u = \Gamma * f$ in terms of the maximum norm of f, and there exist functions $f \in C_c(\mathbb{R}^n)$ for which $u \notin C^2(\mathbb{R}^n)$. In particular, if we measure derivatives in terms of their Holder continuity, we can estimate the $C^{2,\alpha}$-norm of u in terms of the $C^{0,\alpha}$-norm of f.

Example 17.

$$I(\varepsilon) = \int_{B(\varepsilon)} \frac{dx}{|x|^\alpha} = \int_S f(\theta)d\theta \int_\varepsilon^1 \frac{r^{n-1}dr}{r^\alpha} = cJ(\varepsilon) := c\int_\varepsilon^1 \frac{dr}{r^{\alpha-n+1}},$$

where $c = \int_S f(\theta)d\theta$. If $\varepsilon \to 0_+$, $I(\varepsilon)$ has finite limit if $\alpha < n$.

Consider equation $\Delta u_s = |x|^s$. Roughly speaking we try to guess $u_s(x) = c|x|^{s+2}$, $c = c(s, n)$.

But, we have two exeptions:

1. $D_k \ln|x| = \frac{x_k}{|x|^2}$ $D_{kk} \ln|x| = \frac{1}{|x|^2} - 2\frac{x_k^2}{|x|^4}$

$\Delta \ln|x| = (n-2)|x|^{-2}$

2. $\Delta(\ln|x| E_n(x)) = (2-n)|x|^{-n}$; see Example 13.

Check $|\nabla_y \ln|x-y|| \le c|x-y|^{-1}$,

$|\nabla_y |x-y|^{-s}| \le s|x-y|^{-s-1}$, $s \ge 1$.

Lemma 2.7. *If we set for* $n \ge 3$, $u_s(x) = c|x|^{s+2}$, $c = c(s,n) = \frac{s+2}{s+n}$, $s \ne -2$, $s \ne -n$, $u_{-2}(x) = \ln|x|/(n-2)$ *and* $u_{-n}(x) = -E_n(x)\ln|x|/(n-2) = -E_n(x)u_{-2}(x)$ *then* $\Delta u_s = |x|^s$ *for all* $x \in \mathbb{R}^n$, $x \ne 0$.

In order to prove the embedding theorem we first prove:

Let $m = [n/2] + 1$. If $f \in C_0^m(G)$, then

$$|f|_{C(\overline{G})} \le c|f|_{H^m(G)}. \tag{63}$$

Let us first test this result for $n = 3, 4, 5$. The idea is to use the representation theorem and Cauchy-Shwarz inequality.

Recall if $f \in C_0^2(D)$, then for all $x \in D$,

$$f(x) = \int_D \Delta f(y)\,\Gamma_n(x-y)dy = \int_D \Delta^p f(y)\,\Gamma_m(x-y)dy. \tag{64}$$

Set $U_s^x(y) = |x-y|^{-s}$. By definition u_s in Lemma 2.7, when we apply this result we need to pay attention to the case $-s_n = 2-n$ equals $-n$ and -2. Note that $-s_n = 2-n > -n$ and $-s_n = 2-n = -2$ iff $n = 4$.

a) For $n = 3$, $f(x) = c\int_D \Delta f(y)|x-y|^{-1}dy$.

Since U_1^x is in $L^2(D)$ we can prove (63).

b) For $n = 4$, $f(x) = c\int_D \Delta f(y)|x-y|^{-2}dy$. If in addition $f \in C_0^3(D)$, then $f(x) = c\int_D \Delta f(y)\Delta_y \ln|x-y|dy = c\int_D \nabla(\Delta f(y))\nabla_y \ln|x-y|dy$.

By Bunyakovskii- Cauchy-Shwarz inequality, Then

$$|f(x)| \le |f|_{H^3(D)}.$$

c) When $n = 5$, $f(x) = c\int_D \Delta f(y)|x-y|^{-3}dy$ (note that U_3^x is not in $L^2(D)$) and by Lemma 2.7 $\Delta_y|x-y|^{-1} = c|x-y|^{-3}$, $f(x) = c\int_D \Delta f(y)\Delta_y|x-y|^{-1}dy = c\int_D \nabla(\Delta f(y))\nabla_y|x-y|^{-1}$.

Here $m = m(5) = 3$ and $|\nabla_y|x-y|^{-1}| \le U_2^x(y) = |x-y|^{-2}$. Since U_2^x is in $L^2(D)$ we can prove (63).

So our strategy is to find minimal p such that $u_s \in L^2(D)$, where $\Delta^p u_s = \Gamma_n$. If $\nabla u_s \notin L^2(D)$, we use $f = \Delta f(y) * u_s$, but if $\nabla u_s \in L^2(D)$, we can use the formula $f = \nabla(\Delta f(y)) * u_s$.

If $u \in C_0^2$, $v \in C^2$ in $\overline{D}$,

$$\int_D u\Delta v dx = \int_D v\Delta u dx. \tag{65}$$

If an integer $p \geq 2$, $u \in C_0^p$, $v \in C^p$ in $\overline{D}$,

$$\int_D u\Delta^p v dx = \int_D v\Delta^p u dx. \tag{66}$$

Set $\Gamma_s(x) = |x|^{-s}$. For $s = 2p-2$, $s' = s+2p = 4p-2$, $\Delta^p\Gamma_s = c_p\Gamma_{s'}$

Suppose that $f \in C_0^p$, $n = 4p-2$, $n > 2$ and $x \in D$.

Supouse that $n > 2$. Set $U_s^x(y) = |x-y|^{-s}$, $m = m_p = (n-2) - 2(p-1) = n-2p$ and $\alpha = \alpha_p = 2m = 2n-4p$ and $\beta = \beta_p = 2(m_{p+1}+1) = 2(n-2p-1)$. Then

For $f \in C_0^p$, we have

(1) $f(x) = c\langle \Delta f, U_{n-2}^x\rangle = c\langle \Delta^p f, U_{m_p}^x\rangle$ and

(2) for $f \in C_0^{p+1}$, $f(x) = c\langle \Delta^{p+1} f, U_{m_{p+1}}^x\rangle = c\langle \nabla\Delta^p f, \nabla U_{m_{p+1}}^x\rangle$.

For n we choose a convenient p. We will use that $U_{m_p}^x \in L_2$ iff $\alpha < n$, i.e. $n < 4p$ and since $|\nabla U_{m_{p+1}}^x| \leq cU_{m_{p+1}+1}^x$, $\nabla U_{m_{p+1}}^x \in L_2$ if $\beta < n$, i.e. $n < 2(2p+1)$.

For n we choose a convenient p. There are four possibilities $n = 4p-2$, $n = 4p-1$,$n = 4p$ and $n = 4p+1$. We will use (1) if $n < 4p$ and (2) if $n \geq 4p$.

$$f(x) = \int_D \Delta f(y)\,\Gamma_n(x-y)dy = \int_D \Delta^p f(y)\,\Gamma_m(x-y)dy. \tag{67}$$

By Cauchy-Shwarz, $|f(x)|^2 \leq A\int_D |\Delta^p f|^2 dy$, where $A = \int_D \Gamma_m(x-y)^2 dy$.

By Example 17, A is finite, if $\alpha = 2n-4p < n$, i.e. $n < 4p$. Then

$$|f(x)| \leq |f|_{H^{2p}}.$$

$$f(x) = \quad c_{4p}\int_D \nabla(\Delta^p f(y))\nabla(\frac{1}{|x-y|^{2p-2}})dy, \quad n = 4p \tag{68}$$

$$f(x) = \quad c_{4p+1}\int_D \nabla(\Delta^p f(y))\nabla(\frac{1}{|x-y|^{2p-1}})dy, \quad n = 4p+1. \tag{69}$$

Let $m = [n/2]+1$. If $f \in C_0^m(G)$, then

$$|f|_{C(\overline{G})} \leq c|f|_{H^m(G)}. \tag{70}$$

By passing to the limit we conclude that this estimate holds for $f \in H_0^m(G)$. Hence

Theorem 2.9. $H_0^m(G) \subset C(\overline{G})$.

Namely, for $f \in H_0^m(G)$ there is a sequence $f_p \in C_0^m(G)$ which converges to f in $H_0^m(G)$. But, by (77) it follows that $f \in C(\overline{G})$.

Theorem 2.10 ([4], p.155,$ 2.7 Theorem 3 [6])**.** *Let* $k = [n/2]+l+1$ *(*$l = k-[n/2]-1 \geq 0$*).*

If G *be of class* C^k*, then* $H^k(G) \subset C^l(\overline{G})$*. For* $n = 2, 3$*,* $[n/2] = 1$*, and we have* $H^{2+l}(G) \subset C^l(\overline{G})$*. In particular,* $H^2(G) \subset C^0(\overline{G})$ *and* $H^3(G) \subset C^1(\overline{G})$*.*

Rešenje. For $f \in H^k(G)$, and G' such that $\overline{G} \subset G'$ there is F which is an extension of f and finite in G' such that $F \in H_0^k(G')$. △

2.12 Equivalent norm

Set $L_2u = a^{ij}(x)D^2_{ij}u$, $L_1u = b^i(x)D_iu$, $L_0u = cu$, $M = L_2 + L_1$ and $L = L_2 + L_1 + L_0$; consider the equation $Lu = f$. Let G be a bounded C^1 domain in $\mathbb{R}^n$, $a^{ij} \in L^\infty(G)$, $c \in L^\infty(G)$, $d \in L^\infty(S)$, where $S = \partial G$.

The Ermite form is

$$W(u,v) = \int_G (\sum a^{ij}(x)D_iuD_j\overline{v}dx) + \int_G c(x)u(x)\overline{v}(x)dx + \int_S d(x)u(x)\overline{v}(x)dA. \tag{71}$$

Further, let the matrix $A = (a^{ij})$ uniformlly elliptic with constant $c_0 > 0$, and in addition $c \geq 0$, a.e. in G, $d \geq 0$ a.e. in S, such that at least one of these function is not 0 in the corresponding spaces $L^\infty(G)$ or $L^\infty(S)$.

Theorem 2.11. *Under the above condition the Ermite form (71) defined on $H^1(G)$ the scalar product which is equivalent with standard scalar product: $(u,v)_{H^1} = \int_G(\nabla u \nabla \overline{v} + u\overline{v})dx$.*

2.13 Compactness in Sobolev space

Embedding theorems In one space dimension, a function $u : \mathbb{R} \to \mathbb{R}$ which admits a weak derivative $Du \in L_1(\mathbb{R})$ is absolutely continuous (after changing its values on a set of measure zero). On the other hand, if $G \subset \mathbb{R}^n$ with $n \geq 2$, there exist functions $u \in W^{1,p}(G)$ which are not continuous, and not even bounded. This is indeed the case of the function $u(x) = |x|^{-\alpha}$, for ?? $0 < \alpha < \frac{n}{n-p}$. Since

$$D_ku == -\frac{\alpha}{|x|^{\alpha+1}}\frac{x_k}{|x|},$$

$D_ku \in L^p$ if ?? $(\alpha+1)p < n$, ie. $\alpha < \frac{n-p}{p}$.

In several applications to PDEs or to the Calculus of Variations, it is important to understand the degree of regularity enjoyed by functions $u \in W^{k,p}(\mathbb{R}^n)$. We shall prove two basic results in this direction

Morrey

(Gagliardo-Nirenberg). If $p < n$, then every function $u \in W^{1,p}(\mathbb{R}^n)$ lies in the space $L_{p^*}(\mathbb{R}^n)$, with the larger exponent $p^* = p + \frac{p^2}{n-p} = \frac{np}{n-p}$.

Rellich-Kondrachov compactness theorem

Let $G \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary. Assume $1 \leq p < n$. Then for each $1 \leq q < p^*$ one has the compact embedding

$$W^{1,p}(G) \subset\subset L^q(G).$$

The basic approach is as follows:

I - Prove an a priori inequality valid for all smooth functions

II - Extend the embedding to the entire space, by continuity.

2.13.1 Compactness in Hilbert spaces*

Let in addition S be C^1 hyper-surface in $\overline{G}$. Then bounded set in $H^1(G)$ is compact in $L_2(G)$.

First we consider $n = 1$, one-dimensional case. To every function g that is p-integrable on $[0, 1]$, with $1 < p \leq \infty$, associate the function G defined on $[0, 1]$ by

$G(x) = \int_0^x g(t)\,\mathrm{d}t$.

Let F be the set of functions G corresponding to functions g in the unit ball of the space $L_p([0, 1])$. If q is the Hölder conjugate of p, defined by $1/p+1/q = 1$, then Hölder's inequality implies that all functions in F satisfy a Hölder condition with $a = 1/q$ and constant $M = 1$.

It follows that F is compact in $C([0, 1])$. This means that the correspondence $g \to G$ defines a compact linear operator T between the Banach spaces $L_p([0, 1])$ and $C([0, 1])$. Composing with the injection of $C([0, 1])$ into $L_p([0, 1])$, one sees that T acts compactly from $L_p([0, 1])$ to itself. The case $p = 2$ can be seen as a simple instance of the fact that the injection from the Sobolev space $H_0^1(\Omega)$ into $L_2(\Omega)$, for Ω a bounded open set in $\mathbb{R}^d$, is compact.

Let H be a Hilbert Space with inner product $< \ , \ >$, and corresponding norm $||$. Furthermore there is the set $B = \{x \in H : |x| \leq 1\}$.
a) Show, that B is closed and bounded11 regarding (the metric corresponding to) $|x|$.
b) Show, that if H doesn't have a finite dimension, then B isn't compact regarding (the metric corresponding to) $|x|$. (Hint: Look at a countable orthonormal system $\{e_n | n \in N\}$ in H.
c) Give an example of a Hilbert Space H, where B is not compact. In general assumption that a set is closed and bounded does not imply that it is compact - but if a set is compact then it is closed and bounded. If we assume that H has finite dimension a set is closed and bounded if and only if it is compact.

Any closed ball is non-compact in infinite dimensional Hilbert space. The functions $e_n(\theta) = e^{2\pi i n\theta}$ form an orthogonal basis of the Hilbert space $L^2([0, 1])$. The set $\{e_n : n \in N\}$ is bounded, but there is no convergent subsequence of (e_n). The Banach-Alaoglu theorem (also known as Alaoglu's theorem) states that the closed unit ball of the dual space of a normed vector space is compact in the weak* topology. The weak topology is characterized by the following condition: a net (x_λ) in X converges in the weak topology to the element x of X if and only if $f(x_\lambda)$ converges to f(x) in R or C for all f in X^*.

In particular, if (x_n) is a sequence in X, then x_n converges weakly to x if

$\phi(x_n) \to \phi(x)$

as $n \to \infty$ for all $\phi \in X^*$. In this case, it is customary to write

$x_n \xrightarrow{\mathrm{w}} x$

or, sometimes,

$x_n \rightharpoonup x$.

A sequence of $\phi_n \in X^*$ converges to ϕ provided that

$\phi_n(x) \to \phi(x)$

for all x in X. In this case, one writes
$\phi_n \overset{w^*}{\to} \phi$
as $n \to \infty$.

2.13.2 Compactness in L^2

We say that $f \in L^{1,2}(G)$ if $f, \nabla f \in L^2(G)$. We define $(f,g) = \int_G (\nabla f \cdot \nabla \overline{g} + f\overline{g})dx$.

If M is subset of $L^2(-\pi,\pi)$ such that $f \in M$ if $f' \in L^2(-\pi,\pi)$ and $|f'|_2 \leq 1$. Whether M is compact in $L^2(-\pi,\pi)$?

What is difference between $L^{1,2}$ and $W^{1,2}$?
ACL property

Whether $W^{1,2} \subset L^{1,2}$?
Cantor function

The bounded set in $H^1(G)$ is compact in $L_2(G)$.

Theorem 2.12. *Let G be bounded C^1 domain in $\mathbb{R}^n$. Then bounded set in $H^1(G)$ is compact in $L_2(G)$. Let in addition S be C^1 hyper-surface in $\overline{G}$. Then bounded set in $H^1(G)$ is compact in $L_2(G)$.*

Rešenje. Suppose first that $M \subset H_0^1(G)$ is a bounded set in $H^1(G)$ and extend all function from M to be 0 out of G. Set $f_\varepsilon(x) = \int_G f(y)\omega_\varepsilon(x-y)dy$ and $M_\varepsilon = \{f_\varepsilon : f \in M\}$.

Check that $|f - f_\varepsilon|_{L_2(G)} \leq c\varepsilon$, and $|D_i f_\varepsilon| \leq c\varepsilon^{-n-1}\int_G |f(y)|dy \leq C$.

For fixed $\varepsilon > 0$, M_ε is compact in $C(\overline{G})$ and therefore in $L_2(G)$. Let f_p be an arbitrary sequence in M. Corresponding sequence $f_{p,\varepsilon}$ is in M_ε and therefore there is a subsequence (for which we use the same notation) which converges in $L_2(G)$. f_p is Cauchy sequence.

$\triangle$

Theorem 2.13. *Let G be bounded C^1 domain in $\mathbb{R}^n$. Let in addition S be C^1 hyper-surface in $\overline{G}$. Then bounded set in $H^1(G)$ is compact in $L_2(G)$.*

Rešenje. Let S_0 be simple part of S given by equation $x_n = \varphi(x')$, $\varphi \in C^1(\overline{D})$, where D is a domain in hyper-plane $x_n = 0$.

Set $Q = Q_\delta = \{(x', x_n) : x' \in D, \varphi(x') \leq x_n \leq \varphi(x') + \delta\}$, $0 < \delta < \delta_0$.

$x = (x', x_n) \in S_0$ $(x', y_n) \in Q_\delta$

Then $f(x', y_n) - f(x) = \int_{x_n}^{y_n} D_n f(x', t)dt$ and $|f(x)|^2 \leq 2|f(x', y_n)|^2 + 2\delta \int_{x_n}^{x_n+\delta} |D_n f(x', t)|^2 dt$. Hence

$\delta |f(x)|^2 \leq 2\int_{x_n}^{x_n+\delta} |f(x', y_n)|^2 dy_n + 2\delta^2 \int_{x_n}^{x_n+\delta} |D_n f(x', t)|^2 dt$ and

(X1) $\delta_{S_0} |f(x)|^2 dS \leq c_1 \int_Q |f(x)|^2 dx + c_2\delta^2 \int_Q |\nabla f(x)|^2 dx$.

By summing

$|f|^2_{L_2(S)} \leq \frac{c_1}{\delta}|f|^2_{L_2(G)} + c_2\delta |f|^2_{H^1(G)}$.

Let f_p be an arbitrary sequence in M. By Theorem X bounded set in $H^1(G)$ is compact in $L_2(G)$ and therefore there is a subsequence (for which we use the same notation) which converges in $L_2(G)$. f_p is Cauchy sequence. For $\varepsilon > 0$, $|f_p - f_q|^2_{L_2(G)} \leq \varepsilon$. Using (X1) by $\delta = \varepsilon$, it follows that f_p is Cauchy sequence in $L_2(S)$. △

Recall. For the next result see [10] and comments there.

Theorem 2.14. *Let $R = (a,b) \times (0,c)$ be an open rectangle in the upper half-plane. Suppose that*
(a) $F : R \to \mathbb{C}$ is bounded C^1 function, and
(b) $\overline{D}F \in L^p(R)$ for some $p > 1$.
Then $\lim F(x+iy)$ *if $y \searrow 0$ exist a.e. for $x \in (a,b)$.*

Note that (b) is significant generalization of classical assumption that $F \in H^\infty(R)$, ie. $\overline{D}F = 0$. It is not known whether the theorem is true if $p = 1$.

Sobolev spaces are often considered when investigating partial differential equations. It is essential to consider boundary values of Sobolev functions. If $u \in C(\overline{\Omega})$, those boundary values are described by the restriction $u|_{\partial\Omega}$. However, it is not clear how to describe values at the boundary for $u \in W^{k,p}(\Omega)$, as the n-dimensional measure of the boundary is zero. The following theorem resolves the problem:

Trace Theorem. Assume Ω is bounded with Lipschitz boundary. Then there exists a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that

$$Tu = u|_{\partial\Omega} \qquad u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \tag{72}$$

$$\|Tu\|_{L^p(\partial\Omega)} \leq c(p,\Omega)\|u\|_{W^{1,p}(\Omega)} \qquad u \in W^{1,p}(\Omega). \tag{73}$$

Intuitively, taking the trace costs 1/p of a derivative. The functions u in $W^{1,p}(\Omega)$ with zero trace, i.e. $Tu = 0$, can be characterized by the equality

$$W_0^{1,p}(\Omega) = \left\{u \in W^{1,p}(\Omega) : Tu = 0\right\},$$

where

$$W_0^{1,p}(\Omega) := \left\{u \in W^{1,p}(\Omega) : \exists \{u_m\}_{m=1}^{\infty} \subset C_c^\infty(\Omega), \text{such that } u_m \to u \text{ in } W^{1,p}(\Omega)\right\}.$$

In other words, for Ω bounded with Lipschitz boundary, trace-zero functions in $W^{1,p}(\Omega)$ can be approximated by smooth functions with compact support.

3 Sobolev,Elliptic equation

Elliptic equation

Second order linear partial differential equations (PDEs) are classified as either elliptic, hyperbolic, or parabolic. Any second order linear PDE in two variables can be written in the form $Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$, where A, B, C, D, E, F, and G are functions of x and y and where $u_x = \dfrac{\partial u}{\partial x}$ and similarly for $u_{xx}, u_y, u_{yy}, u_{xy}$. A PDE written in this form is elliptic if

$$B^2 - AC < 0,$$

with this naming convention inspired by the equation for a planar ellipse.

The simplest nontrivial examples of elliptic PDE's are the Laplace equation, $\nabla^2 u = u_{xx} + u_{yy} = 0$, and the Poisson equation, $\nabla^2 u = u_{xx} + u_{yy} = f(x, y)$. In a sense, any other elliptic PDE in two variables can be considered to be a generalization of one of these equations, as it can always be put into the canonical form

$$u_{xx} + u_{yy} + \text{ (lower-order terms)} = 0$$

through a change of variables.

We derive the canonical form for elliptic equations in two variables, $u_{xx} + u_{yy} + \text{ (lower-order terms)} = 0$ $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$.

If we set $\phi = \xi + i\eta$, we can define $U = \hat{u} = u \circ \phi^{-1}$ (i.e. $u = \hat{u} \circ \phi$). Often for simlicity of notation we write $u(\xi, \eta) = u[\xi(x, y), \eta(x, y)]$. Applying the chain rule once gives

$u_x = u_\xi \xi_x + u_\eta \eta_x$ and $u_y = u_\xi \xi_y + u_\eta \eta_y$,

a second application gives

$u_{xx} = u_{\xi\xi} {\xi^2}_x + u_{\eta\eta} {\eta^2}_x + 2u_{\xi\eta}\xi_x\eta_x + u_\xi \xi_{xx} + u_\eta \eta_{xx}$,

$u_{yy} = u_{\xi\xi} {\xi^2}_y + u_{\eta\eta} {\eta^2}_y + 2u_{\xi\eta}\xi_y\eta_y + u_\xi \xi_{yy} + u_\eta \eta_{yy}$, and

$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\eta\eta}\eta_x\eta_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_\xi \xi_{xy} + u_\eta \eta_{xy}$.

We can replace our PDE in x and y with an equivalent equation in ξ and η

$$au_{\xi\xi} + 2bu_{\xi\eta} + cu_\eta + \text{ (lower-order terms)} = 0,$$

where

$$a = A{\xi_x}^2 + 2B\xi_x\xi_y + C{\xi_y}^2, \tag{74}$$

$$b = 2A\xi_x\eta_x + 2B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y, and \tag{75}$$

$$c = A{\eta_x}^2 + 2B\eta_x\eta_y + C{\eta_y}^2. \tag{76}$$

To transform our PDE into the desired canonical form, we seek η such that $a = c$ and $b = 0$. This gives us the system of equations

Adding i times the second equation to the first and setting $\phi = \xi + i\eta$ gives the quadratic equation

$$a - c + ib = A{\phi_x}^2 + 2B\phi_x\phi_y + C{\phi_y}^2 = 0.$$

Since the discriminant $B^2 - AC < 0$, this equation has two distinct solutions,

$$\phi_x, \phi_y = \frac{B \pm i\sqrt{AC - B^2}}{A}$$

which are complex conjugates.

Set $\nu_{1,2} = \frac{B\pm i\sqrt{AC-B^2}}{A}$, we transform eq in the form $A(\phi_x - \nu_1\phi_y)(\phi_x - \nu_2\phi_y) = 0$.

Choosing either solution, we can solve for $\phi(x,y)$, and recover ξ and η with the transformations $\xi = \operatorname{Re}\phi$ and $\eta = \operatorname{Im}\phi$. Since ξ will satisfy $a-c=0$ and $b=0$, so with a change of variables from x and y to η and ξ will transform the PDE into the canonical form.

We also can transform eq (1) $\phi_x - \nu_2\phi_y$ into Beltrami equation.

Since $\phi_x = \phi_z + \phi_{\overline{z}}$ and $\phi_y = i(\phi_z - \phi_{\overline{z}})$

$\phi_z = \mu\phi_z$, where $\mu = \frac{i\nu-1}{i\nu+1}$ and $\nu = \nu_2$.

$\nu \in H^-$ iff $\mu \in \mathbb{D}$

Recall

Embedding theorems have important role, cf. \$ 2.7 Theorem 3 [6].

$W_0^{1,p}(G) \subset C^0(\overline{G})$, for $p > n$, [2], p.154.

Let $m = [n/2]+1$. If $f \in C_0^m(G)$, then

$$|f|_{C(\overline{G})} \leq c|f|_{H^m(G)}. \tag{77}$$

By passing to the limit we conclude that this estimate holds for $f \in H_0^m(G)$. Hence

Theorem 3.1. *$H_0^m(G) \subset C(\overline{G})$.*

Namely, for $f \in H_0^m(G)$ there is a sequence $f_p \in C_0^m(G)$ which converges to f in $H_0^m(G)$. But, by (77) it follows that $f \in C(\overline{G})$.

Theorem 3.2 ([4], p.155,\$ 2.7 Theorem 3 [6])**.** *Let $k = [n/2]+l+1$ ($l = k-[n/2]-1 \geq 0$).*

If G be of class C^k, then $H^k(G) \subset C^l(\overline{G})$. For $n = 2,3$, $[n/2] = 1$, and we have $H^{2+l}(G) \subset C^l(\overline{G})$. In particular, $H^2(G) \subset C^0(\overline{G})$ and $H^3(G) \subset C^1(\overline{G})$.

Rešenje. For $f \in H^k(G)$, and G' such that $\overline{G} \subset G'$ there is F which is an extension of f and finite in $\subset G'$ such that $F \in H_0^k(G')$. △

A function (or, more generally, a distribution) is weakly harmonic if it satisfies Laplace's equation

$\Delta f = 0$

in a weak sense (or, equivalently, in the sense of distributions). A weakly harmonic function coincides almost everywhere with a strongly harmonic function, and is in particular smooth. A weakly harmonic distribution is precisely the distribution associated to a strongly harmonic function, and so also is smooth. This is Weyl's lemma.

