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QUASICONFORMALITY OF HARMONIC MAPPINGS BETWEEN JORDAN DOMAINS

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Abstract

Suppose that h is a harmonic mapping of the unit disc onto a $C^{1,\alpha}$ domain D. We give sufficient and necessary conditions in terms of boundary function that h is q.c. We announce some new results and also outline application to existence problem of mean distortion minimizers in the Universal Teichmüller space.

1 Introduction

Throughout this paper, $\mathbb{U}$ will denote the unit disc $\{z : |z| < 1\}$, $\mathbb{T}$ the unit circle, $\{z : |z| = 1\}$ and we will use notation $z = re^{i\theta}$.

By $\partial_\theta h$ and $\partial_r h$ (or sometimes by h'_r and h'_θ), h'_x and h'_y we denote partial derivatives with respect to θ and r, x and y respectively.

Every harmonic function h in $\mathbb{U}$ can be written in the form $h = f + \bar{g}$, where f and g are holomorphic functions in $\mathbb{U}$. Then an easy calculation shows

$\partial_\theta h(z) = i(zf'(z) - \overline{zg'(z)})$, $h'_r = e^{i\theta} f' + \overline{e^{i\theta} g'}$, $h'_\theta + irh'_r = 2izf'$ and therefore rh'_r is the harmonic conjugate of h'_θ. We also use notation $p = f', q = g'$, $\Lambda_h = |f'| + |g'|, \lambda_h = |f'| - |g'|$ and $\mu_h = q/p$.

Let

$$P_r(t) = \frac{1 - r^2}{2\pi(1 - 2r\cos(t) + r^2)}$$

denote the Poisson kernel.

If $\psi \in L^1[0, 2\pi]$ and

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, \psi(t)\, dt,$$

then the function $h = P[\psi]$ so defined is called Poisson integral of ψ.

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If ψ is of bounded variation, define $T_\psi(x)$ as variation of ψ on $[0,x]$, and let $V(\psi)$ denote variation of ψ on $[0,2\pi]$ (see, for example, [38] p.171).

Define
$$h_*(\theta) = h^*(e^{i\theta}) = \lim_{r\to 1} h(re^{i\theta})$$
when this limit exists.

If $\psi \in L^1[0,2\pi]$ (or $L^1[\mathbb{T}]$), then the Cauchy transform $C(\psi)$ is defined as
$$C(\psi)(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\psi(t)e^{it}}{z-e^{it}}\,dt$$
with its kernel
$$K(z,t) = \frac{e^{it}}{z-e^{it}}\,.$$
While the Hilbert transform $H(\psi)$ is defined as
$$H(\psi)(\varphi) = \int_{0_+}^{\pi} \frac{\psi(\varphi+t)-\psi(\varphi-t)}{\tan t/2}dt,$$
where we abuse notation by extending ψ to be 2π periodic, or consider it to be a function from $L^1(\mathbb{T})$. The following property of the Hilbert transform is also sometimes taken as the definition:
If $u = P[\psi]$ and v is the harmonic conjugate of u, then $v_* = H(\psi)$ a.e.

Note that, if ψ is 2π-periodic, absolutely continuous on $[0,2\pi]$ (and therefore $\psi' \in L^1[0,2\pi]$), then
$$h'_\theta = P[\psi']. \tag{1.1}$$
Hence, since rh'_r is the harmonic conjugate of h'_θ, we find
$$rh'_r = P[H(\psi')] \quad \textit{and} \tag{1.2}$$
$$(h'_r)^*(e^{i\theta}) = H(\psi')(\theta)\,\text{a.e.} \tag{1.3}$$
It is clear that
$$K(z,t) + \overline{K(z,t)} - 1 = P_r(\theta - t)\,.$$
For $f : \mathbb{U} \to \mathbb{C}$, define
$$f_*(\theta) = f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$
when this limit exists. For $f : \mathbb{T} \to \mathbb{C}$, define $f_*(\theta) = f^*(e^{i\theta})$.

If f is a bounded harmonic map defined on the unit disc $\mathbb{U}$, then f^* exists a.e., f^* is a bounded integrable function defined on the unit circle $\mathbb{T}$, and f has the following representation
$$f(z) = P[f^*](z) = \int_0^{2\pi} P(r, t-\varphi) f^*(e^{it})dt, \tag{1.4}$$

where $z = re^{i\varphi}$.

A homeomorphism $f\colon D \mapsto G$, where D and G are subdomains of the complex plane $\mathbb{C}$, is said to be K-quasiconformal (K-q.c or k-q.c), $K \geq 1$, if f is absolutely continuous on a.e. horizontal and a.e. vertical line in D and there is $k \in [0,1)$ such that

$$|f_{\bar z}| \leq k|f_z| \quad \text{a.e. on } D, \tag{1.5}$$

where $K = \frac{1+k}{1-k}$ i.e. $k = \frac{K-1}{K+1}$.

