UNIQUE EXTREMALITY OF QUASICONFORMAL MAPPINGS*

By

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Abstract. In this paper, we study the conditions under which unique extremality of quasiconformal mappings occurs and provide a broader point of view of this phenomenon. Additional information is obtained by means of specialized constructions. In particular, we generalize the construction theorem in [BLMM], thus providing a more basic understanding of it. We also generalize the notion of unique extremality and give an analytic characterization of the generalized concept.

1 Introduction

Let f be a quasiconformal map of a region G of the complex plane, and $\mu = f_{\overline{z}}/f_z$ its complex dilatation. The basic problem we consider is to characterize those dilatations μ that are uniquely extremal in their boundary class in the sense that the corresponding mappings are uniquely determined by the requirement that the essential sup of $|\mu|$ be minimal. Research on the problem started with Grötzsch and Teichmüller in the 1930s. We obtain further results in this direction.

During the last several years, important progress has been made in characterizing the conditions under which unique extremality occurs (see [BMM], [BLMM],[M1], [Re9]). In particular, the Characterization Theorem, which gives the characterization of unique extremality in functional-analytic fashion via special sequences of integrable holomorphic functions has found interesting applications.

There are many examples of extremal dilatations with nonconstant modulus, but all examples of uniquely extremal dilatations known up to the papers [BLMM] and [BMM] were of Teichmüller type. Moreover, many results obtained by studying the extremal problems spoke in favor of the conjecture that all uniquely extremal dilatations μ satisfy $|\mu(z)| = \|\mu\|_\infty$, for almost all z. In [BMM] and [BLMM], it was shown that there are uniquely extremal dilatations with nonconstant modulus. Indeed, the form of a uniquely extremal complex dilatation can be very complicated.

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Reich [Re3],[Re9] modified the construction in [BLMM],[BMM] by using Runge's Theorem instead of Mergelyan's Theorem. Because of its highly technical nature, one can miss an intuitive understanding of the construction in [BLMM]. This has motivated the author to continue the study of this subject and of related properties of extremal and uniquely extremal dilatations.

In this paper, we consider uniquely extremal dilatations from a new point of view. Roughly speaking, we study how uniquely extremal dilatations on a domain are determined by their values on special sub-domains. In particular, we present a new construction (Theorem 4.1 below). It is more visual, and a very special case leads to the construction of a uniquely extremal complex dilatation which is of Teichmüller type outside a set K of positive measure with empty interior and has arbitrary values on K. See Section 4 for the corresponding definitions, further details, and a discussion of the significance of our new constructions.

In [M1], we introduced the notion of a uniquely extremal complex dilatation on an extremal set; this can be considered as a generalization of a uniquely extremal complex dilatation if the extremal set is of positive measure. Using this notion, we generalize the results related to uniquely extremal dilatations; see the Equivalence Theorem II for Pairs and Characterization Theorem II in Section 3. In particular, a corollary of these results (Characterization Theorem II for Pairs) finds applications in Section 4. We also provide some simplifications with respect to the corresponding proof in [BLMM] (see Theorem 3.1). Taken together, these results lead to a better understanding of unique extremality.

In Section 2, we discuss some of the background of the subject.

In Section 5, we collect some definitions and results that are required in the sequel.

We have chosen to confine our discussion to subregions of the plane rather than general Riemann surfaces. This enables us to focus on the basics, but still allows for a rich variety of examples.

Part of this paper was published as a Warwick preprint [M5].

2 Definitions, Background; Extremal and uniquely extremal mappings

A. Extremal mappings. In this section, we give basic definitions and state the main result about extremal quasiconformal mappings.

The interested reader can learn more about extremal mappings from the excellent survey articles of Strebel [S5] and Reich [Re9] and the paper by Earle and Li [ELi] (see also [M3]).

The study of extremal mappings has been one of the main topics in the theory of quasiconformal mappings since its earliest days, when Grötzsch solved the extremal problem for two rectangles. In order to discuss such mappings, we recall some familiar definitions.

For a function h Lebesgue integrable on a set $M \subset \mathbb{C}$,

$$\|h\|_M = \iint_M |h| dxdy.$$

A homeomorphism f from a domain G onto another is called quasiconformal (qc) if f is absolutely continuous on lines (ACL) in G and $|f_{\bar{z}}| \leqslant k|f_z|$ a.e. in G, for some real number k, $0 \leqslant k < 1$. In this setting, it is well-known that the partial derivatives f_z, $f_{\bar{z}}$ are locally square integrable and that the directional derivatives satisfy

$$\max |D_\alpha f(z)| \leqslant K \min |D_\alpha f(z)| \tag{2.1}$$

for a.e. $z \in G$, where $K = (1+k)/(1-k)$. Roughly speaking, (2.1) means that at almost all points z of G, infinitesimal circles are mapped onto infinitesimal ellipses with axis ratio $D_f(z) \leqslant K$. It is also well-known that if f is a quasiconformal mapping defined on the region G, then the function f_z is nonzero a.e. in G. The function

$$\mu_f = \frac{f_{\bar{z}}}{f_z}$$

is therefore a well-defined bounded measurable function on G, called the complex dilatation (briefly, dilatation) or Beltrami coefficient of f. In the context of Riemann surfaces, it is usually called a differential instead of a complex dilatation. The L^∞ norm of each Beltrami coefficient is less than one. Conversely, every μ in $L^\infty(G, \mathbb{C})$ with norm less than one is the Beltrami coefficient of some qc mapping whose domain is G. A computation shows that

$$D_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}.$$

The positive number

$$K(f) = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}$$

is called the maximal dilatation of f. We say that f is K-qc if f is a qc mapping and $K(f) \leqslant K$.

Denote by $QC(G)$ (briefly, QC) the family of all quasiconformal mappings from G into $\mathbb{C}$. Let $QC_0(G)$ denote the group of all quasiconformal mappings from $\mathbb{C}$ onto itself that fix every point of $\mathbb{C} \setminus G$ and are homotopic to the identity by a homotopy g_t in which each g_t is a homemorphism of $\mathbb{C}$ onto itself that fixes every point of $\mathbb{C} \setminus G$. Two elements $f, g \in QC(G)$ are equivalent (in the sense of Teichmüller) if $f^{-1} \circ g \in QC_0(G)$. This means that the equivalence class of f is the set

$$Q_f = [f] = \{f \circ (\phi_0|G) : \phi_0 \in QC_0(G)\}.$$

Let $\mathcal{S}$ be a set of qc mappings whose domain is G. The mapping f_0 in $\mathcal{S}$ is said to be extremal in $\mathcal{S}$ if $K(f_0) \leqslant K(g)$ for all g in $\mathcal{S}$. In particular, a qc map f_0 is extremal in its Teichmüller class $[f]$ (abbreviated an EQC map) if $K(f_0) \leqslant K(g)$ for every mapping g in the same class. A qc map f_0 is uniquely extremal in its Teichmüller class $[f]$ if every other mapping g in the same class satisfies $K(f_0) < K(g)$.

We closely follow the approach of Earle-Li [ELi] concerning the definition of QC_0 and extremal quasiconformal mappings on plane regions (see also below). However, we find it more convenient in most of this paper to express the results in terms of extremal and uniquely extremal dilatations. Thus, we refer to a complex dilatation as extremal or uniquely extremal when a mapping with that complex dilatation is extremal or uniquely extremal.

If two elements $f, g \in QC(G)$ are equivalent (in the sense of Teichmüller), we also say that their dilatations $\mu = \mu_f$ and $\nu = \mu_g$ are equivalent (see also the corresponding definition in Section 5). We denote the equivalence class of μ by $[\mu]$.

We also write

$$k_0([f]) = k_0([\mu_f]) = \inf\{\|\mu_g\|_\infty : g \in Q_f\}$$

and

$$K_0([f]) = K_0([\mu_f]) = \inf\{K(g) : g \in Q_f\}.$$

In studying extremal qc mappings of a region, the L^1 norms of functions analytic (holomorphic) in that region play a special role.