Example 18. *1. Find harmonic function in h_0 $\mathbb{U}$ such that $h_0 = 1$ on $(-\pi/2, \pi/2)$ and $h_0 = -1$ on $(\pi/2, 3\pi/2)$.*

Hint: $h_0(z) = \frac{2}{\pi} arg\frac{i+z}{i-z}$. Whether h_0 belong $H^1(\mathbb{U})$? no

2. *Solve equation (1)* $\Delta u = sgn x$ *on the unit disk in* $\mathbb{C}$.

   Set $u_0(x,y) = x^2/2$ *for* $x > 0$ *and* $u_0(x,y) = -x^2/2$ *for* $x < 0$ *and* $h_0 = P[u_0^*]$.

   Check $u_0, h_0 \in H^1(\Omega)$ *and* $u = u_0 - h_0$ *is generalized solution of (1) in* $H_0^1(\mathbb{U})$.

3. *Set* $u_0(x,y) = (x^2+y^2)/4$ *for* $x > 0$ *and* $u_0(x,y) = -(x^2+y^2)/4$ *for* $x < 0$ *and Whether* u_0 *belongs* $H^1(\mathbb{U})$*? no*

4. *Let* $g \in L_2(\Omega)$*, and define* $l(v) = l_g(v) = \int_\Omega g v dx$, $v \in H^1(\Omega)$*. Then* $|l(v)| \le |g|_{L_2(\Omega)} |v|_{L_2(\Omega)}$.

   $|v|_{L_2(\Omega)} \le |v|_{H_0^1(\Omega)}$*. Hence* l_g *is bounded linear functional on* $H^1(\Omega)$ *and in particular on* $H_0^1(\Omega)$.

   Define $l_1(v) = l_g^1(v) = \int_\Omega D_i g v dx$*. For* $v \in H^1(\Omega)$,

   $l_g^1(v) = \int_\Omega D_i g v dx = -\int_\Omega g D_i v dx$ *and* $|l_g^1(v)| \le |g|_{L_2(\Omega)} |v|_{L_2(\Omega)}$*. Since* $|v|_{L_2(\Omega)} \le |v|_{H_0^1(\Omega)}$, l_g^1 *is bounded linear functional on* $H_0^1(\Omega)$.

5. *Let* $g \in H^1(\Omega)$*, and define* $l_2(v) = l_g^2(v) = \int_\Omega \Delta g \, v dx$, $v \in H^1(\Omega)$*. For* $v \in H^1(\Omega)$,

   $l_g^2(v) = -\int_\Omega Dg \cdot Dv dx$ *and therefore* $|l_g^1(v)| \le |g|_{H^1(\Omega)} |v|_{H^1(\Omega)}$.

Green's identity follows easily from the familiar divergence theorem of advanced calculus:

$$\int_D \mathrm{divw}\, dx = \int_S w \cdot n \, d\sigma \, .$$

If $w = vDu$, then $div\, w = Dv \cdot Du + v\Delta u$ and therefore we have (3):

$$\int_S v \frac{\partial u}{\partial n} d\sigma = \int_D (v\Delta u + Dv \cdot Du)\, dx \, .$$

Since $div\, w = \sum_k D_k(vD_k u)$, if $v \in C_0^1(D)$, then

$$\sum_k \int_D D_k(vD_k u) dx = 0$$

and therefore

$$\int_D (v\Delta u)\, dx = -\int_D (Dv \cdot Du)\, dx \, .$$

If u, v are ACL on D and $v \in C_0(D)$, then $\int_D D_k u v dx = -(\int_D u D_k v dx)$.

We consider $\Delta u = f$ and $u = 0$ on S. If $\varphi \in C_c^1(D)$, then $[u,\varphi] = \int_D D\varphi \cdot Du dx = -(f,\varphi)$, where $F(\varphi) = (f,\varphi) = -\int_D f\varphi dx$. The completion of $C_c^1(D)$ with respect to the scalar product $[u,v]$ is Sobolev space H_0^1 (notation $W_0^{1,2}$ is also used). F can be extended to be bounded linear functional on H_0^1. By the

Riesz theorem, there is $u \in H_0^1$ such that $[u, v] = (f, v)$ for all $v \in H_0^1$. Note that $u = 0$ on S.

Let $\Delta u = f$,

$Q = \{x : |x_i| < d, i = 1, 2, \cdots, n\}$, $u \in C^2(Q) \cap C^0(\overline{Q})$, and f bounded in Q.

Using the comparison principle, one can derive

$D_i u(0) \leq \frac{m}{d} \sup_{\partial Q} |u| + \frac{d}{2} \sup_Q |f|$,

$\sup_\Omega d_x |Du(x)| \leq c(\sup_\Omega |u| + \sup_\Omega d_x^2 |f(x)|)$. The integration by parts formula yields that for every $u \in C^k(\Omega)$, where k is a natural number and for all infinitely differentiable functions with compact support $\varphi in C^\infty(\Omega)$,

$$\int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega \varphi D^\alpha u \, dx,$$

where α a multi-index of order $|\alpha| = k$ and Ω is an open subset in R^n. Here, the notation

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} = D_1^{\alpha_1} (\cdots D_n^{\alpha_n} f),$$

is used.

The left-hand side of this equation still makes sense if we only assume u to be locally integrable. If there exists a locally integrable function v, such that

$$\int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega \varphi v \, dx, \qquad \varphi \in C_c^\infty(\Omega),$$

we call v the weak α-th partial derivative of u. If there exists a weak α-th partial derivative of u, then it is uniquely defined almost everywhere. On the other hand, if $u \in C^k(\Omega)$, then the classical and the weak derivative coincide. Thus, if v is a weak α-th partial derivative of u, we may denote it by $D^\alpha u := v$.

That is, the Sobolev space $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \ \forall |\alpha| \leq k\}.$$

The natural number k is called the order of the Sobolev space $W^{k,p}(\Omega)$.

There are several choices for a norm for $W^{k,p}(\Omega)$. The following two are common and are equivalent in the sense of equivalence of norms:

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}, & 1 \leq p < +\infty; \\ \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}, & p = +\infty; \end{cases}$$

ACL-property We denote $R_k^{n-1} = \{x \in R^n : x_k = 0\}$. The projection P_k, given by $P_k x = x - x_k e_k$, is the orthogonal projection of R^n onto R_k^{n-1}.

Let $I = \{x \in \mathbb{R}^n : a_k \leq x_k \leq b_k\}$ be a closed n-interval.

A mapping $f : I \to \mathbb{R}^m$ is said to be absolutely continuous on lines (ACL) if f is continuous and if f is absolutely continuous on almost every line segment in I, parallel to the coordinate axes.

More precisely, if E_k is the set of all $x \in P_k I$ such that the functions $t \to u(x + te_k)$ is not absolutely continuous on $[a_k, b_k]$, then $m_{n-1}(E_k) = 0$ for $1 \le k \le n$.

If Ω is an open set in $\mathbb{R}^n$, a mapping $f : \Omega \to \mathbb{R}^m$ is ACL (absolutely continuous on lines) if $f|I$ is ACL for every closed interval $I \subset \Omega$.

If $f : \Omega \to R$ is continuous we say that $f \in W^{1,p}$ if f is ACL and $D_k f \in L^p$. Absolutely Continuous on Lines (ACL) characterization of Sobolev functions

Theorem 3.3. *Let Ω be an open set in $\mathbb{R}^n$ and $1 \le p \le \infty$. If a function is in $W^{1,p}(\Omega)$, then, possibly after modifying the function on a set of measure zero, the restriction to almost every line parallel to the coordinate directions in $\mathbb{R}^n$ is absolutely continuous; what's more, the classical derivative along the lines that are parallel to the coordinate directions are in $L^p(\Omega)$. Conversely, if the restriction of f to almost every line parallel to the coordinate directions is absolutely continuous, then the pointwise gradient ∇f exists almost everywhere, and f is in $W^{1,p}(\Omega)$ provided f and $|\nabla f|$ are both in $L^p(\Omega)$. In particular, in this case the weak partial derivatives of f and pointwise partial derivatives of f agree almost everywhere. The ACL characterization of the Sobolev spaces was established by Otto M. Nikodym (1933); see (Maz'ya 1985, 1.1.3).*

A stronger result holds in the case $p > n$. A function in $W^{1,p}(\Omega)$ is, after modifying on a set of measure zero, Hölder continuous of exponent $\gamma = 1 - n/p$, by Morrey's inequality. In particular, if $p = +\infty$, then the function is Lipschitz continuous.

Let Ω be an open set in $\mathbb{R}^n$. The Sobolev space$W^{1,2}(\Omega)$ is also denoted by $H^1(\Omega)$. It is a Hilbert space, with an important subspace $H_0^1(\Omega)$ defined to be the closure in $H^1(\Omega)$ of the infinitely differentiable functions compactly supported in Ω. The Sobolev norm defined above reduces here to

$$\|f\|_{H^1} = \left(\int_\Omega \left(|f|^2 + |\nabla f|^2 \right) \right)^{\frac{1}{2}}.$$

When Ω is bounded, the Poincar inequality states that there is a constant $C = C(\Omega)$ such that

$$\int_\Omega |f|^2 \le C^2 \int_\Omega |\nabla f|^2, \quad f \in H_0^1(\Omega).$$

When Ω is bounded, the injection from $H_0^1(\Omega)$ to $L^2(\Omega)$ is compact. This fact plays a role in the study of the Dirichlet problem, and in the fact that there exists an orthonormal basis of $L^2(\Omega)$ consisting of eigenvectors of the Laplace operator (with Dirichlet boundary condition).

When Ω has a regular boundary, $H_0^1(\Omega)$ can be described as the space of functions in $H^1(\Omega)$ that vanish at the boundary, in the sense of traces (see below). When $n = 1$, if $\Omega = (a, b)$ is a bounded interval, then $H_0^1(a, b)$ consists of continuous functions on $[a, b]$ of the form

$$f(x) = \int_a^x f'(t)\,\mathrm{d}t, \qquad x \in [a,b]$$

where the generalized derivative f' is in $L^2(a,b)$ and has 0 integral, so that $f(b) = f(a) = 0$.

the Sobolev embedding theorem. Here k is a non-negative integer and $1 \leq p \leq \infty$. The first part of the Sobolev embedding theorem states that if $k > l$ and $1 \leq p < q \leq \infty$ are two extended real numbers such that $(k-l)p < n$ and:

$$\frac{1}{q} = \frac{1}{p} - \frac{k-\ell}{n},$$

then

$$W^{k,p}(\mathbf{R}^n) \subseteq W^{\ell,q}(\mathbf{R}^n)$$

and the embedding is continuous.

This special case of the Sobolev embedding is a direct consequence of the Gagliardo-Nirenberg-Sobolev inequality.

The second part of the Sobolev embedding theorem applies to embeddings in Hölder spaces $C^{r,\alpha}(\mathbb{R}^n)$. If $(k-r-\alpha)/n = 1/p$ with $\alpha \in (0,1)$, then one has the embedding

$$W^{k,p}(\mathbf{R}^n) \subset C^{r,\alpha}(\mathbf{R}^n).$$

This part of the Sobolev embedding is a direct consequence of Morrey's inequality. Intuitively, this inclusion expresses the fact that the existence of sufficiently many weak derivatives implies some continuity of the classical derivatives.

Recall

$W_0^{1,p}(G) \subset C^0(\overline{G})$, for $p > n$, [2], p.154.

Let $s = [n/2] + 1$. If $f \in C_0^s$, $n \geq 1$, then

$|f|_{C(\overline{G})} \leq c|f|_{H^s(G)}$.

By limit we conclude thatthis estimate holds for $f \in H_0^s(G)$.

Let $k = [n/2] + l + 1$ $(l = k - [n/2] - 1 \geq 0)$.

If G of class C^k, then $H^k(G) \subset C^l(\overline{G})$, [4], p.155.

For $n = 2, 3$, $[n/2] = 1$, and we have $H^{2+l}(G) \subset C^l(\overline{G})$. In particular, $H^2(G) \subset C^0(\overline{G})$ and $H^3(G) \subset C^1(\overline{G})$.

3.1 Dirichlet boundary problem

Given a matrix-valued function $A(x)$ which is symmetric and positive definite for every x, having components a_{ij}, the operator

$$Lu = -\partial_i(a^{ij}(x)\partial_j u) + b^j(x)\partial_j u + cu$$

is elliptic. This is the most general form of a second-order divergence form linear elliptic differential operator. The Laplace operator is obtained by taking $A = I$. These operators also occur in electrostatics in polarized media.

We suppose that operator L uniform (strongly) elliptic, ie.

$$A(x,\xi) = a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad x \in \Omega, \xi \in \mathbb{R}^n\,. \tag{78}$$

If $\zeta = \xi + i\eta \in \mathbb{C}^n$, then $\zeta_i\overline{\zeta_j} = \xi_i\xi_j + \eta_i\eta_j + i(\xi_j\eta_i + \xi_i\eta_j)$
and $A(x,\zeta) = A(x,\xi) + A(x,\eta) \geq \lambda|\xi|^2 + \lambda|\eta|^2 = \lambda|\zeta|^2$.
(1) $Lu = \lambda u$, $u(x) = 0$ on ∂G.

Example 19. *One can check that for $G = [0, L]^n$, $u(x) = (\frac{2}{L})^{n/2} \prod_{i=1}^n \sin\frac{\pi m_i x_i}{L}$, $m_i \in N$ are eigenfunctions with corresponding eigenvalues $\lambda = \frac{\pi^2}{L^2}\sum_{i=1}^n m_i^2$ for Laplace differential operator $Lu = -\triangle u$.*

Given a matrix-valued function $A(x)$ which is symmetric and positive definite for every x, having components a_{ij}, the operator

$$Lu = -\partial_i(a^{ij}(x)\partial_j u) + b^j(x)\partial_j u + cu$$

is elliptic. This is the most general form of a second-order divergence form linear elliptic differential operator. The Laplace operator is obtained by taking $A = I$. These operators also occur in electrostatics in polarized media. Let $G \subset R^n$. Consider

$$Lu = -\partial_i(a^{ij}(x)\partial_j u) + cu \;= f_0 + \sum_{i=1}^n D_i f_i, \tag{79}$$

$$u(x) = g(x) on S\,. \tag{80}$$

In particular consider $Lu = f$. Suppose that $a^{ij} \in C^1(\overline{G})$, $a^{ij} = a^{ji}$, $c, f \in C(\overline{G})$, $g \in C(\overline{\partial G})$, and L strongly elliptic on G. If $u \in C^2(G) \cap C(\overline{\partial G})$, we call u classical solution.

Multiply by $v \in C_0^1(G)$. By partial integration,

$W(u,v) = \int_G a^{ij}(x) D_j u D_i v dx + \int_G cuvdx = l_f(v) = \int_G fv$.

A function $u \in H^1(G)$ we call generalized solution if (1) $W(u,v) = l_f(v)$ for all $v \in C_0^1(G)$).
Since $C_0^1(G)$ is dense in $H_0^1(G)$ it is equivalent with requirement that (1) holds for all $v \in H_0^1(G)$. This is important roe application of Riesz theorem.

This is weak form of equation (79). Note that the weak form can be consider under weak hypothesis $u \in H^1(G)$, $v \in H_0^1(G)$, $f \in L_2(G)$ and $a^{ij}, c \in L_\infty(G)$.
Set $L_2 u = L_2^{div} u = -\partial_i(a^{ij}(x)\partial_j u)$.

We will further assume $a^{ij}, c \in L_\infty(G)$, $a^{ij} = a^{ji}$ a.e. in G, and L strongly elliptic on G a.e.

Theorem 3.4. *Let $c \geq 0$ a.e. in G, $f_k \in L_2(G)$, $k = 0, 1, ..., n$.*
(i) If $g = 0$, The Dirichlet boundary problem has week unique solution in $H_0^1(G)$.
(ii) If g is trace of $H^1(G)$ function, the Dirichlet boundary problem has week unique solution in $H^1(G)$.

Set $[u,v] = [u,v]_{H_0^1} = W(u,v)$ and $l(v) = (f_0, v)_2 - \sum_{i=1}^n (f_i, D_i v)_2$. Then l is a bounded anti linear functional on $H_0^1(G)$ with respect to the scalar product $[,]$.

Consider the equation $[u,v] = l(v)$. By Riesz theorem there is $w \in H_0^1(G)$ such that $[w,v] = l(v)$ for all $v \in H_0^1(G)$.

For (ii) see [6], p.100. For g there is $\tilde{g} \in H^1(G)$ such that $g = tr(\tilde{g})$. Set $\underline{u} = u(x) - \tilde{g}$.

Hence $L\underline{u} = Lu(x) - L\tilde{g}$.

Theorem 3.5. *?? Let $l = k+1-[n/2]$ (that is $k = [n/2]+l-1$). If $f \in H_0^k(G)$, then $f \in C^l(\overline{G})$.*

$H_{loc}^k(G)$ is embedded in $C^l(G)$. If G of class C^k, then $H^k(G) \subset C^l(\overline{G})$.

If G of class C^k, then $H^k(G) \subset C^l(\overline{G})$.

For $n = 3$, $H^{2+l}(G) \subset C^l(\overline{G})$.

A priori estimate

Theorem 3.6 (Corollary, 8.11 [2])**.** *Let G be C^{k+2}, $f \in H^k(G)$, $a_{ij} \in C^k(\overline{G})$, $c \in C^k(\overline{G})$, and $c \geq 0$. Then Dirichlet problem $Lu = f$, $u = 0$ on ∂G has unique generalized solution in $H^{k+2}(G)$.*

Using the Sobolev embedding theorem, one can prove:

Theorem 3.7 (Corollary, 8.11 [2])**.** *Let a^{ij}, b^i, c^i, d, f belong $C^\infty(\Omega)$ and let $u \in W^{1,2}(\Omega)$ be weak solution of strongly elliptic equation $Lu = f$. Then u also belongs to $C^\infty(\Omega)$.*

Corollary 1. *In particular, if G is domain in $\mathbb{R}^n$, $u : G \to \mathbb{C}$ and $\Delta u = 0$ in generalized sense, then u harmonic in G.*

If G is domain in $\mathbb{C}$, $f : G \to \mathbb{C}$ and $f_{\overline{z}} = 0$ in generalized sense, then f holomorphic in G.

Eigen-values There exists an orthonormal set X_k in $L^2(G)$ forms of eigen-functions of L such that for f in $L^2(G)$

$f(x) = \sum_{k=1}^\infty (f, X_k) X_k(x)$.

Set $M = [a^{ij}(x)]$, $m = \inf ess_{x \in \overline{G}} c(x)$ and $\tilde{c}(x) = c(x) - m + 1$.

Def $(u,v)_M = (Mu, v)_e$.

Set $[u,v]_0 = [u,v]_{H_0^1(G)} = \int_G (\sum a^{ij}(x) D_i u D_j \overline{v} dx) + \int_G \tilde{c}(x) u(x) \overline{v} dx$

By partial integration $(Lu, v)_2 = [u,v]_0$, $v \in H_0^1(G)$.

By (1) $(Lu, v)_2 = \lambda (u,v)_2$ and therefore $[u,v]_0 = (\lambda - m + 1)(u,v)_2$.

By $[Au, v]_{H_0^1(G)} = (u,v)_{L_2(G)}$, $v \in H_0^1(G)$, is defined bounded liner operator $A : L_2(G) \to H_0^1(G)$.

Proposition 10. *The restriction of A on $H_0^1(G)$ is self-adjoint operator, positive completely continuous.*

For $u \in L_2(G)$, $l(v) = (u,v)_{L_2(G)}$ is anti-linear functional on $H_0^1(G)$.

3.1.1 Morrey's inequality*

Spherical coordinate system

A spherical coordinate system is a coordinate system for three-dimensional space where the position of a point is specified by three numbers: the radial distance of that point from a fixed origin, its polar angle measured from a fixed zenith direction, and the azimuth angle of its orthogonal projection on a reference plane that passes through the origin and is orthogonal to the zenith, measured from a fixed reference direction on that plane. It can be seen as the three-dimensional version of the polar coordinate system.

The use of symbols and the order of the coordinates differs between sources. In one system frequently encountered in physics (r, θ, φ) gives the radial distance, polar angle, and azimuthal angle, whereas in another system used in many mathematics books (r, φ, θ) gives the radial distance, azimuthal angle, and polar angle. In both systems ρ is often used instead of r.

The surface element spanning from θ to $\theta + d\theta$, and φ to $\varphi + d\varphi$ on a spherical surface at (constant) radius r is

$$\mathrm{d}S_r = r^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi.$$

Thus the differential solid angle is

$$\mathrm{d}\Omega = \frac{\mathrm{d}S_r}{r^2} = \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi.$$

The volume element spanning from r to $r + dr$, θ to $\theta + d\theta$, and φ to $\varphi + d\varphi$ is

$$\mathrm{d}V = r^2 \sin\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\varphi,$$

where the notation is commonly used in physics (ISO convention). Thus, for example, a function $f(r, \theta, \varphi)$ can be integrated over the ball $B(0, R)$ by the triple integral

$$\int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{R} f(r, \theta, \varphi) r^2 \sin\theta \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\varphi.$$

Sometimes it is convenient to switch to spherical coordinates. Write $\mathbb{S}^{n-1}$ for the unit sphere in $\mathbb{R}^n$. Points $y \in B(a, r)$ have the form $y = a + t\omega$, where $0 \le t < r$ and $\omega \in \mathbb{S}^{n-1}$. Moreover, the change of variables formula yields

$$\int_{B(a,r)} f(y) dy = \int_0^r \int_{S^{n-1}} f(x + t\omega) dt d\omega,$$

where $d\omega = t^{n-1} d\omega_0$ and $d\omega_0$ is corresponding measure on $\mathbb{S}^{n-1}$. For example for $n = 3$, $d\omega = r^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi$.

see [8]

Theorem 3.8. *Let $G \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary. Then every function $f \in W^{1,p}(G)$, $n < p < \infty$, coincides a.e. with a function $\tilde{f} \in C^{0,\alpha}(G)$, where $\alpha = 1 - n/p > 0$.*

Theorem 3.9 (Morrey's inequality,see [8]). *Assume $n < p < \infty$ and set $\alpha = 1 - n/p > 0$. Then there exists a constant C, depending only on p and n, such that*

$$|\tilde{f}|_{C^{0,\alpha}(\mathbb{R}^n)} \leq C|f|_{W^{1,p}(\mathbb{R}^n)}$$

for all $\quad f \in C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.

Rešenje. * Let $y, y' \in \mathbb{R}^n$. From an integral estimate on the gradient of the function u, say
(1) $\int_B |\nabla u| \leq c_0$,
where B is a ball containing y and y', we seek a pointwise estimate of the form $|u(y) - u(y')| \leq c_1|y - y'|^\alpha$, $y, y' \in \mathbb{R}^n$.

Set $L(t) = ty + (1-t)y'$, $v = u \circ L$, $u(y) - u(y') = \int_0^1 v'(t)dt$.
(2) $|u(y) - u(y')| \leq \big(\int_0^1 |\nabla u(L(t))|dt\big)|y - y'|$.

However, the integral on the right hand side of (2) only involves values of ∇u over the segment joining y with y'. If the dimension of the space is $n > 1$, this segment has zero measure. Hence the integral in (2) can be arbitrarily large, even if the integral in (1) is small. To address this difficulty, we shall compare both values $u(y), u(y')$ with the average value u_B of the function u over an(n-1)-dimensional ball centered at the midpoint $z = (y + y')/2$, as shown in Figure 8, left. Notice that the difference $|u(y) - u_B|$ can be estimated by an integral of $|\nabla u|$ ranging over a cone of dimension n. In this way the bound (1) can thus be brought into play.

Hint. see http://math.stackexchange.com/questions/593698/the-proof-of-morreys-inequality-in-evans-book

The idea behind the proof is to examine how much the value of $u(x)$ varies from the average value of u on a ball B that contains x. To start, let $B = B(z, r)$ be a ball of radius r and let $x \in B$. Then

$$|u(x) - \overline{u}_B| = \left| u(x) - \frac{1}{|B|}\int_B u(y)dy \right| = \frac{1}{|B|}\int_B |u(x) - u(y)|dy.$$

It is helpful to re-center the integral at x. Since $B \subset B(x, 2r)$ and $|B| = 2^{-n}|B(x, 2r)|$, we have

$$\frac{1}{|B|}\int_B |u(x) - u(y)|dy \leq \frac{2^n}{|B(x,2r)|}\int_{B(x,2r)} |u(x) - u(y)|dy.$$

The problem at this point is to estimate the last integral using the gradient of u. It is convenient to switch to spherical coordinates. Write $\mathbb{S}^{n-1}$ for the unit sphere in $\mathbb{R}^n$. Points $y \in B(x, 2r)$ have the form $y = x + t\omega$, where $0 \leq t < 2r$ and $\omega \in \mathbb{S}^{n-1}$. Moreover, the change of variables formula yields

$$\int_{B(x,2r)} |u(x) - u(y)|dy = I(x, 2r) := \int_0^{2r}\int_{S^{n-1}} |u(x) - u(x + t\omega)|dtd\omega.$$

Fix $\omega \in S^{n-1}$. The fundamental theorem of calculus applied to the function $\phi(s) = u(x + s\omega)$ states that

$$u(x + t\omega) - u(x) = \int_0^t Du(x + s\omega) \cdot \omega ds, 0 < t < 2r.$$

Since $|\omega| = 1$ this leads to

$$f^+(t,\omega) := |u(x) - u(x + t\omega)| \leq \int_0^t |Du(x + s\omega))|ds \leq \int_0^{2r} |Du(x + s\omega))|ds.$$

Hence

$$\int_0^{2r} f^+(t,\omega)dt \leq 2r \int_0^{2r} |Du(x + s\omega))|ds$$

and therefore

$$I(x, 2r) := \int_0^{2r} \int_{S^{n-1}} |u(x) - u(x + t\omega)|dtd\omega \leq 2rI^+(x, 2r),$$

where $I^+(x, 2r) := \int_0^{2r} \int_{S^{n-1}} |Du(x + s\omega))|dsd\omega$.

At this point we can go back to Cartesian coordinates to note first that $I^+(x, 2r) = \int_{B(x,2r)} |Du(y)|dy$ and hence to get

$|u(x) - u_B| \leq \frac{2^{n+1}r}{|B(x,2r)|} \int_{B(x,2r)} |Du(y)|dy$.

Since $|B(x, 2r)| = c_n 2^n r^n$, Holder's inequality implies

$$\int_{B(x,2r)} |Du(y)|dy \leq \left(\int_{B(x,2r)} |Du(y)|^p dy \right)^{1/p} |B(x, 2r)|^{1/p'} \leq |Du|_p (c_n 2^n r^n)^{1/p'},$$

where $1/p' = 1 - 1/p$, and hence, using again that $|B(x, 2r)| = c_n 2^n r^n$,

$|u(x) - u_B| \leq 2c_n^{-1/p} r^\alpha |Du|_p$.

Note if $p = n$, we have $\alpha = 0$ and therefore $|u(x) - u_B| \leq 2c_n^{-1/p} |Du|_p$.