Note that the condition (1.5) can be written as

$$D_f := \frac{\Lambda}{\lambda} = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \leq K, \tag{1.6}$$

where $K = \frac{1+k}{1-k}$ i.e. $k = \frac{K-1}{K+1}$.

Note that if γ is 2π-periodic absolutely continuous on $[0,2\pi]$ (and therefore $\gamma' \in L^1[0,2\pi]$) and $h = P[\gamma]$, then

$$(h_r')^*(e^{i\theta}) = H(\gamma')(\theta)\,\text{a.e.},$$

where H denotes the Hilbert transform.

Let Γ be a curve of $C^{1,\mu}$ class and $\gamma : \mathbb{R} \to \Gamma^*$ be arbitrary topological (homeomorphic) parameterization of Γ and $s(\varphi) = \int_0^\varphi |\gamma'(t)|dt$. It is convenient to abuse notation and to denote by $\Gamma(s)$ natural parameterization.

For $\Gamma(s) = \gamma(\varphi)$, we define $n_\gamma(\varphi) = i\Gamma'\big(s(\varphi)\big)$ and

$$R_\gamma(\varphi, t) = (\gamma(t) - \gamma(\varphi), n_\gamma(\varphi)).$$

For $\theta \in \mathbb{R}$ and $h = P[\gamma]$, define

$$E_\gamma(\theta) = \big((h_r')^*(e^{i\theta}), n_\gamma(\theta)\big) = \big(H(\gamma')(\theta), n_\gamma(\theta)\big)\, a.e. \quad and \tag{1.7}$$

$$\upsilon(z,\theta) = \upsilon_\gamma(z,\theta) = \big(rh_r(z), n_\gamma(\theta)\big), \quad z \in \mathbb{U}\,. \tag{1.8}$$

Note that $\upsilon_*(t,\theta) = \big(H(\gamma_*')(t), n_\gamma(\theta)\big)$ a.e.

1.1 Background

To each mapping (in particular closed curve Γ) given by $\gamma : \mathbb{T} \to \mathbb{C}$, we associate a function $\gamma_* : \mathbb{R} \to \mathbb{C}$ defined by $\gamma_*(t) = \gamma(e^{it})$; we also call $\gamma_* : [0,2\pi] \to \Gamma^*$ a parameterization of Γ.

Harmonic quasiconformal (abbreviated by HQC) mappings are now very active area of investigation (see for example [17, 19]).

Let $\mathcal{D}_1$ (res $\mathcal{D}_2$) be the family of all Jordan domains in the plane which are of class $C^{1,\mu}$ (res $C^{2,\mu}$) for some $0 < \mu < 1$.

In [16] the following result is proved:

Theorem A. Let Ω and Ω_1 be Jordan domains, let $\mu \in (0,1]$, and let $f : \Omega \mapsto \Omega_1$ be a harmonic homeomorphism.

Then
(a) If f is q.c. and $\partial\Omega, \partial\Omega_1 \in \mathcal{D}_1$, then f is Lipschitz;
(b) if f is q.c., $\partial\Omega, \partial\Omega_1 \in \mathcal{D}_1$ and Ω_1 is convex, then f is bi-Lipschitz; and
(c) if Ω is the unit disk, Ω_1 is convex, and $\partial\Omega_1 \in C^{1,\mu}$, then f is quasiconformal if and only if its boundary function is bi-Lipschitz and the Hilbert transform of its derivative is in L^∞.
In [17] it is proved the convexity hypothesis can be dropped if codomain is in $\mathcal{D}_2$:
(b1) if f is q.c., $\partial\Omega \in \mathcal{D}_1$ and $\partial\Omega_1 \in \mathcal{D}_2$, then f is bi-Lipschitz.
Similar results were announced in [32]. These extend the results obtained in [29, 13, 36, 22].

The proof of the part (a) of Theorem A in [16] is based on an application of Mori's theorem on quasiconformal mappings, which has also been used in [36] in the case $\Omega_1 = \Omega = \mathbb{U}$, and the following lemma.

Lemma 1.1. *Let Γ be a curve of class $C^{1,\mu}$ and $\gamma : \mathbb{T} \to \Gamma^*$ be arbitrary topological (homeomorphic) parameterization of Γ. Then*

$$|R_\gamma(\varphi, t)| \le A\, |\gamma((e^{i\varphi}) - \gamma(e^{it})|^{1+\mu}, \tag{1.9}$$

where $A = A(\Gamma)$.

In [18], we prove a version of "inner estimate" for quasi-conformal diffeomorphisms, which satisfies a certain estimate concerning their laplacian. As an application of this estimate, we show that quasi-conformal harmonic mappings between smooth domains (with respect to the approximately analytic metric), have bounded partial derivatives; in particular, these mappings are Lipschitz. Our discussion in [18] includes harmonic mappings with respect to (a) spherical and euclidean metrics (which are approximately analytic) as well as (b) the metric induced by the holomorphic quadratic differential.