Denote by $L^1_a = L^1_a(G)$ the Banach space consisting of all holomorphic functions φ belonging to $L^1 = L^1(G)$ with norm

$$\|\varphi\| = \|\varphi\|_G = \iint\limits_G |\varphi(z)| dxdy < \infty.$$

Instead of L^1_a the notation $\mathcal{A} = \mathcal{A}(G)$ is also used.

In the sequel, we shall assume that the complement $\mathbb{C} \setminus G$ of G contains at least three points. This assumption insures that the space $L^1_a(G)$ has positive dimension.

Let Ω be denote a domain in $\mathbb{C}$ and $L^\infty(\Omega)$ the space of all measurable and essentially bounded functions on Ω. If $\mu \in L^\infty(\Omega)$, we say that μ is a complex dilatation on Ω. Let $\|\mu\|_\infty = \|\mu\|_{\infty,\Omega}$ denote the L^∞-norm of μ on Ω.

Let $\mathcal{M} = \mathcal{M}(\Omega)$ be he open unit ball in L^∞. Thus, if $\mu \in L^\infty(\Omega)$ and $k = \|\mu\|_{\infty,\Omega} < 1$, we write $\mu \in \mathcal{M}$.

For $k \geqslant 0$, let $\overline{\mathcal{M}}_k = \overline{\mathcal{M}}_k(\Omega) = \{\mu \in L^\infty(\Omega) : \|\mu\|_{\infty,\Omega} \leqslant k\}$ be the closed ball of radius k in L^∞.

We say that a sequence φ_n converges c-uniformly on Ω if it converges uniformly on every compact subset of Ω.

It is convenient to write

$$\Lambda_\mu(\varphi) = (\mu, \varphi) = \iint\limits_G \mu\varphi dxdy, \quad \lambda_\mu(\varphi) = \mathrm{Re}\int_G \mu\varphi,$$

where $\mu \in L^\infty(G)$, $\varphi \in \mathcal{A}$; we then say that the linear functional $\Lambda_\mu \in \mathcal{A}^*$ is induced by μ. A function $\eta \in L^\infty$ is an annihilator of $\mathcal{A}$ in L^∞ if $(\eta, \varphi) = 0$ for every $\varphi \in \mathcal{A}$.

We denote by $\mathcal{N} = \mathcal{N}(G)$ the set of all annihilators of $\mathcal{A}$ in L^∞ and we say that $\mu \in L^\infty(G)$ and $\nu \in L^\infty(G)$ are infinitesimally equivalent (belong to the same equivalence class in the tangent space $\mathcal{B} = \mathcal{B}(G)$) if $\mu - \nu \in \mathcal{N}(G)$. By the Hahn-Banach Theorem and the Riesz Representation Theorem, $(L^1_a)^*$ is isometrically isomorphic to the Banach space $\mathcal{B}$ of equivalence classes of elements in L^∞.

We say that $\chi \in L^\infty(G)$ is extremal in its infinitesimal class (or, for short, extremal) and write $\chi \in ED_\alpha$ if the norm of the linear function $\Lambda_\chi \in \mathcal{A}^*$ induced by χ is the same as the sup norm $\|\chi\|_\infty$ of χ. This means that $\|\chi\|_\infty \leqslant \|\mu\|_\infty$ for every complex dilatation infinitesimally equivalent by χ.

For $\mu \in L^\infty$, we denote by $\|\mu\|_*$ the norm of the functional Λ_μ on $\mathcal{A} = L^1_a(G)$.

We say that a $\mu \in L^\infty$ satisfies the Hamilton-Krushkal condition if

$$\|\mu\|_* = \|\mu\|_\infty.$$

We are now ready to state the main result about extremal complex dilatations.

Theorem HKRS (Hamilton-Krushkal and Reich-Strebel)**.** *Let G be a plane region whose complement $\mathbb{C} \setminus G$ contains at least three points. Let f be a qc mapping whose domain is G and $\mu = \mu_f$ its Beltrami coefficient. A necessary and sufficient condition that f be an EQC (extremal) mapping in $[f]$ is that*

$$\|\mu\|_* = \|\mu\|_\infty.$$

The proof that the Hamilton-Krushkal condition is sufficient is based on the Reich–Strebel inequality (also called the Main Inequality). Various forms of this inequality play a major role in the theory of quasiconformal mappings and have many applications. In particular, the generalized Delta Inequality (Theorem 3 in [BLMM]), which is a very convenient tool in the theory of uniquely extremal qc mappings, is proved using the Main Inequality.

A Hamilton sequence for μ_f, is a sequence in $\mathcal{A}$, such that $\|\varphi_n\| = 1$ and

$$\lim_{n\to\infty} (\mu, \varphi_n) = \|\mu\|_\infty.$$

Now we can state the theorem of Hamilton-Krushkal and Reich-Strebel in the form: *f is extremal in its class $[f]$ if and only if μ_f has a Hamilton sequence.*

Theorem HKRS gives, via Hamilton sequences, what may be called an "analytic" method to test for extremality, in distinction to the earlier methods, which were more "geometric" in character. By the Hamilton-Krushkal and Reich-Strebel theorem, $\chi \in \mathcal{M}$ is extremal in its infinitesimal class if and only if it is extremal in its Teichmüller class.

Equivalence Theorem I which follows is the statement parallel to Theorem HKRS for unique extremality.

B. Unique extremality. We say that $\chi \in L^\infty(G)$ is uniquely extremal in its infinitesimal class ($\chi \in HBU_a$) if it is extremal and the linear functional $\Lambda_\chi \in \mathcal{A}^*$ induced by χ,

$$\Lambda_\chi(\varphi) = (\chi, \varphi) = \iint\limits_G \chi\varphi dxdy,$$

has a unique norm-preserving extension from $\mathcal{A}$ to a bounded linear functional on $L^1(G)$. This means that for any $\mu \in L^\infty(G)$ in the same infinitesimal class, $\|\chi\|_\infty < \|\mu\|_\infty$.

It is convenient to write

$$\delta_n = \delta_\mu(\varphi_n; G) = \|\mu\|_\infty \int_G |\varphi_n| dxdy - \mathrm{Re} \int_G \varphi_n \mu dxdy$$

and $\delta_\mu[\varphi_n]$, $\delta_G[\varphi_n]$, $\delta[\varphi_n]$ instead of $\delta_\mu(\varphi_n; G)$ if the meaning of this is clear from the context.

We say that a sequence $\varphi_n \in L^1_a$ is a weak Hamilton sequence for μ if $\delta_\mu[\varphi_n]$ converges to 0.

The next two theorems have been proved by Božin, Lakić, Marković and Mateljević in [BLMM], [BMM] and [MM1].

Theorem A (Equivalence Theorem I)**.** *Let $\chi \in \mathcal{M}$. Then χ is uniquely extremal in its Teichmüller class if and only if χ is uniquely extremal in its infinitesimal class.*

The proof of Equivalence Theorem I is based on estimates which allow us to compare two Beltrami coefficients μ and ν in the same global (Teichmüller) equivalence class and two complex dilatations in the same infinitesimal equivalence class. Equivalence Theorem I was an important step in understanding the notion of uniquely extremal complex dilatation.

The next important step is to analyze the proof of Hahn–Banach theorem and its applications to our setting. In particular, using Equivalence Theorem I, we have obtained the following necessary and sufficient criterion for the unique extremality of a given Beltrami coefficient χ.

Theorem B (Characterization Theorem I, [BLMM], [BMM])**.** *The Beltrami coefficient χ is uniquely extremal if and only if for every admissible variation $\hat{\chi}$ of χ, there exists a sequence φ_n in $L^1_a(G)$ such that*

(a) $\delta[\varphi_n] = \|\varphi_n\|\|\hat{\chi}\|_\infty - \operatorname{Re}\int_G \varphi_n \hat{\chi} \to 0$*;*

(b) $\liminf_{n\to\infty} |\varphi_n(z)| > 0$*, for almost all z in $E(\hat{\chi})$.*

Here, an admissible variation $\hat{\chi}$ of χ is any complex dilatation that does not increase the L^∞-norm of χ, and which is allowed to differ from χ only on the set $E_s = |\chi(z)| \leqslant s < k$, where $k = \|\chi\|_\infty$ and s is a constant, and the extremal set $E(\hat{\chi})$ is the set on which $\hat{\chi}(z) = \|\hat{\chi}\|_\infty$; in this setting, if $\hat{\chi}$ is different from χ only on a set $F \subset E_s$, we say that $\hat{\chi}$ is an admissible variation of χ on F.