Finally, if $x, y \in \mathbb{R}^n$ and let B be a ball with diameter (barely larger than) $|x - y|$ containing both x and y to obtain

$$|u(x) - u(y)| = |u(x) - u_B| + |u(y) - u_B| \leq 2^{1+n/p} c_n^{-1/p} |x - y|^\alpha |Du|_p.$$

△

Note that the above proof shows that
(a) if $u \in W^{1,1}$ then $|u(x) - u_B| \preceq \frac{1}{r^{n-1}}$,
(b) if $u \in W^{1,p}$, $1 \leq p \leq n$, then $|u(x) - u_B| \preceq \frac{1}{r^{n/p-1}}$ and
(c) if $u \in W^{1,p}$, $p > n$, then $|u(x) - u_B| \preceq r^\alpha$.

Example 20. *Suppose that* $u \in W^{1,n}(\mathbb{B})$ *and* $B_x = B(x, d(x)/2)$. *Prove that* $|u(x) - u_{B_x}| \to 0$ *if* $|x| \to 0$.

There is $Eu \in W^{1,p}(\mathbb{R}^n)$ *with support contained inside* $B(0; 2)$.

Explain why we can not derive from those results that u *has continuous extension to* $\overline{\mathbb{B}}$.

Since C^∞ is dense in $W^{1,p}$, Morrey's inequality yields:

Corollary 2 (embedding)**.** *Let $G \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary. Assume $n < p < \infty$ and set $\alpha = 1 - n/p > 0$. Then every function $f \in W^{1,p}$ coincides a.e. with a function $\tilde{f} \in C^{0,\alpha}(G)$. Moreover, there exists a constant C such that*

$$|\tilde{f}|_{C^{0,\alpha}} \leq C|f|_{W^{1,p}} \text{for all} \quad f \in W^{1,p}(G).$$

Hint. Let $\tilde{G} := \{x \in \mathbb{R}^n) : d(x, G) < 1\}$ be the open neighborhood of radius one around the set G. By Theorem XX there exists a bounded extension operator E, which extends each function $f \in W^{1,p}(G)$ to a function $Ef \in W^{1,p}(\mathbb{R}^n)$ with support contained inside $\tilde{G}$. We can find a sequence of functions $g_n \in C^1(\mathbb{R}^n)$ converging to Ef in $W^{1,p}(\mathbb{R}^n)$. By Morrey's inequality

$$|g_n(x) - g_m(y)| \leq c|x-y|^\alpha |g_n - g_m|_{W^{1,p}}.$$

□

By Morrey's inequality, if $G \subset \mathbb{R}^n$ and $w \in W^{1,p}$ with $p > n$, then w coincides a.e. with a Holder continuous function. This by itself does not imply that w should be differentiable in a classical sense. Indeed, there exist Holder continuous functions that are nowhere differentiable. However, for functions in a Sobolev space a much stronger differentiability result holds. Theorem 8.1 (almost everywhere differentiability). Let $G \subset \mathbb{R}^n$ and let $u \in W^{1,p}_{loc}$ for some $p > n$. Then u is differentiable at a.e. point $x \in G$, and its gradient equals its weak gradient.

3.2 Eigenvalues and eigenfunctions*

Recall a compact operator is a linear operator L from a Banach space X to another Banach space Y, such that the image under L of any bounded subset of X is a relatively compact subset of Y. Such an operator is necessarily a bounded operator, and so continuous. A crucial property of compact operators is the Fredholm alternative, which asserts that the existence of solution of linear equations of the form $(\lambda K + I)u = f$ (where K is a compact operator, f is a given function, and u is the unknown function to be solved for) behaves much like as in finite dimensions. The spectral theory of compact operators then follows, and it is due to Frigyes Riesz (1918). It shows that a compact operator K on an infinite-dimensional Banach space has spectrum that is either a finite subset of C which includes 0, or the spectrum is a countably infinite subset of C which has 0 as its only limit point. Moreover, in either case the non-zero elements of the spectrum are eigenvalues of K with finite multiplicities (so that $K - \lambda I$ has a finite-dimensional kernel for all complex $\lambda \neq 0$.

An important example of a compact operator is compact embedding of Sobolev spaces, which, along with the Garding inequality and the LaxMilgram theorem, can be used to convert an elliptic boundary value problem into a Fredholm integral equation. Existence of the solution and spectral properties then follow from the theory of compact operators; in particular, an elliptic boundary value problem on a bounded domain has infinitely many isolated eigenvalues. One consequence is that a solid body can vibrate only at isolated frequencies, given by the eigenvalues, and arbitrarily high vibration frequencies always exist. Completely continuous operators

Let X and Y be Banach spaces. A bounded linear operator $T : X \to Y$ is called completely continuous if, for every weakly convergent sequence (x_n) from X, the sequence (Tx_n) is norm-convergent in Y (Conway 1985, VI.3). Compact operators on a Banach space are always completely continuous. If X is a reflexive Banach space, then every completely continuous operator $T : X \to Y$ is compact.

Hilbert spaces

Consider, for example, the difference between strong and weak convergence of functions in the Hilbert space $L^2(R^n)$. Strong convergence of a sequence $\psi_k \in L^2(R^n)$ to an element ψ means that

$$\int_{\mathbf{R}^n} |\psi_k - \psi|^2 \, \mathrm{d}\mu \to 0$$

as $k \to \infty$. Here the notion of convergence corresponds to the norm on L^2.

In contrast weak convergence only demands that

$$\int_{\mathbf{R}^n} \bar{\psi}_k f \, \mathrm{d}\mu \to \int_{\mathbf{R}^n} \bar{\psi} f \, \mathrm{d}\mu$$

for all functions $f \in L^2$ (or, more typically, all f in a dense subset of L^2 such as a space of test functions, if the sequence ψ_k is bounded). For given test functions, the relevant notion of convergence only corresponds to the topology used in C.

For example, in the Hilbert space $L^2(0, \pi)$, the sequence of functions

$\psi_k(x) = \sqrt{2/\pi} \sin(kx)$

form an orthonormal basis. In particular, the (strong) limit of ψ_k as $k \to \infty$ does not exist. On the other hand, by the RiemannLebesgue lemma, the weak limit exists and is zero.

The Hilbert-Schmidt theorem, also known as the eigenfunction expansion theorem, is a fundamental result concerning compact, self-adjoint operators on Hilbert spaces. In the theory of partial differential equations, it is very useful in solving elliptic boundary value problems. Statement of the theorem

Let $(H, <, >)$ be a real or complex Hilbert space and let $A : H \to H$ be a bounded, compact, self-adjoint operator. Then there is a sequence of non-zero real eigenvalues λ_i, i = 1, ..., n, with n equal to the rank of A, such that $|\lambda_i|$ is monotonically non-increasing and, if $n = +\infty$,

$\lim_{i \to +\infty} \lambda_i = 0$.

Furthermore, if each eigenvalue of A is repeated in the sequence according to its multiplicity, then there exists an orthonormal set $\varphi_i, i = 1, ..., n$, of corresponding eigenfunctions, i.e.

$A\varphi_i = \lambda_i \varphi_i$ for $i = 1, \ldots, n$.

Moreover, the functions φ_i form an orthonormal basis for the range of A and A can be written as

$Au = \sum_{i=1}^{N} \lambda_i \langle \varphi_i, u \rangle \varphi_i$ for all $u \in H$.

f is in M_L if f $C^2(G) \cap C^1(\overline{G})$ and $Lf \in L_2(G)$.

The sets of eigenfunctions is complete in $L_2(G)$

f in M_L

$f(x) = \sum_{k=1}^{\infty} (f, X_k) X_k(x)$ see [6] \$3.5 Example 2

Spectral theorem. For every compact self-adjoint operator T on a real or complex Hilbert space H, there exists an orthonormal basis of H consisting of eigenvectors of T. More specifically, the orthogonal complement of the kernel of T admits, either a finite orthonormal basis of eigenvectors of T, or a countably infinite orthonormal basis $\{e_n\}$ of eigenvectors of T, with corresponding eigenvalues $\{\lambda_n\}$ in $\mathbb{R}$, such that $\lambda_n \to 0$.

$(\lambda - m + 1)Au = u$ λ is eigenvalues for L iff $(\lambda - m + 1)^{-1}$ for $A : H_0^1(G) \to H_0^1(G)$

Set $s_k = \lambda - m + 1$ and $\tilde{u}_k = u_k / s_k$. Then $s_k A u_k = u_k$

$f \in H_0^1(G)$ $Af = \sum (\tilde{Af})_k \tilde{u}_k$ Hence since exists A^{-1}, $f = \sum \tilde{f}_k \tilde{u}_k$

see [6] \$3.5

4 Elliptic differential equations*

Any differential operator exhibiting this property is called a hypoelliptic operator; thus, every elliptic operator is hypoelliptic. The property also means that every fundamental solution of an elliptic operator is infinitely differentiable in any neighborhood not containing 0.

As an application, suppose a function f satisfies the Cauchy-Riemann equations. Since the Cauchy-Riemann equations form an elliptic operator, it follows that f is smooth. General definition

strong and weak derivatives, elliptic equation in the divergence form

Set $L_2 u = a^{ij}(x) D^2_{ij} u$, $L_1 u = b^i(x) D_i u$, $L_0 u = cu$, $M = L_2 + L_1$ and $L = L_2 + L_1 + L_0$; consider the equation $Lu = f$.

Theorem 4.1 (Hopf's Lemma)**.** *Assume $u \in C^2(U) \cap C(\overline{U})$, $Lu \geq 0$ in U, and there exists a point $x_0 \in \partial U$ such that $u(x_0) > u(x)$ for all $x \in U$. Assume finally that U satisfies the interior ball condition at x_0; that is, there exists an open ball $B \subset U$ such that $x_0 \in \partial B$. Then If $c \equiv 0$, then $D_n u(x_0) > 0$ where n is the outer normal to B at x_0.*
If $c \leq 0$, then the same holds if $u(x_0) \geq 0$.

It is clear that we can simply take $U = B$. The idea is to construct an auxiliary function v with $D_n v > 0$ on ∂B and furthermore $u + \varepsilon v$ still reaches maximum at x_0. Without loss of generality, assume B is in fact the ball $B_r(0)$. As we cannot specify where x_0 is, necessarily v should be radially symmetric. Furthermore since we would like $u + \varepsilon v$ to reach maximum at x_0, we should take $v = 0$ on ∂B. Guided by this, we set $v(x) = e^{-\alpha|x|^2} - e^{-\alpha r_0^2}$. Or equivalently we need $Lv \geq 0$. We compute $Mg = e^{-\alpha|x|^2}P$, where $P = 4\alpha^2 \sum a_{ij}x_ix_j - 2\alpha \sum a_i x_i - 2\alpha \sum a_{ii}$

Now it is clear that, no matter what α we choose, $Lv \geq 0$ cannot hold in the whole ball B. However, if we consider the annular region $A = B \setminus B_{r_0/2}(0)$, then we can take α large enough that $Lv \geq 0$ in A. $Mu > 0$ in A if α is sufficiently large. To apply the weak maximum principle, we need (i) $u(x) + \varepsilon v(x) \leq u(x_0)$ for all $|x| = r_0/2$. This is done as follows. Since $u(x_0) > u(x)$ for all $|x| = r_0/2$ as long as ε is small enough we have (i).

We also have $u(x_0) - u(x) \geq \varepsilon(v(x) - v(x_0))$

set $r = |x|$, we find $v'(r) = -2\alpha r e^{-\alpha r^2}$ and therefore $u(x_0) - u(x) \geq c(|x_0| - |x|) = c d_U(x)$.

Example 21. *Let $a \neq 0$. A radial mapping f_a in n-space is given by: $f(X) = f_a(X) = |X|^{a-1}X$, where $X \in \mathbb{R}^n$. Prove*
(i.2) $K_I(f) = |a|$, $K_O(f) = |a|^{n-1}$ if $|a| \geq 1$; in particular $K(f_3) = K_O(f_3) = 3^{n-1}$;
$K_I(f) = |a|^{1-n}$, $K_O(f) = |a|^{-1}$ if $|a| \leq 1$.
In particular, for $n = 3$, $X = (x, y, z) \in \mathbb{R}^3$,
$K_I(f) = |a|$, $K_O(f) = |a|^2$ if $|a| \geq 1$;
Now we consider 3-space.
(i.3) For $a = 3$ set $g = f_3$; then $\partial_k g(0) = 0$, $\partial^2_{ij} g(0) = 0$ and $g^1(X) = x^3 + xy^2 + xz^2$. An easy computation shows that $\Delta g^1 = 10x$ and if $c(X) = -10(x^2 + y^2 + z^2)^{-1}$ then $\Delta g^1 + c g^1 = 0$ and $\Delta g + c g = 0$.

Suppose that F is mapping from a domain $G \subset \mathbb{R}^n$ (in particular, from the unit ball $\mathbb{B} \subset \mathbb{R}^n$) onto a bounded convex domain $D = F(G)$.

To every $a \in \partial D$ we associate a nonnegative function $u = \underline{u}_a = \underline{F}^a$. Since D is convex, for $a \in \partial D$, there is a supporting hyper-plane (a subspace of dimension $n-1$) Λ_a defined by $\Lambda_a = \{w \in \mathbb{R}^n : (w - a, n_a) = 0\}$, where $n = n_a \in T_a\mathbb{R}^n$ is a unit vector such that $(w - a, n_a) \geq 0$ for every $w \in \overline{D}$. Define $u(z) = \underline{F}^a(z) = (F(z) - a, n_a)$.

If a vector valued function f satisfies the equation $Lu = 0$, then $\underline{f}^a$ satisfies the same equation. If $c \leq 0$, we can apply the maximum principle.

Suppose that h satisfies the equation $Lu = 0$ and maps the unit ball $\mathbb{B} \subset \mathbb{R}^n$ onto a bounded convex domain $D = h(\mathbb{B})$, which contains the ball $B(h(0); R_0)$. Then

(i.1) $d(h(z), \partial D) \geq (1 - |z|)\overline{c} R_0, \quad z \in \mathbb{B}$.

$Lu = D_i(a^{ij}(x)D_j + b^i(x)u) + c^i(x)D_i u + d(x)u$

We suppose that operator L uniform (strongly) elliptic, ie.

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad x \in \Omega, \xi \in \mathbb{R}^n\,. \tag{81}$$

$L(u, v) = \int_\Omega [(a^{ij}(x)D_j + b^i(x)u)D_i v - (c^i(x)D_i u + d(x)u)v]\, dx$ for all $v \in C^1_0(\Omega)$.

$\mathcal{L}(u, v) \leq c|u|_{1,2}|v|_{1,2}$

$Lu = g + D_i f^i$, $F(v) = \int_\Omega (f^i D_i v - gv) dx$, $F \in H^*$

$H = W^{1,2}_0$ $I_u v = \int_\Omega uv dx$

Suppose that $\varphi \in W^{1,2}(\Omega)$ $g, f^i \in L^2(\Omega)$, $i = 1, 2, \cdots, n$,
Then generalized Dirichlet problem $Lu = g + D_i f^i$ in Ω, $u = \varphi$ on $\partial\Omega$, has unique solution.

$w = u - \varphi$, $Lw = Lu - L\varphi = \hat{g} + D_i \hat{f}^i$, $w \in W_0^{1,2}$

$\underline{g} = (g, f^1, \cdots, f^n)$

Suppose that $u \in W^{1,2}(\Omega)$ weak solution $Lu = f$,uniform (strongly) elliptic, coefficients a^{ij}, b^i, $i, j = 1, 2, \cdots, n$ uniformly lipschitz continuous in Ω, coefficients $c^i, d \in L^\infty(\Omega)$, $i = 1, 2, \cdots, n$, function f belong $L^2(\Omega)$. Then for every $\Omega' \subset\subset \Omega$, u belong $W^{2,2}(\Omega')$ and the estimate holds

$$|u|_{W^{2,2}(\Omega')} \leq c(|u|_{W^{1,2}(\Omega')} + |f|_{L^2(\Omega)}).$$

$Lu = f$ a.e. in Ω, where $Lu = (a^{ij}(x)D_{ij}u + (D_j a^{ji} + b^i(x) + c^i(x))D_i u + (D_i b^i + d(x))u$.

By induction, coefficients a^{ij}, b^i belong $C^{k,1}(\overline{\Omega})$, coefficients c^i, d belong $C^{k-1,1}(\overline{\Omega})$, function f belong $W^{k,2}(\Omega)$, $k \geq 1$. Then for every $\Omega' \subset\subset \Omega$, u belong $W^{k+2,2}(\Omega')$ and the estimate holds

$$|u|_{W^{k+2,2}(\Omega')} \leq c(|u|_{W^{1,2}(\Omega')} + |f|_{W^{k,2}(\Omega)}).$$

Let $Lu = (a^{ij}(x)D_{ij}u + b^i D_i u + c(x)u$ be uniform (strongly) elliptic, $a^{ij} \in C^{0,1}(\overline{\Omega}), b^i, c \in L^\infty(\Omega)$, $c \leq 0$. Then for $L^2(\Omega)$ and $\varphi \in W^{1,2}(\Omega)$, there is unique $u \in W^{1,2}(\Omega) \cap W^{2,2}_{loc}(\Omega)$ such that $Lu = f$ in Ω and $u - \varphi \in W_0^{1,2}(\Omega)$.

if $\partial\Omega$ enough smooth $\varphi \in W^{2,2}(\Omega)$ the result holds under condition that a^{ij} are continuous on $\overline{\Omega}$. However for discontinuous coefficient the result doe not hold as the following example shows.

$\Delta u + b\frac{x_i x_j}{|x|^2} D_{ij} u = 0$, $b = -1 + \frac{n-1}{1-\lambda}$

for $n > 2(2 - \lambda)$ has two solutions $u_1(x) = 1$ and $u_2(x) = |x|^\lambda$ in $W^{2,2}(B)$, which coincide on S.

Generalizing the maximum principle for harmonic functions which was already known to Gauss in 1839, Eberhard Hopf proved in 1927 that if a function satisfies a second order partial differential inequality of a certain kind in a domain of $\mathbb{R}^n$ and attains a maximum in the domain then the function is constant. The simple idea behind Hopf's proof, the comparison technique he introduced for this purpose, has led to an enormous range of important applications and generalizations.

Proposition 1: If B is positive definite matrix and A non- positive definite matrix, then AB is non positive definite matrix and in particular $tr(AB) \leq 0$.

Let $a : R^2 \to R^+$ be arbitrary function, $u_0(z) = 1 + x^2 + y^2$, $c = \frac{4a}{1+x^2+y^2}$ and $Lu = a\Delta u + cu$.

L is elliptic and $u_0(z) = 1 + x^2 + y^2$ is solution of equation $Lu = 0$.

Theorem 4.2 (Maximum principle). *Let $u = u(x), x = (x_1, \cdots, x_n)$ be a C^2 function in Ω which is continuous on $\overline{\Omega}$ and which satisfies the differential inequality*

$$Lu = \sum_{ij} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial u}{\partial x_i} \geq 0$$

in an open domain Ω, where the symmetric matrix $a_{ij} = a_{ji}(x)$ is locally uniformly positive definite in Ω and the coefficients $a_{ij}, b_i = b_i(x)$ are locally bounded.

Then $\sup_\Omega u = \sup_{b\Omega} u$.

If u has local maximum in $x_0 \in \Omega$, then $a^{ij}(x_0)D_{ij}u(x_0) \leq 0$.

Weak maximum principle

Case 1. Suppose first that $Lu > 0$. If u has local maximum in $x_0 \in \Omega$, then $Du(x_0) = 0$ and matrix $D := D^2u((x_0)$ non-positive and in particular $D_{ii}u(x_0) \leq 0$. Since L is elliptic, the matrix $A = [a_{ij}]$ is positive definite in Ω and in particular $a_{ii}(x_0) > 0$. Hence, by Proposition 1, $Lu(x_0) = \sum_{ij} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) = tr(DA) \leq 0$ which is a contradiction.

Case 2. There is a such that $L(e^{ax_1}) > 0$. Then for every $\epsilon > 0$, $L(u + \epsilon e^{ax_1}) > 0$. By proved in Case 1, $\sup_\Omega(u + \epsilon e^{ax_1}) = \sup_{b\Omega}(u + \epsilon e^{ax_1})$. Letting ϵ tends 0 we obtain $\sup_\Omega u = \sup_{b\Omega} u$.

Theorem 4.3. *Let L be elliptic in Ω and let $c \leq 0$. Suppose that u and v belong $C^2(\Omega) \cap C^0(\overline{\Omega})$ and that $Lu = Lv$ in Ω and $u = v$ on $b\Omega$. Then $u = v$ in Ω. If $Lu \geq Lv$ in Ω and $u \leq v$ on $b\Omega$, then $u \leq v$ in Ω.*

Question. If u takes a maximum value M in Ω whether $u = M$? Yes. It is The strong Maximum principle; the proof is based on Hopf lemma.

It is usually thought that the Hopf maximum principle applies only to linear differential operators L. In particular, this is the point of view taken by Courant and Hilbert's Methods of Mathematical Physics. In the later sections of his original paper, however, Hopf considered a more general situation which permits certain nonlinear operators L and, in some cases, leads to uniqueness statements in the Dirichlet problem for the mean curvature operator and the MongeAmpre equation.

- Mariano Giaquinta, Luca Martinazzi, An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps, 2012 Scuola Normale Superiore Pisa

$u \in C^k(\overline{\Omega})$ if all derivatives of order $\leq k$ have continuous extension to $\overline{\Omega}$.

Let $C^k(\overline{\Omega})$ denote family of functions (mappings) which belong $C^k(\Omega)$ and all derivatives of order $\leq k$ have continuous extension to $\overline{\Omega}$.

A bounded domain Ω in $\mathbb{R}^n$ and its boundary belong to class $C^{k,\alpha}$, $0 \leq \alpha \leq 1$ if for every point $x_0 \in \partial\Omega$ there exists a ball $B = B(x_0)$ and mapping ψ from B onto D such that ([2], p. 95):

(i) $\psi(B \cap \Omega) \subset R^n_+$;
(ii) $\psi(B \cap \partial\Omega) \subset \partial R^n_+$;
(iii) $\psi \in C^{k,\alpha}(B)$, $\psi^{-1} \in C^{k,\alpha}(D)$.

Theorem 6.19 [2]:Let Ω be of class $C^{k,\alpha}$. If $f \in C^{k,\alpha}(\overline{\Omega})$, then $P[f] \in C^{k,\alpha}(\overline{\Omega})$, $k \geq 2$; for k=1 it is proved in Gilbarg-Hörmander in Arch. Ration. Mech. Anal. GERMANY

Gilbarg, David; Hörmander, Lars Intermediate Schauder estimates. Arch. Rational Mech. Anal. 74 (1980), no. 4, 297–318. 35J25 (see also references in [2])

Theorem 4.4 (Lemma 6.18 [2]). *Let boundary of Ω contain part T of class $C^{2,\alpha}$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$ $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, $u = \varphi$ on T, $Lu = f$ on Ω, f and coefficients of L belong $C^\alpha(\overline{\Omega})$.*
Then $u \in C^{2,\alpha}(\Omega \cup T)$.

[?], Ch 7, quasi-linear equation

The linear form

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + F = 0 \tag{82}$$

was considered previously with the possibility of the capital letter coefficients being functions of the independent variables. If these coefficients are additionally functions of u which do not produce or otherwise involve derivatives, the equation is called quasilinear. It must be emphasized that quasilinear equations are not linear, no superposition or other such blessing; however these equations receive special attention. In general, $A = A(x, y, u_x, u_y)$. Recall A priori estimate

Using the Sobolev embedding theorem, one can prove:

Theorem 4.5 (Corollary, 8.11 [2]). *Let a^{ij}, b^i, c^i, d, f belong $C^\infty(\Omega)$ and let $u \in W^{1,2}(\Omega)$ be weak solution of strongly elliptic equation $Lu = f$. Then u also belongs to $C^\infty(\Omega)$.*

Let Ω and Ω' be domain in R^n and $\Omega \subset \Omega'$. Then for every $f \in C^k(\overline{\Omega})$ (resp $f \in H^k(\Omega)$) there is an extension $F \in C^k(\overline{\Omega'})$ (resp $F \in H^k(\overline{\Omega'})$) finite in Ω' such that

$|F|_{H^k(\Omega')} \leq C|f|_{H^k(\Omega)}$.

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https://www.math.psu.edu/bressan/PSPDF/sobolev-notes.pdf

https://terrytao.wordpress.com/2009/04/30/245c-notes-4-sobolev-spaces/

Theorem 4.6 (Exercise 15 (Schauder estimate, Tao [9])). *Let $0 < \alpha < 1$, and let $f \in C^{0,\alpha}(\mathbf{R}^3)$ be a function supported on the unit ball $B(0,1)$. Let u be the unique bounded solution to the Poisson equation $\Delta u = f$ (where $\Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian), given by convolution with the Newton kernel:*

$$u(x) := \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{f(y)}{|x-y|}\, dy.$$

(i) Show that $u \in C^0(\mathbf{R}^3)$.

(ii) Show that $u \in C^1(\mathbf{R}^3)$, and rigorously establish the formula

$$\frac{\partial u}{\partial x_j}(x) = -\frac{1}{4\pi} \int_{\mathbf{R}^3} (x_j - y_j) \frac{f(y)}{|x-y|^3}\, dy$$

for $j = 1, 2, 3$.

(iii) Show that $u \in C^2(\mathbf{R}^3)$*, and rigorously establish the formula*

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \frac{1}{4\pi} \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} [\frac{3(x_i - y_i)(x_j - y_j)}{|x-y|^5} - \frac{\delta_{ij}}{|x-y|^3}] f(y)\, dy$$

for $i, j = 1, 2, 3$*, where* δ_{ij} *is the Kronecker delta. (Hint: first establish this in the two model cases when* $f(x) = 0$*, and when* f *is constant near* x*.)*

(iv) Show that $u \in C^{2,\alpha}(\mathbf{R}^3)$*, and establish the Schauder estimate*

$$\|u\|_{C^{2,\alpha}(\mathbf{R}^3)} \leq C_\alpha \|f\|_{C^{0,\alpha}(\mathbf{R}^3)}$$

where C_α *depends only on* α.

(v) Show that the Schauder estimate fails when $\alpha = 0$*. Using this, conclude that there exists* $f \in C^0(\mathbf{R}^3)$ *supported in the unit ball such that the function* u *defined above fails to be in* $C^2(\mathbf{R}^3)$*. (Hint: use the closed graph theorem.) This failure helps explain why it is necessary to introduce Hölder spaces into elliptic theory in the first place (as opposed to the more intuitive* C^k *spaces).*

4.1 Green-Laplacian formula 1, 2

We will prove:

Proposition 11 (Interior estimate)**.** *Let* $s : \overline{\mathbb{B}} \to \mathbb{R}$ *be a continuous function from the closed unit ball* $\overline{\mathbb{U}}$ *into the real line satisfying the conditions:*

1. s *is* C^2 *on* $\mathbb{U}$*,*

2. $s_b(\theta)$ *is* C^2 *on* $\mathbb{S}$ *and*

3. $|\Delta s| \leq a_0 |\nabla s|^2 + b_0$*, on* $\mathbb{B}$ *for some constants* a_0 *and* b_0 *(the last inequality we will call Poisson-Laplace type inequality or the interior estimate inequality with constants* a_0 *and* b_0*).*

Then the function $|\nabla s|$ *is bounded on* $\mathbb{B}$.