1.2 HQC are bi-Lipschitz

We announce some results obtained in [8] by two of the authors. The results make use of the Gehring-Osgood inequality [10], as we are going to explain.

Let Ω be Jordan domain in $\mathcal{D}_1$, γ curve defined by $\partial\Omega$ and h K-qch from $\mathbb{U}$ onto Ω and $h(0) = a_0$. Then h is L-Lipschitz, where L depends only on $K, dist(a_0, \partial\Omega)$ and $\mathcal{D}_1$ constant C_γ. In [8] we give an explicit bound for the Lipschitz constant.

Let h be a harmonic quasiconformal map from the unit disk onto D in class $\mathcal{D}_1$. Examples show that a q.c. harmonic function does not have necessarily a C^1 extension to the boundary as in conformal case. In [8] it is proved that the corresponding functions E_{h_*} are continuous on the boundary and for fixed θ_0, $\upsilon_{h_*}(z, \theta_0)$ is continuous in z at $e^{i\theta_0}$ on $\mathbb{U}$.

The main result in [8] is

Theorem 1.2. *Let Ω and Ω_1 be Jordan domains in $\mathcal{D}_1$, and let $h : \Omega \mapsto \Omega_1$ be a harmonic q.c. homeomorphism. Then h is bi-Lipschitz.*

It seems that we use a new idea. We reduce proof to the case when $\Omega = H$. Suppose that $h(0) = 0 \in \Omega_1$. We show that there is a convex domain $D \subset \Omega_1$ in $\mathcal{D}_1$ such that $\gamma_0 = \partial D$ touch the boundary of Ω_1 at 0. Since there is qc extension h_1 of h to $\mathbb{C}$, we can apply Gehring-Osgood theorem to $h_1 : \mathbb{C}^* \to \mathbb{C}^*$. This gives estimate for $arg\gamma_1(z)$ for z near 0, where $\gamma_1 = h^{-1}(\gamma_0)$, and we show that there is a domain $D_0 \subset \mathbb{H}$ in $\mathcal{D}_1$ such that $h(D_0) \subset D$. Finally, we combine the convexity type argument and noted continuity of functions E and v to finish the proof.

2 Preliminary results

We first give an extension of Proposition 1 [31]:

Proposition 2.1. *Suppose that $\psi : [0, 2\pi] \to C$ is of bounded variation and $h = P[\psi]$.*

Then

(1) $l(r) = \int_0^{2\pi} |h_\theta(z)| \le V(\psi)$ and $l(r)$ is increasing.

(2) $f', g' \in H^p$ for every $0 < p < 1$

(3) h_r^* exists a.e.

$$\partial_r h(e^{it}) = \lim_{r \to 1_{-0}} \frac{h^*(e^{it}) - h(re^{it})}{1 - r}$$

exists a.e. and $(\partial_r h)^*(e^{it}) = \partial_r h(e^{it})$ a.e. $(\partial_\theta h)^* = \psi'$ a.e.

(4) If ψ is absolutely continuous, then $C[\psi'](z) = izf'(z)$ and $izf'(z)$ is the analytic part of $\partial_\theta h$. Also, $C[\overline{\psi'}](z) = izg'(z)$

(5) If $h = P[\psi]$ is K-q.c., then h_* is absolutely continuous and $h_*'(t) \ne 0$ a.e.

Proof. (1) If ψ is of bounded variation, then

$$\partial_\theta h(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, d\psi(t)$$

and hence

$$l(r) = l(r, h) = \int_0^{2\pi} |h'_\theta(z)| \le \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, dT_\psi(t) d\theta\,,$$

where $T_\psi(x)$ is total variation of ψ on $[0, x]$. Thus $l(r) \le \int_0^{2\pi} dT_\psi(t) = T_\psi(2\pi) = V(\psi)$.

Since h'_θ is harmonic, $|h'_\theta|$ is subharmonic and therefore $l(r)$ is increasing.

(2) Since $h'_\theta \in h^1$, then the Cauchy transform $C[h_\theta] \in H^p$ for every $0 < p < 1$.

We leave proof of (3) to the interested reader.

(4) Note that $d\psi(t) = \psi'(t)dt + d\sigma(t)$, where σ is a singular measure w.r. to Lebesgue measure, i.e., one supported on a set of Lebesgue measure zero. If ψ is absolutely continuous, then $d\psi(t) = \psi'(t)dt$. Hence, we find

$$h'_\theta = P[\psi']$$

and therefore (4). Note that here absolute continuity of ψ is essential.

If $M \subset \mathbb{T}$ is Lebesgue measurable, we denote $|M|$ its Lebesgue measure.

(5) Since $|f'| + |g'| \leq K\,|h_\theta(z)|$, then $f', g' \in H^1$. Hence, we conclude that h_* is absolutely continuous.

