INNER ESTIMATE AND QUASICONFORMAL HARMONIC MAPS BETWEEN SMOOTH DOMAINS

By

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Abstract. We prove a type of "inner estimate" for quasi-conformal diffeomorphisms, which satisfies a certain estimate concerning their Laplacian. This, in turn, implies that quasiconformal harmonic mappings between smooth domains (with respect to an approximately analytic metric), have bounded partial derivatives; in particular, these mappings are Lipschitz. We discuss harmonic mappings with respect to (a) spherical and Euclidean metrics (which are approximately analytic) (b) the metric induced by a holomorphic quadratic differential.

1 Introduction and statement of the main result

1.1 Basic facts and notation. Let $\mathbb{U}$ and $\mathbb{H}$ denote the unit disc and the upper half plane, respectively. By Ω and D we denote simply connected domains. Suppose that γ is a rectifiable curve in the complex plane or on the Riemann sphere S^2. Denote by l the length of γ, and let $\Gamma : [0, l] \to \gamma$ be the natural parameterization of γ, i.e., the parameterization satisfying the condition

$$|\dot{\Gamma}(s)| = 1 \quad \text{for almost all } s \in [0, l].$$

We say that γ is of class $C^{n,\mu}$, for $n \in \mathbb{N}$, $0 < \mu \leq 1$ if Γ is of class C^n and

$$\sup_{t,s} \frac{|\Gamma^{(n)}(t) - \Gamma^{(n)}(s)|}{|t-s|^{\mu}} < \infty$$

We call Jordan domains in $\mathbb{C}$ bounded by $C^{n,\mu}$ Jordan curves $C^{n,\mu}$ domains or smooth domains.

Let $\rho(w)|dw|^2$ be an arbitrary conformal C^1-metric defined on D. If $f : \Omega \to D$ is a C^2 mapping between the Jordan domains Ω and D, the energy integral of f is defined by the formula

$$E[f, \rho] = \int_{\Omega} \rho \circ f (|f_z|^2 + |f_{\bar{z}}|^2) dx\, dy. \tag{1.1}$$

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The stationary points of the energy integral satisfy the Euler–Lagrange equation

$$f_{z\overline{z}} + (\log \rho)_w \circ f f_z\, f_{\bar{z}} = 0, \tag{1.2}$$

and a C^2 solution of this equation is called a **harmonic mapping** (more precisely, a ρ-**harmonic mapping**).

It is known that f is a harmonic mapping if and only if the mapping

$$\Phi = \rho \circ f f_z \overline{f}_{\bar{z}} \tag{1.3}$$

is analytic. If φ is a holomorphic mapping different from 0 and if $\rho = |\varphi|$ on D, we call ρ a φ-metric. We will call the corresponding harmonic mapping φ-**harmonic**. Notice that for $\rho = 1$, a ρ-harmonic mapping is a **Euclidean harmonic function.**

Let $0 \le k < 1$ and $K = \frac{1+k}{1-k}$. An orientation preserving diffeomorphism $f : \Omega \to D$ between two domains $\Omega, D \subset \mathbb{C}$ is called a K or a k-**quasiconformal mapping** (briefly, a q.c. mapping) if it satisfies the condition

$$|f_{\overline{z}}(z)| \le k|f_z(z)| \quad \text{for each } z \in \Omega. \tag{1.4}$$

In this paper, we mainly consider harmonic quasiconformal mappings between smooth domains.

1.2 Background. The first characterization of harmonic quasiconformal mappings with respect to the Euclidean metric for the unit disc was given by O. Martio [17]. Below are several important results in this area.

Theorem P ([22])**.** *If w is a harmonic diffeomorphism of the unit disc onto itself, then the following conditions are equivalent: w is q.c.; w is bi-Lipschitz; the boundary function is bi-Lipschitz and the Hilbert transformation of its derivative is in L^∞.*

Theorem KP ([13] and [9])**.** *An orientation-preserving homeomorphism ψ of the real axis can be extended to a q.c. harmonic homeomorphism of the upper half-plane if and only if ψ is bi-Lipschitz and the Hilbert transformation of the derivative ψ' is bounded.*

Theorem K ([12])**.** *If Ω and Ω' are Jordan domains with $C^{1,\mu}$ boundary ($0 < \mu \le 1$), then every quasiconformal harmonic function from Ω onto Ω' is Lipschitz. If in addition Ω' is convex, then w is bi-Lipschitz. Moreover if $w : \Omega \to \Omega'$ is a harmonic diffeomorphism, where Ω is the unit disc and Ω' is a convex domain with $C^{1,\mu}$ boundary, then the following conditions are equivalent: w is quasiconformal; w is bi-Lipschitz; the boundary function is bi-Lipschitz and* *the Hilbert transformation of its derivative is in L^∞; and therefore a harmonic diffeomorphism in this setting is a quasi-isometry with respect to the corresponding Poincaré distance.*

Concerning quasi-isometry the second author in joint work with M. Knežević obtained the right constant. They proved

Theorem MK1 ([15]). *If f is a $K-$q.c. harmonic diffeomorphism from the upper half plane H onto itself and $f(\infty)=\infty$, then*

$$|z_1-z_2|/K \le |f(z_1)-f(z_2)| \le K|z_2-z_1|, \quad \textit{where } z_1,z_2\in H$$

and f is a $(1/K,K)$ quasi-isometry with respect to the Poincaré distance.