Suppose that

(a1) Let D *be a domain contained in the unit ball and* $X : D \to [-1, 1]$ C^2*-vector function.*

There are positive constants a, b, c, d, e *such that:*

(b1) $\mathfrak{D}(X, K_{n-2}, B) \leq cr^2 + dM_X(x, r)$ *for every ball* $B = B(x, r] \subset D$

(c1) $|\nabla X(x)| \leq eM_X(x, r) + a\mathfrak{D}(X, K_{n-1}, B) + br$.

Theorem 4.7. *Suppose: (a) Let* D *be a domain contained in the unit ball and* $X : D \to [-1, 1]$ C^2*-vector function.*

(b) X *satisfy the Poisson-Laplace differential inequality with parameters* a, b *on* D.

Then there exists a fixed positive constants $c = c(a,b,\alpha,\beta,\gamma)$ such that for every closed ball $B(x_0,R_0)$ in D, the following inequality holds

$$|\nabla X(x_0)| \leq c(1+\frac{M(x_0,R_0)}{R_0}), \tag{83}$$

where $K = M(x_0,R_0) = M_\chi(w_0,R_0) := \max\{|X(x)-X(x_0)| : x \in \overline{B}(x_0,R_0)\}$.

(c) There exists a real-valued function Φ defined on interval $[-r,r]$ of class C^2, where $r = 1+\epsilon$, $\epsilon > 0$, such that $|\nabla\Phi| \leq \alpha$ on $[-1,1]$ and

$\Delta\underline{\Phi} \geq \beta(|\nabla X|^2) - \gamma$,

where $\underline{\Phi} = \Phi \circ X$ and α,β,γ are positive constants.

Then there exists a fixed positive constants $c = c(a,b,\alpha,\beta,\gamma)$ such that for every closed ball $B(x_0,R_0)$ in D, the following inequality holds

$$|\nabla X(x_0)| \leq c(1+\frac{M(x_0,R_0)}{R_0}), \tag{84}$$

where $K = M(x_0,R_0) = M_\chi(w_0,R_0) := \max\{|X(x)-X(x_0)| : x \in \overline{B}(x_0,R_0)\}$.

Proof of Proposition 11. Let $\Phi(x) = e^{\lambda x}$, $x \in \mathbb{R}$. Then $|\Delta\underline{\Phi} = \lambda^2 e^{\lambda s}(|\nabla s|^2 + \Delta/\lambda)$ and therefore hypothesis (c) is satisfied. By an application of Theorem 4.7 we have Proposition 11. □

Hint for Theorem 4.7. Using Green -Laplacian formula for derivatives of X, we estimate $|\nabla X(x_0)|$ by K and a functional $V_X^+(a,r)$ which is integral of $B(a,r)$ and a majorant of the derivative of Newton potential of ΔX and using the hypothesis (b) by the Dirichlet integral of X .

In order to estimate the Dirichlet integral by the oscillation we first use a corollary of a version of Green-Laplacian formula (see (93)) to estimate the majorant of derivative of the Newtonian potential of ΔX from above by the oscillation $M_X(a,r)$. Then using the hypothesis (c) of Theorem 4.7 we estimate $V_X^+(a,r)$ (for convenient choose of $B(a,r)$) from below by the Dirichlet integral of X and we also use the inequality (99).

The inverse of a point P with respect to a reference circle (sphere) $S(O,r)$ with center O and radius r is a point P', lying on the ray from O through P such that $OP \times OP' = r^2$. We also use notation x^* for the inverse of a point x with respect to $S(o,R)$, so we have

$$x^* = J_R x = \frac{R^2}{|x|^2}x. \tag{85}$$

Set $B_R = B(0,R]$. If $x,\xi \in B_R$, $r = |x|$ and $A = \langle\xi,x\rangle$, then $|x-\xi|^2 = |\xi|^2 - 2A + |x|^2$ and $|\xi - x^*|^2 = |\xi|^2 - 2\frac{R^2}{r^2}A + |x^*|^2$.

Then (1) $|x-\xi^*| \geq |\xi - x|$ iff $\frac{R^4}{r^2} - 2\frac{R^2}{r^2}A \geq -2A + r^2$, that is (2) $R^4 - r^4 \geq 2(R^2-r^2)A$.

Since $|A| \geq |\xi||x|$, (2) holds and therefore (1).

$|x| = R$, then $|x||\xi - x^*| = R|x-\xi|$

$$g(x,\xi) = \Gamma(x-\xi) - \Gamma(|x||\xi - x^*|/R), x \neq 0 \quad (86)$$
$$g(x,\xi) = \Gamma(\xi) - \Gamma(R), x = 0. \quad (87)$$

$$G(x,y) = G(y,x), G(x,y) \leq 0, x, y \in \overline{B}_R. \quad (88)$$

Set $G^+(x,y) = |G(x,y)|$. Check that (a) $G(x,y,r) = r^{2-n}G(x/r, y/r)$, $x, y \in \overline{B}_r$,
$D_kG(x,y,r) = r^{2-n}D_kG(x/r,y/r)1/r = r^{1-n}D_kG(x/r,y/r)$ and therefore
$|D_kG(x,y,r)| \leq 2c_nr^{1-n}\frac{1}{|x/r-y/r|^{n-1}} = 2c_n|x-y|^{1-n}$.

(b) $|B_r| = \sigma_n r^n/n$, $G^+(0,y,r) = c_n(\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}})$

(c) $\int_{B_r} \frac{1}{|y|^{n-2}}dy = \sigma_n \int_0^r \rho d\rho = \sigma_n r^2/2$.

Set $E^1(x,\xi) = |\xi - x|^{2-n}$, $E^2(x,\xi) = (|x||\xi - x^*|/R)^{2-n}$, $U_1(x,\xi) = \Gamma(x-\xi)$ and $U_2(x,\xi) = \Gamma(|x||\xi - x^*|/R)$. Partial derivative wr ξ,

$$D_kU_1 = c_n(\xi_k - x_k)|\xi - x|^{-n} = c_n(\xi_k - x_k)E^1(x,\xi)|\xi - x|^{-2} \quad (89)$$
$$D_kU_2 = c_n|\frac{x}{R}|^{2-n}(\xi_k - \overline{x}_k)|\xi - \overline{x}|^{-n} = c_n(\xi_k - \overline{x}_k)E^2(x,\xi)|\xi - \overline{x}|^{-2} \quad (90)$$

By (1), for $|x| \geq R/2$, $|D_kg| \leq c|\xi - x|^{-n+1}$.

$|D_kU_1| \leq c_nE^1(x,\xi)|\xi - x|^{-1}$ and $|D_kU_2| \leq c_nE^2(x,\xi)|\xi - \overline{x}|^{-1}$.

It is clear that $|D_kU_1| \leq c_n|\xi - x|^{1-n}$. By (1) and $E^2(x,\xi) \leq E^1(x,\xi)$, $|D_kU_2| \leq c_n|\xi - x|^{1-n}$ and therefore

$$|D_kG| \leq 2c_n|\xi - x|^{1-n}. \quad (91)$$

and at point $\xi \in S$,
$D_kE(\xi,x) = c_n|\frac{x}{R}|^2(\xi_k - \overline{x}_k)|\xi - x|^{-n}$
$D_i\Gamma(x-y) = \frac{1}{n\omega_n}(x_i - y_i)|x-y|^{-n}$
Since $x_k^* = (J_Rx)_k = \frac{R^2}{|x|^2}x_k$,
$D_kg(\xi,x) = c_n\frac{R^2-|x|^2}{R^2}\xi_k|\xi - x|^{-n}$
Hence

$$D_ng(\xi,x) = \langle Dg, \xi\rangle = c_n\frac{R^2-|x|^2}{R^2}|\xi - x|^{-n} \quad (92)$$

Set $B = B(x_0,R)$ and $S = S(x_0,R)$ and $g = g_B$. Suppose that $u \in C^2(\overline{B})$. Then
(i1) $u(x_0) = \int_S P(x_0,x)u(x)d\sigma(x) + \int_B g(x_0,x)\Delta u(x)dx$.

The function (i2) $u_1(y) = \int_S P(y,x)u(x)d\sigma(x)$ and (i3) $V(y) = \int_B G(y,x)\Delta u(x)dx$. By (i1), $u(y) = u_1(y) + V(y)$. Write $y \in S$ in the form $y = x_0 + R\omega$, where $\omega \in \mathbb{S}$. Note that (i4) for $y \in S$, $g(y,x) = 0$ and therefore $V(y) = 0$.

Since $u_1(y)$ is harmonic in y, by the mean value theorem
(i5) $\int_{\mathbb{S}} u_1(x_0 + R\omega)d\sigma = u_1(x_0)$.

Hence $I = \int_{\mathbb{S}}[u(x_0 + R\omega) - u(x_0)]d\sigma = -V(x_0)$.

Hence $-V(x_0) \le M_u(x_0, R)$ and

$$|V(x_0)| \le M_u(x_0, R). \tag{93}$$

Proof of Theorem 4.7. Differentation of (i3) yields $D_k V(y) = \int_B D_k G(y,x)\Delta u(x)dx$. By (91), $|D_k V(y)| \le \int_B |D_k G(y,x)||\Delta u(x)|dx \le 2c_n \int_B |y-x|^{1-n}|\Delta u(x)|dx$.

In order to estimate the quantity $|\nabla u|^2(x)$ in the ball $B(x_0, R_0)$ we introduce $M = \max_{B(x_0,R_0)}(R_0 - |x-x_0|)|\nabla X|(x)$. Thus $R_0|\nabla X|(w) \le M$. We will show that $M \le c(K + R_0)$, where $K = M_X(w_0, R_0)$.

It is clear that $M = (R_0 - |x_1 - a|)|\nabla X|(x_1)$. Now let $d = R_0 - |x_1 - x_0|$ and let $0 < q < 1$.

If $x \in B(x_1, r)$, then $|x - x_0| \le |x - x_1| + |x_1 - x_0| \le r + R_0 - d$ and therefore $d(x) = R_0 - |x - x_0| \ge R_0 - (r + R_0 - d) \ge d - r$.

If $r = \lambda d$, then $d(x) \ge \lambda d(1-\lambda)$, and $d(1-\lambda)|\nabla u|(x) \le M$.

Set

$$\mathfrak{D}(X, a, r) := \int_{B(a,r)} |\nabla X|^2(x)dx, \tag{94}$$

$$V_X^+(a,r) := V_{\Delta X}^+(a,r) = K_1(\Delta X, a, r) = \mathfrak{D}(X, K_{n-1}, B) = \int_{B(a,r)} \frac{|\Delta X|}{|x-a|^{n-1}}dx, \tag{95}$$

$$I_X(a,r) = K_S^1(|\nabla X|^2) := \int_{S(a,r)} \frac{|\nabla X|^2(x)}{|x-a|^{n-1}}d\sigma, \tag{96}$$

$$G_X(a,r) = G(|\nabla X|^2; a, r) = \mathfrak{D}(X, K_{n-2}, B) := \int_{B(a,r)} G^+(a,x,r)|\nabla X|^2(x)dx, \tag{97}$$

$$G_X^2(a,r) = G_{\Delta X}(a,r) = N(\Delta X, B) := \int_{B(a,r)} G^+(a,x,r)|\Delta X|(x)dx. \tag{98}$$

Set $B_1 = B(a, r_1)$. For $0 < r_1 < r$ and $x \in B_1$, $G^+(a,x,r) = \frac{1}{|x-a|^{n-2}} \ge \frac{1}{r_1^{n-2}}$ and therefore

$$G(|\nabla X|^2; a, r) \ge \frac{1}{r_1^{n-2}}\mathfrak{D}(X, a, r_1). \tag{99}$$

Since using spherical coordinate $\int_B |x-a|^{1-n}dx = c\int_0^r d\rho = cr$,

$I_u(x_1, r) \le c\frac{M^2\lambda}{d(1-\lambda)^2}$. If we set $A(\lambda) = c\lambda(1-\lambda)^2$, we find

$M/d \le \frac{K}{d\lambda} + M^2 A(\lambda)/d + c_0 r$. If we multiple this inequality by d and remember that $d \le R_0$, we obtain $M \le \frac{K}{\lambda} + M^2 A(\lambda) + c_0 R_0^2$. If we set $C(\lambda) = \frac{K}{\lambda} + c_0 R_0^2$, we have $A(\lambda)C(\lambda)$ tends to K, if λ tends 0.

We need to improve this estimate. We can adapt the procedure from [24] or consider V.

?? Since $|\nabla u_1|$ is bounded, then $|\nabla u| \le |\nabla u_1| + |\nabla V| \le c + |\nabla V|$. Hence V satisfies the differential Laplacian-gradient inequality.

If we use the above notation by V instead of u, we obtain $M \le M^2 A(\lambda) + c_0 R_0^2$. $\square$

5 Beltrami equation and Absolute Continuity on Lines

In this section we study differentiability of quasiconformal mappings. Our main goal is to outline a proof that a quasiconformal mapping is differentiable almost everywhere in the sense of the Lebesgue measure. We also generalize the analytic definition of quasiconformal mappings which was earlier studied in the special case of quasiconformal diffeomorphisms only. We sketch some proofs or state only some results. For details we refer to [**?**, **?**, **?**].

Let f be a complex valued function defined on a subinterval $[a, b]$ of $\mathbb{R}$. Suppose that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(a_k) - f(b_k)| < \varepsilon$$

for all $a = a_1 < b_1 \leq \ldots \leq a_n < b_n = b$ such that

$$\sum_{k=1}^{n} |a_k - b_k| < \delta.$$

Then f is said to be *absolutely continuous* in $[a, b]$. Obviously, an absolutely continuous function is continuous. It is easy to see (see [7]) that an absolutely continuous function is differentiable a.e. in $[a, b]$.

Recall that if $f \in L^1[a, b]$ and

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \quad a \leq x \leq b\,,$$

then F is absolutely continuous on the segment $[a, b]$. If F is absolutely continuous on the segment $[a, b]$, then F' exist a.e. on $[a, b]$, $F' \in L^1[a, b]$, and

$$F(x) - F(x_0) = \int_{x_0}^{x} f(t)\mathrm{d}t,$$

where $f = F'$. Then we say $F' = f$ in the distributional sense.

We make the following assumptions: the function F is continuous on the segment $[a, b]$, F is differentiable a.e on the segment $[a, b]$, and $F' \in L^1[a, b]$. This does not imply that, in general, F is absolutely continuous on the segment $[a, b]$. The standard counterexample is the Cantor function.

A continuous real function is said to be *absolutely continuous on lines* (ACL) in $\Omega \subset \mathbb{C}$ if for each rectangle $[a, b] \times [c, d] \subset \Omega$ the functions $x \mapsto u(x + iy)$ are absolutely continuous on $[a, b]$, for almost all $y \in [c, d]$, and the functions $y \mapsto u(x + iy)$ are absolutely continuous on $[c, d]$, for almost all $x \in [a, b]$. If Ω and Ω' are domains in $\mathbb{C}$, a homeomorphism $f\colon \Omega \to \Omega'$ is called ACL if the restriction of f to $\Omega \setminus \{\infty, f^{-1}\infty\}$ is in ACL.

An ACL-mapping $f\colon \Omega \to \mathbb{C}$ has partial derivative a.e. in Ω. The function f is said to be ACL^p, where $p \geq 1$, if the partial derivatives of f are locally L^p-integrable. In this case, we also say f has L^p-generalized derivatives. For convenience, we have restricted ourselves to continuous mappings (see [?]). A homeomorphism $f\colon \Omega \to \Omega'$ is called ACL^p if the restriction of f to $D \setminus \{\infty, f^{-1}\infty\}$ is ACL^p.

5.1 QC

5.1.1 Regularity of Quasiconformal Mappings and the Analytic Definition

Let Ω be a domain and f a sense-preserving homeomorphism of Ω. A quadrilateral $Q = Q(\Omega_1; z_1, z_2, z_3, z_4)$ where $\overline{\Omega_1} \subset \Omega$ is mapped by f onto quadrilateral Q_*. Then the number

$$K(\Omega) = K_{mod}(f;\Omega) := \sup \frac{\mathsf{M}(Q_*)}{\mathsf{M}(Q)},$$

where supremum is taken over all quadrilaterals $\overline{Q} \subset \Omega$ is called the *maximal modulus dilatation* of f in Ω.

Occasionally we write shortly $K_{mod}(\Omega)$ or $K_{mod}(f)$ for *maximal modulus dilatation* of f in Ω.

Next we will study the behavior of the Jacobian of a quasiconformal mapping. Our aim is to give in the general case an analytic definition of quasiconformality that is similar to the one we discussed for the quasiconformal diffeomorphisms. We shall also prove that it is equivalent to the geometric definition. We start by noting that Definition 4 is equivalent to the following:

Definicija 3. *A sense-preserving homeomorphism f of Ω is called quasiconformal (in the geometric sense) if its maximal modulus dilatation is finite. If $K(\Omega) = K_{mod}(\Omega) \leq K < \infty$, then f is called K-quasiconformal (in the geometric sense).*

If f is differentiable at a point z, and its Jacobian $J_f(z)$ does not vanish, we say that z is a *regular* point of f. The dilatation quotient (distortion) at a regular point is then

$$D_f(z) = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|}.$$

Lemma 5.1. *Let Ω be a domain and f a sense-preserving homeomorphism (difeomorphism) of Ω. Between the maximal dilatation and the dilatation quotient of a regular quasiconformal mappings holds*

$$K_{mod}(\Omega) \leq \sup_{z\in\Omega} D_f(z). \tag{100}$$

Proof. Let $K = \sup_{z\in\Omega} D_f(z)$. We have to prove that mod-dilation of an arbitrary quadrilateral Q is at most K.

Consider a quadrilateral Q, $\overline{Q} \subset \Omega$, and its image $Q_* = f(Q)$. Define $M = \mathsf{M}(Q)$ and $M_* = \mathsf{M}(Q_*)$. Now we need to show that

$$M_* \leq K\,M. \tag{101}$$

Suppose that f_1 maps Q conformally onto a rectangle $R = R(M)$ with sides of length 1 and M, preserving the vertices. Let f_2 map the quadrilateral $Q_* = f(Q)$ conformally onto a rectangle $R_* = R(M_*)$. Consider the mapping $g = f_2 \circ f \circ f_1^{-1}$ which maps R onto R_* and thus preserves vertices. Because f is K- quasiconformal, the mapping g is K-quasiconformal and $K(g) = K(f)$. The inequality (101) follows from the Proposition **??**. △

The class of K-quasiconformal diffeomorphisms is not compact. This is one reason to replace the classical definition of Grötzsch with a more general one, which will be given next.

Let $U_z \subset \Omega$ denote a neighborhood of $z \in \Omega$ and $K(U_z) = K_{mod}(U_z)$ the *maximal dilatation* of f in U_z. The number

$$K_{mod}(z) = \inf K(U_z)\,,$$

where the infimum is taken over all $U_z \subset \Omega$ is called the *maximal dilatation* of f at the point z.

Theorem 5.1. *Let f be a K-quasiconformal mapping on Ω in the geometric sense. If f is differentiable at $z_0 \in \Omega$, then*

$$\big(|f_z| + |f_{\overline{z}}|\big)(z_0) \leq K\big(|f_z| - |f_{\overline{z}}|\big)(z_0)\,. \tag{102}$$

Proof. The idea of the proof is to consider a small square centered at z_0 and regard it as a quadrilateral with the vertices. The area and the distance of the sides of $f(Q)$ can be approximated by expressions involving the partial derivatives of f at z. Application of Rengel's inequality then yields a lower estimate for $\mathsf{M}(f(Q))$ from which the inequality follows. Without loss of generality we can suppose that $z_0 = 0$ and we may choose r such that the closure of the domain

$$Q = Q_r = \{z : \, |x|, |y| < r\}$$

is a subset of Ω.

By using a translation and rotation we may assume that $f(z) = ax + i\,by + o(z)$, for some $a, b \in \mathbb{R}$ and $a \geq |b| \geq 0$. Let $Q_* = f(Q)$. Then $u(z) = \pm ar + o(r)$ on the b-sides of Q, and hence $s_b(Q_*) = 2ar + o(r)$. Because $M(Q) = 1$, by the assumptions

$$M(Q_*) \leq K\,M(Q) = K,$$

where $K = K(Q_r)$. By using Rengel's inequality, we estimate $M(Q_*)$ from the below:

$$\frac{s_b^2}{M(Q_*)} \leq M(Q_*).$$

We have the estimate $M(Q_*) = 4abr^2 + o(r^2)$, and hence $s_b^2 \le Km(Q_*)$, i.e.

$$4a^2r^2 + o(r^2) \le 4\,K\,abr^2 + o(r^2).$$

Hence $a \le Kb$. Since $\Lambda_f(0) = a$ and $\lambda_f(0) = b$, the proof follows. $\triangle$

From the proof of Theorem 5.2, we obtain the following estimate for the dilatation of f.

Corollary 3. *For a quasiconformal mapping $f\colon \Omega \to \mathbb{C}$ it holds*

$$D_f(z) \le K_{mod}(z) \tag{103}$$

at every regular point $z \in \Omega$. In particular, $D_f(z) \le K$ at every regular point.

Example 22. *A regular quasiconformal mapping of a domain Ω satisfies*

$$D(z) = K_{mod}(z),$$

at every point $z \in \Omega$.

By Lemma 5.1 and Corollary 3, we have shown the following:

Proposition 12. *If f is a diffeomorphism, then the geometric and analytic definitions of a quasiconformal mapping are equivalent.*

Example 23. *Let $f\colon \Omega \to f(\Omega)$ be a sense-preserving homeomorphism. Prove that*
a) f is K-quasiconformal in the sense of Definition 3 if and only if for every quadrilateral $Q \subset \Omega$,

$$\frac{m(Q)}{K} \le m(f(Q)) \le Km(Q).$$

Note that the condition $m(f(Q)) \le Km(Q)$ implies the double inequality

$$\frac{m(Q)}{K} \le m(f(Q)) \le Km(Q).$$

b) $K(\Omega) = \sup_{z\in\Omega} K_{mod}(z)$.
c) f is K-qc if and only if $K_{mod}(z) \le K$ at every point $z \in \Omega$.

We leave to the reader to verify the following properties of quasiconformal mappings.

(a) f is K-quasiconformal $\Rightarrow$ f^{-1} is K-quasiconformal.

(b) If g is conformal and f is K-quasiconformal, then $f \circ g$ and $g \circ f$ are K-quasiconformal.

(c) If g is K_1-quasiconformal and f is K_2-quasiconformal, then $f \circ g$ and $g \circ f$ are K_1K_2-quasiconformal.

(d) If f is C^1, then the two definitions of quasiconformality agree.

5.1.2 Geometric Definition of Quasiconformality

For more details about qc (quasiconformal) mappings see [33]. Let f be a conformal mapping of a quadrilateral $Q = Q(\Omega; z_1, z_2, z_3, z_4)$ onto a Euclidean rectangle R. If the boundary correspondence is such that f maps the four distinguished points to the vertices of R, then the mapping f is called the *canonical mapping* of Q, and R is called the *canonical rectangle* of Q. Clearly, every conformal equivalence class of quadrilaterals contains rectangles. In addition, all rectangles that can be obtained from each other by sense-preserving similarity transformations obviously belong to the same class. Conversely, it follows from the reflection principle that every conformal mapping between two rectangles is a similarity transformation. Hence, all canonical rectangles of a given quadrilateral Q have the same ratio of sides $\mathsf{M}(Q) = a/b$, where a denotes the length of the images of a-sides and b denotes the length the images of b-sides of the canonical rectangle, respectively. Then the number $\mathsf{M}(Q)$ is called (*conformal*) *modulus* (or module) of the quadrilateral Q.

Recall that a quadrilateral $Q_1 = Q(\Omega; z_1, z_2, z_3, z_4)$ consists of a Jordan domain Ω together with four distinct boundary points given in the positive order. The parts of the boundary arc γ connecting the points z_1, z_2 and z_3, z_4 are called the a-sides, and the two remaining sides the b-sides of the quadrilateral, respectively. Suppose that f maps the quadrilateral Q_1 conformally onto a rectangle with sides of length a and b so that a-sides are mapped onto the a-sides of the rectangle.

Then the conjugate quadrilateral is $Q_2 = Q(\Omega; z_2, z_3, z_4, z_1)$. In other words, it is the same as Q with complementary arcs, the b-sides, considered as its a-sides. It was shown in the previous section that if Γ is the family of curves connecting the b-sides, then the module of Q is

$$M(Q) = \lambda(\Gamma)^{-1} = \frac{b}{a},$$

and the module of the conjugate quadrilateral

$$M(\widetilde{Q}) = M(Q)^{-1} = \frac{a}{b}.$$

Now we are ready to state the geometric definition of quasiconformality.

Definicija 4 (Geometric Definition)**.** *Let $f\colon \Omega \to f(\Omega)$ be a homeomorphism. We say that f is K-quasiconformal if for every quadrilateral $Q \subset \Omega$*

$$\frac{M(Q)}{K} \leq M(f(Q)) \leq K\, M(Q).$$

We will later show that a homeomorphism $f\colon \Omega \to f(\Omega)$ is quasiconformal if and only if $H_f(z)$ is bounded in Ω. We leave to the reader to verify the following basic properties:

(i) If f is K-quasiconformal then f^{-1} is K-quasiconformal.

(ii) If g is conformal and f is K-quasiconformal, then $f \circ g$ and $g \circ f$ are K-quasiconformal.

(iii) If g is K_1-quasiconformal and f is K_2-quasiconformal, then $f \circ g$ and $g \circ f$ are $K_1 K_2$-quasiconformal.

(iv) If f is C^1, then the two definitions of quasiconformality agree.

Example 24. *Prove the properties (i)–(iv).*

Example 25. *The constant $K_1 K_2$ in (iii) is not, in general, sharp. Give an example of K_1-quasiconformal f and g such that $f \circ g$ is K_2-quasiconformal for some $K_2 < K_1$.*

Theorem 5.2. *Let f be a K-quasiconformal mapping on Ω in the geometric sense. If f is differentiable at $z_0 \in \Omega$, then*

$$\big(|f_z| + |f_{\overline{z}}|\big)(z_0) \le K\big(|f_z| - |f_{\overline{z}}|\big)(z_0)\,. \tag{104}$$

Lemma 5.2. *A qc mapping (in the geometric sense) f of a domain Ω is absolutely continuous on lines in Ω.*

From Lemma 5.2, it follows that a quasiconformal mapping has finite partial derivatives almost everywhere. Hence, further conclusions can be derived using the following result of Gehring and Lehto:

Theorem 5.3. *Let $f\colon \Omega \to \mathbb{C}$ be a continuous and open mapping of a plane domain Ω which has finite partial derivatives a.e. in Ω. Then f is differentiable a.e. in Ω.*

Application of this Theorem to the quasiconformal case yields the following basic result:

Lemma 5.3. *A K-quasiconformal mapping f of the domain Ω is differentiable and satisfies the dilatation condition* (104) *a.e. in Ω.*

Note that, by Corollary 3, f satisfies the dilatation condition (103) at its regular points.