Let $A_0 = \{e^{it} : h_*'(t) = 0\}$,

$dh = pdz + qd\overline{z}$ and $E_0 \subset \mathbb{T}$ the set on which p^*, q^* exist and $|p^*| \leq K|q^*|$.

If $z_0 \in A_0 \cap E_0$, then $p = q = 0$ at z_0.

Since p and $\overline{q}$ are analytic on $\mathbb{U}$ and belong to H^1, we conclude that $|A_0 \cap E_0| = 0$. Since $\mathbb{T} \setminus E_0$ has measure 0, we conclude that $|A_0| = 0$ □

3 Characterizations of HQC

3.1 The half plane

By $\mathbb{H}$ we denote the upper-half plane and $\Pi^+ = \{z : \mathrm{Re} z > 0\}$.

The first characterizations of the HQC conditions have been obtained by Kalaj in his thesis research.

In the case of the upper half plane, the following known fact plays an important role, cf for example [24]:

Lemma B. *Let f be an euclidean harmonic $1-1$ mapping of the upper half-plane $\mathbb{H}$ onto itself, continuous on $\overline{\mathbb{H}}$, normalized by $f(\infty) = \infty$ and $v = \mathrm{Im} f$. Then $v(z) = c\,\mathrm{Im} z$, where c is a positive constant. In particular, v has bounded partial derivatives on $\mathbb{H}$.*

The lemma is a corollary of the Herglotz representation of the positive harmonic function v (see for example [4]).

Theorem 3.1. *Let $h : \mathbb{H} \to \mathbb{H}$ be harmonic function. Then h is orientation preserving harmonic diffeomorphism of $\mathbb{H}$ onto itself, continuous on $\mathbb{H} \cup \mathbb{R}$ such that $h(\infty) = \infty$ if and only if there is analytic function $\phi : \mathbb{H} \to \Pi^+$ such that*

$\lim_{z\to\infty} \Phi_1(z) = \infty$, where

$\Phi(z) = \int_i^z \phi(\zeta) d\zeta$, $\Phi_1 = \mathrm{Re}\,\Phi$,

$(*1)$ $h(z) = h^\phi(z) = \Phi_1(z) + icy + c_1$, $c > 0$ and $c_1 \in \mathbb{R}$.

A version of this result is proved in [13].

Let $h = u + iv$. By Lemma B, $u = \mathrm{Re}\,\Phi$ and $v = cy$, where $c > 0$ and Φ is analytic function in $\mathbb{H}$.

Since $\Phi_y' = i\Phi'$ and

$h(z) = h^\phi(z) = (\Phi(z) + \overline{\Phi(z)})/2 + icy + c_1$, we find

$h_y'(z) = (i\Phi'(z) + \overline{i\Phi'(z)})/2 + ic = (i\phi - i\overline{\phi})/2 + ic = -\mathrm{Im}\,\phi(z) + ic$

Hence

(X1) $h_x'(z) = \mathrm{Re}\,\phi(z)$ and $h_y'(z) = -\mathrm{Im}\,\phi(z) + ic$. Since $h_z = (h_x' - ih_y')/2 = \phi/2 + c/2$ is analytic, $-h_y'$ is harmonic conjugate of h_x' and therefore

(X2) $h_y' = H(h_*') = \mathrm{Im}\,\phi(z) - ic$,

where h_* denotes the restriction of h on $\mathbb{R}$.

By $HQC_0(\mathbb{H})$ (respectively $HQC_0^k(\mathbb{H})$) we denote the set of all qc (respectively k-qc) harmonic mappings h of $\mathbb{H}$) onto itself for which $h(\infty)=\infty$.

Theorem 3.2. *Let $\chi:\mathbb{R}\to\mathbb{R}$ be a homeomorphism and $h=P[\chi]$. The following condition are equivalent*
(A1) $h\in HQC_0(\mathbb{H})$
(A2) *there is analytic function $\phi:\mathbb{H}\to\Pi^+$ such that $\phi(\mathbb{H})$ is relatively compact subset of Π^+ and $h=h^\phi$.*

Proof. Suppose (A1). We can suppose that h is K-qc and $c=1$ in the representation $(*1)$. Since $v(z)=\operatorname{Im}\mathrm{h(z)}=\mathrm{y}$, we have $\lambda_h\geq 1/K$. Let $z_0\in\mathbb{H}$ and define the curve $L=\{z:\Phi_1(z)=\Phi_1(z_0)\}$ and denote by l_0 the unit tangent vector to the curve L at z_0. Since $|dh_{z_0}(l_0)|\leq 1$, we have $\Lambda_h\leq K$ on $\mathbb{H}$. Hence absolute values of partial derivatives of h are bounded from above and below by two positive constants. Thus, by (X1) and (X2), ϕ is bounded on $\mathbb{H}$.