Theorem MK2 ([15]). *If f is a $K-$q.c. harmonic diffeomorphism from the unit disc $\mathbb{U}$ onto itself, then f is a $(1/K,K)$ quasi-isometry with respect to the Poincaré distance:*

$$d_h(z_1,z_2)/K \le d_h(f(z_1), \quad f(z_2)) \le K d_h(z_1,z_2),$$

where d_h is hyperbolic distance in the unit disc.

Concerning hyperbolic q.c. harmonic mappings, we present here two results.

Theorem W1 ([28]). *Every harmonic quasi-conformal mapping from the unit disc onto itself is a quasi-isometry of the Poincaré disc.*

Theorem W2 ([28]). *A harmonic diffeomorphism of the hyperbolic plane $\mathbb{H}^2$ is quasiconformal if and only if its Hopf differential (i.e., the function Φ defined in* (1.3)*) is uniformly bounded with respect to the Poincaré metric.*

For the other results in this area, see [19], [15], [16], [27], [24], [25], [21] and [2].

1.3 New results.

The following proposition plays an important role in [9], [13] and [15].

Proposition 1.1. *Let f be an Euclidean harmonic $1-1$ mapping of the upper half-plane $\mathbb{H}$ onto itself, continuous on $\overline{\mathbb{H}}$, normalized by $f(\infty)=\infty$, and let $v=\operatorname{Im} f$. Then $v(z)=cy$, where c is a positive constant. In particular, v has bounded partial derivatives on $\mathbb{H}$.*

Suppose that f is a harmonic Euclidean mapping of the unit disc onto a smooth domain D and ψ is a conformal mapping of D onto $\mathbb{H}$. The composition $\psi\circ f$

is rarely Euclidean harmonic, so we cannot apply Proposition 1.1. However, the composition is harmonic with respect to the metric $\rho(w) = |(\psi^{-1})'(w)|^2$ (see Corollary 4.3). Having this in mind, our idea is to apply the following result instead of Proposition 1.1 in more complicated cases.

Proposition 1.2 (Inner Estimate)**.** (Heinz–Bernstein; see [8])**.** *Let $s : \overline{\mathbb{U}} \to \mathbb{R}$ be a continuous function from the closed unit disc $\overline{\mathbb{U}}$ into the real line satisfying the conditions*
(1) *s is C^2 on $\mathbb{U}$,*
(2) *$s_b(\theta) = s(e^{i\theta})$ is C^2, and*
(3) *$|\Delta s| \le c_0 |\bigtriangledown s|^2$ on $\mathbb{U}$ for some constant c_0.*
Then the function $|\bigtriangledown s| = |\mathrm{grad}\, s|$ is bounded on $\mathbb{U}$.

We refer to this result as the *inner estimate*. Applying this estimate and results of Kellogg–Warschawski (see below), we prove the main result of this paper.

Theorem 1.3. *Let f be a quasiconformal C^2 diffeomorphism from the $C^{1,\alpha}$ Jordan domain Ω onto the $C^{2,\alpha}$ Jordan domain D. If there exists a constant M such that*

$$|\Delta f| \le M |f_z \cdot f_{\bar{z}}|, \quad z \in \Omega, \tag{1.5}$$

then f has bounded partial derivatives. In particular, it is a Lipschitz mapping.

In particular, Theorem 1.3 holds if h is quasiconformal ρ-harmonic and the metric ρ is *approximately analytic,* i.e., $|\bar{\partial}\rho| \le M|\rho|$ on Ω (see Theorems 3.1, 3.3, 3.5 below). Since Euclidean and spherical metrics are approximately analytic, our results can be viewed as extensions of the results in [17], [22], [13], [9] and [12] (mentioned in Subsection 1.2).

The main result is proved in Section 2, and its applications are given in Section 3. In Section 4, we show that the composition of a conformal mapping ψ and a φ-harmonic mapping satisfies certain properties (see Theorem 4.1). In particular, if ϕ is a natural parameter, we obtain a representation of φ-harmonic mappings by means of Euclidean harmonics. We also provide some examples of φ-harmonic mappings and prove that Theorem 3.1 holds for more general domains.

2 The proof of the main result

To prove the main result we need the following two results.

Proposition 2.1 (Kellogg [7])**.** *If the domain $D = Int(\Gamma)$ is $C^{1,\alpha}$ and ω is a conformal mapping of $\mathbb{U}$ onto D, then ω' and $\log \omega'$ are in Lip_α. In particular, $|\omega'|$ is bounded from above and below on $\mathbb{U}$.*

Proposition 2.2 (Kellogg and Warschawski [23, Theorem 3.6])**.** . *If the domain $D = Int(\Gamma)$ is $C^{2,\alpha}$ and ω is a conformal mapping of $\mathbb{U}$ onto D, then $|\omega''|$ has a continuous extension to the boundary. In particular, it is bounded from above on $\mathbb{U}$.*

Proof of Theorem 1.3. Let g be a conformal mapping of the unit disc onto Ω. Let $\tilde{f} = f \circ g$. Since $\Delta\tilde{f} = |g'|^2\Delta f$ and $\tilde{f}_z \cdot \tilde{f}_{\bar{z}} = |g'|^2 f_z \cdot f_{\bar{z}}$, $\tilde{f}$ satisfies the inequality (1.5). We prove the theorem for $\tilde{f}$ and then apply Kellogg's theorem. For simplicity, we write f instead of $\tilde{f}$. Let $t \in \partial\mathbb{U} = \mathbb{T}$ be an arbitrary fixed point.