Let f be a homeomorphism of a domain Ω. Define $\mu_f(E) = m(f(E))$. We say that $\mu'_f(z)$ is the *volume derivative* of f at z, where

$$\mu'_f(z) = \lim_{r\to 0} \frac{m(f\overline{\mathbb{D}(z,r)})}{\pi r^2}.$$

¿From Lebesgue's theorem we obtain (see [7], Theorem 8.6, for a more general result) the following:

- $\mu'_f(z) < +\infty$ a.e.,
- μ'_f is measurable

- The integral
$$\iint\limits_B \mu_f' \,\mathrm{d}\sigma \le m(f(B))$$
for every Borel set $B \subset \Omega$.

If f is differentiable at z, $\mu_f'(z) = |J(z,f)|$.

Our aim is to prove Theorem 5.4 which states that integrating the Jacobian of the quasiconformal mapping over a Lebesgue measurable set gives the measure of the image set. First we need some lemmas.

Lemma 5.4. *If the homeomorphism f of a domain Ω has finite partial derivatives a.e. in Ω, then*
$$\iint\limits_B J_f \,\mathrm{d}x\,\mathrm{d}y \le m(f(B))$$
for every Borel set $B \subset \Omega$. Furthermore f is absolutely continuous in Ω if and only if
$$\iint\limits_E J_f \,\mathrm{d}x\,\mathrm{d}y = m(f(E))$$
for every measurable set $E \subset \Omega$.

Lemma 5.5. *A K-quasiconformal mapping (in the geometric sense) f of a domain Ω has L^2-partial derivatives in Ω.*

Proof. The dilatation condition (103) implies
$$|f_x(z)|^2 \le KJ(z), \text{ and } |f_y(z)|^2 \le KJ(z) \text{ a.e. in } \Omega.$$

It follows by Lemma 5.4 that the Jacobian of an almost everywhere differentiable homeomorphism is locally integrable, and thus f_x and f_y are locally L^2-integrable. △

Lemma 5.6 ([**?**])**.** *A homeomorphism $f = u + iv$ of the domain Ω, which has L^2-partial derivatives in Ω, is locally absolutely continuous in Ω and*
$$\int\limits_E J_f \,\mathrm{d}x\,\mathrm{d}y = m(f(E)) \tag{105}$$
for every Borel (measurable) set $E \subset \Omega$.

Proof. Let $\mu(E) = \int\limits_E J_f \,\mathrm{d}x\,\mathrm{d}y$ and $\mu_f(E) = m(f(E))$. We show first the equality if E is the closed rectangle R with sides parallel to the axes.

Because f is in $ACL(\Omega)$ and the integral (105) depends continuously on E, we can, by enlarging R if necessary, ensure that f is absolutely continuous on ∂R. Then f is also of the bounded variation on ∂R and the image of ∂R under f is consequently rectifiable. By applying generalized Green's formula [**?**]

(note the partial derivatives of u and v belong to L^2) to the domains R and $R_* = f(R)$, we obtain

$$\int\limits_{\partial R} u \,\mathrm{d}v = \iint\limits_{R} J_f \,\mathrm{d}x \,\mathrm{d}y, \quad \text{and} \quad \int\limits_{\partial R_*} u \circ f^{-1} \,\mathrm{d}v(f^{-1}) = \iint\limits_{R_*} \mathrm{d}x \,\mathrm{d}y.$$

By the invariance property of the Stieltjes integral the left sides of the above equation are equal. We note that an open set of the finite plane consists of countably many non-intersecting half-open rectangles with sides parallel to the coordinate axes.

For general Borel set E, for every $\varepsilon > 0$ there exists an open set G_ε, $E \subset G_\varepsilon$, $\overline{G_\varepsilon} \subset \Omega$, with $m(G_\varepsilon \setminus E) < \varepsilon$. Because the integral (105) is a set function, we have

$$\int\limits_E J_f \,\mathrm{d}x \,\mathrm{d}y = \lim_{\varepsilon \to 0} \int\limits_{G_\varepsilon} J_f \,\mathrm{d}x \,\mathrm{d}y = \lim_{\varepsilon \to 0} m(f(G_\varepsilon)) \geq m(f(E)).$$

By Lemma 5.4 the equality holds, and therefore the claim is proved.

Note that it is also true that $\lim\limits_{\varepsilon \to 0} m(f(G_\varepsilon)) = m(f(E))$ and therefore the equality holds. △

Question: Suppose that f is a homeomorphism of the domain Ω, which has L^1-derivatives (or is in ACL) in Ω. Is it true that f is locally absolutely continuous in Ω and the formula (105) holds (see also Lemma 3.3 [**?**])?

Example 26. *Let $0 < \varepsilon < 1/2$, r_k, $k \geq 1$ sequence of rational numbers in $I := (0,1)$ and*

$$J_k = \big(r_k - \varepsilon 2^{-k-1}, r_k + \varepsilon 2^{-k-1}\big) \cap (0,1),$$

where

$$A = \cup_{k=1}^{\infty} J_k, \qquad B = (0,1) \setminus A.$$

Define $f(x) = \int\limits_0^x K_A(t) \,\mathrm{d}t$ and $F(z) = f(x) + iy$. Then $f' = K_A$ a.e. on I, F is homeomorphism on I^2, $J_F(z) = f'(x)$ a.e on I^2, $J_F = 1$ a.e on $A \times I$, $J_F = 0$ a.e on $B \times I$, $m(B \times I) > 0$, $m\big(F(B \times I)\big) = 0$, and F maps a set of positive measure on null-set. Verify that the formula (105) holds for F.

Theorem 5.4. *If f is a quasiconformal mapping of a domain Ω then*

$$\int\limits_E J_f \,\mathrm{d}x \,\mathrm{d}y = m(f(E)) \tag{106}$$

for every measurable set $E \subset \Omega$.

Thus, under a quasiconformal mapping the image area is an absolutely continuous set function. This means that null sets are mapped on null sets, and that the image area can always be represented by (106).

Applying (106) to f^{-1}, we conclude that $J(z) > 0$ almost everywhere. Hence, by Lemma 5.3, we have proved the following:

Lemma 5.7. *A K-quasiconformal mapping f of a domain Ω is regular and satisfies the dilatation condition* (103) *a.e. in* Ω.

Example 27. *Let $f(z) = z + C(x) = (x + C(x), y)$. Then f maps the line $l_y = \{x + iy : 0 \leq x \leq 1\}$ onto $l'_y = \{x + iy : 0 \leq x \leq 2\}$, where $y \in I$. Hence f maps the square $Q = I \times I$ onto the rectangle $R = [0, 2] \times I$, $K_A = 2$, and $K_f = 1$ a.e. on the square $Q = I \times I$. Because the inequality $K_A \leq K_f$ does not hold f does not satisfy qc geometric definition.*

Example 28. *Let $f(z) = z + iC(x)$, where C is singular Cantor function on I. Because $C'(x) = 0$ a.e. on I, i.e. $C'(x) = 0$ a.e. on $I \setminus A$, where $A \subset I$ and $m_1(A) = 0$. Thus, for $z \in A \times I$, $f'_x(z) = 1$, $f'_y(z) = i$, and therefore $\overline{D}f(z) = 0$ and $\mu_f = 0$, $K_f = 1$ a.e. on the square $Q = I \times I$. In spite of this, the mapping f is not quasiconformal. It is not absolutely continuous on any of the segment $l_y = \{x + iy : 0 \leq x \leq 1\}$, $y \in I$.*
Let $f(z) = C(x) + iC(y)$. Check that $f'_x = f'_y = 0$ a.e. on I^2 in the classical sense, but not in distributional (generalized) sense.

Definicija 5. *If a homeomorphism $f \colon \Omega \to \Omega_*$ satisfies*

(i) *f is ACL on Ω*

(ii) *$|f_{\overline{z}}| \leq k\,|f_z|$ almost everywhere in Ω, where $k = \dfrac{K-1}{K+1}$ we say that f is K-quasiconformal in the analytic sense.*

Napomena 2. *This definition is equivalent to the following: f is a homeomorphism and it has locally integrable distributional derivatives which satisfy (ii).*

¿From Lemma 5.2 and Lemma 5.3, it follows that: if a homeomorphism $f \colon \Omega \to \Omega_*$ is K-quasiconformal in the geometric sense then f is K-quasiconformal in the analytic sense. A proof that the converse is true can be based on the fact that the analytic definition is invariant under conformal mapping. For the higher dimensional case by Fuglede's theorem, see [37, p. 95].

Theorem 5.5 (The analytic Definition)**.** *A homeomorphism $f : \Omega \to \Omega_*$ is K-qc iff*

(i) *f is ACL on Ω*

(ii) *$|f_{\overline{z}}| \leq k\,|f_z|$ almost everywhere in Ω, where $k = \dfrac{K-1}{K+1}$.*

5.1.3 Beltrami equation

Quasiconformal mappings are related to the equation

$$\overline{\partial} f = \mu\, \partial f \tag{107}$$

where $\mu \colon \Omega \to \mathbb{C}$ is measurable and $\|\mu\|_\infty < 1$. This equation is called the *Beltrami equation* (the *μ-Beltrami equation*).

A function $f\colon \Omega \to \mathbb{C}$ is said to be an L^p-solution of (107) in Ω if f has L^p-derivatives and (107) holds a.e. in Ω. It is shown that the Beltrami equation $f_{\overline{z}} = \mu f_z$ can be solved for given μ satisfying $\|\mu\|_\infty \leq k < 1$. First we prove that the Beltrami equation (107) has always a homeomorphic solution. It is easier to see the uniqueness of the solution.

Let f and g be quasiconformal mappings of a domain Ω and write $h = f \circ g^{-1}$. Then a direct calculation yields

$$\mu_h(\zeta) = \frac{\mu_f(z) - \mu_g(z)}{1 - \mu_f(z)\overline{\mu_g(z)}} \left(\frac{\partial g(z)}{|\partial g(z)|}\right)^2. \tag{108}$$

Theorem 5.6.

(i) *A 1-qc map is conformal.*

(ii) *If f is a qc of a domain D and $\mu_f = 0$ a.e. onD. Then f is conformal on D.*

Theorem 5.7. *Let f and g be quasiconformal mappings of a domain Ω onto itself whose complex dilatations agree a.e. in Ω. Then $f \circ g^{-1}$ is a conformal mapping.*

Proof. By (108), the complex dilatation of $h = f \circ g^{-1}$ vanishes a.e. in Ω. Hence, it follows from Theorem 5.5 that h is 1-quasiconformal. Then by Theorem **??** and Theorem 5.6(ii) it is conformal. △

In the end of this section, our goal is to prove the following result. In addition, we consider the uniqueness of solutions.

Theorem 5.8 (Existence theorem)**.** *Let μ be a measurable function in a domain Ω with $\|\mu\|_\infty < 1$. Then there is a quasiconformal mapping of Ω whose complex dilatation agrees with μ almost everywhere.*

5.1.4 Integral Transforms

Let $p > 2$. For $h \in L^p(\mathbb{C})$, the *Cauchy transform* is defined by

$$(Ph)(w) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} h(z) \left(\frac{1}{z-w} - \frac{1}{z}\right) \mathrm{d}x\,\mathrm{d}y. \tag{109}$$

For $h \in C_0^1(\mathbb{C})$ we may write

$$(Ch)(w) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} h(z) \left(\frac{1}{z-w}\right) \mathrm{d}x\,\mathrm{d}y = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} h(\zeta + w)\,\frac{1}{\zeta}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{110}$$

Then $(Ph)(w) = (Ch)(w) - (Ch)(0)$. Note that $(Ph)(0) = 0$.

Lemma 5.8. *For $h \in L^p(\mathbb{C})$, $p > 2$, the Cauchy transform (Ph) is Hölder continuous with exponent $1 - 2/p$.*

Proof. By using the Holder inequality, where q is the conjugate exponent of p, we obtain

$$|Ph(w)| = \frac{1}{\pi}\left|\iint h(z)\left(\frac{1}{z-w}-\frac{1}{z}\right)\mathrm{d}x\,\mathrm{d}y\right|$$
$$\leq \frac{|w|}{\pi}\left(\iint |h(z)|^p \mathrm{d}x\mathrm{d}y\right)^{1/p}\left(\iint \frac{1}{(|z-w||z|)^q}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/q}$$

Because q is the conjugate exponent of $p > 2$, we have $q \in (1, 2)$, and therefore the last integral is finite. A change of variables shows that

$$\iint \frac{1}{(|z(z-w)|)^q}\,\mathrm{d}x\,\mathrm{d}y = |w|^{2-2q}\iint \frac{1}{|z(z-1)|^q}\,\mathrm{d}x\,\mathrm{d}y$$

where the value of the last integral is some constant $A = A_p$ depending only on p. Write $K_p = A^{1/p}$. It follows that

$$|(Ph)(w)| \leq |w|^{1-2/p} K_p \|h\|_p.$$

We note that $(Ph)(w_1) - (Ph)(w_2) = (Ph_1)(w_1 - w_2)$, where $h_1(z) = h(z+w_2)$. Then we have

$$|(Ph)(w_1) - (Ph)(w_2)| \leq K_p \|h_1\|_p |w_1 - w_2|^{1-2/p}, \tag{111}$$

and $\|h_1\|_p = \|h\|_p$. △

5.1.5 Green's Formula

If $f \in C^1(\overline{\Omega})$, where Ω is a domain having a finite number of boundary components, each being a regular Jordan curve, and Γ the positively-oriented boundary of G, then Green's formula states that

$$I = \int_\Gamma f\,\mathrm{d}z = \iint_G \mathrm{d}(f\,\mathrm{d}z).$$

Because $\mathrm{d}(f\,\mathrm{d}z) = (\mathrm{d}f \wedge \mathrm{d}z)$, and $\mathrm{d}f = Df(z)\mathrm{d}z + \overline{D}f(z)\mathrm{d}\overline{z}$, we have $\mathrm{d}(f\,\mathrm{d}z) = (Df(z)\,\mathrm{d}z + \overline{D}f(z)\,\mathrm{d}\overline{z}) \wedge \mathrm{d}z = \overline{D}f(z)\,\mathrm{d}\overline{z} \wedge \mathrm{d}z$. Thus we may write

$$I = \int_\Gamma f\,\mathrm{d}z = \iint_G \overline{D}f\,(\mathrm{d}\overline{z} \wedge \mathrm{d}z).$$

Note that $\mathrm{d}\overline{z} \wedge \mathrm{d}z = 2i\,\mathrm{d}x \wedge \mathrm{d}y$, and the second integral is usual the Lebegue integral because $\mathrm{d}x \wedge \mathrm{d}y$ is the Lebegue measure in $\mathbb{R}^2$. It follows that

$$\int_\Gamma f\,\mathrm{d}z = 2\,i\iint_G \overline{D}f\,\mathrm{d}x\,\mathrm{d}y. \tag{112}$$

If $g \in C^1(\overline{G})$, where G is a domain having a finite number of boundary components, each being a regular Jordan curve and Γ is the positively-oriented boundary of G, then Green's formula (112) yields

$$\int_\Gamma g \,\mathrm{d}\overline{z} = \iint_G Dg\,(\mathrm{d}z \wedge \mathrm{d}\overline{z}) = -2\,i \iint_G Dg \,\mathrm{d}x\,\mathrm{d}y. \tag{113}$$

Let $\varepsilon > 0$ and $B_\varepsilon = \{z : |z-w| < \epsilon\}$. Define G_ϵ to be the set $G\backslash\{z : |z-w| < \epsilon\}$. Suppose that $a \in G$. We apply Green's formulas (112) and (113), respectively, with

$$f = \frac{h}{z-a} \text{ and } g = \frac{h}{z-a},$$

where $h \in C^1(\overline{G})$. Let $\gamma = \gamma_\varepsilon$ be the curve defined by

$$\gamma(t) = a + \varepsilon e^{it}, \text{ where } 0 \leq t \leq 2\pi.$$

Because the function $z \mapsto (z-a)^{-1}$ is holomorphic, we have

$$\overline{D}f = f_{\overline{z}} = \frac{(\overline{D}h)(z)}{z-a}, \;\; g_z = Dg(z) = \frac{Dh(z)}{z-a} - \frac{h(z)}{(z-a)^2}. \tag{114}$$

Verify

$$\lim_{\varepsilon \to 0} \int_\gamma f\,\mathrm{d}z = 2\pi i h(a), \;\; \lim_{\varepsilon \to 0} \int_\gamma g\,\mathrm{d}\overline{z} = 0. \tag{115}$$

Hence

$$h(a) = \frac{1}{2\pi i} \int_\Gamma \frac{h(z)\mathrm{d}z}{z-a} - \frac{1}{2\pi i} \iint_G \frac{(\overline{D}h)(z)}{z-a} \mathrm{d}\overline{z} \wedge \mathrm{d}z \tag{116}$$

$$\int_\Gamma \frac{h(z)\mathrm{d}\overline{z}}{z-a} = \iint_G \left(\frac{Dh(z)}{z-a} - \frac{h(z)}{(z-a)^2} \right) \mathrm{d}z \wedge \mathrm{d}\overline{z}. \tag{117}$$

The proof of the formula (117) shows the following: For each $h \in C_0^2$ (space of twice differentiable functions with compact support), the *Hilbert transform* is defined by the Cauchy principal value:

$$(Th)(w) = \lim_{\epsilon \to 0} \left(-\frac{1}{\pi} \iint_{|z-w|>\epsilon} \frac{h(z)}{(z-w)^2} \,\mathrm{d}x\,\mathrm{d}y \right). \tag{118}$$

For $w \in B_R$, we prove

$$\overline{w} = -\frac{1}{\pi} \iint_{B_R} \frac{1}{z-w} \,\mathrm{d}x\,\mathrm{d}y. \tag{119}$$

Example 29. *First, one may apply (116) with $h(z) = \overline{z}$ and to use the indentity $z\overline{z} = R^2$ for $z \in K_R$, to obtain*

$$\int_{K_R} \frac{\overline{z}\,\mathrm{d}z}{z-w} = R^2 \int_{K_R} \frac{\mathrm{d}z}{z(z-w)} = 0\,.$$

Let $h \in C_0^2$ and $\psi(z) = (Th)(z) - h(z_0)\overline{z}$. Then we may show that ψ has at z_0 derivative $\psi'(z_0) = \partial\psi(z_0)$. By direct calculation

$$\psi'(z_0) = \partial\psi(z_0) = (Th)(z_0) = \left(-\frac{1}{\pi} \iint_B \frac{h(z) - h(z_0)}{(z-z_0)^2}\,\mathrm{d}x\,\mathrm{d}y \right), \tag{120}$$

where $B = B_R$ is an arbitrary disk containing the point z_0 and the support of h. Because ψ has at z_0 derivative $\psi'(z_0) = \partial\psi(z_0)$ which is independent of direction, it follows that $\overline{\partial}\psi(z_0)$ vanishes.

This example can also be used for obtaining a proof of the items (a) and (b) of Lemma 5.9 (see [**?**, pp. 155–157]).

Lemma 5.9. *If $h \in C_0^2$, then (Th) is well-defined and $(Th) \in C^1$.*

(a) *$(Ph)_{\overline{z}} = h$,*

(b) *$(Ph)_z = (Th)$, and*

(c) *$\iint_{\mathbb{C}} |(Th)|^2\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathbb{C}} |h|^2\,\mathrm{d}x\,\mathrm{d}y$.*

Remark. Note that (a) solves the d-bar problem $\overline{\partial} f = h$.

Proof. Let

$$(Ch)(w) = -\frac{1}{\pi} \iint_{\mathbb{C}} h(\zeta + w)\,\frac{1}{\zeta}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{121}$$

Then, by formal differentiation,

$$\overline{\partial}(Ch)(w) = -\frac{1}{\pi} \iint_{\mathbb{C}} \overline{\partial}h(\zeta + w)\,\frac{1}{\zeta}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{122}$$

By change of variables, we have $z = w + \zeta$ and

$$(Ph)_{\overline{w}} = -\frac{1}{\pi} \iint \frac{h_{\overline{z}}}{z-w}\,\mathrm{d}x\,\mathrm{d}y \text{ and } (Ph)_w = -\frac{1}{\pi} \iint \frac{h_z}{z-w}\,\mathrm{d}x\,\mathrm{d}y. \tag{123}$$

Now, (a) and (b) follow respectively from formulas (116) and (117). By Green's formula, we have

$$-\frac{1}{2\pi i} \iint \frac{h_{\overline{z}}}{z-w}\,\mathrm{d}z\,\mathrm{d}\overline{z} = \frac{1}{2\pi i} \iint \frac{\mathrm{d}h \wedge \mathrm{d}z}{z-w}.$$

Now define Ω_ϵ to be $\mathbb{C}\backslash\{z : |z-w| < \epsilon\}$. Then

$$(Ph)_{\overline{w}} = \lim_{\epsilon\to 0} \frac{1}{2\pi i} \iint\limits_{\Omega_\epsilon} \frac{\mathrm{d}h \wedge \mathrm{d}z}{z-w} = \lim_{\epsilon\to 0} -\frac{1}{2\pi i} \int\limits_{\partial\Omega_\epsilon} \frac{h}{z-w}\,\mathrm{d}z = h(w)$$

by Green's formula. Note that the change of sign comes from the orientation of the path integral. Similary, we obtain $(Ph)_w = (Th)$.

Finally, we prove (c). First observe that formula (123) can be written in the following form. Recall that if $h \in C_0^2$, by Green's formulas (116) and (117) for the terms in formula (123) hold

$$P(h_{\overline{z}}) = h - h(0), \;\; P(h_z) = (Th) - (Th)(0). \tag{124}$$

Under the assumption $h \in C_0^2$, we can apply (123) to h_z, and by use the second equation (124) we obtain $(Th)_{\overline{z}} = P(h_z)_{\overline{z}} = h_z$ and $(Th)_z = P(h_z)_z = T(h_z) = P(h_{zz}) + (Th)_z(0)$. Thus $(Th) \in C^1$ and $(Ph) \in C^2$.

Because h has a compact support it is immediate that $Ph = O(1)$ and $Th = O(|z|^{-2})$. We have now sufficient information to justify all steps in the calculation. Let K_R denote circle of radius R. Since $(Ph)\overline{(Th)} = O(|z|^{-2})$, and

$$\int\limits_{K_R} Ph\overline{Th}\,\mathrm{d}\,\overline{z}$$

approaches 0 when R tends ∞, by integrating by parts and using parts (a) and (b), we find

$$\iint |(Th)|^2\,\mathrm{d}x\,\mathrm{d}y = -\frac{1}{2i}\iint (Ph)_z(\overline{(Ph)})_{\overline{z}}\,\mathrm{d}z\,\mathrm{d}\overline{z} = \frac{1}{2i}\iint (Ph)(\overline{(Ph)})_{\overline{z}z}\,\mathrm{d}z\,\mathrm{d}\overline{z}.$$

Again, integrating by parts and using parts (a) and (b), we obtain

$$= \frac{1}{2i}\iint (Ph)(\overline{h})_{\overline{z}}\,\mathrm{d}z\,\mathrm{d}\overline{z} = -\frac{1}{2i}\iint \overline{h}(Ph)_{\overline{z}}\,\mathrm{d}z\,\mathrm{d}\overline{z} = \iint |h|^2\,\mathrm{d}x\,\mathrm{d}y. \qquad \Lambda$$

Example 30. *Prove (i) and (ii) directly from (123).*

Now (Th) is defined for $h \in C_0^2$, but since C_0^2 is dense in $L^2(\mathbb{C})$, using the isometry the operator T can be extended to all $h \in L^2(\mathbb{C})$. But we cannot extend P in the same way.

Using the Calderón-Zygmund lemma, we can even define T on $L^p(\mathbb{C})$ for any $p > 1$, but T will no longer be an isometry, although $\|Th\|_p$ is bounded. Approximate $h \in L^p(\mathbb{C})$ by $h_n \in C_0^2$, such that $h_n \to h$ in the L^p-norm. The Calderón-Zygmund lemma says that for $p > 1$, we have

$$\|(Th_n)\|_p \le C_p\|h_n\|_p$$

for some constant C_p, and $C_p \to 1$ as $p \to 2$. It follows that (Th_n) is a Cauchy sequence in $L^p(\mathbb{C})$ and therefore it converges. We define $(Th) = \lim\limits_{n\to\infty}(Th_n)$.

The Calderon-Zygmund lemma says that for $p > 1$,

$$\|Th\|_p \le C_p\|h\|_p$$

for some constant C_p, and $C_p \to 1$ as $p \to 2$.

5.1.6 Absolutely Continuous Functions

Recall that if $f \in L^1[a,b]$ and

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \quad a \le x \le b\,,$$

then F is absolutely continuous on the segment $[a,b]$. If F is absolutely continuous on the segment $[a,b]$, then $F' = f$ exists a.e. on $[a,b]$, $f \in L^1[a,b]$, and

$$F(x) - F(x_0) = \int_{x_0}^x f(t)\mathrm{d}t.$$

Then we say $F' = f$ in the distributional sense.

We make the following assumptions: the function F is continuous on the segment $[a,b]$, F is differentiable a.e on the segment $[a,b]$, and $F' \in L^1[a,b]$. This does not imply that, in general, F is absolutely continuous on the segment $[a,b]$. The standard counterexample is the Cantor function.

Lemma 5.10. *For $h \in L^p(\mathbb{C})$, $p > 2$, then $(Ph)_{\overline{z}} = h$, and $(Ph)_z = (Th)$ in the sense of distributions.*

Proof. Let $\varphi \in C_0^1$ be a test function. If $h \in C_0^2$, then by Lemma 5.9

$$\iint (Ph)\varphi_{\overline{z}} = -\iint \varphi h, \text{ and } \iint (Ph)\varphi_z = -\iint \varphi(Th). \tag{125}$$

Approximate $h \in L^p(\mathbb{C})$ by $h_n \in C_0^2$, such that $h_n \to h$ in the L^p-norm. Replace h with $h_n \in C_0^2$. The right hand members have the right limits since

$$\|Th - Th_n\|_p \le C_p \|h - h_n\|_p.$$

On the left hand side, we know by Lemma 5.8 that $P(h - h_n)$ converges 0 uniformly on compact subsets and since φ has compact support equation (125) holds with h replaced by h. $\triangle$

Remarks. A solution of equation $\overline{D}f = h$ is given by $f = (Ph)$. If g is holomorphic $f = Ph + g$ is also a solution. If f_1 and f_2 are two solutions, then $\overline{D}(f_1 - f_2) = 0$ and by Weyl's lemma $f_1 - f_2$ is holomorphic.