In particular, (A1) implies that h is bi-lipschitz.
Hence there two positive constants s_1 and s_2 such that
$s_1\leq\chi'(x)\leq s_2$, a.e.
Since $\chi'(x)=\operatorname{Re}\phi^*(x)$ a.e. on $\mathbb{R}$ and ϕ is bounded on $\mathbb{H}$, we find
$s_1\leq\operatorname{Re}\phi(z)\leq s_2$, $z\in\mathbb{H}$;
and (A2) follows.
We leave to the reader to prove that
(A2) implies (A1) and
using equation (3.1) below to prove
(A1) implies (A2).

□

It is clear that the conditions (A1) and (A2) are equivalent to
$(A3)$ there is analytic function $\phi\in H^\infty(\mathbb{H})$
and there two positive constants s_1 and s_2 such that $s_1\leq\operatorname{Re}\phi(z)\leq s_2$, $z\in\mathbb{H}$.

Since $\chi'(x)=\operatorname{Re}\phi^*(x)$ a.e. on $\mathbb{R}$ and $H\chi'=\operatorname{Im}\phi^*(x)-ic$ a.e. on $\mathbb{R}$, we get characterization in terms of Hilbert transform:

$(A4)$ χ is absolutely continuous, and there two positive constants s_1 and s_2 such that $s_1\leq\chi'(x)\leq s_2$, a.e. and $H\chi'$ is bounded.

A similar characterization holds for smooth domains and in particular in the case of the unit disk; see Theorems 3.6 and 3.4 below.

From the proof of Theorem 3.3 below, cf [24], it follows that if we set $c=1$ in the representation $(*1)$, then $h=h^\phi\in HQC_0^k(\mathbb{H})$ if and only $\phi(\mathbb{H})$ is in a disk $B_k=B(a_k;R_k)$, where $a_k=\frac{1}{2}(K+1/K)=\frac{1+k^2}{1-k^2}$ and $R_k=\frac{1}{2}(K-1/K)=\frac{2k}{1-k^2}$.
First, we need to introduce some notation:
For $a\in\mathbb{C}$ and $r>0$ we define $B(a;r)=\{z:|z-a|<r\}$. In particular, we write $\mathbb{U}_r$ instead of $B(0;r)$.

Theorem 3.3 ([24], the half plane euclidian-qch version)**.** *Let f be a K-qc euclidian harmonic diffeomorphism from $\mathbb{H}$ onto itself. Then f is a $(1/K, K)$ quasi-isometry with respect to Poincaré distance.*

For higher dimension version of this result see [31, 33, 2].

Proof. We first show that, by pre composition with a linear fractional transformation, we can reduce the proof to the case $f(\infty) = \infty$. If $f(\infty) \neq \infty$, there is the real number a such that $f(a) = \infty$. On the other hand, there is a conformal automorphism A of $\mathbb{H}$ such that $A(\infty) = a$. Since A is an isometry of $\mathbb{H}$ onto itself and $f \circ A$ is a K-qc euclidian harmonic diffeomorphism from $\mathbb{H}$ onto itself, the proof is reduced to the case $f(\infty) = \infty$.
It is known that f has a continuous extension to $\overline{\mathbb{H}}$ (see for example [25]).
Hence, by Lemma B, $f = u + ic\,\mathrm{Im}\,\mathrm{z}$, where c is a positive constant. Using the linear mapping B, defined by $B(w) = w/c$, and a similar consideration as the above, we can reduce the proof to the case $c = 1$. Therefore we can write f in the form $f = u + i\mathrm{Imz} = \frac{1}{2}(F(z) + z + \overline{F(z) - z})$, where F is a holomorphic function in $\mathbb{H}$. Hence,

$$\mu_f(z) = \frac{F'(z) - 1}{F'(z) + 1} \quad \text{and} \quad F'(z) = \frac{1 + \mu_f(z)}{1 - \mu_f(z)}, \quad z \in \mathbb{H}. \tag{3.1}$$

Define $k = \frac{K-1}{K+1}$ and $w = S\zeta = \frac{1+\zeta}{1-\zeta}$. Then, $S(U_k) = B_k = B(a_k; R_k)$, where $a_k = \frac{1}{2}(K + 1/K) = \frac{1+k^2}{1-k^2}$ and $R_k = \frac{1}{2}(K - 1/K) = \frac{2k}{1-k^2}$.

Since f is k-qc, then $\mu_f(z) \in U_k$ and therefore $F'(z) \in B_k$ for $z \in \mathbb{H}$. This yields, first,

$$K + 1 \geq |F'(z) + 1| \geq 1 + 1/K, \quad K - 1 \geq |F'(z) - 1| \geq 1 - 1/K,$$

and then, $1 \leq \Lambda_f(z) = \frac{1}{2}(|F'(z) + 1| + |F'(z) - 1|) \leq K$.