Note that we do not require $\hat{\chi}$ and χ to be equivalent.

We say that $\mu \in L^\infty(G)$ satisfies the Reich condition on a set $S \subset G$, or that φ_n is a Reich sequence for μ on S (relative to G if this is not clear from the context), if

(1) there exists a sequence $\varphi_n \in L^1_a(G)$ such that $\delta_\mu(\varphi_n; G) \to 0$ (i.e., there is a weak Hamilton sequence φ_n for λ_μ), and

(2) $\liminf |\varphi_n(z)| > 0$ a.e. in S.

Thus, by Characterization Theorem I, the Beltrami coefficient χ is uniquely extremal if and only if for every admissible variation $\hat{\chi}$ of χ, there exists a Reich-sequence φ_n in $\mathcal{A}(G)$ on the extremal set $E(\hat{\chi})$.

In particular, if $|\chi|$ is constant on G, χ is uniquely extremal on G if and only if there exists a Reich sequence for χ on G.

The Characterization Theorem gives, via Reich-sequences, what may be called an "analytic" method to test for unique extremality. Roughly speaking, we may say that there is analogue between Theorem HKRS (via Hamilton sequences) and the Characterization Theorem (via Reich-sequences).

Let $\chi \in L^\infty(G)$. It is convenient in some settings shortly to say $\chi \in L^\infty(G)$ is uniquely extremal if the normalization $\chi_k = k\,\chi/||\chi||_\infty$ is uniquely extremal in its Teichmüller class for some $0 < k < 1$ (and hence for every $0 < k < 1$).

Definition 2.1 (complex dilatation of Teichmüller type)**.** Let G be a domain in $\mathbb{C}$. If s is a nonnegative measurable function from G into $[0, 1)$ and ψ is an analytic function, not identically zero, on G, we say that $\mu = s(z)|\psi|/\psi$ is of general Teichmüller type (s,ψ) on G. If, in addition, s is a constant k a.e. on G, we say that μ is of Teichmüller type (k,ψ) on G; and if ψ is an analytic integrable function on G, we say that μ is a Teichmüller complex dilatation.

Thanks to the Characterization Theorem, we can study uniquely extremal dilatations using an infinitesimal cotangent space $\mathcal{A} = L^1_a$, which is the space of holomorphic integrable functions. Using new tools, some properties of uniquely extremal dilatations of general Teichmüller type have been described. In particular, the compactness of certain families of holomorphic functions and the mean value theorem are used to prove the following results [M5].

Theorem C (Second removable singularity theorem)**.** *Let Ω be a bounded domain (multiply connected in general), Ω_∞ the unbounded component of Ω^c and $\Omega_0 = (\Omega_\infty)^c$. Let χ be a uniquely extremal complex dilatation of general Teichmüller type (s,φ) on Ω. Then*

(a) *$\chi = k|\varphi|/\varphi$ a.e. in Ω, where k is a constant.*

If, in addition, χ has a uniquely extremal extension to Ω_0, then

(b) *φ has an analytic extension $\tilde{\varphi}$ from Ω to Ω_0*

(c) *$\chi = k|\tilde{\varphi}|/\tilde{\varphi}$ a.e. in Ω_0.*

If D is a simply-connected domain and K a compact set such that $K \subset D$, we say that (K,D) is a pair.

Theorem D. *Let (K,D) be a pair and $V = D \setminus K$.*

(A) *Suppose that*

- (a) *$|\chi|$ is a constant a.e. on D and χ is HBU_a on D.*

Then

- (b) *there is a Reich sequence consisting of polynomials for χ on D (and, in particular, on V).*

(B) *Suppose that*

- (b_1) *there is a Reich-sequence consisting of polynomials for χ on V; and*
- (c) *χ is a complex dilatation of general Teichmüller type (s,φ) on V.*

Then there is a unique complex dilatation χ_0, which is the uniquely extremal extension of χ to D (and which, consequently, is of Teichmüller type).

Note that the hypothesis in part (B) of this theorem and differs from that Theorem C. Namely, the assumption in Theorem C that χ is uniquely extremal in $D = \Omega_0$, is replaced in Theorem D by the assumption that there is a Reich-sequence consisting of polynomials for χ on V .

3 The equivalence and characterization theorem II

3.1 The Equivalence Theorem II. We first give a very short proof of a version of the generalized infinitesimal delta inequality.

Theorem 3.1 (Generalized infinitesimal delta inequality-Version 2)**.** *Let μ and ν be equivalent in $\mathcal{B}$ and $\|\nu\|_\infty \leqslant k_0 = \|\mu\|_\infty$. Suppose there exist a positive number ε and a measurable set F such that*

$$|\mu(z) - \nu(z)| \geqslant 2\varepsilon, \qquad \textit{a.e. on } F.$$

Then

$$(k_0 - d)\|\varphi\|_F \leqslant \delta_\mu[\varphi], \quad \varphi \in L_a^1,$$

where $d = \sqrt{k_0^2 - \varepsilon^2}$.

Proof. By the parallelogram law, $|\mu+\nu|^2 + |\mu-\nu|^2 = 2(|\mu|^2 + |\nu|^2)$; therefore, by hypothesis, $|\mu+\nu|^2 + 4\varepsilon^2 \leqslant 4k_0^2$ a.e. on F. Hence

$$\left|\frac{\mu+\nu}{2}\right| \leqslant d, \quad \text{a.e. on } F, \tag{3.1}$$

where $d = \sqrt{k_0^2 - \varepsilon^2}$.

Since

$$\delta_\mu[\varphi] = k_0\|\varphi\| - \operatorname{Re}\int_G \frac{\mu+\nu}{2}\varphi, \quad \varphi \in \mathcal{A}, \tag{3.2}$$

using (3.1), we conclude that

$$(k_0 - d)\|\varphi\|_F \leqslant \delta_\mu[\varphi],$$

where

$$\|\varphi\|_F = \iint_F |\varphi| dxdy.$$

□

In order to obtain more insight into the notion of uniquely extremal dilations, we introduce the concept of unique extremality on a subset of a given domain.

Definition 3.1 (Unique extremality on a subset of a given domain)**.** Let W be a subset of a given domain G. Suppose that W has positive Lebesgue 2-dimensional measure. We say that $\chi \in \mathcal{M}$ (respectively, $\chi \in L^\infty(G)$) is uniquely extremal (respectively, satisfies the unique extension property) on W relative (with respect) to G in its Teichmüller class (respectively, in its infinitesimal class) if the hypothesis that μ is equivalent to χ in its Teichmüller class (respectively, in its infinitesimal class) in G together with the condition $\|\mu\|_\infty \leqslant \|\chi\|_\infty$ implies that $\mu = \chi$ a.e. on W.

We are mainly interested in the cases when $G \setminus K$ is the corresponding ring of a pair (K, G) or the extremal set of a dilation.

The proof of the following result uses Lemmas A and B stated in Section 5.

Theorem 3.2 (Equivalence Theorem II for pairs). *Let (K, D) be a pair, $V = D \setminus K$ and $\chi \in \mathcal{M}(D)$.*

Then the following conditions are equivalent:

(a) *χ is uniquely extremal on V with respect to D in its Teichmüller class;*

(b) *χ satisfies the unique extension property on V with respect to D in its infinitesimal Teichmüller class in the tangent space $\mathcal{B}$.*

Proof. Following the same steps as those in the proof of Lemma 3 [BLMM] and using Lemma A, we find that (a) implies (b). It remains to prove that (b) implies (a). Assume (b). Let χ and ν belong to the same class in $\mathcal{T}$. Then there exist qc mappings $f = f^\chi$ and $g = f^\nu$, equivalent in the sense of Teichmüller, with Beltrami coefficients χ and ν, respectively. Let $\alpha = \mathrm{Belt}[f^{-1}] \circ f$ and $\beta = \mathrm{Belt}[g^{-1}] \circ f$. Let $h = g^{-1} \circ f$, and $\eta = \mathrm{Belt}[h]$.