Step 1 (**Local Construction**). In this step, we show that there are two Jordan domains D_1 and D_2 in D with $C^{2,\alpha}$ boundary such that

(i) $D_1 \subset D_2 \subset D$,

(ii) $\partial D \cap \partial D_2$ is a connected arc containing the point $w = f(t)$ in its interior,

(iii) $\emptyset \neq \overline{\partial D_2 \setminus \partial D_1} \subset D$.

Let H_1 be the Jordan domain bounded by the Jordan curve γ_1 which is composed by the following sequence of Jordan arcs:

$$\{y^{1/5} + (2-x)^{1/5} = 1,\ 1 \le x \le 2\}; \quad \{(2-y)^{1/5} + (2-x)^{1/5} = 1,\ 1 \le x \le 2\};$$
$$[(1,2),(-1,2)]; \quad \{(2-y)^{1/5} + (2+x)^{1/5} = 1,\ -2 \le x \le -1\};$$
$$\{y^{1/5} + (2+x)^{1/5} = 1,\ -2 \le x \le -1\} \quad \text{and} \quad [(-1,0),(1,0)].$$

Let H_2 be the Jordan domain bounded by the Jordan curve γ_2 which is composed by the following sequence of Jordan arcs:

$$\{y^{1/5} + (2-x)^{1/5} = 1,\ 1 \le x \le 2\}; \quad [(2,1),(2,2)];$$
$$\{(3-y)^{1/5} + (2-x)^{1/5} = 1,\ 1 \le x \le 2\}; \quad [(1,3),(-1,3)];$$
$$\{(3-y)^{1/5} + (2+x)^{1/5} = 1,\ -2 \le x \le -1\}; \quad [(-2,2),(-2,1)];$$
$$\{y^{1/5} + (2+x)^{1/5} = 1,\ -2 \le x \le -1\} \quad \text{and} \quad [(-1,0),(1,0)].$$

Note that $H_1 \subset H_2 \subset [-2,2] \times [0,3]$, $\partial H_1 \cap \mathbb{R} = \partial H_2 \cap \mathbb{R} = [-1,1]$ and that $\partial H_1, \partial H_2 \in C^3$.

Let Γ be an orientation preserving arc-length parameterization of $\gamma = \partial D$ such that for $s_0 \in (0, \mathrm{length}(\gamma))$, $\Gamma(s_0) = f(t)$. Let $D^* = \overline{\Gamma'(s_0)}D$, $b = \overline{\Gamma'(s_0)}f(t)$ and $\Gamma^* = \overline{\Gamma'(s_0)}\Gamma$. Then there exists $r > 0$ such that $(b, b+ir] \subset D^*$. Since $\gamma^* = \partial D^* \in C^{2,\alpha}$, it follows that, there exist $x_0 > 0$, $\varepsilon > 0$, $y_0 \in (0, r/3)$, a $C^{2,\alpha}$ function $h : [-2x_0, 2x_0] \to \mathbb{R}$, $h(0) = 0$, and the domain $D_2^* \subset D^*$ such that

(1) $\Gamma^*([s_0 - \varepsilon, s_0 + \varepsilon]) = \{b + (x, h(x)) : x \in [-2x_0, 2x_0]\}$,

(2) $D_2^* = \{b + (x, h(x) + y) : x \in [-2x_0, 2x_0],\ y \in (0, 3y_0]\}$.

Let $\Upsilon : [-2,2] \times [0,3] \to D_2^*$ be the mapping defined by

$$\Upsilon(x,y) = b + (xx_0, h(xx_0) + yy_0).$$

Then Υ is a $C^{2,\alpha}$ diffeomorphism.

Take $D_i = \Gamma'(s_0) \cdot \Upsilon(H_i)$, $i = 1,2$. Obviously, $D_1 \subset D_2 \subset D$ and D_1 and D_2 have $C^{2,\alpha}$ boundary. Observe that $f(t) = \Gamma'(s_0)\overline{\Gamma'(s_0)}f(t) = \Gamma'(s_0)\Upsilon(0) \in \Gamma'(s_0)\Upsilon([-1,1]) = \partial D_1 \cap \partial D_2$.

Step 2 (**Application of the Inner Estimate**). Let ϕ be a conformal mapping of D_2 onto H such that $\phi^{-1}(\infty) \in \partial D_2 \setminus \partial D_1$. Let $\Omega_1 = \phi(D_1)$. Then there exist real numbers a,b,c,d such that $a < c < d < b$, $[a,b] = \partial\Omega_1 \cap \mathbb{R}$ and

$$l = \phi^{-1}(\partial\Omega_1 \setminus [c,d]) \subset D.$$

Let $U_1 = f^{-1}(D_1)$ and η be a conformal mapping between the unit disc and the domain U_1. Then the mapping $\hat{f} = \phi \circ f \circ \eta$ is a C^2 diffeomorphism of the unit disc onto the domain Ω_1 such that

(a) $\hat{f}$ is continuous on the boundary $\mathbb{T} = \partial\mathbb{U}$ (it is q.c.) and

(b) $\hat{f}$ is C^2 on the set $T_1 = \hat{f}^{-1}(\partial\Omega_1 \setminus (c,d))$.