So we have $\lambda_f(z) \geq \Lambda_f(z)/K \geq 1/K$.
Thus, we find

$$1/K \leq \lambda_f(z) \leq \Lambda_f(z) \leq K. \tag{3.2}$$

Let λ denote the hyperbolic density on $\mathbb{H}$.
Since $\lambda(f(z)) = \lambda(z)$, $z \in \mathbb{H}$, using (3.2) and the corresponding versions of 3A and 3B for $\mathbb{H}$, cf [24], we obtain

$$\frac{1-k}{1+k}\, d_h(z_1, z_2) \leq d_h(f(z_1), f(z_2)) \leq \frac{1+k}{1-k}\, d_h(z_1, z_2).$$

It also follows from (3.2) that

$$\frac{1}{K}\,|z_2 - z_1| \leq |f(z_2) - f(z_1)| \leq K\,|z_2 - z_1|, \quad z_1, z_2 \in \mathbb{H}.$$

We leave to the reader to prove this inequality as an exercise.
This estimate is sharp (see also [22] for an estimate with some constant $c(K)$). $\square$

3.2 The unit disc

Suppose that h is an orientation preserving diffeomorphism of $\mathbb{H}$ onto itself, continuous on $\mathbb{H} \cup \mathbb{R}$ such that $h(\infty) = \infty$ and χ the restriction of h on $\mathbb{R}$. Recall $h \in HQC_0(\mathbb{H})$ iff there is analytic function $\phi : \mathbb{H} \to \Pi^+$ such that $\phi(\mathbb{H})$ is relatively compact subset of Π^+ and $\chi'(x) = \operatorname{Re}\phi^*(x)$ a.e.
We give similar characterizations in the case of the unit disk and for smooth domains (see below).

Theorem 3.4. *Let ψ be a continuous increasing function on $\mathbb{R}$ such that $\psi(t + 2\pi) - \psi(t) = 2\pi$, $\gamma(t) = e^{i\psi(t)}$ and $h = P[\gamma]$. Then h is q.c. if and only if*
1. $\operatorname{ess\,inf}\psi' > 0$
2. there is analytic function $\phi : U \to \Pi^+$ such that $\phi(U)$ is relatively compact subset of Π^+ *and* $\psi'(x) = \operatorname{Re}\phi^*(e^{i\,x})$ *a.e.*

In the setting of this theorem we write $h = h^\phi$. The reader can use the above characterization and functions of the form $\phi(z) = 2 + M(z)$, where M is an inner function, to produce examples of HQC mappings $h = h^\phi$ of the unit disk onto itself so the partial derivatives of h have no continuous extension to certain points on the unit circle. In particular we can take $M(z) = \exp\frac{z+1}{z-1}$, cf [7].

In the next subsection we extend the above theorem to smooth domains. Note that the proof that a HQC mapping between the unit disk and $\mathcal{D}_1$ domain is bilipshitz is more delicate than in the case of the upper half plane. Instead of Lemma B we use Theorem A and Theorem 1.2.

3.3 HQC and convex smooth codomains

We need the following result related to convex codomains.

Theorem 3.5 ([31])**.** *Suppose that h is a euclidean harmonic mapping from $\mathbb{U}$ onto a bounded convex domain $D = h(\mathbb{U})$, which contains the disc $B(h(0); R_0)$. Then*
(1) $d(h(z), \partial D) \geq (1 - |z|)R_0/2, \quad z \in \mathbb{U}.$
(2) *Suppose that $\omega = h^*(e^{i\theta})$ and $h_r^* = h_r'(e^{i\theta})$ exist at a point $e^{i\theta} \in \mathbb{T}$, and there exists the unit inner normal $n = n_\omega$ at $\omega = h^*(e^{i\theta})$ with respect to ∂D.*
Then $E = (h_r^*, n_{h_*}) \geq c_0$, where $c_0 = \frac{R_0}{2}$.
(3) *In addition to the hypothesis stated in the item* (2)*, suppose that h_*' exists at the point $e^{i\theta}$. Then $|J_h| = |(h_r^*, N)| = \big|(h_r^*, n)\big||N| \geq c_0|N|$, where $N = i\,h_*'$ and the Jacobian is computed at the point $e^{i\theta}$ with respect to the polar coordinates.*

If D is of $C^{1,\mu}$ class, using the result that the function E is continuous [8], we find
(4) $|E| \geq c_0$.