Let X be a Banach space and $A\colon X \to X$ a linear operator such that $\|A\| < 1$. Then the operator $A\colon X \to X$ is surjective, the operator $(I - A)^{-1}$ exists, and

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

It follows that, for every $y \in X$, the equation $x - Ax = y$ has a unique solution $x \in X$. The proof can be based on the following:

$$\sum_{k=0}^{\infty} \|A^k\| \le \sum_{k=0}^{\infty} \|A\|^k < \infty.$$

5.1.7 Solutions of the Beltrami Equation

Suppose that f has locally integrable derivatives in the complex plane $\mathfrak{C}$ and that $f(z) \to 0$ as $z \to \infty$. With the notation

$$(C\omega)(z) \;=\; -\frac{1}{\pi}\iint_{\mathfrak{C}} \frac{\omega(\zeta)}{\zeta - z}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

we then obtain from Green's formula

$$f = C\overline{\partial} f. \tag{126}$$

For a smooth ω with compact support we define the Hilbert transform $H\omega$ of ω by

$$H\omega = \partial(C\omega).$$

By differentiation we obtain an expression for H as the principal value

$$(H\omega)(z) = \lim_{\epsilon \mapsto 0_+} -\frac{1}{\pi}\iint_{A_\epsilon} \frac{\omega(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

where $A_\epsilon = \{\zeta : \epsilon < |\zeta| < \frac{1}{\epsilon}\}$. Instead of H we use also the notation T.

Fix $0 < k < 1$, and let $L^\infty(k, R)$ denote the measurable functions on $\mathfrak{C}$, bounded by k, and supported in the disk B_R. We let $QC^1(k, R)$ denote the continuous differentiable homeomorphisms f of $\mathbb{C}$ such that $\overline{\partial} f = \mu \partial f$, for any $\mu \in L^\infty(k, R)$, normalized so that

$$f(z) = z + O\Big(\frac{1}{z}\Big), \quad \text{as} \quad z \mapsto \infty.$$

Let $f \in QC^1(k, R)$. Then by (126), we have

$$f(z) - z \;=\; C(\overline{\partial} f)(z).$$

Thus, if we set $g = \partial f - 1$ and use $\overline{\partial} f = \mu \partial f$, we obtain

$$g \;=\; H(\overline{\partial} f) = H(\mu \partial f) = H(\mu g) \;+\; H(\mu).$$

In terms of the operator,

$$H_\mu(g) = H(\mu g), \qquad g \in L^p(\mathbb{C}),$$

we obtain the equation

$$(I - H_\mu)g \;=\; H(\mu). \tag{127}$$

If we fix $p = p(k) > 2$ so that $\|H_\mu\| < 1$, then $I - H_\mu$ is invertible. Thus, we can solve the equation (127) for g to obtain

$$g \;=\; (I - H_\mu)^{-1} H(\mu) \in L^p(\mathbb{C}). \tag{128}$$

Now we are ready to give the main result of this section.

Theorem 5.9. *Fix $0 < k < 1$, $R > 0$ and $p = p(k) > 2$ as above. For $\mu \in L^\infty(k, R)$, there is a function f on $\mathfrak{C}$, normalized so that $f(z) = z + O\Big(\dfrac{1}{z}\Big)$ at ∞, with distribution derivatives satisfying the Beltrami equation $\overline{\partial} f = \mu \partial f$.*

Proof. (Outline) Define g by (128), and let

$$f(z) \;=\; z + C(\mu g \;+\; \mu).$$

Because T is the convolution operator with kernel $1/z$ that is locally in L^1, the function f is continuous. Moreover, f is normalized at ∞,

$$\overline{\partial} f = \mu g \;+\; \mu,$$

and

$$\partial f \;=\; 1 + H(\mu g \;+\; \mu) = 1 + g$$

in the distributional sense, so f satisfies the Beltrami equation. △

Recall the Beltrami equation

$$f_{\overline{z}} = \mu f_z \tag{129}$$

where $\|\mu\|_\infty \le k < 1$. We first treat the case where μ has compact support, so that f be analytic in a neighborhood ∞. We shall use a fixed $p > 2$ such that $kC_p < 1$, where C_p is from the Calderón-Zygmund inequality.

Theorem 5.10. *Under the above assumptions, there exists a unique solution of* (129) *such that $f(0) = 0$ and $f_z - 1 \in L^p(\mathbb{C})$. By a solution, we mean f has distributional derivative and is continuous.*

We summarize the following:

Let μ be a measurable function in a domain D with $k = \|\mu\|_\infty < 1$. If $f : D \to D'$ is a solution of of the μ- Beltrami equation (107) and g is is an analytic function on D', then $g \circ f$ is also solution of the same equation.

Let μ be a measurable function on $\mathbb{C}$ with $k = \|\mu\|_\infty < 1$. Then there is a unique quasiconformal mapping f of $\mathbb{C}$, normalized by $f(i) = i$, $f(1) = 1$ and $f(-1) = -1$, whose complex dilatation agrees with μ almost everywhere.

We let f^μ be the solution f of the Beltrami equation (107) normalized by $f(i) = i$, $f(1) = 1$ and $f(-1) = -1$.

Suppose complex dilatation μ has a bounded support and define inductively $h_1 = \mu$, $h_n = \mu H h_{n-1}$, $n = 2, 3, \cdots$.

If we fix $p > 2$ such that $|\mu|_\infty |H|_p < 1$, then
$\lim_{n\to\infty} \sum_{i=1}^n h_i = \overline{D} f$ and it is a function in L^p, that is
(A1) $\quad \overline{D} f = \sum_{i=1}^\infty h_i$ in L^p sense.

Let f be qc with complex dilatation μ and which satisfies the condition $f(z) = z + O(1/z)$ at ∞. Then $f(z) = z + \sum_{i=1}^\infty (C h_i)(z)$. The series absolutely and uniformly converges in the plane.

There is a conjecture that for the norm Hilbert transform is $|H|_p = p - 2$, $p \ge 2$.

6 Appendix 1

6.1 Astala distortion theorm, The Nitsche conjecture

If f is analytic on a domain D, then f with all derivatives is locally bounded. In particular, it is true for conformal mapping.

Astala [**?**]: If f is K-quasiconformal, $f_z \in L^p$ for $p < p(K) = \frac{2K}{K-1}$.

This would follow from the methods outlined in Sec ■ if the conjectural values $||T||_p = p - 1$ for $p \geq 2$ were to be proven.

The example $f(z) = r^{1/K-1}z = r^{1/K}e^{i\theta}$ shows that the esimate is optimal. Since $f'_r = \frac{1}{K}r^{1/K-1}e^{i\theta}$,

and therefore $f_z \in L^p(\mathbb{D})$ iff $p < p(K)$.

Theorem 6.1 (Astala)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be K-quasiconformal, f conformal on $\mathbb{C} \setminus \mathbb{D}$ and $f(z) = z + \frac{a_1}{z} + \ldots$ near ∞.*

Let $E \subset \mathbb{D}$ be measurable, then

$$|f(E)| \leq K\pi^{1-1/K}|E|^{1/K}.$$

The proof is based on:

(i) If $E \subset \mathbb{D}$ with f conformal on $\mathbb{C} \setminus E$, then

$$|f(E)| \leq K\,|E|.$$

(ii) If f is conformal on $\mathbb{C} \setminus E$, then

$$|f(E)| \leq \pi^{1-1/K}|E|^{1/K}.$$

Recent proof of this result by Eremenko and Hamilton [**?**] have distilled Astala's idea's (from dynamics) and now they are fairly straightforward, without using the ideas from dynamics.

XX Mateljevic: If $h\colon \mathbb{D} \overset{\text{onto}}{\longrightarrow} \mathbb{D}$ is a harmonic mapping, $h(0) = 0$, then $h(\mathbb{D}_r) \subset \mathbb{D}_{\hat{r}}$, where $\hat{r} = (1+r)/2$ and $0 < r < 1$.

XX Kovalev: If $h\colon \mathbb{D} \overset{\text{onto}}{\longrightarrow} \mathbb{D}$ is a bijective harmonic mapping, then $|h(\mathbb{D}_r) \leq |\mathbb{D}_r|$, $0 < r < 1$.

Does it hold for injective harmonic mappings (or even arbitrary) $h\colon \mathbb{D} \to \mathbb{D}$?

7 Appendix 2

7.0.1 Stokes theorem

If $f \in L^1(B^n)$, then

$$\int_B f d\nu = n\int_0^1 r^{n-1}dr\int_S f(r\zeta)d\sigma(\zeta). \tag{130}$$

Since $dm = dx = \Omega_n d\nu$ and $\omega_{n-1} = n\Omega_n$, we can rewrite the formula (130) in the form:

$$\int_B f dx = \omega_{n-1} \int_0^1 r^{n-1} dr \int_S f(r\zeta) d\sigma(\zeta). \tag{131}$$

Define $\Delta = D_1^2 + \cdots + D_n^2$. We say that a function u defined on a open set D harmonic function if $u \in C^1(\overline{D}) \cap C^2(D)$ and $\Delta u = 0$ on D. Laplacian commutes with orthogonal transformation; more precisely, if T is orthogonal and $u \in C^2(D)$ then $\Delta(u \circ T) = \Delta(u) \circ T$.

The reader can find details in [1].

see bounded harmonic function, ch 2; and

The Dirichlet Problem and Boundary Behavior, ch 11.

XX

Recall that B denote the unit disc.

The volume of a parallelepiped

The volume of a parallelepiped is the product of the area of its base A and its height h. The base is any of the six faces of the parallelepiped. The height is the perpendicular distance between the base and the opposite face.

An alternative method defines the vectors $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ and $c = (c_1, c_2, c_3)$ to represent three edges that meet at one vertex. The volume of the parallelepiped then equals the absolute value of the scalar triple product $a \cdot (b \times c)$:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})| = |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|$$

Coxeter called the generalization of a parallelepiped in higher dimensions a parallelotope.

Specifically in n-dimensional space it is called n-dimensional parallelotope, or simply n-parallelotope. Thus a parallelogram is a 2-parallelotope and a parallelepiped is a 3-parallelotope.

If S is oriented surface by S^- we denote the opposite oriented surface.

Let $D = \{(x, y, z) : (x, y) \in G, f_1(x, y) < z < f_2(x, y)\}$ and S_k surface defined by $G \ni (x, y) \mapsto (x, y, f_k(x, y)) \in S_k$, $k = 1, 2$. Note that this parameterization of S_1 is not consistent with orientation of S_1 as part of S. If S_1 and S_2 are smooth, we call D elementary domain in z-directions; in similar way we define elementary domain in x and y -directions. We say that domain is simple in z-directions if it is finite union of domains elementary in z-directions, and domain is simple if it is is simple in x, y, z-directions.

By Fubini's theorem

$$\int_D \frac{\partial R}{\partial z}\, dx\, dy\, dz = \int_G dx\, dy \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial R}{\partial z}\, dz = \tag{132}$$

$$\int_G (R(x,y,f_2(x,y)) - R(x,y,f_1(x,y)))dx\, dy = \tag{133}$$

$$\int_{S_2} R\, dx \wedge dy + \int_{S_1^-} R\, dx \wedge dy = \int_S R\, dx \wedge dy\,, \tag{134}$$

where S consists of S_1^-, S_2 and the corresponding cylinder C. We used that the form $dx \wedge dy$ is 0 on C.

Now, let S be a positively oriented, piecewise smooth, simple closed surface in the space $\mathbb{R}^3$, and let D be the region bounded by S. If P, Q and R are functions of (x,y,z) defined on an open region containing D and have continuous partial derivatives there, then:

Again, by Fubini's theorem

$$\int_D (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})\, dx\, dy\, dz = \int_S P\, dy \wedge dz + Q\, dz \wedge dx + R\, dx \wedge dy$$

$$\int_D \frac{\partial R}{\partial z}\, dx\, dy\, dz = \int_S R\, dx \wedge dy\,.$$

Surface-area measure on S

Let S smooth hyper-surface in oriented euclidean space R^m, oriented with continuous unit field of normals $n(x), x \in S$, V -n dimensional volume form in R^m, σ- (m-1) dimensional volume form in S, and let $a_1, \cdots a_{m-1}$ reper in T_xS from the class of orientation given by $n(x)$; then

$V(x)(n, a_1, \cdots, a_{m-1}) = \sigma(x)(a_1, \cdots, a_{m-1})$. Here $n = (n_1, \cdots, n_m)$ is unit normal in x on surface S. Both side are non negative and equal in magnitude because the volume of a parallelepiped defined (spaned) by vectors $n, a_1, \cdots, a_{m-1}$ is the product of the area of its base $A = \sigma(x)(a_1, \cdots, a_{m-1})$ and its height $h = |n| = 1$.

On the other hand,

$$V(x)(n, a_1, \cdots, a_{m-1}) = \begin{vmatrix} n^1 & \dots & n^m \\ a_1^1 & \dots & a_1^m \\ \dots & & \\ a_{m-1}^m & \dots & a_{m-1}^m \end{vmatrix} = \tag{135}$$

$$\sum_{k=1}^m (-1)^{k-1} n_k(x) \sigma_i(x)(a_1, \cdots, a_{m-1}), \tag{136}$$

where $\sigma_k(x) = d\widehat{x}_k = dx_1 \wedge \cdots dx_{k-1} \wedge \widehat{dx_k} \wedge \cdots \wedge dx_m(a_1, \cdots, a_{m-1})$ and the symbol $\widehat{}$ denotes that the corresponding multiple is dropped.

Hence $\sigma(x)(a_1, \cdots, a_{m-1}) = \sum_{k=1}^{m} (-1)^{k-1} n_k(x) dx_1 \cdots dx_{k-1} \widehat{dx_k} \cdots dx_m (a_1, \cdots, a_{m-1})$ and $X \cdot \sigma(x) = \sum_{k=1}^{m} (-1)^{k-1} X^k \sigma_k(x)$.

Using a similar geometric consideraration as the above we conclude, for fixed k, that

$V(x)(e_k, a_1, \cdots, a_{m-1}) = (n(x), e_k) \sigma(x)(a_1, \cdots, a_{m-1})$. Hence

$$n_k \, d\sigma = d\sigma_k = d\widehat{x}_k = (-1)^{k-1} dx_1 \cdots dx_{k-1} \widehat{dx_k} \cdots dx_m \, .$$

If $n_k = (n(x), e_k) = \cos \alpha_k$, then $\cos \alpha_k \sigma = d\widehat{x}_k|_S$ and therefore $\sigma^2 = \sum_{k=1}^{n} (d\widehat{x}_k)^2$.

$X \cdot d\sigma_g = det(X, \cdot)\sqrt{g} = \sum (-1)^{k-1} X^k \sqrt{g} d\sigma_k$,

$d(X \cdot d\sigma_g) = (\sum D_k(X^k \sqrt{g}) V_0 = [\frac{1}{\sqrt{g}} (\sum D_k(X^k \sqrt{g})] V_g$. Hence

$$divX = \frac{1}{\sqrt{g}} (\sum D_k(X^k \sqrt{g}) \, .$$

If x, y, z- the Cartesian (Descartes') coordinate system in $\mathbb{R}^3$,

$$d\sigma = \cos \alpha_1 dy \wedge dz + \cos \alpha_2 dz \wedge dx + \cos \alpha_3 dx \wedge dy$$

orjented area of projection

$$\cos \alpha_1 \, d\sigma = dy \wedge dz \tag{137}$$

$$\cos \alpha_2 \, d\sigma = dz \wedge dx \tag{138}$$

$$\cos \alpha_3 \, d\sigma = dx \wedge dy \tag{139}$$

Here $n(x) = (\cos \alpha_1, \cos \alpha_2, \cos \alpha_3)$.

Integral of function ρ over oriented surface S is

$$\int_S \rho \, \sigma \tag{140}$$

of differential form $\rho\sigma$, where σ is XX area-volume form on S.

Integral XX does not depend on orientation of S because $\sigma_{S^-} = -\sigma_S$.

In functional analysis, the concept of an orthonormal basis can be generalized to arbitrary (infinite-dimensional) inner product spaces (or pre-Hilbert spaces).[4] Given a pre-Hilbert space H, an orthonormal basis for H is an orthonormal set of vectors with the property that every vector in H can be written as an infinite linear combination of the vectors in the basis. In this case, the orthonormal basis is sometimes called a Hilbert basis for H. Note that an orthonormal basis in this sense is not generally a Hamel basis, since infinite linear combinations are required. If B is an orthogonal basis of H, then every element x of H may be written as

$x = \sum_{b \in B} \frac{\langle x, b \rangle}{\|b\|^2} b$.

When B is orthonormal, this simplifies to

$x = \sum_{b \in B} \langle x, b \rangle b$

and the square of the norm of x can be given by
$\|x\|^2 = \sum_{b\in B} |\langle x, b\rangle|^2$.

Even if B is uncountable, only countably many terms in this sum will be non-zero, and the expression is therefore well-defined. This sum is also called the Fourier expansion of x, and the formula is usually known as Parseval's identity. See also Generalized Fourier series.

If B is an orthonormal basis of H, then H is isomorphic to $l^2(B)$ in the following sense: there exists a bijective linear map $F : H \to l^2(B)$ such that
$\langle \Phi(x), \Phi(y)\rangle = \langle x, y\rangle$
for all x and y in H.

The concept of eigenvectors and eigenvalues extends naturally to abstract linear transformations on abstract vector spaces. Namely, let V be any vector space over some field K of scalars, and let T be a linear transformation mapping V into V. We say that a non-zero vector v of V is an eigenvector of T if (and only if) there is a scalar λ in K such that
$T(v) = \lambda v$.

This equation is called the eigenvalue equation for T, and the scalar λ is the eigenvalue of T corresponding to the eigenvector v. Note that $T(v)$ means the result of applying the operator T to the vector v, while λv means the product of the scalar λ by v.

The matrix-specific definition is a special case of this abstract definition. Namely, the vector space V is the set of all column vectors of a certain size $n \times 1$, and T is the linear transformation that consists in multiplying a vector by the given $n \times n$ matrix A.

Some authors allow v to be the zero vector in the definition of eigenvector. This is reasonable as long as we define eigenvalues and eigenvectors carefully: If we would like the zero vector to be an eigenvector, then we must first define an eigenvalue of T as a scalar λ in K such that there is a nonzero vector v in V with $T(v) = \lambda v$. We then define an eigenvector to be a vector v in V such that there is an eigenvalue λ in K with $T(v) = \lambda v$. This way, we ensure that it is not the case that every scalar is an eigenvalue corresponding to the zero vector.

The eigenspaces of T always form a direct sum (and as a consequence any family of eigenvectors for different eigenvalues is always linearly independent). Therefore the sum of the dimensions of the eigenspaces cannot exceed the dimension n of the space on which T operates, and in particular there cannot be more than n distinct eigenvalues.

Any subspace spanned by eigenvectors of T is an invariant subspace of T, and the restriction of T to such a subspace is diagonalizable.

The set of eigenvalues of T is sometimes called the spectrum of T. If λ is an eigenvalue of T, then the operator $T - \lambda I$ is not one-to-one, and therefore its inverse $(T - \lambda I)^{-1}$ does not exist. The converse is true for finite-dimensional vector spaces, but not for infinite-dimensional vector spaces. In general, the operator $T - \lambda I$ may not have an inverse, even if λ is not an eigenvalue.

For this reason, in functional analysis one defines the spectrum of a linear operator T as the set of all scalars λ for which the operator $T - \lambda I$ has no bounded inverse. Thus the spectrum of an operator always contains all its eigenvalues, but is not limited to them.

Let H be a Hilbert space, $L(H)$ be the bounded operators on H. $T \in L(H)$ is a compact operator if the image of each bounded set under T is relatively compact. We list some general properties of compact operators.

If X and Y are Hilbert spaces (in fact X Banach and Y normed will suffice), then $T : X \to Y$ is compact if and only if it is continuous when viewed as a map from X with the weak topology to Y (with the norm topology).

A bounded operator T on a Hilbert space H is said to be self-adjoint if $T = T^*$, or equivalently,

$\langle Tx, y\rangle = \langle x, Ty\rangle, \quad x, y \in H.$

It follows that $< x, Tx >=< Tx, x >$ and $< x, Tx >= \overline{< Tx, x >}$ and therefore $< Tx, x >$ is real for every $x \in H$, thus eigenvalues of T, when they exist, are real.

If λ is the eigenvalue of T corresponding to the eigenvector x then $< Tx, x >= \lambda < x, x >$ and therefore λ is real. If λ_1 and λ_2 are the eigenvalue of T corresponding to the eigenvector v_1 and v_2, then $\lambda_1 < v_1, v_2 >=< Tv_1, v_2 >=< v_1, Tv_2 >= \lambda_2 < v_1, v_2 >$. Hence $< v_1, v_2 >= 0$.

The classification result for Hermitian $n \times n$ matrices is the spectral theorem: If $M = M^*$, then M is unitarily diagonalizable and the diagonalization of M has real entries. Let T be a compact self adjoint operator on a Hilbert space H. We will prove the same statement for T: the operator T can be diagonalized by an orthonormal set of eigenvectors, each of which corresponds to a real eigenvalue.

Theorem 7.1 (Spectral theorem)**.** *Theorem. For every compact self-adjoint operator T on a real or complex Hilbert space H, there exists an orthonormal basis of H consisting of eigenvectors of T. More specifically, the orthogonal complement of the kernel of T admits, either a finite orthonormal basis of eigenvectors of T, or a countably infinite orthonormal basis $\{e_n\}$ of eigenvectors of T, with corresponding eigenvalues $\{\lambda_n\}$ in $\mathbb{R}$, such that $\lambda_n \to 0$.*

In other words, a compact self-adjoint operator can be unitarily diagonalized. This is the spectral theorem.

When H is separable, one can mix the basis $\{e_n\}$ with a countable orthonormal basis for the kernel of T, and obtain an orthonormal basis $\{f_n\}$ for H, consisting of eigenvectors of T with real eigenvalues $\{\mu_n\}$ such that $\mu_n \to 0$.

Corollary. For every compact self-adjoint operator T on a real or complex separable infinite-dimensional Hilbert space H, there exists a countably infinite orthonormal basis $\{f_n\}$ of H consisting of eigenvectors of T, with corresponding eigenvalues $\{\mu_n\}$ in $\mathbb{R}$, such that $\mu_n \to 0$.

The idea

Proving the spectral theorem for a Hermitian $n \times n$ matrix T hinges on showing the existence of one eigenvector x. Once this is done, Hermiticity implies that both the linear span and orthogonal complement of x are invariant subspaces of T. The desired result is then obtained by iteration. The existence of an eigenvector can be shown in at least two ways:

One can argue algebraically: The characteristic polynomial of T has a complex root, therefore T has an eigenvalue with a corresponding eigenvector. Or, The eigenvalues can be characterized variationally: The largest eigenvalue is the maximum on the closed unit sphere of the function $f : R^{2n} \to R$ defined by $f(x) = x^*Tx =< Tx, x >$.

Note. In the finite-dimensional case, part of the first approach works in much greater generality; any square matrix, not necessarily Hermitian, has an eigenvector. This is simply not true for general operators on Hilbert spaces.

The spectral theorem for the compact self adjoint case can be obtained analogously: one finds an eigenvector by extending the second finite-dimensional argument above, then apply induction. We first sketch the argument for matrices.

Since the closed unit sphere S in $\mathbb{R}^{2n}$ is compact, and f is continuous, $f(S)$ is compact on the real line, therefore f attains a maximum on S, at some unit vector y. By Lagrange's multiplier theorem, y satisfies

$\nabla f = \nabla\ y^*Ty = \lambda \cdot \nabla\ y^*y$

for some λ. By Hermiticity, $Ty = \lambda y$.

However, the Lagrange multipliers do not generalize easily to the infinite-dimensional case.

A subset $F \subset C(X)$ is said to be equicontinuous if for every $x \in X$ and every $\varepsilon > 0$, x has a neighborhood U_x such that

$\forall y \in U_x, \forall f \in \mathbf{F} : \qquad |f(y) - f(x)| < \varepsilon.$

A set $F \subset C(X, \mathbb{R})$ is said to be pointwise bounded if for every $x \in X$,

$\sup\{|f(x)| : f \in \mathbf{F}\} < \infty.$

Theorem 7.2 (Arzela-Ascoli)**.** *A family $\mathcal{F}$ of functions with values in a metric space X is normal in the region Ω of the complex plane if and only if*
(i) *for every z in Ω, the values $f(z)$, $f \in \mathcal{F}$, lie in a compact subset of X and*
(ii) *$\mathcal{F}$ is equi-continuous on every compact set $K \subset \Omega$.*

Rešenje. The sufficiency of (ii) together of (i) is proved by Cantor's famous diagonal process. We observe first that there exists an every where dense sequence of points ω_k in Ω, for instance the points with rational coordinates.

A subsequence $g_k = f_{n_k}$ of f_n converges at all points ω_k. Given an $\varepsilon > 0$ we choose $\delta > 0$ such that, for $z, \omega \in K$ and $f \in \mathcal{F}$, $|z - \omega| < \delta$ implies $d(f(z), f(\omega)) < \dfrac{\varepsilon}{3}$. Because K is compact, it can be covered by a finite number of $\dfrac{\delta}{2}$- neighborhoods. We select a point ω_k from each of these neighborhoods. There exists an n_0 such that $n, m > n_0$ implies $d(g_m(\omega_k), g_n(\omega_k)) < \frac{\varepsilon}{3}$ for all these ω_k. For each $z \in K$ one of the ω_k is within distance δ from z; hence $d(g_m(z), g_m(\omega_k)) < \dfrac{\varepsilon}{3}$, $d(g_n(z), g_n(\omega_k)) < \dfrac{\varepsilon}{3}$. The three inequalities yield $d(g_n(z), g_m(z)) < \varepsilon$. △

Let T be a separable topological space and let X be a metric space, and let F be a family of continuous functions from T to X. The family F is normal on a subset $Y \subset T$ if for every sequence $f_n \in F$, there exists a subsequence which converges uniformly on every compact subset of Y.

Theorem 7.3 (Ascoli)**.** *If T is a separable topological space and if X is a compact metric space, then every equicontinuous family $\mathcal{F}$ of mappings $f : T \to X$ is a normal family.*

The Arzela- Ascoli theorem is thus a fundamental result in the study of the algebra of continuous functions. Various generalizations of the above result are possible. For instance: Let T be a compact Hausdorff space and X a metric space. Then a subset F of $C(T, X)$ is compact in the compact-open topology if and only if it is equicontinuous, pointwise relatively compact and closed.

Here pointwise relatively compact means that for each $x \in T$, the set $F_x = \{f(x) : f \in F\}$ is relatively compact in X.

If $M(G)$ is the set of all meromorphic functions we can consider it as a subset of $C(G, \overline{\mathbb{C}})$ and endow it the metric of $C(G, \overline{\mathbb{C}})$.

Let f_n be sequence in $M(G)$ (resp. $H(G)$) and suppose that $f_n \to f$ in $C(G, \overline{\mathbb{C}})$. Then either f is meromorphic (resp. analytic) or $f \equiv \infty$. Hence $M(G) \cup \{\infty\}$ is a complete metric space.

We can adopt simple terminology. We say that $F \subset M(G)$ is normal (in a wider sense or $\overline{\mathbb{C}}$- normal) if
(i) every sequence in F contains a subsequence which converges or tends ∞ uniformly on compact subsets.

Eigendecomposition

A bilinear form on a vector space V is a bilinear map $V \times V \to K$, where K is the field of scalars. In other words, a bilinear form is a function $B : V \times V \to K$ that is linear in each argument separately:

Let V be an n-dimensional vector space with basis $C = \{e_1, ..., e_n\}$. Define the $n \times n$ matrix A by $A_{ij} = B(e_i, e_j)$.