Let $s := \operatorname{Im} \hat{f}$. First, note that (a) implies that s is continuous on $\mathbb{T} = \partial\mathbb{U}$. On the other hand, as $\hat{f} \in C^2$,

(1) $s \in C^2(\mathbb{U})$.

From (b), we obtain that s is C^2 on the set $T_1 = \hat{f}^{-1}(\partial\Omega \setminus (c,d))$. Furthermore, $s = 0$ on $T_2 = \hat{f}^{-1}(a,b)$; and therefore s is C^2 on $T_2 = \hat{f}^{-1}(a,b)$. Hence

(2) s is C^2 on $\mathbb{T} = T_1 \cup T_2$. In other words, the function $s_b : \mathbb{R} \to \mathbb{R}$ defined by $s_b(\theta) = s(e^{i\theta})$ is C^2 in $\mathbb{R}$.

In order to apply the inner estimate, we have to prove that

(3) $|\Delta s(z)| \le c_0 |\bigtriangledown s(z)|^2$, $z \in \mathbb{U}$, where c_0 is a constant.

Now

$$s_z = \frac{\hat{f}_z - \overline{\hat{f}_{\bar z}}}{2i} \quad \text{and} \quad s_{\bar z} = \frac{\hat{f}_{\bar z} - \overline{\hat{f}_z}}{2i}, \tag{2.1}$$

so

$$\overline{s_z} = s_{\bar z};$$

therefore

$$|s_z|^2 = |s_{\bar z}|^2 = \frac{|\bigtriangledown s|^2}{2}. \tag{2.2}$$

Since $\hat{f} = \phi \circ f \circ \eta$, we obtain

$$\frac{\hat{f}_{z\bar{z}}}{\hat{f}_z \cdot \hat{f}_{\bar{z}}} = \left(\frac{\phi''}{\phi'^2} + \frac{1}{\phi'} \frac{\partial \bar{\partial} f}{\partial f \cdot \bar{\partial} f} \right).$$

As $\hat{f}$ is a k-q.c. mapping, we have

$$|\Delta s| = |\mathrm{Im}\, \Delta \hat{f}| \leq |\hat{f}_z| \cdot |\hat{f}_{\bar{z}}| \cdot \left| \frac{\phi''}{\phi'^2} + \frac{1}{\phi'} \frac{\partial \bar{\partial} f}{\partial f \cdot \bar{\partial} f} \right| \leq k |\hat{f}_z|^2 \left| \frac{\phi''}{\phi'^2} + \frac{1}{\phi'} \frac{\partial \bar{\partial} f}{\partial f \cdot \bar{\partial} f} \right|.$$

Using (2.1) and (2.2) respectively, we obtain

$$(1-k)|\hat{f}_z| \leq 2|s_z|,$$

and

$$|\Delta s| \leq \frac{2k}{(1-k)^2} \left| \frac{\phi''}{\phi'^2} + \frac{1}{\phi'} \frac{\partial \bar{\partial} f}{\partial f \cdot \bar{\partial} f} \right| \cdot | \bigtriangledown s|^2. \tag{2.3}$$

Proposition 2.1 and Proposition 2.2 imply that the function $|\phi'|$ is bounded from below by a positive constant C_1 and the function $|\phi''|$ is bounded from above by a constant C_2. With the help of (1.5), we obtain

$$\left| \frac{\phi''}{\phi'^2} + \frac{1}{\phi'} \frac{\partial \bar{\partial} f}{\partial f \cdot \bar{\partial} f} \right| \leq \frac{4C_2 + MC_1}{4C_1^2}. \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$|\Delta s| \leq c_0 | \bigtriangledown s|^2,$$

where

$$c_0 = \frac{2k}{(1-k)^2} \cdot \frac{4C_2 + MC_1}{4C_1^2}.$$

Proposition 1.2 implies that the function $| \bigtriangledown s|$ is bounded by a constant b_t. Since $\hat{f}$ is a $k-$q.c. mapping, we have

$$(1-k)|\hat{f}_z| \leq |\hat{f}_z - \overline{\hat{f}_{\bar{z}}}| \leq 2|s_z| \leq \sqrt{2} b_t.$$

Finally,

$$|\hat{f}_z| + |\hat{f}_{\bar{z}}| \leq \sqrt{2} \frac{1+k}{1-k} b_t.$$

Let $T^t_{c,d} = (f \circ \eta)^{-1}(c,d)$. Observe that c and d depend on the fixed point t. Since $t \in T^t_{c,d}$, we obtain $\mathbb{T} = \bigcup_{t \in \mathbb{T}} T^t_{c,d}$; and therefore there exists a finite set $\{t_1, \ldots, t_n\}$ such that $\mathbb{T} = \bigcup_{i=1}^n T^{t_i}_{c,d}$.