Theorem 3.6. *Suppose that $C^{1,\alpha}$ domain D is convex and denote by γ positively oriented boundary of D. Let $h_0 : \mathbb{T} \to \gamma$ be an orientation preserving homeomorphism and $h = P[h_0]$.*

The following conditions are then equivalent *a) h is K-qc mapping*

b) h is bi-Lipschitz in the Euclidean metric

c) the boundary function h_* is bi-Lipschitz in the Euclidean metric and Cauchy transform $C[h'_*]$ of its derivative is in L^∞.

d) the boundary function h_* is absolutely continuous, $\operatorname{ess\,inf}|h'_*| > 0$ *and Cauchy transform $C[h'_*]$ of its derivative is in L^∞.*

e) the boundary function h_* is bi-Lipschitz in the Euclidean metric and Hilbert transform $H[h'_*]$ of its derivative is in L^∞.

f) the boundary function h_* is absolutely continuous, $\operatorname{ess\,sup}|h'_*| < +\infty$, $\operatorname{ess\,inf}|h'_*| > 0$ *and Hilbert transform $H[h'_*]$ of its derivative is in L^∞.*

Note that here, by our notation, $(h_0)_* = h_*$ and $h_0 = h^*$.

Proof. By the fundamental theorem of Rado, Kneser and Choquet, h is an orientation preserving harmonic mapping of the unit disc onto D.

If D is $C^{1,\alpha}$, it has been shown in [8] that a) implies b) even without hypothesis that D is a convex domain. Note that an arbitrary bi-Lipschitz mapping is quasiconformal. Hence the conditions a) and b) are equivalent.

Hilbert transform of a derivative of HQC boundary function will be in L^∞, and hence a) implies e).

Recall, we use notation $p = f', q = g'$, $\Lambda_h = |f'| + |g'|, \lambda_h = |f'| - |g'|$.

If h_* is absolutely continuous, since $h'_\theta(z) = i(zf'(z) - \overline{zg'(z)})$, we find $C[h'_*](z) = izf'(z)$. It follows that a) implies c) and d).

Since bi-Lipschitz condition implies absolute continuity, c) implies d) and e).

Let us show d) implies a).

Hypothesis $C[h'_*] \in L^\infty$ implies that $f' \in L^\infty$ and therefore since h is orientation preserving and $|f'| \geq |g'|$, we find $g' \in L^\infty$.

This shows that Λ_h is bounded from above.

We will show that $|p^*|$ is bounded from above, $\lambda^*_h = |p^*|(1 - |\mu^*|)$ is bounded from below, and therefore that $(1 - |\mu^*|)$ is bounded from below.

Let $N = i\,h'_*$ and $N = n|N|$.

Since D is a convex domain $|f'|$ and (h^*_r, n) are bounded from below with positive constant (for an outline of proof see [30, 31]).

Condition $C[h'_*] \in L^\infty$ implies that $f' \in H^\infty$. Hence, since $|f'|$ is bounded from below with positive constant, it follows that Λ_h is bounded from above and below with two positive constants.

By assumption d), $|h'_*|$ is bounded essentially from below. Since, $J_h = \Lambda_h \lambda_h$ and by Theorem 3.5

$J^*_h = (h^*_r, N) = (h^*_r, n)|N| \geq c_0|N|$,

where $n = n_{h_*}$ and $N = n|N|$ and $N = i\,h'_*$, we conclude that λ^*_h is bounded from above and below with two positive constants. It follows from $\lambda^*_h = |p^*|(1 - |\mu^*|)$, that $(1 - |\mu^*|)$ is bounded from below with positive constant c_1 and therefore $k_1 = (1 - c_1) \geq |\mu^*|$. By maximum principle, $||\mu||_\infty \leq k_1$.

Note that hypothesis d) implies that $|h'_*|$ is bounded from above and therefore the boundary functions h_* is bi-Lipschitz. Thus, we have that a) and b) follow from d).

Lets prove that f) implies d). This will finish the proof, since e) implies f) and we have already established that d) implies a).

Since the boundary function h_* is absolutely continuous, recall that, by (1.1), we have
$\partial_\theta h(z) = P[h'_*](z) = i(zf'(z) - \overline{zg'(z)})$,
and, by (1.2), that its harmonic conjugate is
$zf'(z) + \overline{zg'(z)} = rh'_r(z) = P[H(h'_*)]$.
Thus if h_* is Lipschitz and $H(h'_*)$ is bounded, then $\partial_\theta h$ and $irh'_r(z)$ are bounded on $\mathbb{U}$ so by adding these two together we conclude that $h'_\theta + irh'_r = 2izf' = 2C[h'_*]$ is bounded and therefore Cauchy transform $C[h'_*]$ is bounded, and d) follows.
Note that we have here $|f'|$ is bounded and therefore all partial derivatives of h are bounded, and
$H(h'_*) = zp^* + \overline{zq^*}$ a.e. on $\mathbb{T}$, where $p = f'$ and $q = g'$. □

A version of the part (a) equivalent to (f) of the main characterization has been stated in [16].