Following the same steps as those in the proof of Lemma 4 [BLMM] and using Lemma B, we find $\alpha = \beta$ a.e. on V, so that h is conformal on V. Since h has boundary values of the identity on ∂D, we see that h is the identity on V; therefore, $f = g$ on V. Thus we have proved that (a) holds. □

3.2 Characterization Theorem II and unique extremality. For a given $\chi \in L^\infty(G)$, it is convenient to mark the extremal vector χ_e defined by $\chi_e = \overline{\chi}$, on E and $\chi_e = 0$, on $G \setminus E$. Thus $\chi_e = K_E \overline{\chi}$, where K_E is the characteristic function of E.

The discussion in [BLMM] shows that χ_e has an important role in characterizations of an uniquely extremal complex dilatation χ.

Let $\mathcal{A} = L^1_a$, and let $\mathcal{A}_e = \mathcal{A}_{\chi_e}$ be the smallest subspace of L^1 which contains $\mathcal{A} \cup \{\chi_e\}$.

In [M1] and [M3], a proof of the following result is outlined.

Theorem 3.3 (Characterization Theorem II). *Let $\chi \in L^\infty$. The following conditions are equivalent:*

(a) *χ satisfies the unique extension property on its extremal set E;*

(b) *satisfies the Reich-condition on its extremal set E.*

Proof. (a) implies (b). Assume (a) and let $\lambda = \lambda_\chi$. Since χ is extremal, $k = \|\chi\|_\infty = \|\lambda\|_*$; and therefore $\|\lambda\|_{\mathcal{A}_e} = k$. We first prove

(c) λ_χ has a unique norm-preserving extension from $\mathcal{A}$ to $\mathcal{A}_e$.

Let $\nu \in L^\infty$ be such that $\lambda_\nu = \lambda$ on $\mathcal{A}$ and $\|\lambda_\nu\|_{A_e} = k$.

By the Hahn-Banach theorem, there exists $\mu \in L^\infty$ such that $\lambda_\mu = \lambda_\nu$ on $\mathcal{A}_e$ and $\|\mu\|_\infty = k$. Hence, by (a), $\mu = \chi$ on E; and we conclude that $\lambda(\chi_e) = \lambda_\mu(\chi_e)$ and $\lambda = \lambda_\mu$ on $\mathcal{A}_e$. Thus $\lambda = \lambda_\mu = \lambda_\nu$ on $\mathcal{A}_e$ and, in particular, $\lambda_\nu = \lambda$ on $\mathcal{A}_e$. This means that (c) holds. Hence, as in [BLMM], we conclude that (b) holds.

An application of the generalized Delta Inequality shows that (b) implies (a). □

Using the considerations in this paper as well as those in [BLMM] and [BMM], one can verify the following results.

Proposition 3.1. (d) *If $\varphi_n \in L^1_a$ is a weak Hamilton sequence for μ and $A = E_s^- = \{z : |\mu(z)| \leqslant s < \|\mu\|_\infty\}$, then $\|\varphi_n\|_A$ converges to* 0.

If χ is uniquely extremal, then

(e) *the set $E_s^- = \{z : |\chi(z)| \leqslant s < k = \|\chi\|_\infty\}$ has empty interior;*

(f) *every annihilator with support on E_s^- is trivial; and*

(g) *every admissible variation $\hat{\chi}$ of χ is uniquely extremal.*

Here is an outline of the proof of (g). Suppose that $\hat{\chi}$ is an admissible variation of χ on $F \subset E_s^-$ and there is an annihilator η such that $\hat{\chi}+\eta \in \overline{\mathcal{M}}_k$, where $k = \|\chi\|_\infty$. Since $\overline{\mathcal{M}}_k$ is convex, $\chi_1 = \chi + \varepsilon\eta \in \overline{\mathcal{M}}_k$ for $\varepsilon > 0$ small enough. Hence, since χ is uniquely extremal, η is trivial.

Theorem 3.4 (Characterization Theorem II for pairs)**.** *Let (K, D) be a pair and $V = D \setminus K$. Suppose that V is an extremal set of $\chi \in \mathcal{M}(D)$. Then*

a) *Then χ is uniquely extremal on V with respect to D if and only if there is a Reich-sequence for χ on V.*

b) *Suppose, in addition, that $|\chi| \leqslant s$ a.e. on K, where $s < \|\chi\|_\infty$. Then χ is uniquely extremal on D if and only if there is a Reich-sequence for χ on V and K has empty interior.*

Proof. Part a) is an immediate corollary of Characterization Theorem II and Equivalence Theorem II for pairs.

It remains to consider part b). Suppose there is a Reich-sequence for χ on V and K has empty interior. Let ν be equivalent to χ in its infinitesimal class and $\eta = \chi - \nu$. By part a), $\chi = \nu$ a.e. on V. Thus η has support on K. Hence, by Lemma R (see Section 5), $\eta = 0$ a.e. on K; and therefore χ is uniquely extremal on D.

If χ is uniquely extremal on D, then by Proposition 3.1 (see also Proposition A in Section 4), K has empty interior. On the other hand, since V is an extremal set of $\chi \in \mathcal{M}(D)$, there is a Reich-sequence for χ on V. □

In a forthcoming paper, we plan to prove the results of this subsection in detail and to elaborate their connections with the subject discussed in this paper and [M5].

4 Constructions

Thanks to the characterization of unique extremality by Reich-polynomials and theorem of Runge and Mergelyan, we can make interesting constructions of uniquely extremal dilatations.

Before we state our results, we introduce a class of sets which is important in our investigation. Let K be a compact subset of $\mathbb{C}$ whose complement is connected in $\mathbb{C}$ (Mergelyan set, M-set) and whose interior is empty. We say that K is a **special Mergelyan set** (**special M-set**). The motivation for this definition is the following celebrated result of Mergelyan.

Theorem E (Mergelyan's Theorem)**.** *Let F be a compact set in the plane whose complement is connected and f a continuous complex function on F which is holomorphic in the interior of F. Then for any $\epsilon > 0$, there exists a polynomial P such that $|f(z) - P(z)| < \epsilon$ for all $z \in F$.*

Let K be a compact subset of a Jordan domain D, containing at least two points, such that $D \setminus K$ is doubly connected. We call (K, D) **a doubly-connected pair**. This notion is convenient for application of the Runge's Theorem. In particular, this means that K is a connected M-set. In connection with Mergelyan's Theorem, note that the hypothesis of that result requires only that K does not separate the plane: the theorem is applicable even if K is not connected.

Let f be a qc mapping on G and let $\mu = \mu_f$ and $k = \|\mu\|_\infty$. Suppose that there is a ball $B \subset G$ such that $\operatorname{ess\,sup}\{|\mu(z)| : z \in B\} < k$. Let $f(B) = B_*$ and let φ and φ_* be conformal mappings from B and B_* onto Π^+, respectively. Define $f_* = \varphi_*^{-1} \circ A_K \circ \varphi$, where $K > 1$. Then $f_* = f$ on ∂B and if K is close to 1 then $\operatorname{ess\,sup}\{|\mu_{f_*}(z)| : z \in B\} < k$. This means f is not uniquely extremal. Thus, we have proved the following result, which was observed by Reich [Re7].

Proposition A. *If $\chi \in HBU_a(G)$ and $k = \|\chi\|_\infty$, then $\operatorname{ess\,sup} |\chi(z)| = k$ over each open set $G_0 \subset G$*

Therefore, the following question arises naturally (see [Re7] and [S5]).

Question. Does $\chi \in HBU_a$ actually imply that $|\chi(z)| = k$ a.e.?