Since the mapping $\eta_i = \eta$ is conformal and maps the circular arc $T_i = (\phi \circ f \circ \eta)^{-1}(a,b)$ onto the circular arc $(\phi \circ f)^{-1}(a,b)$, it can be conformally extended across the arc $T_i' = (\phi \circ f \circ \eta)^{-1}[c,d]$. Hence, there exists a constant A_i such that $|\eta'(z)| \geq 2A_i$ on T_i'. It follows that there exists $r_i \in (0,1)$ such that $|\eta'(z)| \geq A_i$ in $\mathcal{T}_i = \{\rho z : z \in T_i', r_i \leq \rho \leq 1\}$. By Proposition 2.1, the conformal mapping $\phi_i = \phi$ and its inverse have C^1 extensions to the boundary. Therefore, there exists a positive constant B_i such that $|\phi'(z)| \geq B_i$ on some neighborhood of $\phi^{-1}[c,d]$ with respect to D. Thus, the mapping $f = \phi^{-1} \circ \hat{f} \circ \eta^{-1}$ has bounded derivative in some neighborhood of the set $T_{c,d}^{t_i}$, on which it is bounded by the constant

$$C_i = \sqrt{2}\frac{1+k}{1-k}\frac{b_{t_i}}{A_i B_i}.$$

Set $C_0 = \max\{C_1, \dots, C_n\}$. Then

$$|f_z(z)| + |f_{\bar{z}}(z)| \leq C_0 \quad \text{for all } z \in \mathbb{U} \text{ near } \mathbb{T} = \partial\mathbb{U}.$$

As f is diffeomorphism in $\mathbb{U}$, we obtain the desired conclusion. □

3 Applications

Let D be a domain in $\mathbb{C}$ and ρ a conformal metric in D. The Gaussian curvature of the domain is given by

$$K_D = -\frac{1}{2}\frac{\Delta \log \rho}{\rho}.$$

If, in particular, the domain D is the simply connected in $\mathbb{C}$ and the Gaussian curvature $K_D = 0$ on D, then $\Delta \log \rho = 0$. Therefore $\rho = |e^{\omega}|$, where ω is a holomorphic function on D. Thus the metric ρ is induced by the non-vanishing holomorphic function $\varphi(z) = e^{\omega(z)}$ defined on the domain D.

Since $\rho^2 = \varphi\overline{\varphi}$, a short computation yields $2\rho\rho w = \varphi'\overline{\varphi}$, and therefore $2(\log \rho)_w = (\log \varphi)'$. It follows from (1.2) that f is φ-harmonic, then

$$f_{z\overline{z}} + \frac{\varphi'}{2\varphi} \circ f f_z\, f_{\bar{z}} = 0. \tag{3.1}$$

Roughly speaking, *φ-harmonic* maps arise if the curvature of the target is 0.

Theorem 1.3 and (3.1) yield the following result.

Theorem 3.1. *Let f be a φ-harmonic mapping of the unit disc $\mathbb{U}$ onto a $C^{2,\alpha}$ Jordan domain D. If $M = \|(\log \varphi)'\|_\infty < \infty$ and f is quasiconformal, then f has bounded partial derivatives. In particular, it is a Lipschitz mapping.*

Proof. The hypothesis $M = \|(\log \varphi)'\|_\infty < \infty$, along with the equation (3.1) imply that the crucial hypothesis (1.5) of the main theorem is satisfied. □

Definition 3.2. A C^1 function χ satisfying the inequality $|\bar{\partial}\chi| \leq M|\chi|$ in a domain D is said to be approximately analytic in D with constant M.

If a φ-metric satisfies $M = \|(\log\varphi)'\|_\infty < \infty$ on a domain D, then it is approximately analytic. This implies that $|\varphi'| \leq M|\varphi|$ on D. Since $|\varphi|_z \leq |\varphi'|$ and $2\rho_z\rho = |\varphi|_z$, it follows that $2\rho_z\rho = |\varphi|_z \leq |\varphi'| \leq M|\varphi| = M\rho^2$ on D. Thus the metric is approximately analytic in D with the constant $M/2$.

The following theorem, concerning approximately analytic metrics, generalizes Theorem 3.1.

Theorem 3.3. *Let f be a C^2 ρ-harmonic mapping of the unit disc $\mathbb{U}$ onto the $C^{2,\alpha}$ Jordan domain D. If the metric ρ is approximately analytic in D and f is quasiconformal, then f has bounded partial derivatives. In particular, it is a Lipschitz mapping.*

Theorem 3.3 follows directly from Theorem 1.3 (the main result), using the fact that the equation $|\bar{\partial}\chi| = |\partial\chi|$ holds for all real functions χ.

Definition 3.4. Let Ω be a Jordan domain and let $z \in \partial\Omega$. If $r > 0$, then the set $U_z = \Omega \cap \{w : |w - z| < r\}$ is called a **neighborhood of** z.