Theorem 3.7 ([16]). *Let $f : \mathbb{T} \to \gamma$ be an orientation preserving homeomorphism of the unit circle onto the Jordan convex curve $\gamma = \partial\Omega \in C^{1,\mu}$.*

Then $h = P[f]$ is a quasiconformal mapping if and only if

$$0 < \operatorname{ess\,inf} |f'(\varphi)|, \tag{3.3}$$

$$\operatorname{ess\,sup} |f'(\varphi)| < \infty \tag{3.4}$$

and

$$\operatorname*{ess\,sup}_{\varphi} |H(f')(\varphi)| < \infty, \tag{3.5}$$

where

$$H(f')(\varphi) = \int_0^\pi \frac{f'(\varphi + t) - f'(\varphi - t)}{\tan t/2} dt,$$

denotes the Hilbert transformations of f'.

The hypothesis that f is absolutely continuous function was omitted in [16], but it seems to be needed to justify the proof from that paper.

Indeed, it is easy to find an example of a function f satisfying conditions (3.3), (3.4) and (3.5), such that the corresponding harmonic map $h = P[f]$ is not q.c., cf [7].

Our characterization works only for convex domains. If all conditions are kept, but convexity is dropped, then there can be found examples of maps which are not HQC, cf [7]. We can get HQC characterization of general harmonic maps to $C^{1,\alpha}$ domains, if we set apart the condition which depends on the convexity of the domain as one of the requirements.

Theorem 3.8. *Suppose that D is $C^{1,\alpha}$ domain. Let h be a harmonic orientation preserving map of the unit disc onto D and homeomorphism of $\overline{\mathbb{U}}$ onto $\overline{D}$. The following conditions are equivalent* *a*1) *h is K-qc mapping*

*a*2) *the boundary function h_* is absolutely continuous,* $\operatorname{ess\,sup}|h'_*| < +\infty$, $Hh'_* \in L^\infty$ *and* $s_0 = \operatorname{ess\,inf}|(Hh'_*, ih'_*)| > 0$.

We only outline the proof of the this theorem.

Proof. Put $\mu = \mu_h$. Clearly $a2)$ implies $\operatorname{ess\,inf}|h'_*| > 0$. We leave to the reader to check that
$2zp^* = H(h'_*) - ih'_*$, $2zq^* = H(h'_*) + ih'_*$ $\quad J^*_h = (h^*_r, i\,h'_*) = (H(h'_*), i\,h'_*) \geq 0$ a.e. on $\mathbb{T}$ and $J_h > 0$ on $\mathbb{U}$. Hence $|\mu| < 1$ and $\Lambda^*_h \lambda^*_h = J^*_h \geq s_0 > 0$. Similarly like in the proof of the main characterization theorem $a2)$ implies $|\mu^*|_\infty = k < 1$ and so we have $a1)$. The converse is straightforward. □

4 Application to the Universal Teichmüller space

For $\zeta = \xi + i\eta$ we use notation $|d\zeta|^2 = d\xi\, d\eta$. Here we apply our characterization to the problem of minimizing functional

$$\mathbb{K}(f) = \int_U \frac{|f_z(\zeta)|^2 + |f_{\overline{z}}(\zeta)|^2}{|f_z(\zeta)|^2 - |f_{\overline{z}}(\zeta)|^2} |d\zeta|^2$$

over all quasiconformal maps $f : \mathbb{U} \mapsto \mathbb{U}$ with the same boundary condition, i.e. belonging to the same class in the Universal Teichmüller space. Existence of minimizers of functional $\mathbb{K}$ in Teichmüller spaces has been of considerable recent interest. For instance, in [28] it has been proved that minimizers do not exist in the case of punctured disc.

From results in [3], it follows that the minimizer will exist in the Universal Teichmüller class if and only if the inverse map on the boundary induces harmonic quasiconformal map, i.e. if $P[f^{-1}]$ is quasiconformal. Applying our results, we get the following characterization, cf also [3]:

Theorem 4.1. *Let $f : \mathbb{T} \to \mathbb{T}$ be a homeomorphism of $\mathbb{T}$, that satisfies the M-conditon i.e. that has quasiconformal extension to $\mathbb{U}$. Then in the Universal Teichmüller class of f there is minimiser of the functional $\mathbb{K}$ if and only if*
b1) f is bi-Lipschitz and $H[(f^{-1})'] \in L^\infty(\mathbb{T})$ or
b2) f is bi-Lipschitz and $C[(f^{-1})'] \in L^\infty(\mathbb{T})$

Also, we can get the result about minimisers of $\mathbb{K}$ functional of maps from the convex $C^{1,\alpha}$ domains to the unit disc.

Theorem 4.2. *Let D be a convex $C^{1,\alpha}$ domain and $f : \partial D \to \mathbb{T}$ an orientation preserving homeomorphism that has quasiconformal extension to D. Then functional $\mathbb{K}$ is minimised in the class of all qc maps with the same boundary condition if and only if*
b1) f is bi-Lipschitz and $H[(f^{-1})'] \in L^\infty(\mathbb{T})$ or
b2) f is bi-Lipschitz and $C[(f^{-1})'] \in L^\infty(\mathbb{T})$

Note that with the second type of condition we can have more general codomains, applying our theory.

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