The next theorem shows that the answer to the corresponding question, concerning the more general concept of uniquely extremal complex dilatation, is negative.

Proposition 4.1. *Let (K, Δ) be a doubly connected pair and let $V = \Delta \setminus K$. Then there exists $\chi \in L^\infty(\Delta)$ such that χ is zero on K; V is its extremal set, that is, $V = \{z \in \Delta : |\chi(z)| = \|\chi\|_\infty = k > 0\}$; and χ is uniquely extremal on its extremal set V relative to Δ.*

Note that only those cases in which K is of positive measure are of interest.

Although this result can be considered as a corollary of a very special case of Theorem 4.1 (see below) and the Delta Inequality, we state it separately because it has a significant corollary (see Proposition 4.3 and Remark 4, following the statement of Theorem 4.1). Namely, if K is a set with empty interior (i.e., in this setting, a special M-set with connected complement) and with positive Lebesgue two-dimensional measure, then we can use the result to construct an example of an uniquely extremal complex dilatation with nonconstant modulus.

Moreover, it explains the difference between two concepts of unique extremality: more precisely, in the setting described by the proposition, χ is uniquely extremal on Δ if and only if K is a set with empty interior.

Using the Mergelan's theorem instead of Runge's theorem, one can prove that the proposition supposing only that $\Delta \setminus K$ is connected (thus, without assumption that K is connected).

Proposition 4.2. *Let (K, Δ) be a doubly connected pair and let $V = \Delta \setminus K$.*

a) *Then there exists a sequence of polynomials φ_n and $\chi \in L^\infty(\Delta)$ such that χ is zero on K, $\|\chi\|_\infty = k > 0$, V is its extremal set (i.e., $V = E(\chi) = \{z \in \Delta : |\chi(z)| = \|\chi\|_\infty = k > 0\}$), and φ_n is a Reich-sequence for χ on V relative to Δ.*
b) *Moreover, χ described in a) is uniquely extremal on its extremal set V with respect to Δ.*

Proof. Part a) is a very special case of Theorem 4.1. Part b) is an immediate corollary of the part a) of Theorem 3.4. □

Proposition 4.1 follows from Proposition 4.2.

Remark 1. One can also verify that χ defined by the proposition is uniquely extremal on its extremal set V with respect to Δ in its Teichmüller class, using the method of the proof of Theorem V. 3.1 [Re9], which is a special case of the Characterization Theorem. It follows because the proof of Theorem V. 3.1 [Re9] is more elementary then the proof of the generalized Delta Inequality in Teichmüller class.

The situation described in the proposition (in particular, in which K is a set with empty interior and with positive two-dimensional measure) is one that we often encounter. Hence it is convenient to introduce new terminology.

Definition 4.1 (Special sequence of polynomials)**.** Suppose that (K, D) is a pair and K has positive Lebesgue two-dimensional measure. If a sequence of polynomials φ_n uniformly converges to 0 on K and satisfies the Reich-condition on $V = D \setminus K$, we call φ_n **a special sequence of polynomials for the pair** (K, D).

We are mainly concerned in this paper with the case in which (K, D) is a doubly-connected pair. Special sequences of polynomials play an important role in the constructions of uniquely extremal dilatations. Namely, the following special case of Proposition 4.2 is important: if (K, D) is a special doubly-connected pair, then there is a special sequence of polynomials for it.

It is interesting that the following surprisingly simple lemma (concerning special sets) plays a role in the construction of uniquely extremal dilatations with nonconstant modulus.

Lemma R (Annihilators on special sets)**.** *Let $K \subset G$ be a special Mergelyan set and ν an annihilator of $\mathcal{A}(G)$ in L^∞ such that* $\operatorname{supp}\nu \subset K$. *Then $\nu = 0$.*

In the case when G is the unit disk, this lemma was proved by Reich in [Re7] (see also [MM1] and Lemma 6 [BLMM]).

For the reader who likes to have some pictures of special connected Mergelyan sets, the following example may be useful.

Example 1. Let C_0 be a Cantor set on the segment $l = [0, 1]$ with positive one-dimensional Lebesgue measure and $K_0 = \{(x, y) : x \in C_0, |y| \leqslant 1\}$. Then $K = K_0 \cup l$ is a special connected Mergelyan set.

The following result is an immediate corollary of Proposition 4.1 and Lemma R.

Proposition 4.3 ([BLMM])**.** *Let K be a compact set of positive measure with no interior (K, Δ) double connected pair and let $V = \Delta \setminus K$. Then there exists $\chi \in HBU_a$ such that $\chi(z) = 0$ in K and $|\chi(z)| = k > 0$ a.e. in $V = K^c = \Delta \setminus K$.*

Proof. By Proposition 4.1, there exists $\chi \in L^\infty(\Delta)$ such that χ is zero on K; V is its extremal set, i.e. $V = \{z \in \Delta : |\chi(z)| = \|\chi\|_\infty = k > 0\}$; and χ is uniquely extremal on its extremal set V relative to Δ. Hence, if μ is equivalent to χ in its Teichmüller infinitesimal class in Δ and $\|\mu\|_\infty \leqslant \|\chi\|_\infty$, it follows that $\mu = \chi$ a.e.

on V; and therefore $\nu = \mu - \chi$ is an annihilator of $\mathcal{A}$ in L^∞ and has support on K. So, by Lemma R, $\nu = 0$ a.e. on Δ. Thus, χ is uniquely extremal. □

In [BLMM] we showed, using Mergelyan's Theorem, that the result holds if $K \subset \Delta$ is a special Mergelyan set of positive Lebesgue 2-dimensional measure.

We refer to the proof of this result as the construction of a uniquely extremal complex dilatation with nonconstant modulus (briefly, "the construction").

Simplifications of the construction have been given by the author during his lectures at Scoala Normala Superioara Buchurest (SNSB), 2003–2004 (to appear in [M4]).

Theorem F (Runge's Theorem)**.** *Suppose K is a compact set in the plane, $\overline{\mathbb{C}} \setminus K$ is connected, and f is a holomorphic function on Ω (i.e., $f \in \mathcal{H}(\Omega)$), where Ω is some open set containing K. Then there is a sequence $\{P_n\}$ of polynomials such that $P_n \to f$ uniformly on K.*

Outline of new construction. Let (K, D) be a doubly-connected pair. We show that there exists a sequence of Jordan-domains J_n such that

$$J_n \subset Int J_{n+1}, \bigcup_1^\infty \overline{J_k} = D \setminus K. \tag{4.1}$$

In this setting, we say that sequence of Jordan-domains J_n exhausts $D \setminus K$. Namely, since $D \setminus K$ is doubly connected and K contains at least two point, there is a conformal mapping Φ of $\Delta \setminus \overline{B}$ onto $D \setminus K$, where Δ denotes the unit disk and $\overline{B}$ is the closed disc of radius centered at 0. Let $r_n = r + 1/n$, $J_n' = \{\rho e^{i\theta} : r_n < \rho < 1, 0 < \theta < 2\pi - 1/n\}$ and $J_n = \Phi(J_n')$. It is clear that the sequence of Jordan-domains J_n satisfies condition (4.1).

Definition 4.2 (Simply-connected triple)**.** Let $I_r = (r, 1)$ be an interval, $\Lambda = \Phi(I_r)$, $V = D \setminus K$, and $V' = V \setminus \Lambda$. We call (K, D, V') a simply-connected triple (a doubly-connected pair (K, D) with cut Λ). If, in addition, K has empty interior we say that (K, D, V') is a special simply-connected triple and that (K, D) is a special doubly-connected pair.