Theorem 3.5 (Local version)**.** *Let f be a C^2 ρ-harmonic mapping of the unit disc $\mathbb{U}$ onto the $C^{2,\alpha}$ Jordan domain D having a continuous extension $\tilde{f}$ to the boundary such that $\tilde{f}(\partial\mathbb{U}) = \partial D$. If f is quasiconformal in some neighborhood of a point $z_0 \in \mathbb{T} = \partial\mathbb{U}$, and the metric ρ is approximately analytic in some neighborhood of $w_0 = f(z_0)$, then f has bounded partial derivatives and, in particular, is a Lipschitz mapping in a neighborhood of the point z_0.*

Proof. Let $r > 0$ such that f is q.c. in $U_0 = D(z_0, r) \cap \mathbb{U}$. Then $\gamma_0 = f(\mathbb{T} \cap D(z_0, r))$ is a $C^{2,\alpha}$ Jordan arc in ∂D containing w_0. Following the proof of Theorem 1.3, we obtain that the function f has bounded partial derivatives near the arc $\gamma = f(\mathbb{T} \cap \bar{D}(z_0, r/2))$; and therefore, it must have bounded partial derivatives in some neighborhood of the point z_0. □

The harmonic and q.c. mappings between Riemann surfaces

The definition of a quasiconformal and harmonic mapping $f : R \to S$ between the Riemann surfaces R and S with the metrics ϱ and ρ respectively is similar to the definition of those mappings between domains in the complex plane.

If f is a harmonic mapping, then

$$\Phi dz^2 = \rho \circ f\, f_z\, \overline{f}_{\bar{z}}\, dz^2 \tag{3.2}$$

is a holomorphic quadratic differential on R. We call Φ the *Hopf differential* of f and write Φ =Hopf(f).

Lemma 3.6. *Let (S_1, ρ_1) and (S_2, ρ_2) and (R, ρ) be three Riemann surfaces. Let g be an isometric transformation of the surface S_1 onto the surface S_2:*

$$\rho_1(\omega)|d\omega|^2 = \rho_2(w)|dw|^2, \quad w = g(\omega).$$

Then $f : R \to S_1$ is ρ_1-harmonic if and only if $g \circ f : R \to S_2$ is ρ_2-harmonic. In particular, if g is an isometric self-mapping of S_1, then f is ρ_1-harmonic if and only if $g \circ f$ is ρ_1-harmonic.

Proof. If f is a harmonic map, then $\Phi dz^2 = \rho \circ f\, p\, \overline{q}\, dz^2$ is a holomorphic quadratic differential in R, i.e., the mapping $\rho \circ f\, p\, \overline{q}$ is analytic if we consider it as a function of the parameter $z = z(\zeta)$, $\zeta \in R$. Let $\omega = f(z)$, $F = g \circ f$, $P = (g \circ f)_z$ and $Q = (g \circ f)_{\bar{z}}$. Then $P = g'(\omega) \cdot p$ and $Q = g'(\omega) \cdot q$. Since $\rho_1(\omega) = \rho_2(w)|g'(\omega)|^2$, we obtain

$$\rho_2 \circ F\, P\bar{Q} = \rho_2 \circ g \circ f \cdot |g'(\omega)|^2 p\overline{q} = \rho_1 \circ f p\bar{q}.$$

Hence φ_1 =Hopf$(g \circ f)$ is a holomorphic differential, i.e., $g \circ f$ is harmonic with respect to the metric ρ_2. □

The remaining part of this section deals with the Riemann sphere. However, most of the arguments work for an arbitrary compact Riemann surface.

We call the metric ρ defined on $S^2 = \overline{\mathbb{C}}$ by

$$\rho|dz|^2 = \frac{4|dz|^2}{(1 + |z|^2)^2}$$

the **spherical metric**. The corresponding distance function is

$$d_S(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}, \quad d_S(z, \infty) = \frac{2}{\sqrt{(1 + |z|^2)}}. \tag{3.3}$$

The orientation preserving isometries of the Riemann sphere S^2 with respect to the spherical metric are given by Möbius transformations of the form

$$g(z) = \frac{az + b}{\bar{a} - \bar{b}z}, \quad a, b \in \mathbb{C}, \ |a|^2 + |b|^2 \neq 0. \tag{3.4}$$

The Euler–Lagrange equation for spherical harmonic mappings is

$$f_{z\bar{z}} + \frac{2\bar{f}}{1 + |f|^2}\, f_z \cdot f_{\bar{z}} = 0. \tag{3.5}$$

It is easy to verify that the spherical density is approximately analytic in $\mathbb{C}$ with constant 1; more precisely, one has

$$\frac{\rho_{\overline{z}}}{\rho} = -\frac{2z}{1+|z|^2}.$$

If f is a diffeomorphism of the Riemann sphere (or of a compact Riemann surface M) onto itself, then f is a quasi-isometry with respect to the corresponding metric and, consequently, is quasiconformal.

It is natural to ask what can be said for harmonic q.c. diffeomorphisms defined in some sub-domain of the Riemann sphere.

Using Theorem 3.3, Lemma 3.6 and the isometries defined by (3.4), we can prove the following

Proposition 3.7. *Let the domains $\Omega, D \subset \overline{\mathbb{C}}$ have $C^{1,\alpha}$ and $C^{2,\alpha}$ Jordan boundary on $S^2 = \overline{\mathbb{C}}$, respectively. Then any q.c. spherical harmonic diffeomorphism of Ω onto D is Lipschitz with respect to the spherical metric.*

4 Representation of φ-harmonic mappings

If $\varphi(w_0) \neq 0$, then there is a neighborhood V of w_0 and a branch of $\sqrt{\varphi}$ in V such that $\phi = \int \sqrt{\varphi(z)}\, dz$ is conformal on V. We refer to $\phi = \int \sqrt{\varphi(z)}\, dz$ as the natural parameter on V defined by φ.