If the $n \times 1$ matrix x represents a vector v with respect to this basis, and analogously, y represents w, then:

$$B(\mathbf{v}, \mathbf{w}) = x^{\mathrm{T}} A y = \sum_{i,j=1}^{n} a_{ij} x_i y_j.$$

Let B be bilinear form on V and $C = \{e_1, ..., e_n\}$ basis in V and A matrix wrt C. Let $C' = \{e'_1, ..., e'_n\}$ be another basis in V $e'_k = Se_k$, S is an invertible matrix. Now the new matrix representation for the symmetric bilinear form is given by

$$A' = S^{\mathsf{T}} A S.$$

A basis $C = \{e_1, ..., e_n\}$ is orthogonal with respect to B if and only if:

$$B(e_i, e_j) = 0 \ \forall i \neq j.$$

When the characteristic of the field is not two, V always has an orthogonal basis say $C^* = \{e^*_1, ..., e^*_n\}$. This can be proven by induction.

A basis C^* is orthogonal if and only if the matrix representation $A^* = diag(\lambda_1, ..., \lambda_n)$ is a diagonal matrix.

If $x = \sum x_k e_k = \sum x^*_k e^*_k$, then $B(x, x) = a^{ij} x_i x_j = \sum \lambda_k (x^*_k)^2$. Further if $V = \mathbb{R}^n$ and $\lambda = \min\{\lambda_1, ..., \lambda_n\}$ and $\Lambda = \max\{\lambda_1, ..., \lambda_n\}$, then

$\lambda|x|^2 B(x,x) \leq \Lambda|x|^2$.

The trace of an n-by-n square matrix A is defined to be the sum of the elements on the main diagonal (the diagonal from the upper left to the lower right) of A, i.e.,

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

where a_{ii} denotes the entry on the ith row and ith column of A. The trace of a matrix is the sum of the (complex) eigenvalues, and it is invariant with respect to a change of basis. This characterization can be used to define the trace of a linear operator in general. Note that the trace is only defined for a square matrix (i.e., $n \times n$).

The trace is similarity-invariant, which means that A and $P^{-1}AP$ have the same trace. This is because

$$\operatorname{tr}\left(P^{-1}AP\right) = \operatorname{tr}\left(P^{-1}(AP)\right) = \operatorname{tr}\left((AP)P^{-1}\right) = \operatorname{tr}\left(A\left(PP^{-1}\right)\right) = \operatorname{tr}(A).$$

If A is symmetric and B is antisymmetric, then

$$\operatorname{tr}(AB) = 0.$$

If A is a linear operator represented by a square n-by-n matrix with real or complex entries and if $\lambda_1, ..., \lambda_n$ are the eigenvalues of A (listed according to their algebraic multiplicities), then

$$\operatorname{tr}(A) = \sum_i \lambda_i$$

This follows from the fact that A is always similar to its Jordan form, an upper triangular matrix having $\lambda_1, ..., \lambda_n$ on the main diagonal. In contrast, the determinant of A is the product of its eigenvalues; i.e.,

$$\det(A) = \prod_i \lambda_i$$

More generally,

$$\operatorname{tr}\left(A^k\right) = \sum_i \lambda_i^k.$$

Let V be a finite dimensional vector space over some field K, and let $(,)$ be a nondegenerate bilinear form on V.

We then have for every linear endomorphism A of V, that there is a unique endomorphism A^* of V such that $(Ax, y) = (x, A^*y)$ for all x and $y \in V$.

The existence and uniqueness of such an A^* requires some explanation, but I will take it for granted.

An endomorphism of a vector space V is a linear map, $A : V \to V$.

Proposition: Given an endomorphism A of a finite dimensional vector space V equipped with a nondegenerate bilinear form $(,)$, the endomorphisms A and A^* have the same set of eigenvalues.

Proof: Let λ be an eigenvalue of A. And let v be an eigenvector of A corresponding to λ (in particular, v is nonzero). Let w be another arbitrary vector. We then have that: $(v\lambda, w) = (\lambda v, w) = (Av, w) = (v, A^*w)$. This implies that $(v, \lambda w - A^*w) = 0$ for all $w \in V$. Now either λ is an eigenvalue of A^* or not. If it isn't, the operator $\lambda w - A^*w$ is an automorphism of V since $\lambda I - A^*$ being singular is equivalent to λ being an eigenvalue of A^*. In particular, this means that $(v, z) = 0$ for all $z \in V$. But since $(,)$ is nondegenerate, this implies that $v = 0$. A contradiction. λ must have been an eigenvalue of A^* to begin with. Thus every eigenvalue of A is an eigenvalue of A^*. The other inclusion can be derived similarly. We are working over a real vector space and considering the dot product as your bilinear form. Now consider an endomorphism T of $\mathbb{R}^n$ which is given by $T(x) = Ax$ for some $n \times n$ matrix A. It just so happens that for all $y \in \mathbb{R}^n$ we have $T^*(y) = A^t y$. Since T and T^* have the same eigenvalues, so do A and A^t.

In linear algebra, a symmetric $n \times n$ real matrix M is said to be positive definite if $z^T Mz$ is positive for every non-zero column vector z of n real numbers. Here z^T denotes the transpose of z.

Let $x_1, \ldots, x_n$ be a list of n linearly independent vectors of some complex vector space with an inner product $\langle \cdot, \cdot \rangle$. It can be verified that the Gram matrix M of those vectors, defined by $M_{ij} = \langle x_i, x_j \rangle$, is always positive definite. Conversely, if M is positive definite, it has an eigendecomposition $P^{-1}DP$ where P is unitary, D diagonal, and all diagonal elements $D_{ii} = \sqrt{\lambda_i}$ of D are real and positive. Let E be the real diagonal matrix with entries so $D = E^2$ and $M = (EP)^*EP$. Its leading principal minors are all positive. The kth leading principal minor of a matrix M is the determinant of its upper-left k by k submatrix. It turns out that a matrix is positive definite if and only if all these determinants are positive. This condition is known as Sylvester's criterion, and provides an efficient test of positive-definiteness of a symmetric real matrix. Namely, the matrix is reduced to an upper triangular matrix by using elementary row operations, as in the first part of the Gaussian elimination method, taking care to preserve the sign of its determinant during pivoting process. Since the k-th leading principal minor of a triangular matrix is the product of its diagonal elements up to row k, Sylvester's criterion is equivalent to checking whether its diagonal elements are all positive. This condition can be checked each time a new row k of the triangular matrix is obtained. It has a unique Cholesky decomposition. The matrix M is positive definite if and only if there exists a unique lower triangular matrix L, with real and strictly positive diagonal elements, such that $M = LL^*$. This factorization is called the Cholesky decomposition of M.

Suppose that B is positive definite. Then there is $B^{1/2}$. Look, the eigenvalues of a symmetric matrix are real, thus the eigenvectors too. So the Jordan form is real $B = USU^T$ with S being diagonal positive and U orthogonal. Then construct $B^{1/2}$ as $US^{1/2}U^T$.

If we call $B^{1/2}$ the symmetric matrix such that $B^{1/2}B^{1/2} = B$ (i.e. the standard square root of a positive definite matrix) then $AB = AB^{1/2}B^{1/2} = B^{-1/2}(B^{1/2}AB^{1/2})B^{1/2}$, that is AB is similar to the positive definite matrix $B^{1/2}AB^{1/2}$, sharing all eigenvalues. It makes the eigenvalues of AB be positive if A is positive definite and non positive if A is non-positive definite.

By definition $I := x^T B^{1/2}AB^{1/2}x = z^T Az$ for $z = B^{1/2}x$. Hence if A is positive definite (non-positive definite) $I > 0$ ($I \leq 0$) $z = B^{1/2}x \neq 0$ iff $x \neq 0$. Thus positive definite.

Proposition: If B is positive definite matrix and A non- positive definite matrix, then AB is non positive definite matrix and in particular $tr(AB) \leq 0$.

8 Motivacija

Svetlik

Razmotrimo diferencijalnu jednačinu

$$y'(x) + p(x)y(x) = q(x), \qquad x \in [a, b], \tag{141}$$

pri čemu su $p, q : [a, b\,] \to \mathbb{R}$ zadate neprekidne funkcije a $y : [a, b\,] \to \mathbb{R}$ nepoznata funkcija. Kako se u jednačini (141) javlja prvi izvod funkcije y, prirodno je zahtevati da rešenje tražimo u klasi diferencijabilnih funkcija na $[a, b\,]$. Međutim kako je

$$y'(x) = -p(x)y(x) - q(x),$$

a funkcije p, q i y su neprekidne, sledi da i y' mora biti neprekidna. Dakle, ako postoji diferencijabilna funkcija y takva da zadovoljava jednačinu (141) onda je ta funkcija i neprekidno diferencijabilna tj. pripada klasi $C^1[a, b\,]$.

Neka je φ proizvoljna funkcija klase $C^1[a, b\,]$ takva da je $\varphi(a) = \varphi(b) = 0$. Ako levu i desnu stranu jednačine (141) pomnožimo sa $\varphi(x)$ dobijamo jednačinu

$$y'(x)\varphi(x) + p(x)y(x)\varphi(x) = q(x)\varphi(x),$$

a nakon integraljenja i jednačinu

$$\int_a^b y'(x)\varphi(x)dx + \int_a^b p(x)y(x)\varphi(x)dx = \int_a^b q(x)\varphi(x)dx. \tag{142}$$

Ako u integralu $\displaystyle\int_a^b y'(x)\varphi(x)dx$ primenimo parcijalnu integraciju dobijamo

$$\int_a^b y'(x)\varphi(x)dx = y(x)\varphi(x)\Big|_{x=a}^{b} - \int_a^b y(x)\varphi'(x)dx = -\int_a^b y(x)\varphi'(x)dx.$$

Otuda se jednačina (142) svodi na jednačinu

$$\int_a^b (-y(x)\varphi'(x) + p(x)y(x)\varphi(x))dx = \int_a^b q(x)\varphi(x)dx. \tag{143}$$

Primetimo da se u jednačini (143) ne pojavljuje izvod nepoznate funkcije y, kao i da ima smisla tražiti rešenje iste u klasi $C[a,b]$. Takođe, na osnovu prethodnog izvođenja, sledi da ako je y rešenje jednačine (141) onda je y rešenje i jednačine (143).

Razmotrimo sada i jednačinu

$$y'(x) = \operatorname{sgn} x, \qquad x \in [-1,1], \tag{144}$$

gde je $y : [-1,1] \to \mathbb{R}$ nepoznata funkcija. Primetimo da funkcija sgn ima prekid prve vrste u tački 0. S druge strane prvi izvod neke funkcije ne može imati prekide prve vrste. Otuda jednačina (144) nema rešenja u klasi diferencijabilnih funkcija na $[-1,1]$ tj. ne postoji funkcija $y : [-1,1] \to \mathbb{R}$ takva da je $y'(x) = \operatorname{sgn} x$ za svako $x \in [-1,1]$.

Međutim umesto jednačine (144) analagno kao u prvom primeru možemo posmatrati jednačinu

$$\int_{-1}^{1} -y(x)\varphi'(x)dx = \int_{-1}^{1} \operatorname{sgn} x\, \varphi(x)dx, \tag{145}$$

pri čemu je φ proizvoljna funkcija klase $C^1[-1,1]$ takva da je $\varphi(-1) = \varphi(1) = 0$. Pri tome prirodno je rešenje jednačine (145) tražiti u klasi $C[-1,1]$. Neposrednom proverom se utvrđuje da je funkcija $y(x) = |x| + c$ (c realna konstanta) rešenje jednačine (145) u klasi $C[-1,1]$. Otuda to rešenje nazivamo i *slabo rešenje* jednačine (144).

Suppose that φ is $C^1[-1,1]$ such that $\varphi(-1) = \varphi(1) = 0$. By partial integration,

$\int_{-1}^1 |x|'\varphi(x)dx = -\int_{-1}^1 y(x)\varphi'(x)dx = \int_{-1}^0 |x|'\varphi(x)dx + \int_0^1 |x|'\varphi(x)dx = \int_{-1}^1 \operatorname{sgn} x\, \varphi(x)dx$

9 surfaces

Supp that in ngb V of point $y_0 \in \mathbb{R}^n$ surface S is given by $\psi = 0$, where $\psi \in C^2(V)$ and $|D\psi| > 0$ in V. The unit normal in $y \in S \cap V$ is given by $\nu = D\psi/|D\psi|$. Let $L = (d\nu)_{y_0}$ be linear operator which can be identified with matrix $A = d\nu = [D_i(D_j\psi/|D\psi|)]$ with respect the standard system $C = \{e_1, ..., e_{n-1}, e_n\}$ in $\mathbb{R}^n$. There is orth system $C' = \{e'_1, ..., e'_{n-1}, e'_n\}$, $e'_1, ..., e'_{n-1} \in T_{y_0}S$, $e'_n = \nu(y_0)$ wrt A in which the corresponding matrix A' has diagonal form $A' = diag(k_1, ..., k_{n-1}, k_n)$.

Let L' be the restriction of $L = (d\nu)_{y_0}$ on $T_{y_0}S$. Since $\nu \cdot \nu = 1$, we have $D_i\nu \cdot \nu = 0$. Hence since $D_i\nu \in T_{y_0}S$, $D_i\nu_n = 0$ and in particular $k_n = D_n\nu_n = 0$ wrt the system C'. It is known that $tr(L)$ is independent of coordinates. So (1) $tr(A) = tr(A_*) = \sum_{i=1}^n \kappa_i = \sum_{i=1}^n D_i[D_i\nu](y_0)$.

The mean curvature is defined $H(y_0) = \frac{1}{n-1}\sum_{i=1}^n D_i[D_i\psi/|D\psi|]$.

In particular if S is graph in $\mathbb{R}^{n+1}$ over a domain G, given by $x_{n+1} = u(x_1, ..., x_n)$, where $u \in C^2(G)$ is a real valued function, we can consider $\psi = u(x_1, ..., x_n) - x_{n+1}$.

If we set $A = \hat{A} = \sqrt{1+|Du|^2}$, then $D_j\psi = D_ju = A\nu_j$, $AD_jA = \sum_{k=1}^n D_juD_{jk}u$ and $D_jA = \sum_{k=1}^n \nu_jD_{jk}u$. Hence

$$D_i\nu_j = \frac{D_{ij}u}{A} - D_ju\frac{D_iA}{A^2} = \frac{D_{ij}u}{A} - \frac{1}{A}\sum_{k=1}^n \nu_j\nu_kD_{ik}u.$$

Set $u_{ik} = D_{ik}u$, $U = [u_{ik}]$ and denote the columbs of U with U^1, ... U^n. If we add to j-th columb of the matrix $[\frac{D_{ij}u}{A}]$, the linear combination of $\frac{1}{A}\sum_{k=1}^n \nu_j\nu_kU^k$, then the determinant of new matrix is $(1-\nu_j^2)\det[\frac{D_{ij}u}{A}]$. If we do this procedure respectively for $j = 1, ..., n$, we can conclude that

$$\det[D_i\nu_j] = \det[\frac{D_{ij}u}{A}] = \frac{\det D^2u}{A^{n+2}}.$$

Set $N = [D_i\nu_j]$ $t_{kj} = \nu_k\nu_j$, $T = [t_{kj}]$, $r_{kj} = \delta_{kj} - \nu_k\nu_j$ and $R = [r_{kj}]$. Since $R = I - T$ $AN = U - UT = U(I-T) = UR$ and therefore $A^n\det N = \det U \det R$
$\det R = A^{-2}$

Now an application of (1) yields $H(y_0) = \frac{1}{n}\sum_{i=1}^n D_i[D_iu/A]$.

The Gaussian curvature is

$$K(x_0) = \det(L') = \Pi\kappa_i = \det[D_i\nu_j(y_0)] = \frac{\det D^2u}{A^{n+2}}.$$

10 Minimal surfaces

$p = u_x$ $q = u_y$

Finding the extrema of functionals is similar to finding the maxima and minima of functions. The maxima and minima of a function may be located by finding the points where its derivative vanishes (i.e., is equal to zero). The extrema of functionals may be obtained by finding functions where the functional derivative is equal to zero. This leads to solving the associated EulerLagrange equation.

Consider the functional

$J[u] = \int_G L(x, y, u, p, q)dxdy$, where $L(x, y, u, p, q)$ is twice continuously differentiable with respect to its arguments.

$h(t) = J[u+tv] = L(x, y, u+tv, p+tv_x, qtv_y)dxdy$ Since $h'(t) = \int_G L_tdxdy$ and $L_t = vL_u + v_xL_p + v_yL_q$, $h'(t) = \int_G L_tdxdy = \int_G(vL_u + v_xL_p + v_yL_q)dxdy$ and in particular $h'(0) = I := \int_G(vL_u + v_xL_p + v_yL_q)dxdy$.

By partial integration, $I = \int_G v(L_u - (L_p)_x - (L_q)_y)dxdy$ and therefore

(1) $L_u - (L_p)_x - (L_q)_y = 0$.

For example, if $f(x, y)$ denotes the displacement of a membrane above the domain D in the x, y plane, then its potential energy is proportional to its surface area:

$$U[\varphi] = \iint_D \sqrt{1 + \nabla\varphi\cdot\nabla\varphi}\,dx\,dy.$$

Plateau's problem consists of finding a function that minimizes the surface area while assuming prescribed values on the boundary of D; the solutions are called minimal surfaces. The EulerLagrange equation for this problem is nonlinear:

$$\varphi_{xx}(1+\varphi_y^2)+\varphi_{yy}(1+\varphi_x^2)-2\varphi_x\varphi_y\varphi_{xy}=0.$$

Here

See Courant (1950) for details.

Here $L=A=\sqrt{1+p^2+q^2}$ and by (1) $div(\nabla f/A)=0$.

Dirichlet's principle

It is often sufficient to consider only small displacements of the membrane, whose energy difference from no displacement is approximated by

$$V[\varphi]=\frac{1}{2}\iint_D \nabla\varphi\cdot\nabla\varphi\,dx\,dy.$$

$L=p^2+q^2$, $L_p=2p$, $(L_p)_x=2u_{xx}$ and by (1), $u_{xx}+u_{yy}=0$

The functional V is to be minimized among all trial functions f that assume prescribed values on the boundary of D. If u is the minimizing function and v is an arbitrary smooth function that vanishes on the boundary of D, then the first variation of $V[u+\varepsilon v]$ must vanish:

$$\frac{d}{d\varepsilon}V[u+\varepsilon v]|_{\varepsilon=0}=\iint_D \nabla u\cdot\nabla v\,dx\,dy=0.$$

Provided that u has two derivatives, we may apply the divergence theorem to obtain

$$\iint_D \nabla\cdot(v\nabla u)\,dx\,dy=\iint_D \nabla u\cdot\nabla v+v\nabla\cdot\nabla u\,dx\,dy=\int_C v\frac{\partial u}{\partial n}ds,$$

where C is the boundary of D, s is arclength along C and $\partial u/\partial n$ is the normal derivative of u on C. Since v vanishes on C and the first variation vanishes, the result is

$$\iint_D v\nabla\cdot\nabla u\,dx\,dy=0$$

for all smooth functions v that vanish on the boundary of D. The proof for the case of one dimensional integrals may be adapted to this case to show that

$$\nabla\cdot\nabla u=0$$

in D.

The difficulty with this reasoning is the assumption that the minimizing function u must have two derivatives. Riemann argued that the existence of a smooth minimizing function was assured by the connection with the physical problem: membranes do indeed assume configurations with minimal potential energy. Riemann named this idea the Dirichlet principle in honor of his teacher Peter Gustav Lejeune Dirichlet. However Weierstrass gave an example of a variational problem with no solution: minimize.

11 QCH

In this section we present the results from Filomat 31:10 (2017), 3023-3034, https://doi.org/10.2298/FIL1710023M.

Sažetak

In [18], we study the growth of gradients of solutions of elliptic equations, including the Dirichlet eigenfunction solutions on bounded plane convex domain. Several results related to Bi-Lipschicity of quasiconformal harmonic (qch) mappings with respect to quasi-hyperbolic and euclidean metrics, are proved. In connection with the subject, we announce a few results concerning the so called interior estimate, including Proposition 13. In addition, a short review of the subject is given.

11.1 Introduction

There is a numerous literature related to the subject, see [33] and the literature cited there and in this paper. Here we give a short review of the subject, announce and consider new results. In particular, here our discussion is related to the following items:

(i) In Section 11.3, we outline a proof of result of Bozin- Mateljevic which gives an answer to an intriguing problem probably first posed by Kalaj and which states that Quasiconformal and HQC mappings between Lyapunov Jordan domains are co-Lipschitz.

(ii) In [35], Li Peijin, Jiaolong Chen, and Xiantao Wang proved the gradient of Quasiconformal solutions of poisson equations are bounded under some hypothesis. In Section 11.4 we announce some results related to the local version of the interior estimate(see for example Proposition 22), and discuss whether their result holds without the hypothesis that the radial derivative is bounded.

(iii) In Section 11.6, the author shows that the Dirichlet eigenfunction solutions on bounded plane convex domain have bounded gradients. Our considerations gives contribution to the problem posed in communication of the author with Yacov Sinai.

(iv) Bi-Lipschicity property of Harmonic K-quasiconformal maps with respect to k-metrics (quasi-hyperbolic metrics) in space is subject of of Section 11.7.

(v) In Section 11.8, we extend a result of Tam and Wan, [38], 1998. More precisely, we prove if f is K-qc hyperbolic harmonic mappings of $\mathbb{H}^n$ with respect to the hyperbolic metric with $K < 3^{n-1}$, then f is a quasi-isometry.

Concerning the items (i), (ii) and (iii) we only outline some proofs. Note that in Section 11.4, we only announce the following result:

Proposition 13 (Local version of Interior estimate)**.** *Let $s : \overline{\mathbb{U}} \to \mathbb{R}$ be a continuous function from the closed unit disc $\overline{\mathbb{U}}$ into the real line satisfying the conditions:*

1. *χ is $C^{1,\alpha}$ on $\mathbb{U}$, then the the corresponding version of this result holds.*
2. *$|\Delta\chi| \le a_0|\nabla\chi|^2 + b_0$, on $V = V(r) = \mathbb{U} \cap B(w_0, r)$, where $w_0 = e^{it_0}$, for some constants a_0 and b_0 (the last inequality we will call the interior estimate inequality) and*
3. *$\underline{\chi_b(\theta)} = \chi(e^{i\theta})$ is $C^{1,\alpha}$ on the interval $l = \{e^{it} : t \in (t_0 - e, t_0 + e)\} = \underline{V(r)} \cap \mathbb{T}$.*
   Then there is $0 < r_1 < r$ such that the function $|\nabla\chi|$ is bounded on $V(r_1) = \mathbb{U} \cap B(w_0, r_1)$.

See also Proposition 22[1] in Section 11.4 which is more complete statement.

In Section 11.2 we shortly consider the background of the subject and in Section 11.9, we collect some definitions.

11.2 Bacground

For a function h, we use notation $\partial h = \frac{1}{2}(h'_x - ih'_y)$ and $\overline{\partial} h = \frac{1}{2}(h'_x + ih'_y)$; we also use notations Dh and $\overline{D}h$ instead of ∂h and $\overline{\partial} h$ respectively when it seems convenient.

We use the notation $\lambda_f = l_f(z) = |\partial f(z)| - |\bar\partial f(z)|$ and $\Lambda_f(z) = |\partial f(z)| + |\bar\partial f(z)|$, if $\partial f(z)$ and $\bar\partial f(z)$ exist.

Throughout the paper we denote by Ω, G and D open subsets of $\mathbb{R}^n$, $n \ge 1$.

Let $B(x,\ r) == B^n(x,\ r) = \{z \in \mathbb{R}^n : |z-x| < r\}$, $S^{n-1}(x,\ r) = \partial B^n(x,\ r)$ (abbreviated $S(x,\ r)$) and let $\mathbb{B}^n$, $\mathbb{S} = \mathbb{S}^{n-1}$ stand for the unit ball and the unit sphere in $\mathbb{R}^n$, respectively. In particular, by $\mathbb{D}$ (or $\mathbb{U}$) we denote the unit disc $\mathbb{B}^2$ and $\mathbb{T} = \partial\mathbb{D}$ we denote the unit circle $\mathbb{S}^1$ in the complex plane. For a domain D in $\mathbb{R}^n$ with non-empty boundary, we define the distance function $d = d_D = \mathrm{dist}(D)$ by $d(x) = d(x; \partial D) = \mathrm{dist}(D)(x) = \inf\{|x-y| : y \in \partial D\}$; and if f maps D onto $D' \subset \mathbb{R}^n$, in some settings it is convenient to use short notation $d^* = d^*(x) = d_f(x)$ for $d(f(x); \partial D')$. It is clear that $d(x) = \mathrm{dist}(x,\ D^c)$, where D^c is the complement of D in $\mathbb{R}^n$.

Proposition 14 (Proposition 5 [29])**.** *If h is a harmonic univalent orientation preserving k-qc mapping of domain D onto D', then*

$$d(z)\Lambda_h(z) \le 16\, K\, d_h(z), \text{and} \quad d(z)\lambda_h(z) \ge \frac{1-k}{4} d_h(z)\,. \tag{146}$$

Proposition 15 (Corollary 1, Proposition 5 [29])**.** *Every e-harmonic quasiconformal mapping of the unit disc (more generally of a strongly hyperbolic domain) is a quasi-isometry with respect to hyperbolic distances.*

[1] We believe that this result will find further application.

From Proposition 14 directly follows next result (Proposition 16). [2]

Proposition 16 ([27]). *Every e-harmonic quasi-conformal mapping of a domain different from* $\mathbb{C}$ *is a quasi-isometry with respect to quasi-hyperbolic distances.*

The next theorem concerns harmonic maps onto a convex domain. For the planar version of Theorem 11.1 cf. [28, 29], also [33], pp. 152-153. The space version was communicated on International Conference on Complex Analysis and Related Topics (Xth Romanian-Finnish Seminar, August 14-19, 2005, Cluj-Napoca, Romania), by Mateljević and stated in [29], cf. also [31].

Theorem 11.1 (Theorem 1.3, [29]). *Suppose that* h *is an Euclidean harmonic mapping from the unit ball* $\mathbb{B}^n$ *onto a bounded convex domain* $D = h(\mathbb{B}^n)$*, which contains the ball* $h(0) + R_0\mathbb{B}^n$*. Then for any* $x \in \mathbb{B}^n$

$$d(h(x), \partial D) \geq (1 - \|x\|)R_0/2^{n-1}.$$

Although the proofs of the above results are not difficult, it turns out that they have further impact on the subject. We will shortly discuss it in this paper.

We use a distortion property of quasiconformal maps to prove that for n-dimensional Euclidean harmonic quasiconformal mappings with $K_O(f) < 3^{n-1}$, Jacobian is never zero.