Suppose that (K, D) is a pair and χ is of general Teichmüller type (s, φ) on $V = D \setminus K$. In this setting, the proof of Theorem D (see also Theorem C) tells us that if $\chi \neq 0$ a.e. on V and has a Reich-sequence consisting of polynomials on V, then the corresponding normalized sequence of polynomials P_n converges c-uniformly on D and gives the analytic continuation $\tilde{\varphi}$ of φ to D. Since $\tilde{\varphi}$ is not identically zero on D, we can define $\tilde{\chi} = k\frac{|\tilde{\varphi}|}{\tilde{\varphi}}$. It is not difficult to verify that P_n is a Reich-sequence for $\tilde{\chi}$ on D and therefore that $\tilde{\chi} \in HBU_a(D)$ (see Theorem D). In particular, if χ is uniquely extremal on D and $\chi = 0$ on K, then $\chi = 0$ on D a.e. or K is a finite set.

If, in addition, the normalized sequence of polynomials P_n is special for χ and the pair (K, D), then we first conclude that $\tilde{\varphi}$ is zero on K. Therefore, χ cannot be of Teichmüller type on V unless K is a finite set or $s = 0$ a.e. on V.

The two next lemmas are immediate corollaries of the Runge and Mergelyan theorem respectively.

Lemma Ru. *Let K be a M-set and G be a Jordan domain such that $\overline{G} \cap K = \emptyset$. Suppose that ψ_0 is a holomorphic function on K, and φ_0 is a holomorphic function on $\overline{G}$. Then, for every $\varepsilon > 0$, there exists a polynomial P such that*
(1) $|P - \psi_0| < \varepsilon$ *on* K
(2) $|P - \varphi_0| < \varepsilon$ *on* G.

Lemma Ru-M. *Let K be a special M-compact set and G be a Jordan domain such that $\overline{G} \cap K = \emptyset$. Suppose that f is a continuous function on K and φ_0 a holomorphic function on $\overline{G}$. For every $\varepsilon > 0$, there exists polynomial P such that*
(1) $|P - f| < \varepsilon$ *on* K
(2) $|P - \varphi_0| < \varepsilon$ *on* G.

Concerning Lemma Ru, we note that we can consider it the germ of a new construction. If ψ is a holomorphic function on K, we can modify the original construction from [BLMM] (see also the proof of Theorem 4.1 and Reich's survey [Re9]) to prove that there is a sequence of polynomials φ_n which converge uniformly to ψ on K and satisfy the Reich-condition on $V = D \setminus K$ for some complex dilatation μ.

Theorem 4.1 (Construction Theorem)**.** *Let (K, D, V') be a simply-connected triple and ψ_0 a holomorphic function on K.*

a) *There is a sequence of polynomials φ_n which converge uniformly to ψ_0 on K and converge c-uniformly to a holomorphic nonzero function φ_o on V', and which is a Reich-sequence on V.*
b) *If, in addition, ψ_0 is different from 0 a.e. in K, and if χ is defined on D by $\chi = \frac{|\psi_0|}{\psi_0}$ on K and $\chi = \frac{|\varphi_0|}{\varphi_o}$ on V', then the sequence φ_n is a Reich-sequence for χ on D, and therefore χ is HBU_a on D.*

The situation described in the theorem is often encountered. Hence it is convenient to introduce new terminology.

Definition 4.3 ((φ_o, ψ_0)-special sequence of polynomials)**.** We call the sequence φ_n described in Theorem 4.1 a (φ_0, ψ_0)**-special sequence of polynomials** for the simply-connected triple (K, D, V').

Before proving the theorem, we state some remarks and corollaries which provide some indication of its significance.

Remarks on and corollaries to Theorem 4.1

1. The theorem holds if we only suppose that ψ_0 is continuous on K and holomorphic in the interior of K.

2. The assumption that φ_n is a Reich-sequence of polynomials on V' which converges c-uniformly to a not identically zero function φ_o holomorphic on V', implies that φ_o is different from 0 on V'.

3. (*Holomorphic function as germ of uniquely extremal complex dilatation.*) If the holomorphic function ψ_0 is different from 0 a.e. in K and χ is defined on K by $\chi = \frac{|\psi_0|}{\psi_0}$, then it follows from part b) of the theorem that χ has an extension which is uniquely extremal on D.
Namely, if we extend χ to D by $\chi = \frac{|\varphi_0|}{\varphi_o}$ on V', where the function φ_0 is defined by the sequence φ_n on V', then φ_n is a Reich-sequence for χ on D. Therefore, by a special case of Characterization Theorem (Proposion B, see Section4), χ is HBU_a on D, i.e., χ is uniquely extremal.
In this way, we can roughly state that every ψ_0 is a germ of a uniquely extremal complex dilatation on D.
Continuous function as germ of uniquely extremal complex dilatation.
In particular, if K is a special set, f is a continuous function on K which is different from 0 a.e. in K, and if $\chi_0 = \frac{|f|}{f}$ on K, then using the method of the proof of Theorem 4.1 and Mergelyan's Theorem instead of Runge's, one can show that χ_0 has a uniquely extremal extension to D (see Corollary 2).

4. (*Construction of uniquely extremal complex dilatation with nonconstant modulus.*) If $\psi_o \equiv 0$, and χ is defined on D by $\chi = |\varphi_0|/\varphi_o$ on V' and $\chi = 0$ on K, using the Delta Inequality in the tangent space $\mathcal{B}$ (as in [BLMM]), one can show that χ is uniquely extremal on V relative to D (see Proposition 4.2).
If, in addition, the set K has empty interior and positive two-dimensional measure, then using Lemma R, which states roughly that annihilators with support on special sets vanish, one can show that χ is uniquely extremal (HBU_a) on D (see Proposition 4.3). This gives an example of uniquely extremal complex dilatation with nonconstant modulus.
Note that one can verify that χ defined in this item is uniquely extremal by means of Theorem V. 3.1 [Re9], which is a special case of the Characterization theorem.

That makes sense because the proof of this theorem is much simpler than the proof of the Characterization Theorem.

5. If ψ_0 has no analytic continuation to D, then by Theorem C, the complex dilatation χ is not of Teichmüller type on V (i.e., the function φ_o has no analytic continuation to V). Thus we can construct a uniquely extremal complex dilatation μ on D which is of Teichmüller type on both V' and K, but not on $K \cup V'$.

The next statement is an immediate corollary of Theorem 4.1.

Corollary 1 (HBU_a continuation of holomorphic function)**.** *Let (K, D) be a doubly-connected pair, ψ_0 a holomorphic function on K, which is nonzero 0 a.e. in K. If $\chi_0 = \frac{|\psi_0|}{\psi_0}$ on K, then there exists $\chi \in HBU_a$ on D such that $\chi = \chi_0$ on K.*

Corollary 2 (HBU_a continuation of continuous function)**.** *Let (K, D) be a special doubly-connected pair, and let f be a continuous function on K which is nonzero a.e. in K. If $\chi_0 = \frac{|f|}{f}$ on K, then there exists $\chi \in HBU_a$ on D such that $\chi = \chi_0$ on K.*

Using Mergelyan's Theorem as in [BLMM], one can show that the Correlary 1 holds if (K, D) is a pair and K a special M-set.

The proof of Corollary 2 is similar to the proof of Theorem 4.1 below. The only difference is in the application of Mergelyan's Theorem (instead of Runge's Theorem), which gives

(2″) $\varphi_n - f = 0(1)$ on K.

The proof of the part $b)$ of Theorem 4.1 is based on the following important corollary of the Characterization Theorem.

Proposition B. *If $|\chi|$ is a constant on G, then χ is uniquely extremal if and only if χ satisfies the Reich-condition on G.*

Proof of Theorem 4.1. Inductively, we can find a sequence of polynomials $\{\varphi_n\}$ and a sub-sequence V_n of the sequence J_n such that

(1) $\int\int_{\Lambda_n} |\varphi_n| = 0(1)$, where $\Lambda_n = V \setminus V_{n+1}$.

(2) $\varphi_n - \psi_0 = 0(1)$ on K

(3) $|\varphi_{n+1} - \varphi_n| < \frac{1}{2^n}$ on V_{n+1}

Note that we have used Lemma Ru in the inductive procedure. After application of Lemma Ru on the pair K and V_n, we construct function φ_n. It seems that we have difficulties because we do not control φ_n on the canal $\Lambda_{n-1} = V \setminus V_n$. Since $\bigcup J_k = V'$, there is m such that $\int_{V \setminus J_m} |\varphi_n| = 1/2^n$. Set $V_{n+1} = J_m$.