Theorem 4.1. *If f is φ-harmonic and ψ is conformal on the co-domain of f, then the mapping $F = \psi \circ f$ satisfies the equation*

$$F_{z\bar{z}} = \left[\frac{\psi''(w)}{\psi'(w)^2} - \frac{\varphi'(w)}{2\psi'(w)\cdot\varphi(w)}\right] \cdot F_z \cdot F_{\bar{z}}, \tag{4.1}$$

where $w = f(z)$.

Proof. Since ϕ is analytic, $F_z = \psi'(w) \cdot f_z$ and $F_{\bar{z}} = \psi'(w) \cdot f_{\bar{z}}$. Hence $F_{z\bar{z}} = \psi''(w) f_z f_{\bar{z}} + \psi'(w) f_{z\bar{z}}$. On the other hand, f is φ-harmonic, and therefore

$$f_{z\overline{z}} = -\frac{1}{2}\frac{\varphi'}{\varphi} \circ f \cdot f_z\, f_{\bar{z}}.$$

Now (4.1) follows easily. □

Observe that if $\varphi = 1$, then the φ-metric reduces to the Euclidean metric. So if f is a Euclidean harmonic mapping, then

$$F_{z\bar{z}} = \frac{\psi''}{\psi'^2} F_z \cdot F_{\bar{z}}\,. \tag{4.2}$$

Corollary 4.2. *Let φ be an analytic function such that there exists a branch of $\int \sqrt{\varphi(z)}\,dz$ in some domain D. If $f : \Omega \to D$ is φ-harmonic and*

$$\phi = \int \sqrt{\varphi(z)}\,dz,$$

then the mapping $F = \phi \circ f$ is harmonic with respect to the Euclidean metric.

Proof. It is easy to see that

$$\frac{\phi''(w)}{\phi'(w)^2} - \frac{\varphi'(w)}{2\phi'(w)\cdot\varphi(w)} = 0.$$

It follows from (4.1) that $F_{z\bar{z}} \equiv 0$. Hence F is harmonic. □

Using (4.2) we obtain

Corollary 4.3. *Let h be a Euclidean harmonic mapping, and let ψ be conformal on the range of h; and let $\varphi = ((\psi^{-1})')^2$. Then the mapping $\hat{h} = \psi \circ h$ is φ-harmonic.*

Recall that if f is φ-harmonic and ϕ is the natural parameter defined by φ, then the mapping $F = \phi \circ f$ is Euclidean harmonic. Applying Theorem K (see the Introduction and also [12, Theorem 3.1]) to the $C^{1,\alpha}$ domain $D' =$ the $\phi(D)$ and the Euclidean harmonic mapping $F = \phi \circ f$ (note that ϕ is not 1-1 in general), we can prove that Theorem 3.1 holds for more general domains.

Theorem 4.4. *Let f be a φ-harmonic mapping of the $C^{1,\alpha}$ domain Ω onto the $C^{1,\alpha}$ Jordan domain D. If $M = \|(\log\varphi)'\|_\infty < \infty$ and f is quasiconformal, then f has bounded partial derivatives. In particular, f is a Lipschitz mapping.*

Assume that $\varphi(z) \neq 0$ and that the natural parameter $\phi(z) = \int \sqrt{\varphi(z)}\,dz$ is well-defined on the domain D. Let ϕ map D onto the convex domain $D' = \phi(D)$.

By the definition of the φ-metric, we have

$$d(z,w) = \inf_{z,w\in\gamma\subset D} \int_\gamma \sqrt{|\varphi(\zeta)|}|d\zeta|.$$

Since

$$\sqrt{|\varphi(\zeta)|}|d\zeta| = |d(\phi(\zeta))|,$$

by the chain rule we obtain

$$d(z,w) = \inf_{A,B\in\gamma'\subset D'} \int_{\gamma'} |d\xi|,$$

where $A = \phi(z)$, $B = \phi(w)$, $\xi = \phi(\zeta)$, and $D' = \phi(D)$. It follows that the segment $[A,B]$ (which belongs to D', as D' is convex), is the curve that minimizes the previous functional. Hence $d(z,w) = |A - B| = |\phi(z) - \phi(w)|$. We have proved the following

Proposition 4.5. *If $D' = \phi(D)$ is convex, then ϕ transforms the φ-metric to the Euclidean metric; i.e., the distance function defined by the φ-metric is given by the formula*

$$d(z,w) = |\phi(z) - \phi(w)| .$$

Example 4.6. Let $\varphi_0(w) = 1/(w-c_0)^2$. Consider the harmonic maps between two domains Ω and D with respect to the metric density

$$\rho_0(w) = |\varphi_0(w)| = 1/|w - c_0|^2, \quad w \in D, \tag{4.3}$$

on D, where $c_0 \notin \overline{D}$ is a given point. If $D' = \log(D - \{c_0\})$ is a convex domain, then the metric defined by (4.3) is

$$d_0(z,w) = \left|\log \frac{z - c_0}{w - c_0}\right| .$$

It is easy to verify that the conformal mappings

$$A(z) = c_0 + re^{i\alpha(1-\varepsilon)}(z - c_0)^{\varepsilon}, \quad r \in \mathbb{R}^+,\ \varepsilon = \pm 1, \tag{4.4}$$

describe the orientation preserving isometries of the domain

$$D_\alpha = \mathbb{C} \setminus \{c_0 + te^{i\alpha}, t \in \mathbb{R}^+\}$$

with respect to the metric d_0 given by (4.3).