Theorem 11.2. *[32, 19] Suppose that* $h : \Omega \mapsto \mathbb{R}^n$ *is a harmonic quasiconformal map. If* $K_O(h) < 3^{n-1}$*, then its Jacobian has no zeros.*

Theorem 11.3. *Suppose* h *is a harmonic* K*-quasiconformal mapping from the unit ball* $\mathbb{B}^n$ *onto a bounded convex domain* $D = h(\mathbb{B}^n)$*, with* $K < 3^{n-1}$*. Then* h *is co-Lipschitz on* $\mathbb{B}^n$*.*

We can generalize this result:

Theorem 11.4. *Suppose that* $f : D_1 \longrightarrow D_2$*, where* $D_1, D_2 \subset \mathbb{R}^n$ *and the complement* D_1 *has at least one point, is a harmonic* K*-quasiconformal mapping with* $K_O(f) < 3^{n-1}$*, (or and that* f *belongs to a non-zero Jacobian family of harmonic maps), then* f *is bi-Lipschitz with respect* k*-metrics.*

This theorem is stated as Theorem 11.12 in Section 11.7 and it is also proved by Shadia Shalandi [36].

In particular,

(A) f is Lipschitz with respect to k-metrics.

Note that (A) holds more generally without the hypothesis that f belongs to a non-zero Jacobian family, cf. [34].

Theorem 11.5 ([34]). *Suppose that* $\Omega \subset \mathbb{R}^n$*,* $f : \Omega \to \mathbb{R}^n$ *is* K*-qr and* $\Omega' = f(\Omega)$*. Let* $\partial\Omega'$ *be a continum containing at least two distinct point.*
If f *is a vector harmonic map, then* f *is Lipschitz w.r. to quasi-hyperbolic metrics on* Ω *and* Ω'*.*

[2] In that time, the author did not realized that quasi-hyperbolic metrics have important applications and did not state this version which due to V. Manojlovic.

11.3 Quasiconformal and HQC mappings between Lyapunov Jordan domains

Although the following two statements did not get attention immediately after their publication, it turns out surprisingly that they have an important role in the demonstration of Theorem 11.8 (co-Lip), [20].

Proposition 17 (Corollary 1, Proposition 5 [29]; see also [27])**.** *Every e-harmonic quasi-conformal mapping of the unit disc (more generally of a strongly hyperbolic domain) is a quasi-isometry with respect to hyperbolic distances.*

Theorem 11.6 ([28])**.** $(ii.1)$ *Suppose that* $h = f + \overline{g}$ *is a Euclidean orientation preserving harmonic mapping from* $\mathbb{D}$ *onto bounded convex domain* $D = h(\mathbb{D})$*, which contains a disc* $B(h(0); R_0)$ *. Then* $|f'| \geq R_0/4$ *on* $\mathbb{D}$*.*
$(ii.2)$ *Suppose, in addition, that* h *is qc. Then* $l_h \geq (1-k)|f'| \geq (1-k)R_0/4$ *on* $\mathbb{D}$
$(ii.3)$ *In particular,* h^{-1} *is Lipschitz.*

Further Kalaj [39] proved that

Theorem 11.7. *Suppose* $h : D_1 \to D_2$ *is a hqc homeomorphisam, where* D_1 *and* D_2 *are domains with* $C^{1,\mu}$ *boundary.*
(a)Then h *is Lipschitz.*
(b)If in addition D_2 *is convex, then* h *is bi-Lipschitz.*

But this theorem in mind the following question:
Question 1: whether Quasiconformal and HQC mappings between Lyapunov Jordan domains is co-Lipschitz?
is natural.

The proof of the part (a) of Theorem 11.7 in [39] is based on an application of Mori's theorem on quasiconformal mappings, which has also been used previously by Miroslav Pavlović in [40] in the case $D_1 = D_2 = \mathbb{U}$, and a geometric lemma related to Lypunov domains. It seems that using local version of the interior estimate, Proposition 22, one can prove that the theorem holds if D_2 has C^2 boundary.

Note that our proof of Proposition 22 is not based on Mori's theorem on quasiconformal mappings, and a natural question arises:

Question 2. Whether a proof of Theorem 11.7(a) can be based on Proposition 22.

It seems that we can use

Proposition 18. *Every Lyapunov domain is exhausted by a monotonous sequence of* C^∞*-domains which are Lyapunov - uniformly bounded.*

Theorem C. Let f be harmonic K-qc map of $\mathbb{U}$ onto a domain G which is exhausted by a monotonous sequence of domains which are Lyapunov- uniformly bounded. Then f is L- Lipschitz and we have explicit estimate for L. Outline of proof of Theorem C:

Rešenje. Let ω conformal and f be harmonic K-qc map of $\mathbb{U}$ onto $C^{1,\mu}$ domain G, $r_n = 1 - 1/n$, $U_n = D(0; r_n)$, $G_n = \omega(U_n)$, γ_n boundary of G_n, $V_n = f^{-1}(G_n)$, ψ_n conformal mapping of U onto V_n such that $\psi_n(0) = 0$ and $\psi_n'(0) > 0$ and $f_n = f \circ \psi_n$.

Let f_n be L_n-Lipschitz on U. An application of Theorem XX shows then there are constants c_0 and L_0 such that

Lyp$\gamma_n \leq c_0$ and $L_n \leq L_0$.

Since $\psi_n'(z)$ converges to 1 for every $z \in \mathbb{U}$, f is L_0-Lipschitz on $\mathbf{U}$. △

As an application of Gehring-Osgood inequality[22] concerning qc mappings and quasi-hyperbolic distances, in the particular case of punctured planes, we prove

Proposition 19. *Let f be a K-qc mapping of the plane such that $f(0) = 0$, $f(\infty) = \infty$ and $\alpha = K^{-1}$. If $z_1, z_2 \in \mathbb{C}^*$, $|z_1| = |z_2|$ and $\theta \in [0, \pi]$ (respectively $\theta^* \in [0, \pi]$) is the measure of convex angle between z_1, z_2 (respectively $f(z_1), f(z_2)$), then*

$$\theta^* \leq c \max\{\theta^\alpha, \theta\},$$

where $c = c(K)$. In particular, if $\theta \leq 1$, then $\theta^* \leq c\theta^\alpha$.

We shortly refer to this result as (GeOs-BM). Through the paper we frequently consider the setting ($\mathbb{U}_{qc}$): Let $h : \mathbb{U} \to D$ be K-qc map, where $\mathbb{U}$ is the unit disk and suppose that D is Lyapynov domain. Under this hypothesis, using (GeOs-BM), we prove that for every $a \in \mathbb{T} = \{|z| = 1\}$, there is a special Lyapunov domain U_a, of a fixed shape, in the unit disk $\mathbb{U}$ which touches a and a special, convex Lyapunov domain $lyp(D)_b^-$, of a fixed shape, in D such that $lyp(D)_b^- \subset h(U_a) \subset H_b$, where H_b is a half-plane H_b, which touches $b = h(a)$. We can regard this result as "good local approximation of qc mapping h by its restriction to a special Lyapunov domain so that codomain is locally convex". In addition if h is harmonic, using it, we prove that h is co-Lip $\mathbb{U}$:

Theorem 11.8. *Suppose $h : \mathbb{U} \to D$ is a hqc homeomorphisam, where D is a Lyapanov domain with $C^{1,\mu}$ boundary. Then h is co-Lipschitz.*

It settles an open intriguing problem in the subject and can be regarded as a version of Kellogg- Warschawski theorem for hqc.

11.4 Quasiconformal solutions of poisson equations

In [35], Li Peijin, Jiaolong Chen, and Xiantao Wang proved the gradient of Quasiconformal solutions of poisson equations are bounded under some hypothesis. Using local version of the interior estimate, Proposition 22, we outline an argument that their result holds without hypothesis (ii.2) (see below)[3].

[3] At this point it may seem that we use a heuristic approach, but we hope to fill details in a forth-coming paper.

We introduce the following hypothesis:
(i.1) Let g be a function from $\mathbb{D}$ to $\mathbb{C}$ with a continuous extension to the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$, and
(i.2) let $f : \mathbb{S} \to \mathbb{C}$ be a bounded integrable function on $\mathbb{S}$ and
(i.3) let Ω be a Jordan domain with C^2 boundary.

Further, let
(ii.1) $\mathcal{D}_{\mathbb{D}\to\Omega}(g)$ denote the family of solutions $w : \mathbb{D} \to \Omega$ of the Poisson equation $\Delta w = g$, where $w|S = f \in L^1(\mathbb{S})$ and each w is a sense-preserving diffeomorphism. In [35], it is proved if Ω is C^2 domain,
(ii.2) $|\partial u/\partial r|$ is bounded on $\mathbb{D}$, where $u = P[f]$, and
(ii.3) w belongs to $\mathcal{D}_{\mathbb{D}\to\Omega}(g)$, then w is Lip on $\mathbb{D}$.

For example, it seems natural to ask whether we can weaken (or remove) the hypothesis that $|\partial u/\partial r|$ is bounded and whether we can weaken the hypothesis that g is continuous on the closure of the unit disk. We suggest the procedure to drop the hypothesis (i1).

We study to which extent conformal theory can be extended to harmonic qc mappings. It turns out that the following result is very useful.

Proposition 20 (Interior estimate)**. (Heinz-Bernstein, see Theorem 4' [24])**. *Let $s : \overline{\mathbb{U}} \to \mathbb{R}$ be a continuous function from the closed unit disc $\overline{\mathbb{U}}$ into the real line satisfying the conditions:*

1. *s is C^2 on $\mathbb{U}$,*
2. *$s_b(\theta) = s(e^{i\theta})$ is C^2 and*
3. *$|\Delta s| \le a_0|\nabla s|^2 + b_0$, on $\mathbb{U}$ for some constants a_0 and b_0 (the last inequality we will call the interior estimate inequality).*

Then the function $|\nabla s|$ is bounded on $\mathbb{U}$.

We call Theorem 4' [24], the interior estimate of Heinz-Bernstein.

Proposition 21. *If w belongs $\mathcal{D}_{\mathbb{D}\to\Omega}(g)$, then $|\nabla w|$ is bounded on $\mathbb{D}$.*

By hypotheses (i.1) and (ii.1), $|\Delta w|$ is bounded on $\mathbb{D}$, and w satisfies the Poisson type inequality on $\mathbb{D}$. It seems the idea[4] behind the proof is to use local coordinates ψ to make the part of boundary of the image to lay on $\mathbb{R}$ (a hyperplane if we work in space) whose 2-th coordinate is 0 and then to apply inner estimate on 2-th coordinate of function $\psi \circ u$, which is 0 on the the part of boundary of the unit disk $\mathbb{D}$. An application of Proposition 20 (the interior estimate of Heinz-Bernstein) (more precisely the local version of Interior estimate, Proposition 22 below) yields the proof.

[4]We discussed this shortly as a new idea at Workshop on Harmonic Mappings and Hyperbolic Metrics, Chennai, India, Dec. 10-19, 2009, see Course-materials [41].

11.5 Further results related to the interior estimate

We can refine the methods of the proof of Theorem 4' [24] to derive:

Proposition 22 (Local version of Interior estimate). *Let $s : \overline{\mathbb{U}} \to \mathbb{R}$ be a continuous function from the closed unit disc $\overline{\mathbb{U}}$ into the real line satisfying the conditions:*

1. *χ is C^2 on $\mathbb{U}$,*
2. *$|\Delta\chi| \leq a_0|\nabla\chi|^2 + b_0$, on $V = V(r) = \mathbb{U} \cap B(w_0, r)$, where $w_0 = e^{it_0}$, for some constants a_0 and b_0 (the last inequality we will call the interior estimate inequality) and*
3. *$\chi_b(\theta) = \chi(e^{i\theta})$ is C^2 on the interval $l = \{e^{it} : t \in (t_0 - e, t_0 + e)\} = \overline{V(r)} \cap \mathbb{T}$.*
4. *More generally $\chi \in L^1(0, 2\pi)$, $\chi' \in L^\infty(l)$ and the Hilbert transform of $H(\chi') \in L^\infty(l)$.*

Then there is $0 < r_1 < r$ such that the function $|\nabla\chi|$ is bounded on $V(r_1) = \mathbb{U} \cap B(w_0, r_1)$.

If χ is $C^{1,\alpha}$ on $\mathbb{U}$, then the the corresponding version of this result holds.

The proof of this result will appear elsewere. Using Proposition 22, one can prove:

Theorem 11.9. *Let f be a quasiconformal C^2 diffeomorphism from the plane domain Ω onto the plane domain G. Let $\gamma_\Omega \subset \partial\Omega$ and $\gamma_G = f(\gamma_\Omega) \subset \partial G$ be $C^{1,\alpha}$ respectively C^2 Jordan arcs. If for some $\tau \in \gamma_\Omega$ there exist positive constants r, a and b such that*

$$|\Delta f| \leq a|\nabla f|^2 + b\,, \quad z \in \Omega \cap D(\tau, r), \tag{147}$$

then f has bounded partial derivatives in $\Omega \cap D(\tau, r_\tau)$ for some $r_\tau < r$. In particular it is a Lipschitz mapping in $\Omega \cap D(\tau, r_\tau)$.

Under the stronger hypothesis that γ_G is $C^{2,\alpha}$ this is proved in [25](and it has been used there as the main tool).

11.6 the boundary regularity of Dirichlet Eigenfunctions

In communication with Yakov Sinai[5] (April 2016, Princeton) the following question appeared:

Question S-M. What can we say about the boundary regularity of Dirichlet Eigenfunctions on bounded domains which are C^2 except at a finite number of corners[6].

We have discussed the subject with Pier Lamberti who informed about numerous literature related to this subject and in particular about items 1)-3).

[5] Abel prize Laureate 2014

[6] we address this question as Y. Sinai's question or shortly S-M question

1) the eigenfunctions of the Dirichlet Laplacian are always bounded, not matter what the boundary regularity is.

2) the gradient of the eigenfunctions may not be bounded. The typical situation in the plane is as follows. If you have a corner with angle β, then the gradient is bounded around it if $\beta \leq \pi$ and unbounded if $\beta > \pi$.

3) An example: if Ω is a circular sector in the plane with central angle β, then for all $n \in \mathbb{N}$, $\nabla\varphi_n \in L^\infty(\Omega)$ if $0 < \beta \leq \pi$; if $\pi < \beta < 2\pi$ then for all $n \in \mathbb{N}$, $\nabla\varphi_n \in L^p(\Omega)$ for all $1 \leq p < \frac{2\beta}{\beta-\pi}$ and there exists an infinite number of eigenfunctions φ_n such that $\nabla\varphi_n \notin L^p(\Omega)$ if $p \geq \frac{2\beta}{\beta-\pi}$.
This example is discussed in Example 6.2.5 in E.B. Davies, Spectral theory and differential operators, Cambridge University Press, Cambridge, 1995.
For example, if Ω is C^2, using the so called interior estimate one can show that the Dirichlet eigenfunctions are Lipschitz.

In standard spectral theory for differential operators, the eigenvalue problem for the Dirichlet Laplacian is defined as follows:

Find $u \in W_0^{1,2}(\Omega)$ (eigenfunction) and $\lambda \in \mathbb{R}$ (eigenvalue) such that

$$\int_\Omega \nabla u \cdot \nabla\varphi dx = \lambda \int_\Omega u\varphi dx$$

for all functions $\varphi \in W_0^{1,2}(\Omega)$.

We will call eigenfunctions in the above sense eigenfunctions for the Dirichlet Laplacian (in SSTM; in standard spectral theory meaning) if there is possibility of misunderstanding.

By w we denote a unique solution to the Dirichlet problem $L(\partial_x)w = f$, $w \in W_0^{m,2}(\Omega)$, where $f \in W^{1-m,q}(\Omega)$ with $q \in (2,\infty)$. Here $W_0^{l,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the Sobolev space $W^{l,p}(\Omega)$, $1 < p < \infty$, and $W^{-l,p'}(\Omega)$ with $p' = p/(p-1)$ is the dual of $W_0^{l,p}(\Omega)$. The operator $L(\partial_x)$ is strongly elliptic and given by $L(\partial_x) = \sum_{0\leq k\leq 2m} a_k\partial_1^k\partial_2^{2m-k}$. We can apply Kozlov-Mazya result:

Theorem 11.10 (Theorem 2, [26])**.** *Let u be a solution of the Dirichlet problem for elliptic equations of order $2m$ with constant coefficients in an arbitrary bounded plane convex domain G. Then m-th order derivatives of u are bounded if the coefficients of the equation are real.*

Laplacian is elliptic equation of order 2. An application of Theorem 11.10 yield

Theorem 11.11. *Suppose that the Dirichlet eigenfunction solution $w \in W_0^{1,2}(\Omega)$ and Ω is bounded plane convex domain, then we have that gradient of w is bounded.*

11.7 Bi-Lipschicity of quasiconformal harmonic mappings in n-dimensional space with respect to quasi-hyperbolic metrics

By the distortion property of qc (see [21], p. 383, [37], p. 63), there are the constants C_* and c_* depend on n and K only, such that

$$B(f(x), c_* d_*) \subset f(B_x) \subset B(f(x), C_* d_*), \quad x \in G, \tag{148}$$

where $d_*(x) := d(f(x)) = d\big(f(x),\ \partial G'\big)$ and $d(x) := d(x,\ \partial G)$.

For definition od a non-zero Jacobian family see Definition 11.2 in Section 11.9 below.

Using our considerations in [34, 30, 32, 19], we can give a short proof of the following result:

Theorem 11.12. *Suppose that $f : D_1 \longrightarrow D_2$, where $D_1, D_2 \subset \mathbb{R}^n$ and the complement of D_1 has at least one point, is a harmonic K-quasiconformal mapping with $K_O(f) < 3^{n-1}$, (or and that f belongs to a non-zero Jacobian family of harmonic maps), then f is bi-Lipschitz with respect k-metrics.*

In particular,
(A) f is Lipschitz with respect to k-metrics.

Note that (A) holds more generally without the hypothesis that f belongs to a non-zero Jacobian family, cf. [34].

It seems that there is a simple proof of Theorem 11.12.

Rešenje. Set $r = r(z) = d(z, \partial D_1)$ and $R = R(fz) = d(f(z), \partial D_2)$. Then, by (148) (see also [30]), there is a constant c such that

$f(B(z, r(z)) \supset B(fz, cR(z))$, $z \in D_1$. There there is a constant c_0 such that $r(z)\lambda_f(z) \geq cR(z)$, $z \in D_1$, and hence
(B) f is co-Lipschitz with respect to to quasi-hyperbolic metrics.

The following result completes the proof.

Theorem 11.13 ([34]). *Suppose that $\Omega \subset \mathbb{R}^n$, $f : \Omega \to f(\Omega)$ is harmonic and K-qc.*

Then h is pseudo-isometry w.r. to quasi-hyperbolic metrics on Ω and $\Omega' = f(\Omega)$. In particular, it is Lipschitz with respect to k-metric.

△

By $J(z) = J_f(z) = J(f, z)$ we denote the Jacobian determinant of f at z. The above consideration also shows that
(C) $r(z)J^{\frac{1}{n}}(z) \approx R(z)$, $z \in D_1$.

After writing a version of this manuscript we received information about Shadia Shalandi work, see [36]. She also proved Theorem 11.12. Note that in her formulation the hypothesis that the complement of D_1 has at least one point is missing.

11.8 On harmonic K-quasiconformal map on $\mathbb{H}^n$

Given Riemannian manifolds (M, g), (N, h) and a map $\phi : M \to N$, the energy density of $e(\phi)$ at a point x in M is defined as

$$e(\phi) = \frac{1}{2}\|d\phi\|^2 .$$

The energy density can be written more explicitly as

$$e(\phi) = \frac{1}{2}\operatorname{trace}_g \phi^* h.$$

The energy of ϕ on a compact subset K of M is

$$E_K(\phi) = \int_K e(\phi)\, dv_g = \frac{1}{2}\int_M \|d\phi\|^2\, dv_g,$$

where dv_g denotes the measure on M induced by its metric.

Using the Einstein summation convention, if the metrics g and h are given in local coordinates by $g = \sum g_{ij}dx^i dx^j$ and $h = \sum h_{\alpha\beta}du^\alpha du^\beta$, the right hand side of this equality reads

$$e(\phi) = \frac{1}{2}g^{ij}h_{\alpha\beta}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}.$$

We define the tension field $\tau(u)$ of u by coordinates

$$\tau(u)^\nu = \Delta_g u^\nu + g^{ij}\Gamma'^{\nu}_{\alpha\beta}\circ u\, u_i^\alpha u_j^\beta\,, \tag{149}$$

where Δ_g is the Laplace-Beltrami operator on M and $\Gamma'^{\nu}_{\alpha\beta}$ are the Christoffel symbols on N. The Euler-Lagrange equation for this energy functional is the condition for the vanishing of the tension, which is, in local coordinates given by (149), $\tau(u)^\nu = 0$.

If $\mathbb{H}^m$ is identified as $\{(x^1, ..., x^m) : x^m > 0\}$ with the metric:

$$\frac{1}{(x^m)^2}((dx^1)^2 + \cdots (dx^m)^2)$$

then the tension field of u is given by

$$\tau(u)^\nu = (x^m)^2\big(\Delta_0 y^\nu - \frac{m-2}{(x^m)^2}y^\nu_m - \frac{2}{(y^m)^2}\langle\nabla_0 y^\nu, \nabla_0 y^m\rangle\big)$$

for $1 \le \nu \le m-1$, and

$$\tau(u)^m = (x^m)^2\big(\Delta_0 y^m - \frac{m-2}{(x^m)^2}y^m_m + \frac{1}{(y^m)^2}(\sum_{\nu=1}^{m-1}|\nabla_0 y^\nu|^2 - |\nabla_0 y^m|^2\big),$$

where ∇_0 is the Euclidean gradient and Δ_0 is the Euclidean Laplacian.

In [32], we proved:

Proposition 23. *Suppose that f has continuous partial derivatives up to the order 3 at the origin 0 and that $f : U(0) \to \mathbb{R}^n$ is K-qc, where $U(0)$ is a neighborhood of 0 in $\mathbb{R}^n$. If $K_O(f) < 3^{n-1}$, then $J(f,0) \neq 0$.*
In particular, if g is analytic (more generally $C^{(3)}(U(0))$ or g only has partial derivatives up to the order 3), and if g is K-qc with $K_O(g) < 3^{n-1}$, then $J(g,0) \neq 0$.

The result of the next proposition is based on Proposition 23.

Proposition 24. *Let F be K-qc hyperbolic harmonic mappings of $\mathbb{H}^n$ with respect to the hyperbolic metric. If $K < 3^{n-1}$, then f is a quasi-isometry.*

Tam and Wan, [38], 1998, proved the result if $K < 2^{n-1}$.

Rešenje. We follow their argument. Suppose that there is a sequence of points $x_n \in \mathbb{H}^n$, such that $e_n(F)(x_n) \to 0$ as $n \to \infty$. Let $o \in \mathbb{H}^n$ be a fixed point and A_n and B_n be isometry such that $A_n(o) = x_n$ and $B_n(F(x_n)) = o$. Then $u_n = B_n \circ F \circ A_n$ are harmonic maps such that $e(u_n)(o) \to 0$ as $n \to \infty$. A subsequence of u_n converges uniformly to a K-qc hyperbolic harmonic mappings u with $u(o) = o$ and $e(u)(o) = 0$. This contradicts the statement of a version of Proposition 23 for C^3 mapping, cf. also [31]. △

11.9 Appendix

11.10 Some definitions and results

Let $\Omega \in \mathbb{R}^n$ and $\mathbb{R}^+ = [0, \infty)$ and $f, g : \Omega \to \mathbb{R}^+$. If there is a positive constant c such that $f(x) \le c\, g(x)$, $x \in \Omega$, we write $f \preceq g$ on Ω. If there is a positive constant c such that

$$\frac{1}{c}\, g(x) \le f(x) \le c\, g(x)\,, \quad x \in \Omega\,,$$

we write $f \approx g$ (or $f \asymp g$) on Ω.

Let $G \subset \mathbb{R}^2$ be a domain and let $f : G \to \mathbb{R}^2$, $f = (f_1, f_2)$, be a harmonic mapping. This means that f is a map from G into $\mathbb{R}^2$ and both f_1 and f_2 are harmonic functions, i. e. solutions of the two-dimensional Laplace equation

$$\Delta u = 0\,. \tag{150}$$

The above definition of a harmonic mapping extends in a natural way to the case of vector-valued mappings $f : G \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$, defined on a domain $G \subset \mathbb{R}^n$, $n \ge 2$. Let h be a harmonic univalent orientation preserving mapping on a domain D, $D' = h(D)$ and $d_h(z) = d\,(h(z), \partial D')$. If $h = f + \overline{g}$ has the form, where f and g are analytic, we define $\lambda_h(z) = D^-(z) = |f'(z)| - |g'(z)|$, and $\Lambda_h(z) = D^+(z) = |f'(z)| + |g'(z)|$.

Theorem 11.14. *Let (f_j), $f_j : \Omega \mapsto \overline{\mathbb{R}^n}$, be a sequence of K-quasiconformal maps, which converges pointwise to a mapping $f : \Omega \mapsto \overline{\mathbb{R}^n}$. Then there are*

three possibilities:
A. f is a homeomorphism and the convergence is uniform on compact sets.
B. f assumes exactly two values, one of which at exactly one point; covergence is not uniform on compact sets in that case.
C. f is constant.

Definition 11.1. *We say that a family $\mathcal{F}$ of maps from domains in $\mathbb{R}^n$ to $\mathbb{R}^n$ is RHTC-closed if the following holds:*

- *(Restrictions) If $f : \Omega \mapsto \mathbb{R}^n$ is in $\mathcal{F}$, $\Omega' \subset \Omega$ is open, connected and nonempty, then $f|_{\Omega'} \in \mathcal{F}$.*
- *(Homothety) If $f : \Omega \mapsto \mathbb{R}^n$ is in $\mathcal{F}$, $a \in \mathbb{R}$, $a > 0$ then $g : \Omega \mapsto \mathbb{R}^n$ and $h : a\Omega \mapsto \mathbb{R}^n$ are in $\mathcal{F}$, where $g(x) = af(x)$ and $h(x) = f(x/a)$.*
- *(Translations) If $f : \Omega \mapsto \mathbb{R}^n$ is in $\mathcal{F}$, $t \in \mathbb{R}^n$, then $g : \Omega \mapsto \mathbb{R}^n$ and $h : t + \Omega \mapsto \mathbb{R}^n$ are in $\mathcal{F}$, where $g(x) = t + f(x)$ and $h(x) = f(x - t)$.*
- *(Completeness) If $f_j : \Omega \mapsto \mathbb{R}^n$, $j \in \mathbb{N}$ are in $\mathcal{F}$, (f_j) converges uniformly on compact sets to $g : \Omega \mapsto \mathbb{R}^n$, where g is non-constant, then $g \in \mathcal{F}$.*

For instance, families of harmonic maps and of gradients of harmonic functions are RTHC-closed. Also, due to Theorem 11.14, for any given $K \geq 1$, a subfamily of K-quasiconformal members of a RTHC-closed family is also RTHC-closed.

Definition 11.2. *We say that a family $\mathcal{F}$ of harmonic maps from domains in $\mathbb{R}^n$ to $\mathbb{R}^n$ is non-zero Jacobian closed, if it is RHTC-closed and Jacobians of all maps in the family have no zeros.*

Note that uniform convergence on compact sets in the case of harmonic maps implies convergence of higher order derivatives, via Hölder and Schauder apriori estimates (see [23], pp. 60, 90). This is related to elliptic regularity and holds for more general elliptic operators, and not just Laplacian, so that this method applies in that more general setting too.

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