Note that we do not yet have control of φ_n on the set $\Lambda'_n = V_{n+1} \setminus V_n$.

It is interesting that another application of Lemma Ru enables to overcome this difficulty. Namely, for given φ_n, there is a polynomial φ_{n+1} such that $|\varphi_{n+1}-\varphi_n| < 1/2^n$ on V_{n+1} and $\varphi_{n+1} - \psi_0 = 0(1)$ on K. Thus, we modify φ_n by φ_{n+1} on V_{n+1}.

By the triangle inequality and (3), it follows that

$$|\varphi_m - \varphi_n| < \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots = \frac{1}{2^{n-1}}$$

on V_{n+1} ($m \geqslant n \geqslant 1$) and therefore φ_n forms a Cauchy sequence, which converges c-uniformly to a holomorphic function φ_0 on V'. In particular, $|\varphi_n - \varphi_0| = 1/2^{n-1}$ on V_{n+1} and hence

(3′) $\varphi_n - \varphi_0 = 0(1)$ on V_{n+1}, and $|\varphi_n| - |\varphi_0| = 0(1)$ on V_{n+1}.

One can choose φ_1 such that $\varphi_1 \equiv 2$ on $\mathbb{C}$.

Since $|\varphi_1(z) - \varphi_0(z)| \leqslant 1$ for all $z \in V_2$, φ_0 does not vanish in V_2; therefore, φ_0 can only have isolated zeros in V'. Suppose that ψ_0 is different from 0 a.e. in K. Define χ on D as follows $\chi = \frac{|\psi_0|}{\psi_0}$ on K and $\chi = \frac{|\varphi_0|}{\varphi_0}$ on V'.

By (2), it follows first that $|\varphi_n| = |\psi_0| + 0(1)$ on K and then $\frac{|\psi_0|}{\psi_0}\varphi_n = \frac{|\psi_0|}{\psi_0}(\psi_0 + 0(1)) = |\psi_0| + 0(1)$ on K. Therefore,

(2′) $\delta_n(K) = \delta_\chi(\varphi_n; K)$ tends to 0.

In a similar way, if we substitute $\varphi_n = \varphi_0 + 0(1)$ and $|\varphi_n| = |\varphi_0| + 0(1)$ in $\delta_n(V_{n+1}) = \delta_\chi[\varphi_n] = \delta_\chi(\varphi_n; V_{n+1})$, since

$$\frac{|\varphi_0|}{\varphi_0}\varphi_n = \frac{|\varphi_0|}{\varphi_0}(\varphi_0 + 0(1)) = |\varphi_0| + 0(1) \quad \text{ on } \ V_{n+1},$$

we get

(3′) δ_n tends 0.

Since $\delta_n(V) = \delta_n(V_{n+1}) + \delta_n(\Lambda_n)$, using (1) and (3″), we see that

(4) $\delta_n(V)$ tends 0.

Since $\delta_n(D) = \delta_n(K) + \delta_n(V)$, it follows from (2′) and (4) it follows that $\delta_n(D)$ tends to 0. Therefore φ_n is a Reich-sequence for χ on D and hence, by Proposition B, $\chi \in HBU_a$ on D. Part b) of Theorem 4.1 is proved.

In particular, φ_n is a Reich-sequence for χ on V; and part a) of Theorem 4.1 is also proved if ψ_0 is different from 0 a.e. in K.

Combining (2) and that $\delta_n(V)$ tends 0, we obtain that the part a) also holds if $\psi_0 \equiv 0$ on K. □

Question. If χ is smooth (only continuous) and uniquely extremal is χ of Teichmüller type?

Remark 2. If G is a ring domain and K belongs to a bounded component of G^c, we cannot apply Runge's Theorem on the pair K and G in the general situation, so canals $\Lambda_n = D \setminus J_{n+1}$ have an important role in the construction.

Also, one can modify our construction [BLMM] so that the corresponding sequence $\mu_n(z)$ is a Cauchy sequence on all of D i.e., $\mu(z) = \lim \mu_n(z)$ exists for every $z \in D$. In addition, we can choose polynomials P_k such that $P_k = 1 + 0(1)/k^2$ uniformly on V_k, where V_k is the corresponding exhaustion of V. We can verify that $\prod P_k$ converges c-uniformly to a holomorphic function φ on V'. Hence $\mu = \frac{|\varphi|}{\varphi}$ on V'.

Using Runge's Theorem, we can prove the following result.

Lemma Ma. *Let K be a M-set and G a Jordan domain such that $\overline{G} \cap K = \emptyset$ and let $B = B_R$ contains $F = K \cup \overline{G}$. For given positive numbers p, q and ε, there exists a polynomial P which has no zeros in B_R such that*
(a$_1$) $|P| < p$ *on K,*
(a$_2$) $q < |P|$ *on $\overline{G}$,*
(a$_3$) $|1 - |P|/P| < \varepsilon$ *on $F = K \cup \overline{G}$.*

Remark. By Runge's Theorem, there is a polynomial Q such that $|Q| < \ln p$ on K and $\ln q < |Q|$ on $\overline{G}$ and which is close to $\ln p$ and $\ln q$ on K and G, respectively. For suitable Q close to constant functions on K and $\overline{G}$ the entire function $\phi = e^Q$ satisfies the above conditions. Hence, there is a nonvanishing entire function which satisfies the above conditions. Let $\delta = \min\{|\phi(z)| : z \in B_R\}$ and choose $0 < \varepsilon < \delta/2$. Let $P = P_n$ be a partial sum of the Taylor series of ϕ such that $|P(z) - \phi(z)| < \varepsilon$ for every $z \in B_R$. Moreover, we can choose ε small enough such that P satisfies condition (a$_3$) of the theorem.

Using only (a$_1$) and (a$_2$) of Lemma Ma, we can prove the following proposition.

Proposition 4.4. *Let K be a M-set. There is a sequence of polynomials P_n which tends uniformly to 0 on K such that $P_n(z) \to \infty$ for every $z \in \mathbb{C} \setminus K$.*

5 Appendix

5.1 Teichmüller class It is useful to give the definition of a Teichmüller class using the solution of the Beltrami equation.

Let $L^\infty = L^\infty(G)$ be the space of essentially bounded complex-valued measurable functions on G and $\mathcal{M} = \mathcal{M}(G)$ the open unit ball in L^∞. For any μ in $\mathcal{M}$ there exists a quasiconformal solution $f = f^\mu : G \mapsto \mathbb{C}$ of the Beltrami equation

$$\overline{\partial} f = \mu \partial f,$$

unique up to a postcomposition by a conformal transformation.

Two elements μ_0 and μ_1 in $\mathcal{M}$ are equivalent if we can choose the corresponding solutions f^{μ_0} and f^{μ_1} to be equivalent; in the other words, two elements μ_0 and μ_1 in $\mathcal{M}$ are equivalent if there are equivalent qc mappings f_0 and f_1 in G, whose Beltrami coefficients are μ_0 and μ_1, respectively. It is known that this definition is equivalent to the corresponding definition given in Section 2.

It is very difficult to recognize this equivalence by direct comparison of μ_0 and μ_1. Because we cannot solve the global problem, we study the local one for infinitesimal deformations. For this, the roles of Theorem HKRS and the Characterization theorem are crucial.

For given $\mu \in \mathcal{M}$, the equivalence class $[\mu]$ contains at least one element μ_0 such that

$$\|\mu_0\|_\infty = \inf\{\|\nu\|_\infty : \nu \in [\mu]\}.$$

Such a μ_0 is referred to as an extremal complex dilatation, and $f_0 = f^{\mu_0}$ as an extremal quasiconformal mapping (abbreviated EQC mapping). We say μ_0 is uniquely extremal in its Teichmüller class if every other complex dilatation ν in the same class satisfies $\|\mu_0\|_\infty < \|\nu\|_\infty$.