Let f be φ_0-harmonic between simply connected domains Ω and D, where $D \subset D_\alpha$ for some α. The natural parameter is $\phi_0(w) = \pm\log(w - c_0)$. As an application of Corollary 4.2, we see that $F(z) = \log(f(z) - c_0)$ is a harmonic function defined on Ω. Therefore,

$$f(z) - c_0 = e^{g_0(z)+\overline{h_0(z)}} = g_1(z)\cdot\overline{h_1(z)} = \big(\sqrt{c_0} - \frac{1}{\sqrt{c_0}}g(z)\big)\cdot\big(-\overline{\sqrt{c_0}} + \frac{1}{\overline{\sqrt{c_0}}}\overline{h(z)}\big),$$

which yields the representation

$$f(z) = g(z) + \overline{h(z)} - {c_0}^{-1}g(z)\cdot\overline{h(z)}, \tag{4.5}$$

where g and h are analytic mappings of Ω into $\mathbb{C}\setminus\{c_0\}$.

It is easy to see that the family of mappings defined by (4.5) is closed under transformations given by (4.4) (see Lemma 3.6).

The above example provides the motivation for the following theorem.

Theorem 4.7. *Let g and h be analytic functions and let $f = g + \overline{h} - {c_0}^{-1}g\overline{h}$, $c_0 \neq 0$, be a diffeomorphism of the $C^{1,\beta}$ domain Ω onto the $C^{1,\alpha}$ Jordan domain D such that $c_0 \in \overline{\mathbb{C}}\setminus\overline{D}$. If f is q.c. mapping, then it has bounded partial derivatives, and the analytic functions g' and h' are bounded.*

Proof. The case $c_0 = \infty$ is proved by Theorem 4.4; therefore, we can assume that $c_0 \neq \infty$. Put

$$g_1 = \sqrt{c_0} - \frac{1}{\sqrt{c_0}} g \quad \text{and} \quad h_1 = -\overline{\sqrt{c_0}} + \frac{1}{\overline{\sqrt{c_0}}} h.$$

Then $f - c_0 = g_1 \cdot \overline{h}_1$. Since $f(z) \neq c_0$ it follows that $h_1(z) \neq 0$ and $g_1(z) \neq 0$. The mapping $F = \log(f - c_0)$, which can be written as $F = \log g_1 + \overline{\log h_1}$ on Ω, is a harmonic mapping of Ω onto $C^{1,\alpha}$ domain $D' = \log(D - c_0)$. Then Theorem 4.4 implies that there exists a constant M such that

$$|h_1'/h_1|^2 + |g_1'/g_1|^2 < M. \tag{4.6}$$

Thus $(\log h_1)'$ is bounded on Ω, and consequently $\log h_1$ has a continuous extension to $\overline{\Omega}$. Therefore, h_1 has a continuous and non-vanishing extension to $\overline{\Omega}$. The same holds for g_1.

The inequality (4.6) implies that h_1' and g_1' are bounded mappings. Thus h' and g' are bounded. □

Example 4.8. A harmonic mapping u with respect to the hyperbolic metric on the unit disk satisfies the equation

$$u_{z\bar{z}} + \frac{2\bar{u}}{1 - |u|^2} u_z \cdot u_{\bar{z}} = 0.$$

As far as we know, this equation cannot be solved using known methods of PDE. However, we can produce some examples. More precisely, we characterize real hyperbolic harmonic mappings.

Let

$$\varphi_1(w) = 4/(1 - w^2)^2.$$

Using the natural parameter, i.e., a branch of $\phi_1(w) = \log \frac{1+w}{1-w} = 2 \tanh^{-1}(w)$, one can verify that f is φ_1-harmonic if and only if $f = \tanh g$, where g is Euclidean harmonic. Since the metric $\rho = |\varphi(w)|$ coincides with the Poincaré metric

$$\lambda = 4/(1 - |w|^2)^2,$$

for real w we obtain that *f is real λ-harmonic (hyperbolic harmonic) if and only if $f = \tanh g$, where g is real Euclidean harmonic.* Since the mappings

$$w = e^{i\varphi} \frac{z - a}{1 - \bar{a}z}, \quad (|a| < 1),$$

are the isometries of the Poincaré disc, it follows from Lemma 3.6 that if h is real harmonic defined on some domain Ω, then the function

$$w = e^{i\varphi} \frac{\tanh(h(z)) - a}{1 - \bar{a} \tanh(h(z))} \quad (|a| < 1) \tag{4.7}$$

is harmonic with respect to the hyperbolic metric. Note that the mappings given by (4.7) map Ω into circular arcs orthogonal to the unit circle $\mathbb{T}$.

Moreover, if a circle S orthogonal to the unit circle $\mathbb{T}$ is given, we can use (4.7) to describe all λ-harmonic mappings between Ω and $S \cap \mathbb{U}$.

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