The Teichmüller space $\mathcal{T} = \mathcal{T}(G)$ may be represented as the space of equivalence classes of Beltrami coefficients μ in the unit ball $\mathcal{M}(G)$.

5.2 The Uniform Convergence Theorem and the Generalized Delta Inequality in $\mathcal{T}$ We say that a Reich-sequence ψ_n for $\mu = k|\psi|/\psi$ is normalized at a point $z_0 \in G$ if $\psi_n(z_0) \to \psi(z_0) \neq 0$.

Theorem G (Uniform Convergence Theorem, [BLMM]). *Let χ be a complex dilatation of Teichmüller type (k, φ) defined by an analytic function φ on G. Suppose that χ is uniquely extremal on G. Then every normalized Reich-sequence φ_n for χ on G converges uniformly on compact subsets of G to φ.*

Suppose μ and ν belong to the same class in $\mathcal{T}$. Then there exist qc mapings $f = f^\mu$ and $g = f^\nu$ equivalent in the sense of Teichmüller with Beltrami coefficients μ and ν, respectively.

Theorem H (Generalized Delta Inequality, [BLMM]). *Let $h = g^{-1} \circ f$, $\chi =$ $\mathrm{Belt}[h]$ and $I = I(\varphi) = \int_G |\chi|^2 |\varphi|$. In addition, suppose that $\|\nu\|_\infty \leqslant k = \|\mu\|_\infty$. Then*

$$I(\varphi) \leqslant C\delta_\mu(\varphi), \quad \varphi \in \mathcal{A},$$

where C is a constant which depends only on $k = \|\mu\|_\infty$.

We outline a proof. Let $\alpha = \mathrm{Belt}[f^{-1}] \circ f$ and $\beta = \mathrm{Belt}[g^{-1}] \circ f$.

Note that

$$\chi = \frac{\mu}{\alpha}\frac{\alpha-\beta}{1-\overline{\alpha}\beta}$$

and

$$|\chi|^2 = \Big|\frac{\alpha-\beta}{1-\overline{\alpha}\beta}\Big|^2.$$

Define $\chi_1 = \chi|\alpha|/\mu$. One can verify that

$$\operatorname{Re}\chi_1 - |\chi|^2 = A|\chi|^2 - B_1(|\beta|-|\alpha|), \tag{5.1}$$

where $A = 2^{-1}|\alpha|^{-1}(1-|\alpha|)^2$ and $B_1 = 2^{-1}|\alpha|^{-1}(|\alpha|+|\beta|)(1-|\alpha|^2)|1-\overline{\alpha}\beta|^{-2}$.

Define

$$J[\varphi] = \iint_G \operatorname{Re}\frac{(\chi_1|\varphi| - \chi\varphi)}{1-|\chi|^2}dx\,dy$$

and

$$J^+[\varphi] = \iint_G \frac{|\chi|}{|\mu|(1-|\chi|^2)}\big||\alpha||\varphi| - \mu\varphi\big|dx\,dy.$$

An easy consequence of the Main Inequality yields

$$\iint_G \frac{\operatorname{Re}\chi_1 - |\chi|^2}{1-|\chi|^2} \leqslant J[\varphi] \leqslant J^+[\varphi]. \tag{5.2}$$

Let $\tau = \tau[\varphi] = \int_G (k-|\mu|)|\varphi|$.

Suppose first that $|\mu|$ is bounded from below by a positive constant s for almost every z in G.

Using the Cauchy-Schwarz inequality and the identity

$$\big||w| - w\big|^2 = 2|w|(|w| - \operatorname{Re} w),$$

we obtain

$$J^+ \leqslant cI^{1/2}\delta_\mu^{1/2}. \tag{5.3}$$

The proof now follows from (5.1), (5.2), (5.3), and an argument which appears in the proof of Theorem 3 in [BLMM], pp. 315–317. □

Proposition 5.1. *Assume the hypothesis of Theorem H and the above notation. If φ_n is a weak Hamilton sequence (in particular, a Reich-sequence) for μ, then $J[\varphi_n]$ and $I[\varphi_n]$ converge to 0.*

Proof. If φ_n is a weak Hamilton sequence (in particular, a Reich-sequence) for μ, then, by the inequality (5.3), $J[\varphi_n]$ converges 0. Since $\tau[\varphi_n]$ converges 0, it follows that $I[\varphi_n]$ converges 0. □

5.3 Lemma R In this subsection, we outline a proof of Lemma R, which roughly states that annihilators with support on special sets vanish.

Lemma R (Annihilators on special sets)**.** *Let $K \subset G$ be a special Mergelyan set and ν an annihilator of $\mathcal{A}(G)$ in L^∞ such that* $\operatorname{supp}\nu \subset K$. *Then $\nu = 0$.*

Outline of proof of Lemma R. Let $\nu \in L^\infty(G)$ be such that $\operatorname{supp}\nu \subset K$. Put $\nu = 0$ on $\mathbb{C} \setminus G$, so that $\nu \in L^\infty(\mathbb{C})$, and let P denote the Cauchy integral operator defined on [Ah, p. 85]. By hypothesis, $K_\infty = \mathbb{C} \setminus K$ is connected, so K_∞ is the unbounded component of K^c. We first show that if ν is an annihilator with respect to polynomials, then $P\nu = 0$ on K_∞. There is a disc $B = B_R$ such that $K \subset B_R$. Let $\Gamma = \Gamma_R$ be positively-oriented boundary of B_R. Then by the generalized Green formula and [Ah, Lemma 3, p. 90], we have

$$\int_\Gamma P\nu(z)z^n dz = \int_B \overline{\partial}\big(P\nu(z)\, z^n\big) d\overline{z} \wedge dz = 2\,i \int_B \nu(z)\, z^n\, dxdy = 0, \quad n \geqslant 0. \tag{5.4}$$

Therefore, since $P\nu$ is an analytic function on K_∞, $P\nu = 0$ in a neighborhood of ∞ and therefore on the unbounded component of K, i.e., on K_∞. Since $P\nu$ is continuous on $\mathbb{C}$ and every point of K is a limit point of K_∞, it follows that $P\nu = 0$ on $\mathbb{C}$. Another application of [Ah, Lemma 3, p. 90], shows that ν is zero a.e. on $\mathbb{C}$.

5.4 Lemma A, B and Equivalence Theorem III We outlined proofs of the following lemmas in [M1].

Lemma A. *Let μ be uniquely extremal on its extremal set E in its Teichmüller class, and let $F \subset E$ be a compact set of positive measure on which $|\mu| = \|\mu\|_\infty < 1$. Then for each $r > 0$, there is a unit vector $\varphi \in \mathcal{A}$ such that*

$$\delta_\mu[\varphi] \leqslant C(k)\, r \int_F |\varphi|. \tag{5.5}$$

Lemma B. *Let μ satisfy the unique extension property on its extremal set E, and let $K \subset E$ be a compact set of positive measure on which $|\mu| = \|\mu\|_\infty = k$. Then for each $r > 0$, there is a unit vector $\varphi \in \mathcal{A}$ such that*

$$\delta_\mu[\varphi] \leqslant kr \int_K |\varphi|.$$

We announce the following result, whose proof can be based on applying a procedure similar to that in [BLMM].

Theorem 5.1 (Equivalence Theorem III)**.** *Let $\chi \in \mathcal{M}$ and let $W \subset G$ be a set of positive measure.*

(A) *Suppose that*

(a) χ *is uniquely extremal on* W *with respect to* G *in its Teichmüller class. Then*

(b) χ *satisfies the unique extension property on* W *with respect to* G *in its infinitesimal Teichmüller class in the tangent space* $\mathcal{B}$.

(B) *Moreover if (b) holds, then*

(c) $\tilde{\chi}$ *is uniquely extremal on* $f(W)$ *with respect to* $f(G)$ *in its Teichmüller class, where* f *is a quasiconformal mapping with domain* G *and Beltrami coefficient* χ *and* $\tilde{\chi}$ *is the Beltrami coefficient of* f^{-1}.

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